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Algebraic Geometry I

Exercise Sheet 10

Exercise 1:
Let k be a field. Compute X_{red} for the following schemes.

1. $X = \operatorname{Spec}({}^{k[x]}/_{(x^3)})$
2. $X = \operatorname{Spec}({}^{\mathbb{Z}}/_{18\mathbb{Z}})$
3. $X = V(x^2 - y) \cap V(y) \subset \operatorname{Spec}(k[x,y]) = \mathbb{A}^2_k$
4. $X = V(x^2 - y) \cap V(x^2 + (y-1)^2 - 1) \subset \operatorname{Spec}(k[x,y]) = \mathbb{A}^2_k$

Which of the X and X_{red} are irreducible? Which are integral?

Exercise 2:
Let A be a ring and let $R = A[x_0, x_1, \ldots, x_n]$. We prescribe degrees q_i to the x_i for $i = 0, 1, \ldots, n$. With these degrees R is a graded ring (but with a different grading than in the lecture where all the q_i equal 1). The scheme $\mathbb{P}_A(q_0, q_1, \ldots, q_n) := \operatorname{Proj} R$ is called *weighted projective space*. From now on let $A = k$ be a field.

1. Let $p, q \in \mathbb{Z}_{>0}$ coprime and let $\deg x_0 = p$ and $\deg x_1 = q$. Let $R = k[x_0, x_1]$ be the graded ring with these degrees. Show that $(R_{x_0})_0 = k[\frac{x_1^p}{x_0^q}] \cong k[u]$ and $(R_{x_1})_0 = k[\frac{x_0^q}{x_1^p}] \cong k[u^{-1}]$. Conclude that $\mathbb{P}_k(p,q) \cong \mathbb{P}^1_k$.
2. Let $q_0 = 1$, $q_1 = 1$ and let $q_2 = 2$ and consider $R = k[x_0, x_1, x_2]$ with $\deg x_i = q_i$ for $i = 0, 1, 2$. Show that both $(R_{x_0})_0$ and $(R_{x_1})_0$ are isomorphic to $k[u,v]$ but $(R_{x_2})_0 \cong {}^{k[x,y,z]}/_{(xy-z^2)}$. Indeed one can show that ${}^{k[x,y,z]}/_{(xy-z^2)} \not\cong k[u,v]$ and that $\mathbb{P}_k(1,1,2) \not\cong \mathbb{P}^2_k$.

Exercise 3:
Let k be a field. Let $X = \coprod_{n\geq 1} \operatorname{Spec}({}^{k[x_n]}/_{(x_n^n)})$ and let $U_n = \operatorname{Spec}({}^{k[x_n]}/_{(x_n^n)})$ for $n \geq 1$

1. Show that $\mathcal{O}_X(X) = \prod_{n\geq 1} {}^{k[x_n]}/_{(x_n^n)}$.
2. Let $s = (x_1, x_2, x_3, \ldots) \in \mathcal{O}_X(X)$. Show that the image $(s|_{U_n})_{\text{red}} \in (\mathcal{O}_X(U_i))_{\text{red}}$ of $s|_{U_n} \in \mathcal{O}_X(U_n)$ equals 0.
3. Show that the image $s_{\text{red}} \in \mathcal{O}_X(X)_{\text{red}}$ of s does not equal 0. So in general $U \mapsto \mathcal{O}_X(U)_{\text{red}}$ is not a sheaf.
4. Is X in this example an affine scheme?

Exercise 4:
Let R be a graded ring which is an integral domain and let $X = \operatorname{Proj} R$. Show that

$$K(X) = \{\frac{g}{h} : g, h \in R, \deg g = \deg h\} \subset \operatorname{Frac} R.$$