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Algebraic Geometry I

Exercise Sheet 11

Exercise 1:

1. Let $f_X : X \to S$ and $f_Y : Y \to S$ be morphisms in a category and assume that the fiber product exists. Show that $X \times_S Y$ is unique up to unique isomorphism.

2. Let R be a ring. Show that you have the following R-valued points of the fiber product

$$X \times_S Y(R) = X(R) \times_{S(R)} Y(R)$$

where $X(R) \times_{S(R)} Y(R)$ is the fiber product in the category of sets.

Exercise 2:
Assume K/k is a finite field extension which is Galois. Show that $K \otimes_k K \cong \prod_{g \in \mathrm{Gal}(K/k)} K$. Describe the underlying topological space of $\operatorname{Spec} K \otimes_{\operatorname{Spec} k} \operatorname{Spec} K$.

Exercise 3:
Let X and Y be integral schemes. A rational map $f : X \dashrightarrow Y$ is called *birational* if there is a rational map $g : Y \dashrightarrow X$ that is inverse to f. In this case X and Y are said to be *birational*.

1. Show that being birational is an equivalence relation.

2. Show that if $f : X \dashrightarrow Y$ is birational then it is dominant (i.e. the closure of the image is Y). In particular, f maps the generic point of X to the generic point of Y and we get an induced map of function fields $f^{\#} : K(Y) \to K(X)$.

3. Show that if $f : X \dashrightarrow Y$ is birational then the induced map of function fields is an isomorphism of fields. (For “nice” schemes this is an equivalence, but we have not introduced all the adjectives for nice yet.)

4. Show that the following schemes are birational, for this let k be a field:
   - $\mathbb{A}^2_k$
   - $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$
   - $\mathbb{P}^2_k$

   What is the function field for all three examples isomorphic to?

Exercise 4:
Let M, M', M'', N, P and Q be A-modules. Also let $\mathfrak{a} \subset A$ be an ideal.

1. Show that
$$\mathrm{Hom}_A(M \otimes_A N, P) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)).$$

2. Show that
$$M' \to M \to M'' \to 0$$
is an exact seqeuence of A-modules if and only if
$$0 \to \mathrm{Hom}_A(M'', Q) \to \mathrm{Hom}_A(M, Q) \to \mathrm{Hom}_A(M', Q)$$
is an exact sequence of A-modules for any A-module Q.

3. Conclude that
$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$
is exact. Hint: Consider $Q = \mathrm{Hom}_A(N, P)$.

4. Deduce that $A/\mathfrak{a} \otimes_A N \cong N/\mathfrak{a}N$.

5. Tensoring is not left exact: Let $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$. Show that this does not stay injective after tensoring with the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$.