RECOLLECTIONS ON COMMUTATIVE ALGEBRA

SUMMER 2025

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0. Commutative algebra

The amount of commutative algebra used during the lectures roughly corresponds to [AM69, Chapters 1–3], which some readers might know from past lectures or past seminars in algebra. The author highly recommends reading these chapters. Here we collect some important properties, but do not give full details or references:

§0.1 Spectrum of a ring
§0.2 Radical of ideals
§0.3 Cayley–Hamilton
§0.4 Nakayama's lemma
§0.5 Tensor products
§0.6 Base change

A great source to look up specific definitions, properties and proofs is also the Stacks Project [Sta18] - just google some keywords and add the words "stacks project". Further results from commutative algebra will be discussed during the lectures whenever needed.

0.1. **Spectrum of a ring.** All rings are assumed to be unital and commutative. Let A be a ring. Recall that an ideal $\mathfrak{p} \subset A$ is called *prime* if $A/\mathfrak{p}$ is a domain, i.e., $\mathfrak{p} \neq A$ and if $a, b \in A$ with $ab \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ (or both). An ideal $\mathfrak{m} \subset A$ is called *maximal* if $A/\mathfrak{m}$ is a field ($\implies$ $\mathfrak{m}$ is prime), i.e., $\mathfrak{m} \neq A$ and for any ideal I in A containing $\mathfrak{m}$ one has $I = \mathfrak{m}$ or $I = A$. Recall that every ring $A \neq 0$ has a maximal ideal, and that A is called *local* if it has exactly one maximal ideal.

Definition 0.1. The spectrum of A is the set

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subset A \text{ prime ideal}\}.$$

By the discussion above, $\mathrm{Spec}(A)$ is empty if and only if A is the zero ring.

Exercise 0.2. Show the following statements:

(1) Let $\varphi\colon A \to B$ be a ring homomorphism. Then, taking the preimage $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$ induces a map of sets $\mathrm{Spec}(\varphi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.
(2) Let A be a ring and I an ideal in A. Then, the map $\varphi\colon A \to A/I, a \mapsto a \mod I$ induces an injection

$$\mathrm{Spec}(\varphi)\colon \mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}(A),$$

whose image consists of all prime ideals $\mathfrak{p} \subset A$ that contain I.

For every $\mathfrak{p} \in \mathrm{Spec}(A)$, the localization $A_{\mathfrak{p}} := A[(A \backslash \mathfrak{p})^{-1}]$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. It is called the *local ring of A at $\mathfrak{p}$*.

Exercise 0.3. Show that the map $A \to A_{\mathfrak{p}}, a \mapsto \frac{a}{1}$ induces an injection $\mathrm{Spec}(A_{\mathfrak{p}}) \hookrightarrow \mathrm{Spec}(A)$ with image the prime ideals $\mathfrak{p}' \subset A$ with $\mathfrak{p}' \subset \mathfrak{p}$.

Definition 0.4. The field

$$\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \mathrm{Frac}(A/\mathfrak{p})$$

is called the *residue field of A at $\mathfrak{p}$*.

This allows to establish a (very rough) dictionary

$$\text{(elements of } A) \;\leftrightarrow\; \text{(functions on } \mathrm{Spec}(A)) \tag{0.1}$$

as follows: For an element $f \in A$ and some $\mathfrak{p} \in \mathrm{Spec}(A)$, we denote by

$$f(\mathfrak{p}) := f \mod \mathfrak{p} \in \kappa(\mathfrak{p})$$

the *value of f at $\mathfrak{p}$*. Note that the residue fields $\kappa(\mathfrak{p})$ for varying $\mathfrak{p}$ are not isomorphic, so the definition comes at the expense of allowing the "target of the function" to vary. Making (0.1) precise is the content of Algebraic Geometry, leading to the notion of so-called schemes. Here we only point out the following property:

Exercise 0.5. Let A be a ring and $f \in A$. Show that $f \in A^{\times}$ if and only if $f(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. (Hint: Consider $\mathrm{Spec}(A/fA)$.)

Example 0.6. One has $\mathrm{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime number}\} \cup \{(0)\}$. The localizations at (p) and (0) are $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\}$ and $\mathbb{Q}$ respectively. The residue fields are $\mathbb{F}_p$ and $\mathbb{Q}$ respectively.

