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Algebraic Geometry I

Exercise Sheet 2

Let k be an algebraically closed field and denote by $\mathbb{A}^n(k) = k^n$ the affine n-space over k together with its Zariski topology.

Exercise 1:
Let $X \subset \mathbb{A}^n(k)$ be closed. Show the following:

(1) If $X = X_1 \cup \ldots \cup X_r$ is the decomposition into irreducible components, then

$$\dim X = \max_{i=1}^{r}\{\dim X_i\}.$$

(2) Assume X is irreducible. Denote by $k(X)$ the fraction field of the coordinate ring $\mathcal{O}(X)$. Show that $\dim X = \operatorname{trdeg}(k(X)/k)$, i.e., the transcendence degree of $k(X)$ over k.

Exercise 2:
Let $f \in k[T_1, \ldots, T_n]$ be a non-constant polynomial, and denote by $V(f) \subset \mathbb{A}^n(k)$ its vanishing locus. Show that $\dim V(f) = n - 1$.

Exercise 3:
Let $X \subset \mathbb{A}^n(k)$ be a closed subset and $x \in X$ a point. Let $\mathfrak{m}_x \subset \mathcal{O}(X)$ be the maximal ideal of all functions vanishing at x. Consider the following maps between k-vector spaces:

$$\operatorname{Der}(X, x) \underset{\psi}{\overset{\phi}{\leftrightarrows}} \operatorname{Hom}_{k\text{-vs}}(\mathfrak{m}_x/\mathfrak{m}_x^2, k),$$

defined by $\psi(d) = d|_{\mathfrak{m}_x} \mod \mathfrak{m}_x^2$ and $\phi(t)(f) = t(f - f(x) \mod \mathfrak{m}_x^2)$ for all $f \in \mathcal{O}(X)$ respectively. Show that the maps ϕ, ψ are well-defined, k-linear and mutually inverse to each other.

Exercise 4:
Let $X \subset \mathbb{A}^2(k)$ be the plane curve defined by

$$X := \{(x, y) \in \mathbb{A}^2(k) \mid y^2 = g(x)\}$$

for a non-constant polynomial $g \in k[T]$.

(1) Assume that $\operatorname{char}(k) \neq 2$. Show that X is regular if and only if g has only simple roots. Also show that X is irreducible in this case.

(2) Does part (1) still hold for $\operatorname{char}(k) = 2$?

Remark: For different polynomials g, the above curves specialize to examples you may know from other lectures or even from school: *parabolas* and *conics* for $\deg(g) = 1$ and $\deg(g) = 2$ respectively, *elliptic curves* for $\deg(f) = 3$ and so-called *hyperelliptic curves* for $\deg(g) \geq 4$. It is instructive to use your favorite computer algebra system to plot some of the curves for $k = \mathbb{R}$ and particular choices of g. Note that the pictures may not be accurate because $\mathbb{R}$ is not algebraically closed. It is also fun to look at the examples $g = T^2$, $g = T^3$ and $g = T^2(T + 1)$ respectively, which are examples of non-regular (possibly non-irreducible) curves according to part (1).