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Arc descent for constructible adic sheaves

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Abstract

In this thesis, we give a detailed proof that the functor which assigns to a scheme X the derived ∞-category of constructible Λ-sheaves on the pro-étale site of X satisfies arc-descent for certain condensed rings Λ. In particular, we consider the cases where Λ is an algebraic extension of a disconnected local field, its ring of integers, the ring of finite adeles of a global field or the profinite completion of the ring of integers of a global field. Furthermore, we will show that the formation of the category of constructible sheaves is well-behaved under localization in the coefficient ring.

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Chapter 1

Introduction

All schemes are assumed to be qcqs. Throughout this thesis, we will use cohomological notation.

In [CS] the notion of a *condensed set/group/ring/...* was introduced and defined as a sheaf of sets/groups/rings/... on the pro-étale site (c. f. [BS]) of an algebraically closed point. This is equivalently a sheaf on the category of profinite sets with finite jointly surjective maps as covers.

Any T_1-topological space T gives rise to a condensed set via the following construction: We identify T with the sheaf that sends a profinite set S to the set of continuous functions $\mathcal{C}(S,T)$. If T is a topological group/ring/... then its associated condensed set is in fact a condensed group/ring/...

The functor $T \mapsto (S \mapsto \mathcal{C}(S,T))$ is faithful and fully faithful when restricted to the full subcategory of such T which are compactly generated as topological spaces [CS, Proposition 1.7]. Hence, we do not strictly distinguish between the topological rings and their associated condensed rings in the following.

In [HRS], condensed rings were used to define *constructible sheaves* on the pro-étale site of arbitrary qcqs schemes. Let X be a scheme. For any condensed ring Λ, there is a sheaf of rings on $X_{\text{proét}}$, denoted by Λ_X, given by pullback of the canonical map of sites $X_{\text{proét}} \to *_{\text{proét}}$. Denote by $\mathrm{D}(X,\Lambda)$ the derived ∞-category of Λ_X-sheaves. This is a symmetric monoidal closed stable ∞-category which allows the following definition.

Definition 1.1. Let X be a scheme and Λ a condensed ring.

1. A sheaf $M \in \mathrm{D}(X,\Lambda)$ is called *lisse* if it is dualizable (c. f. [Lur17, Chapter 4.6.1]).

2. A sheaf $M \in \mathrm{D}(X,\Lambda)$ is called *constructible* if for any open affine $U \subset X$, there

exists a finite subdivision of U into constructible locally closed subschemes $U_i \subset U$ such that each $M|_{U_i}$ is lisse. That is, M becomes dualizable on a constructible stratification.

The full subcategories of $\mathrm{D}(X, \Lambda)$ of lisse and constructible Λ-sheaves, respectively, are denoted by

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda).$$

For now, let Λ be one of the following topological rings, viewed as a condensed ring:

1. Λ is an algebraic extension of a disconnected local field or its ring of integers.
2. Λ is the profinite completion of the ring of integers of a global field K.
3. Λ is the ring of finite adeles for some global field K.
4. Λ is the ring of adeles for a function field over some finite field.[1]

The goal of this thesis is to provide a detailed proof of the following theorem.

Theorem 1.2. *The functor $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ is an arc-sheaf of ∞-categories.*

The case of coefficients in algebraic extensions of $\mathbb{Q}_\ell$ or its ring of integers was proven in [HS, Theorem 2.2]. However, the proof is rather sketchy, hence our aim is to fill in some of the details. It turns out that the same proof works for algebraic extensions of a disconnected local field as well. With this result one can conclude the case of (finite) adeles, using the work of [HRS], where it was shown that constructible sheaves are well-behaved under sequential limits and filtered colimits [HRS, Proposition 3.19 and Proposition 3.20].

Additionally, we will show that in our cases the constructible sheaves are well-behaved under localization in the following sense.

Theorem 1.3. *Let $\mathcal{O}$ be the ring of integers of Λ in case 1. and the profinite completion of the ring of integers in the cases 3. and 4.. Then, the fully faithful functor*

$$\mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}) \otimes_{\mathcal{O}} \Lambda \to \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$$

is an equivalence.

[1] We use the convention that every global field admits a place at ∞. Hence, we state this case separately, although one could say that case 3. already includes this case.

In Chapter 2, we start by studying the rings of continuous functions $\mathcal{C}(S,E)$, where S is an extremally disconnected profinite set and E is a local ring or its ring of integers. We are interested in this case, because extremally disconnected profinite sets correspond one-to-one to affine qcqs w-contractible pro-étale covers of an algebraically closed point. We will show that every finitely generated ideal in $\mathcal{C}(S,E)$ is principal and additionally projective (Theorem 2.10 and Corollary 2.11) using [Vec83, Theorem 1]. We conclude that the rings $\mathcal{C}(S,\hat{\mathcal{O}}_K), \mathcal{C}(S,\mathbb{A}_K)$ and $\mathcal{C}(S,\mathbb{A}_{K,\text{fin}})$ for a global field K have these properties as well (Corollary 2.15, Theorem 2.17).

Next, we take a look at the derived category of these rings of continuous functions in Chapter 3. Recall that the *perfect complexes* are those elements $A^\bullet \in \mathrm{D}(R)$ for a ring R which are quasi-isomorphic to a bounded complex of finite projective modules. For any multiplicative set $S \subset R$ there is a functor $\mathrm{Perf}_R[S^{-1}] \to \mathrm{Perf}_{S^{-1}R}$ given by $A^\bullet \mapsto A^\bullet \otimes^{\mathbf{L}}_R S^{-1}R$. By $\mathrm{Perf}_R \otimes_R S^{-1}R$ we denote its essential image. Using the fact that finite projective modules over semi-hereditary rings are a finite direct sum of finitely generated ideals [Lam99, Theorem 2.29], we show that for an extremally disconnected profinite set S, a disconnected local field E with ring of integers $\mathcal{O}_E$ and a global field K, the functors

$$\mathrm{Perf}_{\mathcal{C}(S,\mathcal{O}_E)} \otimes_{\mathcal{O}_E} E \xrightarrow{\cong} \mathrm{Perf}_{\mathcal{C}(S,E)}$$

$$\mathrm{Perf}_{\mathcal{C}(S,\hat{\mathcal{O}}_K)} \otimes_{\hat{\mathcal{O}}_K} \mathbb{A}_{K,\text{fin}} \xrightarrow{\cong} \mathrm{Perf}_{\mathcal{C}(S,\mathbb{A}_{K,\text{fin}})}$$

are equivalences. If K is a global field with characteristic $p > 0$, then

$$\mathrm{Perf}_{\mathcal{C}(S,\hat{\mathcal{O}}_K)} \otimes_{\hat{\mathcal{O}}_K} \mathbb{A}_K \xrightarrow{\cong} \mathrm{Perf}_{\mathcal{C}(S,\mathbb{A}_K)}$$

is an equivalence as well (Theorem 3.7).

In Chapter 4, we briefly recall some sheaf theory. We define sites, sheaves, ∞-sheaves and topoi as well as the étale and pro-étale site on a scheme and in particular condensed sets. At last, we state the definition of the arc-topology.

In Chapter 5, we will finally introduce constructible sheaves on a scheme. In this chapter, we will prove Theorem 1.2 and Theorem 1.3 using the equivalence proven in Chapter 3.

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Chapter 2

Rings of continuous functions

In this section we will prove that the ring of continuous functions $\mathcal{C}(S, E)$ for a extremally disconnected profinite set S and a local field E or its ring of integers is *Bézout*, i. e. every finitely generated ideal is principal. Then one can easily show that every finitely generated ideal is even projective. We do not require a Bézout ring to be a domain.

We will follow [Vec83] and in addition show that we can apply the proof to further topological rings, in particular the ℓ-adic integers $\mathbb{Z}_\ell$, the profinite integers $\hat{\mathbb{Z}}$ and the ring of adeles of a global field $\mathbb{A}_K$.

All rings are assumed to be unitary and commutative.

2.1 Functions with values in a local field

We start with some definitions [Vec83, p. 643].

Definition 2.1. Let X be a topological space and E be a Hausdorff topological ring.

1. We say a continuous function $f\colon X \to E$ is *bounded*, if $f(X) \subset E$ is relatively compact. We denote the subring of bounded continuous functions by $\mathcal{C}^*(X, E)$.

2. For $f \in \mathcal{C}(X, E)$ we write $V(f) := f^{-1}(\{0\})$ for its *zeroset*. Its complement is called the *cozeroset of* f and denoted by $D(f)$.

3. A subspace $A \subset X$ is called C^*_E*-embedded* if every bounded continuous function $f\colon A \to E$ can be extended to a continuous function $F\colon X \to E$.

4. The space X is called EF*-space* if all cozerosets are C^*_E-embedded.

We will show that the ring of continuous functions with values in a local field E is Bézout for an EF-space. To apply this theorem to our setting, we have to prove that extremally disconnected profinite sets are EF-spaces. We will use the following theorem [Eng89, Theorem 3.2.1].

Theorem 2.2. *Let A be a dense subspace of a topological space X and f a continuous function over A to a compact Hausdorff space Y. The function f has a continuous extension over X if and only if for every pair B_1 and B_2 of disjoint closed subsets of Y the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in X.*

Proof sketch. The full proof can be found in [Eng89, Theorem 3.2.1].

The necessity of the condition is clear. For sufficiency, let $f\colon A \to Y$ be continuous satisfying the condition in the theorem. For $x \in X$ denote the family $\mathcal{B}(x)$ of all neighbourhoods of x in X and consider the family $\mathcal{F}(x) := (\overline{f(A \cap U)})_{U \in \mathcal{B}(x)}$. The intersection of finitely many elements of $\mathcal{F}(x)$ is non-empty. Since Y is compact Hausdorff, the intersection $\bigcap_{V \in \mathcal{F}(x)} V$ is non-empty [Eng89, Theorem 3.1.1].