Exercise 0.7. Let k be a field and denote by $k[T]$ the polynomial ring in an indeterminate T. Describe the spectrum, the localizations and the residue fields for analogously as in Example 0.6 for $k[T]$. Also, study how this simplifies if k is algebraically closed. (Hint: Use that $k[T]$ is a principal ideal domain.)

For an A-module M and $\mathfrak{p} \in \mathrm{Spec}(A)$, we extend the above notation by defining $M_{\mathfrak{p}} := M[(A \backslash \mathfrak{p})^{-1}]$ to be the localization of M at the multiplicative subset $A \backslash \mathfrak{p}$.

Definition 0.8. The support of an A-module M is the set

$$\mathrm{supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}.$$

Example 0.9. Let $n \in \mathbb{Z}$, $n \neq 0$. Then, we have

$$\mathrm{supp}(\mathbb{Z}/n\mathbb{Z}) = \{(p) \mid p \text{ prime number dividing } n\}.$$

0.2. **Radical of ideals.** Let A be a ring.

Definition 0.10. For an ideal $I \subset A$, the set

$$\sqrt{I} = \{a \in A \mid \exists n \geq 1 : a^n \in I\}$$

is called the *radical of* I. The radical of $I = (0)$ is also called the *Nilradical of* A.

We leave it to the reader to check that $\sqrt{I}$ defines an ideal in A. One always has $I \subset \sqrt{I}$ with equality if and only if 0 is the only nilpotent element in A/I. However, if $I = \mathfrak{p}$ is a prime ideal, then $\sqrt{\mathfrak{p}} = \mathfrak{p}$.

Exercise 0.11. Show the following statements:

(1) Let A be a principal ideal domain and $a \in A$ non-zero. Let $a = u \cdot p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$ with $r \in \mathbb{Z}_{\geq 0}$, $u \in A^\times$, p_i pairwise non-associated prime elements in A and $e_i \in \mathbb{Z}_{\geq 1}$ for $i = 1, \ldots, r$. Then, one has $\sqrt{(a)} = (p_1 \cdot \ldots \cdot p_r)$.
(2) Let A be a Noetherian ring and I an ideal in A. Then, there exists some $m \in \mathbb{Z}_{\geq 1}$ such that $(\sqrt{I})^m \subset I$.

Proposition 0.12. *Let A be a ring and I an ideal in A. Then, one has*

$$\sqrt{I} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}, \tag{0.2}$$

where the intersection runs over all prime ideals $\mathfrak{p} \subset A$ that contain I. In particular, one has

$$\sqrt{0} = \bigcap_{\mathfrak{p}} \mathfrak{p}, \tag{0.3}$$

where the intersection runs over all prime ideals $\mathfrak{p} \subset A$.

Proof. Taking the preimage of subsets along the map $A \to A/I, a \mapsto a \mod I$ induces a bijection between prime ideals in A/I and prime ideals in A that contain I, see Exercise 0.2(2). So, replacing A by A/I it suffices to prove (0.3). We leave the inclusion "$\subset$" to the reader and prove "$\supset$". Let $x \in A$ be not nilpotent. We need to show that there exists a prime ideal $\mathfrak{p} \subset A$ with $x \notin \mathfrak{p}$. Set $\Sigma := \{\mathfrak{a} \subset A \text{ ideal} \mid \forall n \in \mathbb{N} : x^n \notin \mathfrak{a}\}$. Since x is not nilpotent, we have $(0) \in \Sigma$ and so $\Sigma \neq \varnothing$. We define a partial order on Σ by the inclusion of ideals. One checks that every chain has an upper bound given by the set theoretic union (check that this is an ideal). By Zorn's lemma, Σ has a maximal element $\mathfrak{p}$. We claim that $\mathfrak{p}$ is a prime ideal. Let $f, g \in A \backslash \mathfrak{p}$. Then, $(f) + \mathfrak{p}, (g) + \mathfrak{p} \notin \Sigma$ by maximality of $\mathfrak{p}$. So, there exists $m, n \in \mathbb{N}$ with $x^m \in (f) + \mathfrak{p}$ and $x^n \in (g) + \mathfrak{p}$, hence $x^{n+m} \in (fg) + \mathfrak{p}$. This shows that $(fg) + \mathfrak{p} \notin \Sigma$, i.e., $fg \notin \mathfrak{p}$. □

