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Algebraic Geometry I

Exercise Sheet 1

Let k be an algebraically closed field and denote by $\mathbb{A}^n(k) = k^n$ the affine n-space over k together with its Zariski topology.

Exercise 1:
(1) Let $Z \subseteq \mathbb{A}^n(k)$ be a finite set. Prove that Z is Zariski closed in $\mathbb{A}^n(k)$.
(2) Show that the Zariski topology on $\mathbb{A}^2(k)$ is not given by the product topology on $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$.

Exercise 2:
Consider the following algebraic set $X := V(\mathfrak{a})$ in $\mathbb{A}^3(k)$ (with coordinates x, y, z), where $\mathfrak{a} := (xz - y^2, x^2 - y)$. Show that $X = V(x, y) \cup V(y - x^2, z - x^3)$ is the decomposition into irreducible components.

Exercise 3:
Let $X \subseteq \mathbb{A}^n(k)$ be an algebraic set with

$$I(X) := \{f \in k[T_1, \dots, T_n] \mid f(x_1, \dots, x_n) = 0 \text{ for all } x = (x_1, \dots, x_n) \in X\}.$$

and coordinate ring $A := \mathcal{O}(X) = k[T_1, \dots, T_n]/I(X)$. Let $f \in A$ be any element. Prove that

$$D(f) := \{x = (x_1, \dots, x_n) \in X \mid f(x_1, \dots, x_n) \neq 0\}$$

is open in X. Show that

$$D(f) \subseteq \mathbb{A}^{n+1}(k), \quad x \mapsto (x_1, \dots, x_n, f(x_1, \dots, x_n)^{-1})$$

is again an algebraic set with coordinate ring $\mathcal{O}(D(f))$ given by the localisation $A[f^{-1}]$.

Exercise 4:
Identify the space $M_{2,2}(k)$ of 2×2-matrices over k with $\mathbb{A}^4(k)$ (with coordinates a, b, c, d). Define ideals

$$\mathfrak{a} := (a^2 + bc, d^2 + bc, (a + d)b, (a + d)c) \subseteq k[a, b, c, d]$$

$$\mathfrak{b} := (ad - bc, a + d) \subseteq k[a, b, c, d]$$

and denote their vanishing loci by $V(\mathfrak{a})$ and $V(\mathfrak{b})$ respectively. Prove that

$$X := V(\mathfrak{a}) = V(\mathfrak{b}) = \{A \in M_{2,2}(k) \mid A \text{ is nilpotent}\}$$

i.e., that a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is nilpotent if and only if $A^2 = 0$ if and only if the determinant and the trace of A are zero. Moreover, prove that $\operatorname{rad} \mathfrak{a} = \mathfrak{b}$, but $\mathfrak{a} \neq \mathfrak{b}$.
The affine algebraic set X is called the *nilpotent cone.*