Next, one shows that the intersection consists of a single point and defines $F\colon X \to Y$ via $F(x) := \bigcap_{V \in \mathcal{F}(x)} V$. One has $F(x) = f(x)$ for $x \in A$ in this case. One finishes the proof, by proving the continuity of F. □

Corollary 2.3. *Let S be an extremally disconnected, profinite set and E a non-discrete, locally compact Hausdorff ring. Then S is an EF-space.*

Proof. Let $U \subset S$ be a cozeroset and $f : U \to E$ be a bounded continuous function. Since E is Hausdorff, $U \subset S$ is open and since S is extremally disconnected, $\overline{U} \subset S$ is open and closed. Hence, the characteristic function $\chi_{\overline{U}}$ on $\overline{U}$ is continuous. Thus, it suffices to show, that we can extend f to $\tilde{f}\colon \overline{U} \to E$ because then $\tilde{f}\chi_{\overline{U}}$ extends f to the whole space.

Since S is extremally disconnected Hausdorff and $\overline{U} \subset S$ is closed, $\overline{U}$ is extremally disconnected as well. Since f is bounded, $\overline{f(U)} \subset E$ is compact. Without loss of generality assume $U \subset S$ is dense and E is compact. Let $B_1, B_2 \subset E$ be disjoint closed subsets. Since E is compact and Hausdorff, we find disjoint open neighborhoods $B_i \subset V_i$. The preimages $f^{-1}(V_i) = f^{-1}(V_i) \cap U$ are open and disjoint in S. Since S is extremally disconnected, these have disjoint closures in S [Gle58, Lemma 2.2]. By Theorem 2.2, f extends to a function on S. □

Definition 2.4. Let X be a topological space and E be a topological ring with absolute value. Let $I \subset \mathcal{C}(X, E)$ (or $I \subset \mathcal{C}^*(X, E)$) be an ideal and $U \subset X$ a subset. One defines

$$V(I) := \bigcap_{f \in I} V(f)$$
$$I_U := \{f \in I \mid f(U) = 0\}.$$

The ideal I is called

- *universally decomposable*, if for every open partition $X \setminus V(I) = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$ we have a decomposition $I = I_{U_1} \oplus I_{U_2}$. This is equivalent to $I \subset I_{U_1} + I_{U_2}$,
- *absolutely convex*, if for all $f \in \mathcal{C}(X, E), g \in I$ with $|f| \leq |g|$ we have $f \in I$.

We recall that two subsets A, B of a topological space X are called *separated by a function*, if there exists a continuous function $f\colon X \to \mathbb{R}$, such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Correspondingly, we call these subsets *E-separable* for a topological ring E, if there exists a continuous function $f\colon X \to E$, such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Lemma 2.5. *Let X be a topological space and E be a topological Hausdorff domain. A principal ideal $(f) \subset \mathcal{C}(X, E)$ (or $\mathcal{C}^*(X, E)$) is universally decomposable if and only if the members of an open partition $\{U, V\}$ of $D(f)$ are E-separable.*

Proof. " $\Rightarrow$ ": Let $D(f) = U \cup V$ be an open partition. Since (f) is universally decomposable, we have $(f) \subset (f)_U + (f)_V$, i. e. $f = fg + fh$ with $(fg)(U) = 0$ and $(fh)(V) = 0$. We conclude $(fg)_{|V} = f_{|V}$ and hence $g_{|V} = 1$ (since E is an integral domain). Because of $f(x) \neq 0$ for all $x \in U$ we have $g_{|U} = 0$ and g separates U and V.

" $\Leftarrow$ ": Let $D(f) = U \cup V$ be an open partition and $g \in \mathcal{C}(X, E)$ such that $g_{|U} = 0$ and $g_{|V} = 1$. Then we have $f \in (fg) + (f(1 - g))$. Since $(fg)_{|U} = (f)_{|U}$ and $(f(1 - g))_{|V} = (f)_{|V}$ this finishes the proof. □

Next, we define what sort of fields we are interested in. We follow the definition of [Wei95, p. 20].

Definition 2.6. A *local field* is a non-discrete, locally compact topological field.

Remark. In [Wei95, Chapter I, Theorem 5 and Theorem 8] it is proven that every local field is isomorphic (as topological fields) to one of the following:

- the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. These are the connected local fields,

- a finite extension of the p-adic numbers $\mathbb{Q}_p$. These are the disconnected local fields of characteristic 0,
- the field of formal power series over a finite field $\mathbb{F}_q((T))$. These are the disconnected local fields of characteristic p.

In particular, we see that a local field is complete with respect to an absolute value. The absolute value of a disconnected local field is *non-Archimedean.*

Remark. Local fields arise naturally as completions of *global fields*, i. e. finite extensions of $\mathbb{Q}$ or $\mathbb{F}_q(T)$ (c. f. [Neu99, Chapter II: Theorem 4.2 and Proposition 5.2]).

Definition 2.7. The *ring of integers* of a non-Archimedean local field E with normalized absolute value $|\cdot|$ is the ring

$$\mathcal{O} := \{a \in E \mid |a| \leq 1\}.$$

Lemma 2.8. *Let X be a topological space and E a local field or ring of integers in a local field. For every $f, g \in \mathcal{C}(X, E)$ there exist $\varphi, \psi \in \mathcal{C}(X, E)$ such that $V(f\varphi + g\psi) = V(f) \cap V(g)$.*

Proof. The inclusion " $\supset$ " is always true for any φ and ψ, so we only need to consider the other inclusion. If E is connected (i. e. $E = \mathbb{R}$ or $E = \mathbb{C}$), we can choose the complex conjugates $\varphi = \overline{f}$ and $\psi = \overline{g}$.

If E is a disconnected local field, we know that the normalized absolute value of the non-zero elements is given by $|a| = q^{-\nu_p(a)}$ where q is the cardinality of the residue field and ν_p is a discrete valuation. We choose some $\alpha \in E$ with $|\alpha| = q^{-1}$. We set $\varphi = f$ and $\psi = \alpha g$. Because of the multiplicativity of the absolute value, we have

$$\begin{aligned} |(f\varphi)(x)| &= |f(x)^2| = |f(x)|^2 \in \{q^{-2n} \mid n \in \mathbb{Z}\} \cup \{0\}, \\ |(g\psi)(x)| &= |\alpha||g(x)^2| = q^{-1}|g(x)|^2 \in \{q^{-2n-1} \mid n \in \mathbb{Z}\} \cup \{0\} \end{aligned}$$

for all $x \in X$. In particular, $f\varphi + g\psi = 0$ implies

$$\begin{aligned} &(f\varphi)(x) = -(g\psi)(x) \\ \Rightarrow\ &|f(x)|^2 = q^{-1}|g(x)|^2 \end{aligned}$$

for all $x \in X$. We conclude

$$\begin{aligned} |f(x)|^2 &\in \big(\{q^{-2n} \mid n \in \mathbb{Z}\} \cup \{0\}\big) \cap \big(\{q^{-2n-1} \mid n \in \mathbb{Z}\} \cup \{0\}\big) = \{0\}, \\ q^{-1}|g(x)|^2 &\in \big(\{q^{-2n} \mid n \in \mathbb{Z}\} \cup \{0\}\big) \cap \big(\{q^{-2n-1} \mid n \in \mathbb{Z}\} \cup \{0\}\big) = \{0\}. \end{aligned}$$

Since the kernel of $|\cdot|$ is trivial, it follows $f(x) = g(x) = 0$.

If the functions have values in the ring of integers of E, then f and α have values in the ring of integers as well. Hence, the statement is also true for rings of integers. □

The next theorem was proven in [Vec83, Theorem 1] for local fields E, but in fact, we can extend the proof to our setting.

Theorem 2.9. *Let E be a local field or its ring of integers and X an EF-space. Then the following holds:*

1) All ideals in $\mathcal{C}(X, E)$ are absolutely convex.

2) All ideals in $\mathcal{C}(X, E)$ are universally decomposable.

3) Every open partition of any E-cozeroset is E-separable.

4) $\mathcal{C}(X, E)$ is Bézout.

Proof. 1): Let $I \subset \mathcal{C}(X, E)$ be an ideal and $f \in \mathcal{C}(X, E), g \in I$ with $|f| \leq |g|$. First, consider the case where E is a local field. We define $\tilde{h}\colon D(g) \to E$ by $\tilde{h} := f/g$. By assumption, $\tilde{h}$ is bounded and can be extended to some $h \in \mathcal{C}(X, E)$. Then we have $f = g \cdot h$. If E is the ring of integers of a non-Archimedean local field, we can again define $\tilde{h} := f/g$ as function to the fraction field of E. Since $|f/g| \leq 1$ we have even $\tilde{h}\colon D(g) \to E$. We extend $\tilde{h}$ again and get $f = g \cdot h$.

2), 3): Let $I \subset \mathcal{C}(X, E)$ be an ideal, $S \setminus V(I) = U \cup W$ an open partition. Pick $f \in I$ and set

$$f_1(x) := \begin{cases} f(x) & x \in V(I) \cup U \\ 0 & x \in W \end{cases}.$$

This function is continuous and we set $f_2 := f - f_1$. We have $|f_1| \leq |f|$ and $|f_2| \leq |f|$, respectively. By 1) we have $f_1, f_2 \in I$. Additionally, $f_{2|U} = 0$ and $f_{2|W} = f$. This means $f_1 \in I_W, f_2 \in I_U$ and $f = f_1 + f_2 \in I_U + I_W$. With Lemma 2.5 we get 3).

4): Let $f, g \in \mathcal{C}(S, E)$. We will show that (f, g) is principal. The general case follows by induction.