Corollary 0.13. *Let A be a ring and $I \subset A$ be an ideal. Then, the map $A/I \to A/\sqrt{I}, a \mod I \mapsto a \mod \sqrt{I}$ induces a bijection*

$$\operatorname{Spec}(A/\sqrt{I}) \xrightarrow{1:1} \operatorname{Spec}(A/I).$$

Proof. This follows from 0.2 and Exercise 0.2(2). □

0.3. **Cayley–Hamilton.** Let A be a ring. Let $u\colon M \to N$ be a map of A-modules. Assume that M, N are finitely generated. Let $(m_1, \dots, m_r)$ and $(n_1, \dots, n_s)$ be systems of generators for M and N respectively. Then, for all $j = 1, \dots, r$, there exist $t_{1j}, \dots, t_{sj} \in A$ such that

$$u(m_j) = \sum_{i=1}^{s} t_{ij} n_i. \tag{0.4}$$

This defines a matrix $T = (t_{ij}) \in \mathrm{Mat}_{s\times r}(A)$.

Remark 0.14.
(1) The matrix T is not uniquely determined by u, only if $(n_1, \dots, n_s)$ is a basis of N.
(2) Not every matrix in $\mathrm{Mat}_{s\times r}(A)$ defines a linear map u by (0.4), only if $(m_1, \dots, m_r)$ is a basis of M.

Theorem 0.15 (Cayley–Hamilton)**.** *Let M be a finitely generated A-module with generators $(m_1, \dots, m_r)$. Let $u\colon M \to M$ be an A-linear map and $T \in \mathrm{Mat}_{r\times r}(A)$ the matrix of u with respect to $(m_1, \dots, m_r)$. Denote by $\chi_T := \det(XI_r - T) \in A[X]$ the characteristic polynomial of T, and write*

$$\chi_T = X^r + a_1 X^{r-1} + \ldots + a_{r-1} X + a_r.$$

Then, one has

$$\chi_T(u) = u^r + a_1 u^{r-1} + \ldots + a_{r-1} u + a_r I_r = 0 \in \mathrm{End}_A(M).$$

Moreover, if $I \subset A$ is an ideal with $u(M) \subset IM$, then one can choose T such that $a_i \in I^i$ for all $i = 1, \dots, r$.

Remark 0.16. It is also possible to give a proof by reduction to the case where A is a field, see [Sta18, 05G6]. Here we give a direct proof.

Reminder 0.17. Let $r \in \mathbb{N}$, $T \in \mathrm{Mat}_{r\times r}(A)$. Then, there exists $S \in \mathrm{Mat}_{r\times r}(A)$ with

$$ST = TS = \det(T) I_r,$$

where $I_r \in \mathrm{Mat}_{r\times r}(A)$ denotes the identity matrix. Namely, take $S = (s_{ij})$ with $s_{ij} = \det(T_{ji})$ where $T_{ji} \in \mathrm{Mat}_{(r-1)\times(r-1)}(A)$ arises from T by deleting the j-th row and the i-th column. The matrix S is called the *adjoint* of T.

Proof of Theorem 0.15. If $u(M) \subset IM$, then we can choose the entries of T in (0.4) to lie in I. Since a_i is a sum of i-fold products of the entries, it is contained in I^i.

Next, let us write ${}^tT = (t_{ij})$ for the transposed of T. So, we have $u(m_j) = \sum_{i=1}^{r} t_{ji} m_i$ and thus

$$\sum_{i=1}^{r} (u\delta_{ji} - t_{ji}) m_i = 0. \tag{0.5}$$

Consider the matrix $C(X) := (X\delta_{ji} - t_{ji}) = XI_r - {}^tT \in \mathrm{Mat}_{r\times r}(A[X])$. Let $D(X) = (d_{kj}(X))$ be the adjoint of C, hence

$$D(X)C(X) = \chi_T(X) I_r \tag{0.6}$$

using that $\chi_T = \chi_{{}^tT}$. The map $f \mapsto f(u)$ induces a homomorphism of commutative A-algebras

$$A[X] \to A[u] := \{f(u) \in \mathrm{End}_A(M) \mid f \in A[X]\}.$$

Thus, we get $C(u), D(u) \in \mathrm{Mat}_{r\times r}(A[u])$. Multiplying (0.5) with $d_{kj}(u)$ and applying $\sum_j$ gives

$$0 = \sum_i \sum_j d_{kj}(u)(u\delta_{ji} - t_{ji})m_i = \chi_T(u)m_k$$

for all $k = 1, \ldots, r$ by using (0.6) for the second equality. Since the m_k generate M, this shows $\chi_T(u) = 0 \in \mathrm{End}_A(M)$. □

