Prof. Sabrina Pauli
Prof. Timo Richarz
Aaron Rauchfuß

Summer 2025

Algebraic Geometry I

Exercise Sheet 4

Exercise 1:

Let k be an algebraically closed field.

(1) Determine all open subsets of $\operatorname{Spec} k[T]$ in the Zariski topology.

(2) If k is countable (e.g., $k = \overline{\mathbb{F}_p}$ or $k = \overline{\mathbb{Q}}$), show there exists a homeomorphism $\operatorname{Spec} k[T] \cong \operatorname{Spec} \mathbb{Z}$, but no isomorphism of rings between $k[T]$ and $\mathbb{Z}$.

Exercise 2:

Let $\varphi : A \to B$ be a ring homomorphism, and let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ denote the map attached to φ.

(1) Let $\mathfrak{b} \subseteq B$ be an ideal. Prove that

$$\overline{f(V(\mathfrak{b}))} = V(\varphi^{-1}(\mathfrak{b})),$$

where $\overline{(-)}$ denotes the topological closure.

(2) Assume that φ is surjective. Prove that f induces a homomorphism from $\operatorname{Spec} B$ onto $V(\ker(\varphi))$.

(3) Prove that the image of f is dense in $\operatorname{Spec} A$ if and only if every element of $\ker(\varphi)$ is nilpotent.

Exercise 3:

Let $(I, \leq)$ be a partially ordered set. An *inductive system* of sets indexed by I is a family of sets X_i, $i \in I$, together with maps $\varphi_{ji} : X_i \to X_j$ for all pairs $i, j \in I$ with $i \leq j$, such that

$$\varphi_{ii} = \mathrm{id}_{X_i}, \qquad \varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$$

for all $i \leq j \leq k$ in I. (In other words, this is just a functor from the *category* $(I, \leq)$ to the category of sets, Set.)
A set C together with maps $\psi_i : X_i \to C$ such that $\psi_j \circ \varphi_{ji} = \psi_i$ for all $i \leq j$ (a *cocone* aka *inductive cone* for the system) is called a *colimit* (or *inductive limit*) if it satisfies the following universal property: For every set T together with maps $\xi_i : X_i \to T$ such that $\xi_j \circ \varphi_{ji} = \xi_i$ for all $i \leq j$ (i.e., another cocone), there exists a unique map $\chi : C \to T$ such that $\chi \circ \psi_i = \xi_i$ for all $i \in I$. The colimit is also denoted by $\mathrm{colim}_{i \in I} X_i$ or by $\varinjlim_{i \in I} X_i$. (The maps φ_{ji} and ψ_i are thus usually elided.)

(1) Suppose that I is *directed* (also called *filtered*), i.e., for all $i, j \in I$, there exists $k \in I$ with $i, j \leq k$. Let (X_i, φ_{ji}) be an inductive system of sets, let U be the disjoint union

$$U = \bigsqcup_{i \in I} X_i,$$

and consider the following relation on U: For $x, y \in U$, say $x \in X_i$, $y \in X_j$, we set $x \sim y$ if and only if there exists $k \geq i, j$ with $\varphi_{ki}(x) = \varphi_{kj}(y)$. Prove that $\sim$ is an equivalence relation and that the set $U/\!\sim$ of equivalence classes together with the natural maps $X_i \to U/\!\sim$ is a colimit of the system (X_i, φ_{ji}). (We say that the colimit is a directed/filtered colimit.)

(2) Suppose again that I is directed, and additionally that each X_i is a subset of the set X, that $i \leq j$ if and only if $X_i \subseteq X_j$, and that in this case the map φ_{ji} is the inclusion map. Prove that the union $C := \bigcup_{i \in I} X_i$ with the inclusion maps $X_i \hookrightarrow C$ is a colimit of the X_i.

Exercise 4:

Let I be a partially ordered set.

(1) Define the notion of *colimit* with index set I in the category of abelian groups, Ab.

(2) Suppose that I is directed. Prove that all colimits over I in Ab exist. (In fact, any *functor* $X : I \to \text{Ab}$, where I is a small *category* has a colimit, but you don't need to prove this, though you're welcome to do so.)

(3) Suppose that I is directed. Prove that the functor colim_i is exact on abelian groups, i.e.: Let (A_i, φ_{ji}), (B_i, ψ_{ji}), (C_i, ξ_{ji}) be inductive systems of abelian groups indexed by I. Suppose that for each $i \in I$, we are given short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

that are compatible with the maps $\varphi_{ji}, \psi_{ji}, \xi_{ji}$ in the sense that the obvious squares commute. Prove that these sequences induce a sequence

$$0 \to \text{colim}_i A_i \to \text{colim}_i B_i \to \text{colim}_i C_i \to 0$$

that is again exact.

(4) Give an example that shows that colim_i is not exact on abelian groups, if I is the poset $(1 \geq 0 \leq 2)$ (i.e., 1 and 2 are incomparable).