For $E = \mathbb{R}$ or $E = \mathbb{C}$ we use the absolute convexity to get

$$\left.\begin{array}{l} |f| \leq |f| + |g| \\ |g| \leq |f| + |g| \end{array}\right\} \Rightarrow f, g \in (|f| + |g|)$$

and

$$\left.\begin{array}{l}||f|| \leq |f| \\ ||g|| \leq |g|\end{array}\right\} \Rightarrow |f|, |g| \in (f, g)$$

and hence $(f, g) = (|f| + |g|)$.

Now, let E be a disconnected local field or its ring of integers. We set

$$\begin{aligned} U &:= \{x \in S \mid |f(x)| < |g(x)|\} \\ W &:= \{x \in S \mid |f(x)| > |g(x)|\} \\ \Delta &:= \{x \in S \mid |f(x)| = |g(x)|\} \end{aligned}$$

and show that Δ is a cozeroset.

We know that the norm of every non-zero element $a \in E$ is of the form $|a| = q^{-n}$ for some $n \in \mathbb{Z}$. For every $n \in \mathbb{Z}$ we choose a $\beta_n \in E$ such that $|\beta_n| = q^{-n}$. We define

$$\hat{f}(x) := \begin{cases} \beta_n & , \text{for } |f(x)| = q^{-n} \\ 0 & , \text{for } f(x) = 0 \end{cases} .$$

Then $\hat{f} \in \mathcal{C}(S, E)$ and $|\hat{f}| = |f|$. Analogously, we define $\hat{g}$. Then $\Delta = V(\hat{f} - \hat{g})$ and U, W are an open partition of $S \setminus \Delta$. By 3), there exists some $e \in \mathcal{C}(S, E)$ with $e_{|U} = 0$ and $e_{|W} = 1$. We set $h := ef + (1 - e)g$. We have $h_{|V(e)} = g_{|V(e)}$ and $h_{|V(1-e)} = f_{|V(1-e)}$ and conclude

$$\begin{aligned} &|f(x)| = |h(x)| && \text{for } x \in V \\ &|f(x)| < |g(x)| = |h(x)| && \text{for } x \in U, \end{aligned}$$

i. e. $|f| \leq |h|$. Analogously, we get $|g| \leq |h|$. By absolute convexity, we get $f, g \in (h)$. On the other hand, we have $h \in (f, g)$, so all in all $(f, g) = (h)$. □

Remark. In [Vec83] it is shown, that the conditions in the theorem are indeed equivalent and even equivalent to S being an EF-space. But we do not need equivalence for our work, so for us this implication suffices.

We can conclude the following corollaries.

Corollary 2.10. *Let S be an extremally disconnected profinite set and E be a local field or its ring of integers. Then $\mathcal{C}(S, E)$ is a Bézout ring.*

Proof. Corollary 2.3 and Theorem 2.9. □

Corollary 2.11. *Let S, E be as in Corollary 2.10. Then every finitely generated ideal in $\mathcal{C}(S, E)$ is principal and projective.*

Proof. We use a proof from [HRS, p. 21]. By Theorem 2.9 it suffices to prove the statement for principal ideals. Let $f \in \mathcal{C}(S, E)$. Consider the exact sequence

$$0 \to \operatorname{Ann} f \hookrightarrow \mathcal{C}(S, E) \xrightarrow{\cdot f} (f) \to 0. \tag{2.1}$$

Let $U := D(f)$. Then $U \subset S$ is open and $\overline{U}$ is closed and open. Let e be the characteristic function of $S \setminus \overline{U}$, then e is continuous and $ef = 0$, i. e. $e \in \operatorname{Ann} f$. Let $g \in \mathcal{C}(S, E)$ such that $gf = 0$. Then $g_U = 0$ since E is a domain. Since $g^{-1}(0) \subset S$ is closed, we even have $g_{\overline{U}} = 0$ and $g = g \cdot e \in (e)$. Thus, $\operatorname{Ann} f = (e)$.

Since $e^2 = e$, we obtain a retract $\mathcal{C}(S, E) \to \operatorname{Ann} f$ via $g \mapsto g \cdot e$. So the sequence (2.1) splits and (f) is a direct summand of the free module $\mathcal{C}(S, E)$ and hence projective. □

Remark 2.12. We do not need E to be a local field or its ring of integers in the proof above. The proof works for any T_1-domain.

2.2 Functions with values in the adele ring

Now, let K be a *global field*, i. e. a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(T)$ and let $\mathcal{O}_K$ denote its ring of integers. For each place ν, we denote by K_ν the completion of K with respect to ν and for non-Archimedean places $\mathcal{O}_{K\nu}$ its ring of integers, respectively. Recall that the *ring of finite adeles* $\mathbb{A}_{K,\text{fin}}$ is defined as the restricted product of the K_ν with respect to $\mathcal{O}_{K\nu}$ (c. f. [Wei95, Chapter IV]). In other words, we have

$$\mathbb{A}_{K,\text{fin}} := \left\{ (x_\nu)_\nu \in \prod_{\nu < \infty} K_\nu \mid x_\nu \in \mathcal{O}_{K\nu} \text{ for almost all } \nu \right\}$$

as a set, and for the topology we define a basis in the following way: For each finite subset S of non-Archimedean places and each open $U_\nu \subset K_\nu$ with $\nu \in S$ we define

$$\prod_{\nu \in S} U_\nu \times \prod_{\nu \notin S} \mathcal{O}_{K\nu}$$

to be open. Then these sets form a basis of the topology.

The *ring of adeles* $\mathbb{A}_K$ is defined as the product

$$\mathbb{A}_K := \mathbb{A}_{K,\text{fin}} \times \prod_{\nu = \infty} K_\nu$$

equipped with the product topology. One could also define $\mathbb{A}_K$ as the restricted product over all places, since there are only finitely many places at infinity and the restricted product topology does not depend on finitely many factors.

Remark. The topology of $\mathbb{A}_{K,\text{fin}}$ is constructed in such a way that $\prod_{\nu<\infty} \mathcal{O}_{K\nu}$ is open in $\mathbb{A}_{K,\text{fin}}$. In particular, if K is an algebraic number field and $\mathcal{O}_K$ its ring of integers, then the profinite completion

$$\hat{\mathcal{O}}_K \cong \prod_{\nu<\infty} \mathcal{O}_{K\nu}$$

is open and compact. For a function field K over a finite field, we define $\hat{\mathcal{O}}_K$ this way.

Remark. To make the proof of Theorem 2.17 more readable, we use the convention that the absolute value on $\mathbb{F}_p(T)$ given by $|f/g|_\infty := p^{-(\deg f - \deg g)}$ is a place at infinity. This absolute value is the unique absolute value on $\mathbb{F}_p(T)$ with $|T|_\infty > 1$ [Wei95, Chapter II, Theorem 2]. In particular, every global field admits places at infinity.

Nevertheless, one could also define $|\cdot|_\infty$ on $\mathbb{F}_p(T)$ as a finite place. In this case, the statements about the finite adeles (e. g. Lemma 2.16) stay true.

We cite the following theorem.

Theorem 2.13 (Strong Approximation Theorem)**.** *Let K be a global field and ν_0 be any place of K. Then K is dense in the restricted product*

$$\prod_{\nu\neq\nu_0}' K_\nu.$$

In particular, K is dense in $\mathbb{A}_{K,\text{fin}}$.

Proof. [Wei95, Chapter IV, Corollary 2] □

Next, we want to prove that for an extremally disconnected profinite set S the rings $\mathcal{C}(S, \hat{\mathcal{O}}_K)$ and $\mathcal{C}(S, \mathbb{A}_K)$ are also Bézout rings. First, we need a proposition.

Proposition 2.14. *Let $(A_i)_{i\in I}$ be a family of Bézout rings. then the product $\prod_{i\in I} A_i$ is also a Bézout ring.*

Proof. Let $f = (f_i)_i, g = (g_i)_i \in \prod_{i\in I} A_i$ and let (f, g) be the ideal generated by f and g. Since each A_i is Bézout, we can find $h_i \in A_i$ such that $(h_i) = (f_i, g_i)$ in A_i. Set $h := (h_i)_i$. Then $(f, g) = (h)$ in $\prod_{i\in I} A_i$. □

Corollary 2.15. *Let S be an extremally disconnected profinite set. Then $\mathcal{C}(S, \hat{\mathcal{O}}_K)$ is Bézout.*

Proof. By the universal property of the product topology and Theorem 2.9 we get

$$\mathcal{C}(S, \hat{\mathcal{O}}_K) = \mathcal{C}(S, \prod_{\nu<\infty} \mathcal{O}_{K\nu}) = \prod_{\nu<\infty} \mathcal{C}(S, \mathcal{O}_{K\nu}).$$

□

For $\mathbb{A}_K$ we need the following lemma.

Lemma 2.16. *We have*

$$\mathbb{A}_{K,\mathrm{fin}} = \bigcup_{q\in\mathcal{O}_K} q^{-1}\hat{\mathcal{O}}_K.$$

and every compact subset $A \subset \mathbb{A}_{K,\mathrm{fin}}$ is contained in some $q^{-1}\hat{\mathcal{O}}_K$.

Proof. Let $(a_\nu)_\nu \in \mathbb{A}_{K,\mathrm{fin}}$ and let $S := \{\nu_1, \ldots, \nu_m\}$ be the finite set of places, such that $a_{\nu_i} \notin \mathcal{O}_{K_\nu}$ for $i = 1, \ldots, m$. Since $K \subset \mathbb{A}_{K,\mathrm{fin}}$ is dense, there exists a $q \in K$ with

$$\begin{aligned} |q|_{\nu_i} &\leq |a_{\nu_i}|_{\nu_i}^{-1} < 1 && \text{for } i = 1, \ldots, m, \\ |q|_\nu &\leq 1 && \text{for any } \nu \notin S. \end{aligned}$$

In particular, we have $q \in \mathcal{O}_K$. Then we have $|qa_\nu|_\nu \leq 1$ for all places ν, i. e. $q(a_\nu)_\nu \in \hat{\mathcal{O}}_K$ or equivalently, $(a_\nu)_\nu \in q^{-1}\hat{\mathcal{O}}_K$.