Corollary 0.18. *Let A be a ring and M a finitely generated A-module. Let $I \subset A$ be an ideal such that $M = IM$. Then, there exists some $f \in 1 + I$ with $fM = 0$.*

Proof. Apply Theorem 0.15 to $u = \mathrm{id}_M$ to get $f \cdot \mathrm{id}_M = 0$ with $f := 1 + a_1 + \ldots + a_r$ and $a_i \in I^i \subset I$. This shows $fM = 0$. □

Exercise 0.19. Let A be a ring and M a finitely generated A-module. Let $u\colon M \to M$ be an A-linear endomorphism. Assume that u is surjective. Show that u is an isomorphism. (Hint: Consider M as an $A[X]$-module via $X \cdot m := u(m)$ for all $m \in M$.)

Is every injective endomorphism of a finitely generated module an automorphism?

0.4. **Nakayama's lemma.**

Definition 0.20. Let A be a ring. Then, the ideal

$$\mathrm{Jac}(A) = \bigcap_{\mathfrak{m} \subset A \text{ maximal ideal}} \mathfrak{m}$$

is called the *Jacobson radical of A*.

Proposition 0.21. *Let A be a ring and I be an ideal in A. Then, one has $I \subset \mathrm{Jac}(A)$ if and only if $1 + I \subset A^\times$.*

Proof. First, let $I \subset \mathrm{Jac}(A)$. We argue by contraction. So, assume there exists an $x \in I$ such that $1 + x \notin A^\times$. Then, $A/(1+x) \neq 0$ and there exists a maximal ideal $\mathfrak{m} \subset A$ with $1 + x \in \mathfrak{m}$. Since $x \in \mathrm{Jac}(A) \subset \mathfrak{m}$, it follows $1 \in \mathfrak{m}$ ↯.

Conversely, let $1 + I \subset A^\times$. Assume $I \not\subset \mathrm{Jac}(A)$. Then, there exists $x \in I$ and a maximal ideal $\mathfrak{m}$ with $x \notin \mathfrak{m}$. Thus, $(x) + \mathfrak{m} = A$, i.e., there exists $y \in A$, $v \in \mathfrak{m}$ such that $xy + v = 1$. This implies $1 + (-xy) \in \mathfrak{m}$ and $-xy \in I$, so $1 + I \not\subset A^\times$ ↯. □

Exercise 0.22. Let A be a ring and I an ideal in A with $I \subset \mathrm{Jac}(A)$. Consider the map $\varphi\colon A \to A/I, a \mapsto a \mod I$. Show that an element $a \in A$ is a unit if and only if $\varphi(a)$ is a unit in A/I. Deduce that for a local ring A with maximal ideal $\mathfrak{m}$ one has $A^\times = A \backslash \mathfrak{m}$.

Exercise 0.23. Let $\varphi\colon A \to B$ be a ring map such that the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Then, an element $f \in A$ is a unit if and only if $\varphi(f) \in B$ is a unit. (Hint: Use Exercise 0.23.)

Lemma 0.24 (Nakayama's lemma)**.** *Let A be a ring and $u\colon N \to M$ be a map of A-modules. Let $I \subset A$ be an ideal with $I \subset \mathrm{Jac}(A)$. Assume that M is finitely generated. Then, the map $u\colon N \to M$ is surjective if and only if the induced map*

$$\bar{u}\colon N/IN \to M/IM, \quad n \mod IN \mapsto u(n) \mod IM$$

is surjective.

Proof. If u is surjective, so is $\bar{u}$ as one checks readily (without assuming that M is finitely generated). Conversely, assume that $\bar{u}$ is surjective. Then, one has

$$0 = \operatorname{coker}(\bar{u}) = \operatorname{coker}(u)/I\operatorname{coker}(u),$$

i.e., $\operatorname{coker}(u) = I\operatorname{coker}(u)$. Since M is finitely generated, so is $\operatorname{coker}(u)$. Hence, Corollary 0.18 shows that $f \cdot \operatorname{coker}(u) = 0$ for some $f \in 1 + I$. Since $1 + I \subset A^\times$ by Proposition 0.21, the element f is invertible and we get $\operatorname{coker}(u) = 0$, i.e., u is surjective. □

Exercise 0.25. Let A be a ring and M a finitely generated A-module. Let I be an ideal in A with $I \subset \operatorname{Jac}(A)$. If $M = IM$, then $M = 0$.

