**Prof. Sabrina Pauli**
**Prof. Timo Richarz**
**Aaron Rauchfuß**

**Summer 2025**

# Algebraic Geometry I

## Exercise Sheet 6

**Exercise 1:**

Let $X$ be a topological space and let $\mathcal{F}$ be the presheaf

$$U \mapsto \{f\colon U \to \mathbb{R} \mid f \text{ continuous and bounded } \}$$

on $X$. Describe the sheafification $\tilde{\mathcal{F}}$ of $\mathcal{F}$.

**Exercise 2:**

Let $X \subseteq \mathbb{C}$ be an open subset.

1. Show that sending $U \subseteq X$ open to $\mathcal{O}_X(U) := \{f\colon U \to \mathbb{C} \text{ holomorphic}\}$ defines a sheaf on $X$.
2. For a holomorphic function $f\colon U \to \mathbb{C}$ let $f'$ be its derivative. Show that $f \mapsto f'$ defines a surjective morphism $D\colon \mathcal{O}_X \to \mathcal{O}_X$ of sheaves. Give an example of an open set $X \subseteq \mathbb{C}$ such that $D\colon \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ is not surjective.

**Exercise 3:**

Given two presheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space $X$, one may form a presheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ by letting the sections over an open set $U$ be given by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathrm{PSh}(X)}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\mathcal{F}|_U$ and $\mathcal{G}|_U$ denote the restrictions of $\mathcal{F}$ and $\mathcal{G}$ to $U$, and $\mathrm{Hom}_{\mathrm{PSh}(X)}$ denotes morphisms of presheaves of sets on $U$. If $V \subset U$, the restriction map sends $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ to the restriction $\varphi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$. Show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf whenever $\mathcal{G}$ is a sheaf.

**Exercise 4:**

Let $X$ be a topological space and let $(U_i)_i$ be an open covering of $X$. For all $i$, let $\mathcal{F}_i$ be a sheaf on $U_i$. Assume that for each pair $i, j$ of indices we are given isomorphisms $\varphi_{ij} : \mathcal{F}_{j|U_i \cap U_j} \to \mathcal{F}_{i|U_i \cap U_j}$ satisfying for all $i, j, k$ the "cocycle condition"

$$\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk} \quad \text{on } U_i \cap U_j \cap U_k.$$

Show that there exists a sheaf $\mathcal{F}$ on $X$ together with isomorphisms $\psi_i : \mathcal{F} \xrightarrow{\sim} \mathcal{F}_{|U_i}$ for all $i$, such that $\psi_i \circ \varphi_{ij} = \psi_j$ on $U_i \cap U_j$ for all $i, j$. Show that $\mathcal{F}$ and the $\psi_i$ are determined up to unique isomorphism by these conditions.

*Remark:* We say that the sheaf $\mathcal{F}$ is obtained by *gluing the* $\mathcal{F}_i$ via the *gluing data* $\varphi_{ij}$, and that the sheaves on $X$ form a *stack*.