Now, let $A \subset \mathbb{A}_{K,\mathrm{fin}}$ be compact. For each $x \in A$ let $q_x \in K$ be such that $x \in q_x^{-1}\hat{\mathcal{O}}_K$. Then

$$A \subset \bigcup_{x\in A} q_x^{-1}\hat{\mathcal{O}}_K$$

is an open cover of A and by compactness, there exists a finite subcover

$$A = \bigcup_{i=1}^{k} q_i^{-1}\hat{\mathcal{O}}_K.$$

Since $|q|_\nu \leq |\tilde{q}|_\nu$ for $\tilde{q}|q$, we have $\tilde{q}^{-1}\hat{\mathcal{O}}_K \subset q^{-1}\hat{\mathcal{O}}_K$ in this case. We define $q := \prod_{i=1}^k q_i$ and conclude $A \subset q^{-1}\hat{\mathcal{O}}_K$. □

Now we can finally prove the Bézout property for $\mathcal{C}(S, \mathbb{A}_K)$.

Theorem 2.17. *Let S be an extremally disconnected profinite set and K be a global field. Then $\mathcal{C}(S, \mathbb{A}_{K,\mathrm{fin}})$ and $\mathcal{C}(S, \mathbb{A}_K)$ are Bézout rings and every finitely generated ideal is projective.*

Proof. By Remark 2.12 every principal ideal is projective, so it suffices to check the Bézout property.

At first, we consider $\mathcal{C}(S, \mathbb{A}_{K,\mathrm{fin}})$, as the Bézout property for $\mathcal{C}(S, \mathbb{A}_K)$ then follows from Proposition 2.14, Theorem 2.9 and

$$\mathcal{C}(S, \mathbb{A}_K) = \mathcal{C}(S, \mathbb{A}_{K,\mathrm{fin}} \times \prod_{\nu=\infty} K_\nu) = \mathcal{C}(S, \mathbb{A}_{K,\mathrm{fin}}) \times \prod_{\nu=\infty} \mathcal{C}(S, K_\nu).$$

Let $f, g \in \mathcal{C}(S, \mathbb{A}_{K,\mathrm{fin}})$. Since S is compact, we know by Lemma 2.16 that there exists some $q \in K$, such that

$$f(S) \cup g(S) \subset q^{-1}\hat{\mathcal{O}}_K.$$

Since $\hat{\mathcal{O}}_K$ and $q^{-1}\hat{\mathcal{O}}_K$ are homeomorphic, we know by Corollary 2.15 that $(f, g) = (h)$ is in $\mathcal{C}(S, q^{-1}\hat{\mathcal{O}}_K)$ and hence $(f, g) = (h)$ in $\mathcal{C}(S, \mathbb{A}_{K,\mathrm{fin}})$. The general case follows by induction. □

Chapter 3

Perfect complexes

Throughout this thesis, we will use cohomological notation.

First, let R be an arbitrary commutative, unital ring. Denote Mod_R the category of R-modules and by $\mathrm{D}(R)$ the derived category of complexes of R-modules equipped with its symmetric monoidal structure $\otimes_R^{\mathbf{L}}$. We start with the following lemma.

Lemma 3.1. *For $A^\bullet \in \mathrm{D}(R)$ the following are equivalent:*

1. *$A^\bullet$ is quasi-isomorphic to a complex of finite length of finite projective R-modules.*

2. *$A^\bullet$ is* compact *in $\mathrm{D}(R)$, i. e. $\mathrm{Hom}_{\mathrm{D}(R)}(A^\bullet, -)$ commutes with direct sums.*

3. *$A^\bullet$ is dualizable.*

Proof. [Sta22, Tag 07LT], [Sta22, Tag 0FNK]. □

Definition 3.2. A complex $A^\bullet \in \mathrm{D}(R)$ is called *perfect*, if it satisfies one of the equivalent properties mentioned in Lemma 3.1. We denote $\mathrm{Perf}_R \subset \mathrm{D}(R)$ the full subcategory of perfect objects.

An R-module M is called *perfect*, if the complex $M[0]$ concentrated in degree 0 is perfect.

Let $S \subset R$ be a multiplicative subset. We consider the elements of S as morphisms of complexes via multiplication. We denote $\mathrm{Perf}_R[S^{-1}]$ the localization of Perf_R with respect to S (c. f. [Sta22, Tag 04VB]). The goal of this chapter is to introduce a fully faithful embedding $\mathrm{Perf}_R[S^{-1}] \to \mathrm{Perf}_{S^{-1}R}$ and to prove, that it is an equivalence for the rings of continuous functions introduced in the previous chapter.

Consider the functor

$$\mathrm{D}(R) \to \mathrm{D}(S^{-1}R) \quad A^\bullet \mapsto A^\bullet \otimes^{\mathbf{L}}_R S^{-1}R.$$

For any complex $A^\bullet \in \mathrm{D}(R)$ the complex $A^\bullet \otimes^{\mathbf{L}}_R S^{-1}R$ is obtained via localizing each A^i with respect to S. Since the localization of a finite projective module is a finite projective module over the localized ring, this functor restricts to a functor

$$\mathrm{Perf}_R \to \mathrm{Perf}_{S^{-1}R}\,.$$

By the universal property of the localization of categories this functor factors through the functor

$$F\colon\ \mathrm{Perf}_R[S^{-1}] \to \mathrm{Perf}_{S^{-1}R}\,. \tag{3.1}$$

More concrete, a complex $A^\bullet \in \mathrm{Perf}_R[S^{-1}]$ can be considered as a complex of $S^{-1}R$-modules via

$$\frac{a}{s}A^\bullet = s^{-1}(a \cdot A^\bullet).$$

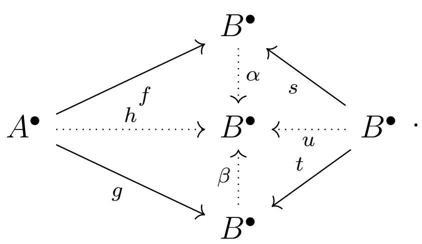
Lemma 3.3. *For $A, B \in \mathrm{Perf}_R$ one has*

$$\mathrm{Hom}_{\mathrm{Perf}_R[S^{-1}]}(A, B) = \mathrm{Hom}_{\mathrm{Perf}_R}(A, B) \otimes_R S^{-1}R.$$

Proof. Since S is a multiplicative system in the categorial sense [Sta22, Tag 04VC], morphisms in $\mathrm{Hom}_{\mathrm{Perf}_R[S^{-1}]}(A, B)$ are given by equivalence classes of pairs of morphisms $f\colon A \to B, s : B \to B$ denoted $s^{-1}f$ ([Sta22, Tag 04VD] and [Sta22, Tag 04VG]). Consider the map

$$\varphi\colon\ \mathrm{Hom}_{\mathrm{Perf}_R[S^{-1}]}(A, B) \to \mathrm{Hom}_{\mathrm{Perf}_{S^{-1}R}}(A, B) \otimes_R S^{-1}R$$
$$s^{-1}f \mapsto f \otimes \frac{1}{s}.$$

To check that this is well-defined, pick another representative $t^{-1}g$ of the morphism. By definition, there exists another morphism $u^{-1}h$ and a commutative diagram



From this we get

$$uf = \alpha \circ sf = s\alpha \circ f = sh$$

and

$$ug = th$$

analogously. Hence, we have

$$f \otimes \frac{1}{s} \cdot = f \otimes \frac{u}{su} \cdot = uf \otimes \frac{1}{su} \cdot = sh \otimes \frac{1}{su} \cdot = h \otimes \frac{1}{u} \cdot$$

and

$$g \otimes \frac{1}{t} \cdot = h \otimes \frac{1}{u} \cdot .$$

To construct the inverse map

$$\psi \colon \operatorname{Hom}_{\operatorname{Perf}_{S^{-1}R}}(A, B) \otimes_R S^{-1}R \to \operatorname{Hom}_{\operatorname{Perf}_R[S^{-1}]}(A, B)$$

note, that the R-linear maps $S^{-1}R \to S^{-1}R$ are precisely the multiplications with elements of $S^{-1}R$. This follows from the universal property of the localization. Hence, we define

$$\psi(f \otimes \frac{a}{s}) := s^{-1}(af).$$

One checks, that φ and ψ are inverse to each other. □

Proposition 3.4. *The functor* $F \colon \operatorname{Perf}_R[S^{-1}] \to \operatorname{Perf}_{S^{-1}R}$ *is fully faithful.*

Proof. Let $P, Q \in \operatorname{Perf}_R$ be perfect complexes. By Lemma 3.3 we have

$$\operatorname{Hom}_{\operatorname{Perf}_R[S^{-1}]}(P, Q) = \operatorname{Hom}_{\operatorname{Perf}_R}(P, Q) \otimes_R S^{-1}R.$$

Let $P^\vee = R\operatorname{Hom}_{\mathrm{D}(R)}(P, R)$, where $R\operatorname{Hom}_{\mathrm{D}(R)}(-,-)$ is the derived hom. By [Sta22, Tag 07VI], we have

$$R\operatorname{Hom}_{\mathrm{D}(R)}(P, Q) = P^\vee \otimes^{\mathbf{L}}_R Q,$$

because P is perfect. Since localization is exact, it commutes with H^n for every $n \in \mathbb{Z}$. In particular,

$$\begin{aligned}
\operatorname{Hom}_{\mathrm{D}(R)}(P, Q) \otimes_R S^{-1}R &= H^0(R\operatorname{Hom}_{\mathrm{D}(R)}(P, Q)) \otimes_R S^{-1}R \\
&= H^0(R\operatorname{Hom}_{\mathrm{D}(R)}(P, Q) \otimes_R S^{-1}R) \\
&= H^0((P^\vee \otimes^{\mathbf{L}}_R Q) \otimes_R S^{-1}R) \\
&= H^0((P^\vee \otimes_R S^{-1}R) \otimes^{\mathbf{L}}_{S^{-1}R} (Q \otimes_R S^{-1}R)) \\
&= H^0((P \otimes_R S^{-1}R)^\vee \otimes^{\mathbf{L}}_{S^{-1}R} (Q \otimes_R S^{-1}R)) \\
&= H^0(R\operatorname{Hom}_{\mathrm{D}(S^{-1}R)}(P \otimes_R S^{-1}R, Q \otimes_R S^{-1}R)) \\
&= \operatorname{Hom}_{\mathrm{D}(S^{-1}R)}(P \otimes_R S^{-1}R, Q \otimes_R S^{-1}R).
\end{aligned}$$

□

Notation. We denote $\mathrm{Perf}_R \otimes_R S^{-1}R$ the essential image of the fully faithful embedding $F\colon \mathrm{Perf}_R[S^{-1}] \to \mathrm{Perf}_{S^{-1}R}$.