Corollary 0.26. *For every finitely generated A-module and prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, one has $M_\mathfrak{p} = 0$ if and only if $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} = 0$.*

Proof. This follows from Exercise 0.25 applied to the finitely generated module $M_\mathfrak{p}$ over the local ring $A_\mathfrak{p}$ and its Jacobson radical $I = \mathfrak{p}A_\mathfrak{p}$. □

0.5. **Tensor products.** Let A be a ring and M, N, P be A-modules. Recall that a map $\beta\colon M \times N \to P$ is called *A-bilinear* if for all $m \in M$, $n \in N$ the maps $\beta(m, -)$ and $\beta(-, n)$ are A-linear.

Definition 0.27. Let M, N be A-modules. A *tensor product of M and N* is an A-module $M \otimes_A N$ together with a A-bilinear map $\tau\colon M \times N \to M \otimes_A N, (m, n) \mapsto m \otimes n$ such that the following universal property holds: For every A-module P and every A-bilinear map $\beta\colon M \times N \to P$ there exists a unique map $\sigma\colon M \otimes_A N \to P$ such that $\beta = \sigma \circ \tau$, i.e., the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[rr, "\beta"] \arrow[d, "\tau"] & & P \\ M \otimes_A N \arrow[urr, dashed, "\exists!\sigma"'] & & \end{tikzcd}$$

Properties 0.28. *For the following basic properties, the reader is referred to* [AM69, Proposition 2.12ff.]*:*

(1) *The pair $(M \otimes_A N, \tau)$ exists and is unique up to unique isomorphism. One puts*

$$M \otimes_A N := \operatorname{Free}_A\{m \otimes n \mid m \in M, n \in N\}/\operatorname{Span}_A\{(3a)\text{–}(3c)\},$$

where $m \otimes n$ are formal symbols, $\operatorname{Free}_A\{-\}$ is the free A-module generated on these symbols and $\operatorname{Span}_A\{-\}$ denotes its submodule generated by the relations (3a)–(3c) *below. The map $\tau\colon M \times N \to M \otimes_A N, (m, n) \mapsto m \otimes n$ is given by $\tau(m, n) = m \otimes n$.*

(2) *If $(m_i)_{i \in I}$ and $(n_j)_{j \in J}$ is a generating system of M and N respectively, then $(m_i \otimes n_j)_{i \in I, j \in J}$ is a generating system of $M \otimes_A N$. Note that an arbitrary element in $M \otimes_A N$ is a finite sum of the form $\sum_{i,j} a_{ij} m_i \otimes n_j$ for some $a_{ij} \in A$.*

(3) *The bilinearity of τ means that for all $m, m' \in M$, $n, n' \in N$ and $a \in A$:*
   (a) $(m + m') \otimes n = m \otimes n + m' \otimes n$
   (b) $m \otimes (n + n') = m \otimes n + m \otimes n'$
   (c) $(am) \otimes n = a(m \otimes n) = m \otimes (an)$

Lemma 0.29. *Let $u\colon M \to M'$ and $v\colon N \to N'$ be maps of A-modules. Then, there exists a unique map of A-modules*

$$u \otimes v\colon M \otimes_A N \to M' \otimes_A N'$$

with $(u \otimes v)(m \otimes n) = u(m) \otimes v(n)$ for all $m \in M$, $n \in N$.

Proof. Consider the following diagram:

$$\begin{tikzcd} M \times N \arrow[rr, "u\times v"] \arrow[d, "\tau"] & & M' \times N' \arrow[d, "\tau'"] \\ M \otimes_A N \arrow[rr, dashed, "\exists! u\otimes v"] & & M' \otimes_A N' \end{tikzcd}$$

Since the composition $\tau \circ (u \times v)$ is A-bilinear, we get the existence of a unique map $u \otimes v$ as indicated. □

Lemma 0.30. *Let M, N, P be A-modules.*

(1) There exists a unique isomorphism

$$(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$$

such that $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$ for all $m \in M$, $n \in N$, $p \in P$.

(2) There exists a unique isomorphism

$$M \otimes_A N \cong N \otimes_A M$$

such that $m \otimes n \mapsto n \otimes m$ for all $m \in M$, $n \in N$.

