Prof. Sabrina Pauli
Prof. Timo Richarz
Aaron Rauchfuß

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Algebraic Geometry I

Exercise Sheet 5

Exercise 1:

Let I be a directed poset. Let $J \subseteq I$ be a *cofinal* subset, i.e., for all $i \in I$ there exists $j \in J$ with $j \geq i$. Give J the induced partial order. Let $(X_i)_{i \in I}$ be a direct system indexed by I (in some category $\mathcal{C}$). By restriction, we then regard $(X_j)_{j \in J}$ as a direct system indexed by J. Prove that the structure maps $X_j \to \varinjlim_{i \in I} X_i$ for $j \in J$ give rise to an isomorphism

$$\varinjlim_{j \in J} X_j \xrightarrow{\sim} \varinjlim_{i \in I} X_i.$$

Exercise 2:

Let X be a topological space and let $\mathcal{B}$ be a basis of the topology of X. Let $\mathrm{Sh}_{\mathcal{B}}(X)$ be the category of sheaves on the basis $\mathcal{B}$ as defined in the lecture. Prove that the functors

$$(-)_{|\mathcal{B}^{\mathrm{op}}} : \mathrm{Sh}(X) \to \mathrm{Sh}_{\mathcal{B}}(X), \quad (\mathcal{F} : \mathrm{Ouv}(X)^{\mathrm{op}} \to \mathrm{Set}) \mapsto (\mathcal{F}_{|\mathcal{B}^{\mathrm{op}}} : \mathcal{B}^{\mathrm{op}} \to \mathrm{Set})$$

and

$$(-)^{e\mathcal{B}} : \mathrm{Sh}_{\mathcal{B}}(X) \to \mathrm{Sh}(X), \quad \mathcal{F} \mapsto (U \mapsto \varprojlim_{V \subseteq U, V \in \mathcal{B}} \mathcal{F}(V))$$

are inverse equivalences of categories. (I.e., the two round-trip composite functors are naturally isomorphic to the identities.) Prove that $\mathcal{F}^{e\mathcal{B}}$ (not a standard notation) is the unique sheaf such that $(\mathcal{F}^{e\mathcal{B}})_{|\mathcal{B}^{\mathrm{op}}}$ is isomorphic to $\mathcal{F}$.

Exercise 3:

Let A be a ring, $X := \mathrm{Spec}(A)$ and let M be an A-module. Show that the assignment

$$D(f) \mapsto M[f^{-1}]$$

defines a sheaf of abelian groups on the basis $\mathcal{B} := \{D(f) \subseteq X \mid f \in A\}$ of X.
Hint: Follow the proof for $M = A$ which was presented in the lecture.

Exercise 4:

Give an example of a topological space X, a surjective map $\mathcal{F} \to \mathcal{G}$ of sheaves on X, and an open $U \subseteq X$ where the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.