The functor F is not always an equivalence, i. e. essentially surjective. To get a feeling for essential surjectivity of this functor, we consider some examples.

Example 3.5.

1. Let R be any ring, $\mathfrak{p} \subset R$ a prime-ideal and $S = R \setminus \mathfrak{p}$. Then $S^{-1}R$ is a local ring and thus any finite projective $S^{-1}R$-module is finite free. This follows from [AM69, Proposition 2.8]. We conclude that F is essentially surjective in this case.

2. Let R be a principal ideal domain and $S \subset R \setminus \{0\}$ any multiplicative subset. Then $S^{-1}R$ is also a principal ideal domain. Thus, all finite projective modules over R or $S^{-1}R$ are finite free, because submodules of free modules over PIDs are again free. In this case, F is again essentially surjective.

3. We can generalize the former example even further. If R is a Dedekind domain and $S \subset R$ a multiplicative subset, then $S^{-1}R$ is again a Dedekind domain. Any finite projective module M over some Dedekind domain is isomorphic to a direct sum $P \oplus Q$, where P is a projective module of rank ≤ 1 and Q is a free module [Ser58, Proposition 7]. Hence, it suffices to check, that the induced map $\mathrm{Pic}(R) \to \mathrm{Pic}(S^{-1}R)$ on the Picard groups is surjective. To check this, recall that the Picard group of a Dedekind domain is canonically isomorphic to its ideal class group. It is easy to check, that every fractional ideal of $S^{-1}R$ comes from a fractional ideal of R.[1] We conclude that the map of ideal class groups induced by the localization is surjective. Hence, F is essentially surjective.

4. ([BS, Paragraph before Definition 1.9]) Let R be the local ring of the nodal cubic curve at the origin over an algebraically closed field k of characteristic 0, i. e.

$$R = \left(k[X,Y]/(Y^2 - X^3 + X^2)\right)_{(x,y)}.$$

Since R is a local ring, the ring of formal power series $R[[T]]$ is again a local ring and all finite projective modules over $R[[T]]$ are free. But the localization

[1] A fractional ideal in R, resp. $S^{-1}R$ is a finitely generated R-submodule, resp. $S^{-1}R$-submodule of the mutual fraction field K. If a finitely generated $S^{-1}R$-submodule $N \subset K$ is generated by $m_1, \dots, m_n \in K$, then $M := Rm_1 + \cdots + Rm_n$ is a fractional ideal of R such that $N \cong M \otimes_R S^{-1}R$.

at $S := \{1, T, T^2, \dots\}$, the ring of formal Laurent series $R((T))$, has a non-trivial Picard group, which means that the functor $\mathrm{Perf}_{R[[T]]}[T^{-1}] \to \mathrm{Perf}_{R((T))}$ is not essentially surjective. To see this, it suffices to check that $H^1_{\text{ét}}(R, \mathbb{Z})$ is non-trivial [BČ22, Equation (1.2.2)]. To see that $H^1_{\text{ét}}(R, \mathbb{Z})$ is non-trivial, let Y be the projective nodal cubic. The normalization of Y is given by $\mathbb{P}^1_k \to Y$ and we can identify Y with $\mathbb{P}^1_k$ where to points are glued together. Now, consider countably many copies $(\mathbb{P}^1_{k,i})_{i\in\mathbb{Z}}$ of $\mathbb{P}^1_k$. We construct a scheme $Y_\infty \to Y$ by glueing the point $0 \in \mathbb{P}^1_{k,i}$ to the point $\infty \in \mathbb{P}^1_{k,i+1}$ for each $i \in \mathbb{Z}$. The resulting scheme $Y_\infty \to Y$ is a non-trivial étale $\mathbb{Z}$-torsor. Base changing this torsor with the localization map $\operatorname{Spec} R \cong \operatorname{Spec} \mathcal{O}_{Y,0} \to Y$ yields again a non-trivial étale $\mathbb{Z}$-torsor and we conclude $H^1_{\text{ét}}(R, \mathbb{Z}) \neq 0$.

Next, we want to prove, that F is essentially surjective in the case where R is one of the rings of continuous functions mentioned in Chapter 2. We start with a lemma.

Lemma 3.6. *Let X be a compact Hausdorff space, R be a compact Hausdorff topological ring, $S \subset R$ a countable subset of non-zero divisors. Equip the localization $S^{-1}R = \operatorname{colim}_{s\in S} R$ with the colimit topology. Then we have*

$$\mathcal{C}(X, R) \otimes_R S^{-1}R = \mathcal{C}(X, S^{-1}R).$$

Proof. This follows from [BS, Lemma 4.3.7]. We will adapt the proof to our case.

Without loss of generality, we can assume S to be of the form $S = \{s_i \mid i \in \mathbb{N}\}$ with $s_i | s_{i+1}$. For each $i \in \mathbb{N}$ we have

$$R[s_i^{-1}] = \operatorname{colim}\left(R \overset{s_i\cdot}{\to} R \overset{s_i\cdot}{\to} \dots\right)$$

equipped with the colimit topology. Since R is compact Hausdorff, $(s_i)^n R \subset R$ is closed for all $n \in \mathbb{N}$ and hence, $R \subset R[s_i^{-1}]$ is closed. Analogously, we get $R[s_i^{-1}] \subset R[s_j^{-1}]$ closed whenever $i < j$. In total, we get a countable tower of closed immersions of Hausdorff spaces with $S^{-1}R = \operatorname{colim}_i R[s_i^{-1}]$ and we are in the setting of [BS, Lemma 4.3.7]. For the sake of completeness, we will cite its proof. For readability, we set $Y_i := R[s_i^{-1}], Y := S^{-1}R$.

We have to show that each $f\colon X \to Y$ factors through some Y_i. Assume, there exists a map $f\colon X \to Y$ with $f(X) \not\subset Y_i$ for all $i \in \mathbb{N}$. After reordering of S we may assume, there exists a sequence $(x_i)_{i\in\mathbb{N}}$ with $x_i \in Y_i \setminus Y_{i-1}$. By the universal property of the Stone-Čech compactification, the map $i \mapsto x_i$ extends to a map $\varphi\colon \beta\mathbb{N} \to X$. By replacing f with $f \circ \varphi$, we may assume $X = \beta\mathbb{N}$.

Pick any $x \in Y \setminus f(\mathbb{N})$. Then x lies in Y_j for some $j \in \mathbb{N}$. For any $i \geq j$, the set $Y_i \cap f(\mathbb{N})$ is finite. Thus, there exists an open neighbourhood $\tilde{U}_i \subset Y_i$ of x such that $\tilde{U}_i \cap f(\mathbb{N}) = \emptyset$. Set $U_j := \tilde{U}_j$ and inductively

$$U_{i+1} := (\tilde{U}_{i+1} \cap (Y_{i+1} \setminus Y_i)) \cup U_i$$

for any $i > j$. Then $U_i \subset Y_i$ is open, $U_i \cap f(\mathbb{N}) = \emptyset$ and $U_{i+1} \cap Y_i = U_i$ for all $i \geq j$. The union $U := \bigcup_i U_i \subset Y$ is an open neighbourhood of x that misses $f(\mathbb{N})$. By density of $\mathbb{N} \subset X$ this means $x \notin \operatorname{im} f$. We conclude $f(X) = f(\mathbb{N})$.

At last, pick any open neighbourhood $x_1 \in U_1 \subset Y_1$. We can extend U_1 inductively in the following way: For $i > 1$ choose open subsets $x_1 \in \tilde{U}_i \subset Y_i$ with $x_i \notin \tilde{U}_i$. Set

$$U_i := (\tilde{U}_i \cap (Y_i \setminus Y_{i-1})) \cup U_i.$$

Then each $U_i \subset Y_i$ is an open neighbourhood of x_1 such that $U_{i+1} \cap Y_i = U_i$ and $x_i \notin U_i$. Since X is compact, $f(\mathbb{N})$ must be finite, which is a contradiction. □

Theorem 3.7. *Let S be an extremally disconnected profinite set.*

1. *Let E be a non-Archimedean local field, $\mathcal{O}_E$ its ring of integers. Then the functor*

$$\mathrm{Perf}_{\mathcal{C}(S,\mathcal{O}_E)} \otimes_{\mathcal{O}_E} E \xrightarrow{\cong} \mathrm{Perf}_{\mathcal{C}(S,E)}$$

   is an equivalence.