(3) One has $M \otimes_A A \cong M$ given by $m \otimes a \mapsto am$ for all $m \in M$, $a \in A$.

Proof. (1): This is left to the reader.

(2): We consider the following diagram:

$$\begin{tikzcd} M \times N \arrow[rr, "\varphi\colon (m,n)\mapsto(n,m)", "\cong"'] \arrow[d, "\tau"] & & N \times M \arrow[d, "\tau'"] \\ M \otimes_A N \arrow[rr, dashed, shift left, "\exists!\sigma"] & & N \otimes_A M \arrow[ll, dashed, shift left, "\exists!\rho"] \end{tikzcd}$$

Since $\tau' \circ \varphi$ and $\tau \circ \varphi^{-1}$ are A-bilinear, there exist unique maps σ and ρ respectively. One necessarily has $\rho \circ \sigma = \mathrm{id}$ and $\sigma \circ \rho = \mathrm{id}$.

(3): The inverse map is given by $m \mapsto m \otimes 1$. □

Remark 0.31. The functor $(-) \otimes_A N$ is left adjoint to the functor $\mathrm{Hom}_A(N, -)$, both viewed as endofunctors on the category of A-modules. More precisely, for all A-modules M, N, P, there are bijections

$$\begin{aligned} \mathrm{Hom}_A(M \otimes_A N, P) &\overset{u\mapsto u\circ\tau}{=} \{\beta\colon M \times N \to P \ A\text{-bilinear maps}\} \\ &\overset{\beta\mapsto\left(m\mapsto(n\mapsto\beta(m,n))\right)}{=} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \end{aligned} \tag{0.7}$$

that are functorial in M, N and P. Functorial in N means that a map $v\colon N \to N'$ of A-modules induces a diagram

$$\begin{tikzcd} \mathrm{Hom}_A(M \otimes_A N, P) \arrow[rr, "\cong", "(0.7)"'] & & \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \\ \mathrm{Hom}_A(M \otimes_A N', P) \arrow[rr, "\cong", "(0.7)"'] \arrow[u, "u'\mapsto u'\circ(\mathrm{id}_M\otimes v)"] & & \mathrm{Hom}_A(M, \mathrm{Hom}_A(N', P)) \arrow[u, "w\mapsto(m\mapsto w(m)\circ v)"'] \end{tikzcd}$$

that commutes as one verifies. We leave it to the reader to spell out the functoriality in M and P, which requires writing down similar diagram and checking their commutativity.

Corollary 0.32. *Let N be an A-module. Then, the functor $(-) \otimes_A N \colon \mathrm{Mod}_A \to \mathrm{Mod}_A$ commutes with colimits. In particular, the following hold:*

(1) If $(M_i)_{i \in I}$ is a family of A-modules, then the canonical map

$$\Big(\bigoplus_{i \in I} M_i\Big) \otimes_A N \xrightarrow{\cong} \bigoplus_{i \in I} (M_i \otimes_A N)$$

is an isomorphism. In other words, the functor $(-) \otimes_A N$ commutes with direct sums (=coproducts in Mod_A).

(2) If $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ is an exact sequence of A-modules, then the sequence

$$M' \otimes_A N \xrightarrow{u \otimes \mathrm{id}_N} M \otimes_A N \xrightarrow{v \otimes \mathrm{id}_N} M'' \otimes_A N \longrightarrow 0$$

is exact. In other words, the functor $(-) \otimes_A N$ commutes with finite colimits (and Mod_A is an abelian category).

Proof. This follows from Remark 0.31 because left adjoint functors commute with colimits (and the category of A-modules admits all colimits). $\square$

Example 0.33. For $0 \neq n \in \mathbb{Z}$, we consider the exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Tensoring with $(-) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ induces the sequence

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{n=0} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

In particular, we see that tensoring does not preserve injective maps in general, i.e., if $u \colon M \to M'$ is injective, then $u \otimes \mathrm{id}_N \colon M \otimes_A N \to M' \otimes_A N$ is not injective in general.

Exercise 0.34. Show the following statements:

(1) Let $u \colon M \to M'$, $v \colon N \to N'$ be surjective maps of A-modules. Then, the map $u \otimes v \colon M \otimes_A N \to M' \otimes_A N'$ is surjective with kernel

$$\ker(u \otimes v) = \mathrm{Span}_A\{m \otimes n \mid m \in \ker(u) \text{ or } n \in \ker(v)\},$$

where $\mathrm{Span}_A\{-\}$ denotes the A-submodule generated by $(-)$. Deduce that for an ideal $I \subset A$ one has $M \otimes_A A/I = M/IM$.

