

Algebraic Geometry I  
Exercise Sheet 1

Let  $k$  be an algebraically closed field and denote by  $\mathbb{A}^n(k) = k^n$  the affine  $n$ -space over  $k$  together with its Zariski topology.

**Exercise 1:**

- (1) Let  $Z \subseteq \mathbb{A}^n(k)$  be a finite set. Prove that  $Z$  is Zariski closed in  $\mathbb{A}^n(k)$ .
- (2) Show that the Zariski topology on  $\mathbb{A}^2(k)$  is not given by the product topology on  $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$ .

**Exercise 2:**

Consider the following algebraic set  $X := V(\mathfrak{a})$  in  $\mathbb{A}^3(k)$  (with coordinates  $x, y, z$ ), where  $\mathfrak{a} := (xz - y^2, x^2 - y)$ . Show that  $X = V(x, y) \cup V(y - x^2, z - x^3)$  is the decomposition into irreducible components.

**Exercise 3:**

Let  $X \subseteq \mathbb{A}^n(k)$  be an algebraic set with

$$I(X) := \{f \in k[T_1, \dots, T_n] \mid f(x_1, \dots, x_n) = 0 \text{ for all } x = (x_1, \dots, x_n) \in X\}.$$

and coordinate ring  $A := \mathcal{O}(X) = k[T_1, \dots, T_n]/I(X)$ . Let  $f \in A$  be any element. Prove that

$$D(f) := \{x = (x_1, \dots, x_n) \in X \mid f(x_1, \dots, x_n) \neq 0\}$$

is open in  $X$ . Show that

$$D(f) \subseteq \mathbb{A}^{n+1}(k), \quad x \mapsto (x_1, \dots, x_n, f(x_1, \dots, x_n)^{-1})$$

is again an algebraic set with coordinate ring  $\mathcal{O}(D(f))$  given by the localisation  $A[f^{-1}]$ .

**Exercise 4:**

Identify the space  $M_{2,2}(k)$  of  $2 \times 2$ -matrices over  $k$  with  $\mathbb{A}^4(k)$  (with coordinates  $a, b, c, d$ ). Define ideals

$$\mathfrak{a} := (a^2 + bc, d^2 + bc, (a + d)b, (a + d)c) \subseteq k[a, b, c, d]$$

$$\mathfrak{b} := (ad - bc, a + d) \subseteq k[a, b, c, d]$$

and denote their vanishing loci by  $V(\mathfrak{a})$  and  $V(\mathfrak{b})$  respectively. Prove that

$$X := V(\mathfrak{a}) = V(\mathfrak{b}) = \{A \in M_{2,2}(k) \mid A \text{ is nilpotent}\}$$

i.e., that a matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is nilpotent if and only if  $A^2 = 0$  if and only if the determinant and the trace of  $A$  are zero. Moreover, prove that  $\text{rad } \mathfrak{a} = \mathfrak{b}$ , but  $\mathfrak{a} \neq \mathfrak{b}$ .

The affine algebraic set  $X$  is called the *nilpotent cone*.