2. *Let K be a global field. Set*

$$\hat{\mathcal{O}}_K := \prod_{\nu<\infty} \mathcal{O}_{K_\nu}.$$

   Then the functor

$$\mathrm{Perf}_{\mathcal{C}(S,\hat{\mathcal{O}}_K)} \otimes_{\hat{\mathcal{O}}_K} \mathbb{A}_{K,\mathrm{fin}} \xrightarrow{\cong} \mathrm{Perf}_{\mathcal{C}(S,\mathbb{A}_{K,\mathrm{fin}})}$$

   is an equivalence.

3. *If K is in particular a function field over a finite field and*

$$\hat{\mathcal{O}}_K := \prod_{\nu\leq\infty} \mathcal{O}_{K_\nu},$$

   then the functor

$$\mathrm{Perf}_{\mathcal{C}(S,\hat{\mathcal{O}}_K)} \otimes_{\hat{\mathcal{O}}_K} \mathbb{A}_K \xrightarrow{\cong} \mathrm{Perf}_{\mathcal{C}(S,\mathbb{A}_K)}$$

   is also an equivalence.

Proof. By Proposition 3.4 and Lemma 3.6 it suffices to show essential surjectivity. Consider case 1. Let $A^\bullet$ be a complex of finite length of finite projective $\mathcal{C}(S,E)$-modules. By Corollary 2.11 $\mathcal{C}(S,E)$ is *semihereditary*, i. e. every finitely generated ideal is projective. By [Lam99, Theorem 2.29] each A^i is isomorphic to a finite direct sum of finitely generated ideals

$$A^i \cong \bigoplus_{n=1}^{n_i} I_n^i$$

for some $n_i \geq 1$.

Let $(f_1, \dots, f_m) \subset \mathcal{C}(S,E)$ be a finitely generated ideal. Since S is compact, there exists some $q \in \mathcal{O}_E$ such that $f_1(S) \cup \dots \cup f_m(S) \subset q^{-1}\mathcal{O}_E$. Note that the collection of all tuples (i,n) with $I_n^i \neq 0$ is finite. For each tuple, we choose some $q_{i,n} \in \mathcal{O}_E$ in such a manner and we set $q := \prod_{i,n} q_{i,n}$. For $I_n^i = (f_1^{i,n}, \dots, f_{m_{i,n}}^{i,n})$ we set

$$J_n^i := \mathcal{C}(S,\mathcal{O}_E)qf_1^{i,n} + \dots + \mathcal{C}(S,\mathcal{O}_E)qf_{m_{i,n}}^{i,n}$$

and

$$\tilde{A}^i := \bigoplus_{n=1}^{n_i} J_n^i.$$

This way, we obtain a bounded complex $\tilde{A}^\bullet$ of finite projective $\mathcal{C}(S,\mathcal{O}_E)$-modules, such that

$$\tilde{A}^\bullet \otimes_{\mathcal{O}_E} E \cong A^\bullet.$$

The same proof works for the other cases as well. Therefore use Theorem 2.17 instead of Theorem 2.11 and Lemma 2.16 to get a fitting $q \in \hat{\mathcal{O}}_K$. $\square$

Chapter 4

Sheaf theory

In this chapter, we will briefly recall definitions of sheaves, sites and topoi and introduce the Grothendieck topologies used in Chapter 5.

Definition 4.1. Let $\mathcal{C}$ be a small category.

- A *family of morphisms with fixed target* is given by an object $U \in \mathcal{C}$ and a family of morphisms $\{U_i \to U\}_{i \in I}$.
- A *coverage* on $\mathcal{C}$ is a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target, called *coverings* of $\mathcal{C}$, satisfying the following axioms:
    1. If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
    2. If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{C})$.
    3. If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.
- A *site* is a category $\mathcal{C}$ equipped with a coverage $\mathrm{Cov}(\mathcal{C})$.

Definition 4.2. Let $\mathcal{C}$ be a site and $\mathcal{A}$ be a category that has small limits.

1. A *presheaf* on $\mathcal{C}$ with values in $\mathcal{A}$ is a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. A *morphism of presheaves* is a natural transformation between functors. We denote the category of $\mathcal{A}$-valued presheaves by $\mathrm{PSh}(\mathcal{C}, \mathcal{A})$.
2. A presheaf F is called *sheaf* if for every $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ the diagram
$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$
is an equalizer diagram. A *morphism of sheaves* is a morphism of presheaves. We denote the category of $\mathcal{A}$-valued sheaves by $\mathrm{Sh}(\mathcal{C}, \mathcal{A})$.

3. If $\mathcal{A}$ is an ∞-category, a presheaf F is called an ∞-*sheaf* if F carries finite disjoint unions to finite products and for every $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ we have
$$F(U) \simeq \lim \left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots \right).$$
Another way to write this limit is by using the Čech nerve $Y^{\bullet/X}$:
$$F(U) \simeq \lim_{\Delta} F(Y^{\bullet/X}).$$

Definition 4.3. Let $\mathrm{Sch}^{\mathrm{op}}_{\mathrm{qcqs},R}$ be the category of qcqs schemes over some ring R equipped with some coverage. Suppose $\mathcal{A}$ is an ∞-category that has filtered colimits. Let $F\colon \mathrm{Sch}^{\mathrm{op}}_{\mathrm{qcqs},R} \to A$ be an ∞-sheaf. We call F *finitary* if whenever $\{Y_i\}_{i\in I}$ is a tower of qcqs R-schemes (indexed over some cofiltered partially ordered set I) with affine transition maps, then we have
$$\varinjlim_I F(Y_i) \xrightarrow{\simeq} F(\varprojlim_I Y_i).$$

In order to define a morphism of sites, we need to consider the following:

Definition 4.4. Let $\mathcal{C}, \mathcal{D}$ be sites. A functor $u\colon \mathcal{C} \to \mathcal{D}$ is called *continuous* if

1. for every $\{V_i \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{D})$ we have $\{u(V_i) \to u(V)\} \in \mathrm{Cov}(\mathcal{C})$ and
2. for every $\{V_i \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{D})$ and for every morphism $T \to V$ in $\mathcal{D}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

Remark. Suppose we are given two sites $\mathcal{C}, \mathcal{D}$ and a continuous functor $u\colon \mathcal{D} \to \mathcal{C}$. This functor induces a functor
$$\begin{aligned} u^p\colon \mathrm{PSh}(\mathcal{C}, \mathcal{A}) &\to \mathrm{PSh}(\mathcal{D}, \mathcal{A}) \\ \mathcal{F} &\mapsto \mathcal{F} \circ u. \end{aligned}$$
This functor admits a left adjoint u_p [Sta22, Tag 00VC]. We denote u^s the restriction of u^p to the subcategory of sheaves of $\mathcal{C}$. Since u is continuous, $u^s(\mathcal{F})$ is a sheaf whenever $\mathcal{F}$ is a sheaf. This functor admits a left adjoint as well, denoted u_s and given by $u_s : \mathcal{G} \mapsto (u_p\mathcal{G})^{\#}$, i. e. it sends $\mathcal{G}$ to the sheafification of $u_p\mathcal{G}$ [Sta22, Tag 00WU].

Definition 4.5. A *morphism of sites* $f\colon \mathcal{C} \to \mathcal{D}$ is given by a continuous functor $u\colon \mathcal{D} \to \mathcal{C}$ (in the other direction) such that the functor $u_s\colon \mathrm{Sh}(\mathcal{D}, \mathrm{Set}) \to \mathrm{Sh}(\mathcal{C}, \mathrm{Set})$ is exact. We write $f^{-1} := u_s$ and $f_* := u^s$.

Definition 4.6. A *topos* is a category which is equivalent to the category of sheaves of sets on a site. A *morphism of topoi* $u\colon \mathcal{C} \to \mathcal{D}$ is a pair of adjoint functors $(u^*\colon \mathcal{D} \to \mathcal{C}, u_*\colon \mathcal{C} \to \mathcal{D})$, u^* left adjoint, such that u^* commutes with finite limits.

Remark 4.7. Given a site $\mathcal{C}$, one can lift the topology to the presheaf category $\mathrm{PSh}(\mathcal{C})$ the following way. First of all, we equip $\mathrm{Sh}(\mathcal{C})$ with the canonical topology [GVA72, Exp. II, Definition 2.5]. This topology is characterized by the fact that sheaves on $\mathrm{Sh}(\mathcal{C})$ are exactly the representable sheaves. Hence, we have a canonical isomorphism of topoi [Mak77, Lemma 1.3.14]

$$\mathrm{Sh}(\mathcal{C}) \simeq \mathrm{Sh}_{\mathrm{can}}(\mathrm{Sh}(\mathcal{C})).$$

A map of sheaves $F \to G$ is a cover if and only if $F \to G$ is an epimorphism. A family of morphisms $\{F_i \to G\}_{i\in I}$ is called a cover if $\coprod_i F_i \to G$ is an epimorphism.

Next, we declare a map of presheaves $F \to G$ to be a cover if and only if the induced map of sheaves $F^{\#} \to G^{\#}$ is a cover. This gives $\mathrm{PSh}(\mathcal{C})$ the finest subcanonical topology such that covers in $\mathcal{C}$ give rise to covers in $\mathrm{PSh}(\mathcal{C})$ [GVA72, Exp. II, §5].

A similar result holds for ∞-topoi [Lur09, Section 6.2.4].

The following sites will be of particular interest for us.

Definition 4.8. Fix a scheme X. An *étale cover* of X is a family of étale morphisms $\{f_i\colon U_i \to X\}_{i\in I}$ such that $X = \bigcup_{i\in I} f_i(U_i)$.

The *étale site* of X, denoted $X_{\text{ét}}$, consists of the subcategory of Sch/S consisting of étale morphisms $U \to X$ equipped with étale covers. We denote $X_{\text{ét}}^{\mathrm{aff}}$ the full subcategory of affine schemes in $X_{\text{ét}}$.