(2) Let I, J be ideals in a ring A. Then, there is a canonical isomorphism

$$A/I \otimes_A A/J \cong A/(I + J).$$

Deduce that for $m, n \in \mathbb{Z}$ one has $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ if and only if m, n are prime to each other. Note that this is equivalent to $\mathrm{supp}(\mathbb{Z}/m\mathbb{Z}) \cap \mathrm{supp}(\mathbb{Z}/n\mathbb{Z}) = \varnothing$, see Example 0.9.

Reminder 0.35. Let M be an A-module. Then, M is called

(1) *free* if $M \cong \oplus_{i \in I} A =: A^{(I)}$ for some set I. In this case, the cardinality $\mathrm{rank}_A(M) := \#I$ depends only on M. It is called the *rank of M*.

(2) *projective* if M is a direct summand of a free A-module, i.e., there exists a free A-module E such that $M \oplus N \simeq E$ for some A-module N. Equivalently, for every short exact sequence of A-modules

$$0 \longrightarrow K \xrightarrow{i} N \xrightarrow{p} M \longrightarrow 0$$

there exists $s\colon M \to N$ such that $p \circ s = \mathrm{id}_M$. In this case, $K \oplus M \simeq N, (k, m) \mapsto i(k) + s(m)$.

Exercise 0.36. Let M, N be A-modules. Show the following properties:

(1) If M is free of rank r and N is free of rank s, then $M \otimes_A N$ is free of rank rs.
(2) If M, N are projective, then so is $M \otimes_A N$.
(3) If M, N are finitely generated, then so is $M \otimes_A N$. (Hint: An A-module M is finitely generated if and if there exists a surjection $A^r \to M$ for some $r \in \mathbb{N}$.)

0.6. **Base change.** Let $\rho\colon A \to B$ be a map of rings. We also say that B is a (commutative) *A-algebra with structure map ρ*. If ρ is understood, then we simply say that B is an *A-algebra*. Equivalently, B is an A-module together with an A-bilinear, commutative, unital map $B \times B \to B$.

Remark 0.37. The base change of an module or algebra is defined as follows:

(1) Let M be an A-module. Then, $B \otimes_A M$ becomes a B-module by scalar multiplication on the first factor:
$$B \times (B \otimes_A M) \to B \otimes_A M,$$
$$(b, b' \otimes m) \mapsto bb' \otimes m$$
(2) Let C be an A-algebra. Then, $B \otimes_A C$ becomes a B-algebra with multiplication
$$B \otimes_A C \times B \otimes_A C \to B \otimes_A C,$$
$$(b_1 \otimes c_1, b_2 \otimes c_2) \mapsto b_1 b_1 \otimes c_1 c_2$$
and structure map $B \to B \otimes_A C, b \mapsto b \otimes 1$. Note that the situation is symmetric in B and C, i.e., $B \otimes_A C$ is also a C-algebra.

We call $B \otimes_A M$ and $B \otimes_A C$ the *base change* of the A-module M and the A-algebra C respectively.

Properties 0.38. *Let $\rho\colon A \to B$ be a ring map. The following are important:*

(1) The map
$$B \otimes_A A[T_1, \ldots, T_n] \xrightarrow{\cong} B[T_1, \ldots, T_n]$$
$$b \otimes \sum_{i_1,\ldots,i_n \geq 0} a_{i_1 \ldots i_n} T_1^{i_1} \cdots T_n^{i_n} \mapsto \sum_{i_1,\ldots,i_n \geq 0} b\rho(a_{i_1 \ldots i_n}) T_1^{i_1} \cdots T_n^{i_n}$$
is an isomorphism of B-algebras for all $n \in \mathbb{N}$. An analogous statement holds for polynomial rings in infinitely many variables.

(2) Let I be an ideal in A and consider the projection $A \to A/I, a \mapsto a \mod I$. Then, the map $B = B \otimes_A A \to B \otimes_A A/I$ induces an isomorphism of B-algebras
$$B/IB \cong B \otimes_A A/I,$$
where IB is the ideal in B generated by $\rho(I)$.

