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Summer 2025

Algebraic Geometry I

Exercise Sheet 3

Let k be an algebraically closed field and denote by $\mathbb{A}^n(k) = k^n$ the affine n-space over k together with its Zariski topology.

Exercise 1:
Let x, y be coordinates on $\mathbb{A}^2(k)$ and $g \in k[T]$ be a polynomial. We denote by $\mathbb{C}$ the field of complex numbers. Consider the following maps:

(1) $f_1 \colon \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C}),\ x \mapsto \exp(x)$

(2) $f_2 \colon \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C}),\ x \mapsto \begin{cases} x+1 & \text{if } x \in \mathbb{Q}[i] \\ x & \text{else} \end{cases}$

(3) $f_3 \colon \mathbb{A}^1(k) \to V(x^3 - y^2),\ x \mapsto (x^2, x^3)$

(4) $f_4 \colon V(g(x) - y) \to \mathbb{A}^1(k),\ (x, y) \mapsto x$

Which maps are continuous for the Zariski topology? Which maps are morphisms of affine algebraic sets, and which are isomorphisms?

Exercise 2:
Let $f \colon X \to Y$ be a map of algebraic sets over k.

(1) Show that f induces a k-linear map $T_x f \colon T_x X \to T_{f(x)} Y$ on tangent spaces. The set $\operatorname{Ram}(f) = \{x \in X \mid T_x f \text{ not injective}\}$ is called the *ramification locus of f*.

(2) Let $f \colon \mathbb{A}^1(k) \to \mathbb{A}^1(k)$ be given by $f(x) = x^n$ for some $n \in \mathbb{N}$. Determine the ramification locus of f.

Exercise 3:
Assume that $\operatorname{char}(k) \neq 2$. Let $X \subset \mathbb{A}^2(k)$ be the plane curve defined by

$$X := \{(x, y) \in \mathbb{A}^2(k) \mid y^2 = g(x)\}$$

for a non-constant polynomial $g \in k[T]$ with pairwise different roots $\lambda_1, \ldots, \lambda_d \in k$ where $d = \deg(g)$.

(1) Show that the composition of the inclusion with the projection on the first coordinate

$$f \colon X \subset \mathbb{A}^2(k) \xrightarrow{(x,y) \mapsto x} \mathbb{A}^1(k)$$

is a surjective map between regular, irreducible curves.

(2) Show that one has

$$\mathrm{Ram}(f) = \{(\lambda_1, 0), \ldots, (\lambda_d, 0)\}$$

for the ramification locus of f defined in Exercise 3.

Hint for (1): Sheet 2, Exercise 4.