Definition 4.9. Let X, Y be schemes. A map $f\colon Y \to X$ is called *weakly étale* if f is flat and the diagonal map $\Delta_f\colon Y \to Y \times_X Y$ is flat. Denote $X_{\text{proét}}$ the category of weakly étale X-schemes. We equip $X_{\text{proét}}$ with a coverage by declaring the coverings to be exactly the coverings in the fpqc topology, i. e. a family $\{Y_i \to Y\}_{i\in I}$ of maps in $X_{\text{proét}}$ is covering family if any $U \subset Y$ is mapped onto by an open affine in $\bigsqcup_{i\in I} U_i$. We call $X_{\text{proét}}$ the *pro-étale site* of X.

Remark. The pro-étale site of a scheme is introduced and studied in [BS]. In this thesis, we will ignore the set-theoretic issues. To avoid those issues, one could fix an uncountable strong limit cardinal κ with $|X| < \kappa$ and restrict $X_{\text{proét}}$ to those $Y \to X$ with $|Y| < \kappa$ (c. f. [BS, Remark 4.1.2]).

Remark. Since any étale covering is also a weakly étale covering [Sta22, Tag 022C] and any étale map is weakly étale, we obtain a morphism of topoi

$$\nu\colon \operatorname{Sh}(X_{\text{proét}}) \to \operatorname{Sh}(X_{\text{ét}})$$

induced by the canonical morphism of sites.

Definition 4.10. Fix a scheme X. A scheme $U \in X_{\text{proét}}$ is called *pro-étale affine* if we can write $U = \varprojlim_i U_i$ with $U_i \in X_{\text{ét}}^{\text{aff}}$ for a small cofiltered diagram. The full subcategory of $X_{\text{proét}}$ spanned by pro-étale affines is denoted $X_{\text{proét}}^{\text{aff}}$

Example 4.11. For an algebraically closed field k we can describe $(\operatorname{Spec} k)_{\text{proét}}^{\text{aff}}$ explicitly. A map of schemes $X \to \operatorname{Spec} k$ is étale if and only if $X \cong \coprod_S \operatorname{Spec} k$ for some set S [Sta22, Tag 02GL]. For X affine, this set S is finite. Hence, we obtain an equivalence of categories via

$$\begin{aligned}(\operatorname{Spec} k)_{\text{ét}}^{\text{aff}} &\xrightarrow{\cong} \text{FinSet}\\ X &\mapsto X(k)\\ \coprod_S \operatorname{Spec} k &\leftarrow\!\shortmid S.\end{aligned}$$

With the embedding of the étale site into the pro-étale site, we obtain a functor $\text{FinSet} \to (\operatorname{Spec} k)_{\text{proét}}^{\text{aff}}$. By [BS, Lemma 4.1.8], $(\operatorname{Spec} k)_{\text{proét}}^{\text{aff}}$ has all small limits, thus we get a functor $\text{Pro}(\text{FinSet}) \to (\operatorname{Spec} k)_{\text{proét}}^{\text{aff}}$ via right Kan extension. By definition of $(\operatorname{Spec} k)_{\text{proét}}^{\text{aff}}$, this functor is an equivalence. Via this identification a covering in $(\operatorname{Spec} k)_{\text{proét}}^{\text{aff}}$ corresponds to a finite jointly surjective family of maps of profinite sets. By [BS, Remark 4.2.5], the topos $\operatorname{Sh}((\operatorname{Spec} k)_{\text{proét}})$ is equivalent to $\operatorname{Sh}((\operatorname{Spec} k)_{\text{proét}}^{\text{aff}})$. Finally, we obtain the definition of condensed sets introduced in [CS].

Definition 4.12. A *condensed set (group, ring etc.)* is a sheaf of sets (groups, rings etc.) on the pro-étale site of an algebraically closed point $*_{\text{proét}}$. This is a functor

$$T : \text{Pro}(\text{FinSet}) \to \text{Sets} / \text{Groups} / \text{Rings} \ldots$$

satisfying $T(\emptyset) = *$ and the two following conditions (which are equivalent to the sheaf condition):

1. For any profinite sets S_1, S_2 the canonical map

   $$T(S_1 \sqcup S_2) \to T(S_1) \times T(S_2)$$

   is a bijection.

2. For any surjection $S' \to S$ of profinite sets with fiber product $S' \times_S S'$ with projections p_1, p_2 the map
$$T(S) \to \{x \in T(S') \mid p_1^*(x) = p_2^*(x) \in T(S' \times_S S')\}$$
is a bijection.

As before, this definition provides set theoretic issues. These can be solved by only considering profinite sets with cardinality less than a fixed uncountable strong limit cardinal. Afterwards, one takes a colimit over the poset of all uncountable strong limit cardinals (c. f. [CS, Appendix to Lecture II]).

Remark. For any scheme $Y \in X_{\text{proét}}$ and any profinite set S we can define a scheme $Y \times S \in X_{\text{proét}}$ as follows (c. f. [BS, Example 4.1.9]): Let $S = \varprojlim_i S_i$. For each S_i consider the constant sheaf with value S_i. In the Zariski topology, this sheaf is the sheaf of locally constant functions $T \to S_i$ with S_i viewed as a discrete space [Sta22, Tag 03P5]. This Zariski sheaf is representable by $\coprod_{S_i} \operatorname{Spec} \mathbb{Z}$. As any representable presheaf satisfies the sheaf condition for the fpqc topology [Sta22, Tag 03O3] and any étale covering is a fpqc covering [Sta22, Tag 03PH], we conclude that $\coprod_{S_i} \operatorname{Spec} \mathbb{Z}$ represents the (étale) constant sheaf with value S_i. Via base change we obtain $\underline{S_i} := \coprod_{S_i} X \to X$. Then we set $\underline{S} := \varprojlim_i \underline{S_i}$ and finally $Y \times S := Y \times_X \underline{S}$.

Definition 4.13. For any scheme X there is a morphism of sites denoted
$$p_X \colon X_{\text{proét}} \to *_{\text{proét}}$$
given by the limit preserving functor which sends a profinite set $S = \lim_i S_i$ to $\lim_i X \times S_i$. For a condensed ring Λ we denote $\Lambda_X := p_X^{-1}\Lambda$.

The condensed rings we are interested in arise as follows (c. f. [HRS, Example 3.1]):

Example 4.14. Any topological T_1-ring Λ induces a sheaf of rings on $*_{\text{proét}}$ via
$$S \mapsto \mathcal{C}(S, \Lambda).$$
By abuse of notation we donote this condensed ring Λ.

Let X be a scheme and $p_X \colon X_{\text{proét}} \to *_{\text{proét}}$ be the morphism of sites mentioned in Definition 4.13. Let $U \in X_{\text{proét}}$ and $S = \varprojlim_i S_i$ be a profinite set. We have
$$\begin{aligned}\operatorname{Hom}_{X_{\text{proét}}}(U, X \times S) &= \operatorname{Hom}_{X_{\text{proét}}}(U, \varprojlim_i (X \times S_i)) = \varprojlim_i \operatorname{Hom}_{X_{\text{proét}}}(U, X \times S_i) \\ &= \varprojlim_i \operatorname{Hom}_{X_{\text{proét}}}(U, \coprod_{S_i} X) = \varprojlim_i \mathcal{C}_X(U, \coprod_{S_i} X) \\ &= \varprojlim_i \mathcal{C}(U, S_i) = \mathcal{C}(U, S) = \mathcal{C}(\pi_0(U), S)\end{aligned}$$

because every continuous function $U \to S$ factors uniquely through $\pi_0(U)$ since S is totally disconnected. This calculation shows, that the index category used to calculate the inverse image presheaf [Sta22, Tag 00VC] has the final object U. We conclude that Λ_X is the sheafification of the presheaf given by $U \mapsto \mathcal{C}(\pi_0(U), \Lambda)$ (for $U \in X_{\text{proét}}$). For Λ totally disconnected, this presheaf is already a sheaf by [BS, Lemma 4.2.12].

The last topology relevant for this thesis is the *arc-topology*, which was introduced in [BM21].

Definition 4.15. 1. An *extension* of valuation rings is a faithfully flat map of valuation rings $V \to W$ (equivalently, an injective local homomorphism).

2. A map of qcqs schemes $Y \to X$ is an *arc-cover* if for any valuation ring V with $\operatorname{rank} V \leq 1$ and map $\operatorname{Spec} V \to X$, there is an extension of rank ≤ 1 valuation rings $V \to W$ and a map $\operatorname{Spec} W \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec} W & \dashrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} V & \longrightarrow & X \end{array}$$

3. The *arc-topology* on the category of schemes is the Grothendieck topology where the covering families $\{f_i \colon Y_i \to X\}_i$ are those families with the following property: for any affine open $V \subset X$, there exist finitely many indices $i_1, \ldots, i_m$ and affine opens $U_{i_j} \subset f_{i_j}^{-1}(V)$ such that the induced map $\bigsqcup_{j=1}^m U_{i_j} \to V$ is an arc-cover in the sense of (2).

Chapter 5

Arc descent for constructible adic sheaves

Recall that all schemes are assumed qcqs. We use the notation of [HRS, Section 3.1] and recall some of the facts given there. For any scheme X let $\mathrm{D}(X,\Lambda)$ denote the derived ∞-category of the abelian category of sheaves of Λ_X-modules on $X_{\text{proét}}$ [Lur17, Definition 1.3.5.8 and Remark 1.3.5.10]. This ∞-category is stable and comes with a monoidal structure denoted by $-\otimes_{\Lambda_X}-$ ([Lur17, §4.8.1]) and inner homomorphisms denoted by $\underline{\mathrm{Hom}}_{\Lambda_X}(-,-)$ such that

$$R\operatorname{Hom}_{\Lambda_X}(-,-) = R\operatorname{Hom}_{\mathrm{D}(X,\Lambda)}(-,-) = R\Gamma(X, \underline{\mathrm{Hom}}_{\Lambda_X}(-,-)).$$

After passing to the homotopy category, the tensor product becomes the classical derived tensor product.