Properties 0.38 (1) and (2) allow for a description in the general case: Let C be an A-algebra. Choose generators $(c_\lambda)_{\lambda\in\Lambda}$ of C as an A-algebra. We get a surjective homomorphisms of A-algebra

$$\pi\colon A[(T_\lambda)_{\lambda\in\Lambda}] \to C,\ \ T_\lambda \mapsto c_\lambda.$$

Set $I := \ker(\pi)$, so $C \cong A[(T_\lambda)_{\lambda\in\Lambda}]/I$. Then, we compute:

$$\begin{aligned} B\otimes_A C &\cong \big(B\otimes_A A[(T_\lambda)_{\lambda\in\Lambda}]\big)\otimes_{A[(T_\lambda)_{\lambda\in\Lambda}]} A[(T_\lambda)_{\lambda\in\Lambda}]/I \\ &\overset{(1)}{\cong} B[(T_\lambda)_{\lambda\in\Lambda}]\otimes_{A[(T_\lambda)_{\lambda\in\Lambda}]} A[(T_\lambda)_{\lambda\in\Lambda}]/I \\ &\overset{(2)}{\cong} B[(T_\lambda)_{\lambda\in\Lambda}]/IB[(T_\lambda)_{\lambda\in\Lambda}] \end{aligned}$$

Example 0.39. Let $\rho\colon A\to B$ be a ring map. Let $C := A[T_1,\dots,T_n]/(f_1,\dots,f_r)$ for some $n\in\mathbb{N}$ and $f_1,\dots,f_r\in A[T_1,\dots,T_n]$. Then, we have

$$B\otimes_A C \cong B[T_1,\dots,T_n]/(\rho(f_1),\dots,\rho(f_r)),$$

where for $f=\sum_{i_1,\dots,i_n\geq 0} a_{i_1\dots i_n}T_1^{i_1}\cdots T_n^{i_n}\in A[T_1,\dots,T_n]$ we write

$$\rho(f) := \sum_{i_1,\dots,i_n\geq 0}\rho(a_{i_1\dots i_n})T_1^{i_1}\cdots T_n^{i_n}\in B[T_1,\dots,T_n].$$

As concrete examples, we consider the following special cases:

(1) Let $\rho\colon A := \mathbb{Z}\to\mathbb{F}_p =: B$ and $C := \mathbb{Z}[i]$. Then, $\mathbb{Z}[T]/(T^2+1)\cong\mathbb{Z}[i], T\mapsto i$ induces an isomorphism of $\mathbb{F}_p$-algebras

$$\mathbb{F}_p\otimes_{\mathbb{Z}}\mathbb{Z}[i]\cong\mathbb{F}_p[T]/(T^2+1).$$

(2) Let $\rho\colon A := \mathbb{R}\to\mathbb{C} =: B$ and $C := \mathbb{C} = \mathbb{R}[i] = \mathbb{R}[T]/(T^2+1)$. Then,

$$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} = \mathbb{C}[T]/(T^2+1) = \mathbb{C}[T]/(T+i)\times\mathbb{C}[T]/(T-i)\cong\mathbb{C}\times\mathbb{C},$$

where we use the Chinese remainder theorem for the 2nd identification.

The following lemma gives some permanence properties for the base change of modules:

Lemma 0.40. *Let M be an A-module, B an A-algebra and κ some cardinal. If M is a free of rank κ (respectively finitely generated, respectively projective) A-module, then so is the B-module $B\otimes_A M$.*

Proof. First, assume $M\cong A^{(I)} := \oplus_{i\in I}A$ for some set I with $\#I=\kappa$. Then, $B\otimes_A M = \oplus_{i\in I}(B\otimes_A A) = B^{(I)}$ by Corollary 0.32(1) and Lemma 0.30(3). Hence, $B\otimes_A M$ is free of rank κ.

Next, assume M is finitely generated and pick a surjection $A^r\to M$ for some $r\in\mathbb{N}$. Then, the induced map $B^r = B\otimes_A A^r\to B\otimes_A M$ is surjective as well by Corollary 0.32(2). Hence, $B\otimes_A M$ is a finitely generated B-module.

Finally, assume M is projective and pick some free A-module E with $M\oplus N\cong E$ for some A-module N. Since direct sums commute with tensor products, one gets as A-modules

$$\big(B\otimes_A M\big)\oplus\big(B\otimes_A N\big) = B\otimes_A E$$

which is checked to be B-linear. As $B\otimes_A E$ is a free B-module, we see that $B\otimes_A M$ is projective. ∎

References

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