Let $f\colon Y \to X$ be a morphism of schemes. Then $\Lambda_Y = f^{-1}\Lambda_X$ and the ordinary pullback and pushforward of sheaves induce an adjunction

$$f^* = f^{-1}\colon\ \mathrm{D}(X,\Lambda) \leftrightarrows \mathrm{D}(Y,\Lambda) : f_*,$$

with f^* exact and symmetric monoidal. For a morphism of condensed rings $\Lambda \to \Lambda'$, the forgetful functor $\mathrm{D}(X,\Lambda') \to \mathrm{D}(X,\Lambda)$ admits a symmetric monoidal left adjoint

$$\mathrm{D}(X,\Lambda) \to \mathrm{D}(X,\Lambda'), \quad M \mapsto M \otimes_{\Lambda_X} \Lambda'_X. \tag{5.1}$$

As before, we denote the essential image of the functor (5.1) by $\mathrm{D}(X,\Lambda) \otimes_{\Lambda_X} \Lambda'_X$.

Definition 5.1. 1. A subset of a qcqs topological space is called *constructible* if it is a finite Boolean combination of quasi-compact open subsets. That is, it can be written as a finite union of subsets of the form $U \cap V^{\mathsf{c}}$ for quasi-compact open U, V.

2. A *stratification* of a topological space X is a decomposition $X = \coprod_{i \in I} X_i$ into locally closed subsets $X_i \subset X$.

Definition 5.2. Let X be a scheme and Λ a condensed ring.

1. A sheaf $M \in \mathrm{D}(X, \Lambda)$ is called *lisse* if it is dualizable (c. f. [Lur17, Chapter 4.6.1]).

2. A sheaf $M \in \mathrm{D}(X, \Lambda)$ is called *constructible* if, for any open affine $U \subset X$, there exists a finite subdivision of U into constructible locally closed subschemes $U_i \subset U$ such that each $M|_{U_i}$ is lisse. That is, M becomes dualizable on a constructible stratification.

The full subcategories of $\mathrm{D}(X, \Lambda)$ of lisse, resp. constructible Λ-sheaves are denoted by

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda).$$

Remark. The subcategories $\mathrm{D}_{\mathrm{lis}}(X, \Lambda), \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ inherit the symmetric monoidal structure from $\mathrm{D}(X, \Lambda)$.

For any map of schemes $f\colon Y \to X$, the pullback f^* preserves lisse and constructible sheaves. For lisse sheaves, this follows immediately from f^* being symmetric monoidal. For a constructible sheaf M, one covers both $X = \bigcup_i U_i$ and $Y = \bigcup_i V_i$ by finitely many (not necessarily distinct) affine opens, such that $f(V_i) \subset U_i$. In this case, $f|_{V_i}$ is continuous in the constructible topology [Sta22, Tag 0A2S] and the preimage of the constructible stratification of U_i such that M becomes dualizable is again a constructible stratification.

Also, for a map of condensed rings $\Lambda \to \Lambda'$, the functor $(-) \otimes_{\Lambda_X} \Lambda'_X$ preserves lisse, resp. constructible sheaves.

We cite the following results for later use.

Lemma 5.3. *In the ∞-category Cat_n of small $(n, 1)$-categories, filtered colimits commute with Δ-indexed limits, where Δ is the simplex category.*

Proof. [BM21, Example 3.6, Lemma 3.7]. □

Lemma 5.4. *Let X be a scheme and Λ a condensed ring. Then one has*

$$\mathrm{D}_{\mathrm{cons}}(X, \Lambda) = \bigcup_{a \leq b} \mathrm{D}^{[a,b]}_{\mathrm{cons}}(X, \Lambda).$$

Proof. [HRS, Lemma 3.16, Corollary 3.17]. □

The following lemma allows us to simplify the task of testing the descent condition for constructible sheaves.

Lemma 5.5. *Let X be a scheme and $\Lambda = \mathrm{colim}_i \Lambda_i$ a filtered colimit of condensed rings. If the functor $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Lambda_i)$ is an arc sheaf for all i, then $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ is an arc sheaf as well.*

Proof. By [HRS, Proposition 3.20], the natural functor

$$\mathrm{colim}_i \, \mathrm{D}_{\mathrm{cons}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$$

is an equivalence. Assume that all the categories with Λ_i-coefficients satisfy arc descent. For all $a, b \in \mathbb{Z}, a \leq b$, the full subcategory $\mathrm{D}^{[a,b]}_{\mathrm{cons}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ is a $(b-a+1)$-category. Since the colimit is filtered, we conclude that for any arc-cover $Y \to X$

$$\begin{aligned} \mathrm{D}^{[a,b]}_{\mathrm{cons}}(X, \Lambda) &= \mathrm{colim}_i \, \mathrm{D}^{[a,b]}_{\mathrm{cons}}(X, \Lambda_i) \\ &= \mathrm{colim}_i \lim_{\Delta} \mathrm{D}^{[a,b]}_{\mathrm{cons}}(Y^{\bullet/X}, \Lambda_i) \\ &= \lim_{\Delta} \mathrm{colim}_i \, \mathrm{D}^{[a,b]}_{\mathrm{cons}}(Y^{\bullet/X}, \Lambda_i) \\ &= \lim_{\Delta} \mathrm{D}^{[a,b]}_{\mathrm{cons}}(Y^{\bullet/X}, \Lambda) \end{aligned}$$

by Lemma 5.3. By Lemma 5.4, we are done. □

5.1 Coefficients in disconnected local fields

For this section, ℓ will be a fixed prime number. Recall, that we do not strictly distinguish between a topological T_1-ring and its associated condensed ring. The goal of this section is to show that, in the case of Λ being a local field or its ring of integers or the ring of finite adeles of a global field, the functor $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ is an arc-sheaf of ∞-categories. The case of L-coefficients with L an algebraic extension of $\mathbb{Q}_\ell$ is proven in [HS, Theorem 2.2], here we add some details. First, we cite the following theorem [BM21, Theorem 5.13, Remark 5.9].

Theorem 5.6. *Let X be a qcqs scheme and Λ a finite ring. Then the functor $X \mapsto \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \Lambda)$ is an arc-sheaf.*

The ∞-category $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \Lambda)$ is defined analogously to $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$. A sheaf $M \in \mathrm{D}(X_{\text{ét}}, \Lambda)$ in the derived ∞-category of étale sheaves of $\underline{\Lambda}$-modules is called *lisse*, if it is dualizable with respect to its monoidal structure $- \otimes_{\underline{\Lambda}} -$, and M is called *constructible*, if there exists a finite subdivision of X into constructible locally closed subschemes $X_i \subset X$ such that each $M|_{X_i}$ is lisse.

We use this to prove arc descent for $\mathcal{O}_L$-coefficients.

Theorem 5.7. *Let L be an algebraic extension of $\mathbb{Q}_\ell$ or $\mathbb{F}_p((T))$ for some prime p and let $\mathcal{O}_L \subset L$ its ring of integers. Then the functor $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_L)$ is an arc-sheaf.*

Proof. We start with the case $L/\mathbb{Q}_\ell$. Since $L = \operatorname{colim} L'$ where L' runs over all finite extensions of $\mathbb{Q}_\ell$, we have $\mathcal{O}_L = \operatorname{colim} \mathcal{O}_{L'}$. By Lemma 5.5, it suffices to prove the claim for finite extensions.

Now, assume L to be a finite extension of $\mathbb{Q}_\ell$, i. e. a disconnected local field. Then $\mathcal{O}_L$ is a discrete valuation ring with finite and discrete residue field. We conclude that $\mathcal{O}_L/\ell^n$ is finite and discrete for every $n \in \mathbb{Z}$ with $n \geq 1$. By [HRS, Proposition 3.40] and Theorem 5.6, we conclude that

$$X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_L/\ell^n) \cong \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \mathcal{O}_L/\ell^n)$$

is an arc sheaf. Since $\mathcal{O}_L$ is ℓ-adically complete [Sta22, Tag 090T] and the canonical maps $\mathcal{O}_L/\ell^{n+1} \to \mathcal{O}_L/\ell^n$ are surjective and with nilpotent kernel, (c. f. [HRS, Section 3.4.1]) we use [HRS, Proposition 3.19], to conclude that

$$X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_L) \cong \lim_n \mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_L/\ell^n)$$

is an arc sheaf. This follows from the fact that the inclusion functor $\mathrm{Sh}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ preserves limits for every small site $\mathcal{C}$. [Lur09, Definition 6.2.2.6, Proposition 6.2.2.7].

For $L/\mathbb{F}_p((T))$ one just has to exchange ℓ with T and does the same proof again. $\square$

Before we continue to prove that constructible sheaves with coefficients in algebraic extensions of disconnected local fields satisfy arc descent, we still need to do some preparation. We start by proving the following proposition [HS, Proposition 2.3].

Proposition 5.8. *Let L be an algebraic extension of $\mathbb{Q}_\ell$, X be a scheme and let $A \in \mathrm{D}_{\mathrm{cons}}(X, L)$ be fixed. Let F be the functor taking an X-scheme X' to the ∞-category consisting of pairs (A_0, α) with $A_0 \in \mathrm{D}_{\mathrm{cons}}(X', \mathcal{O}_L)$ and an isomorphism $\alpha\colon A_0[\ell^{-1}] \xrightarrow{\cong} A|_{X'}$. Then F is a finitary arc-sheaf and admits a section over an étale cover of X.*

Precisely, the ∞-category $F(X')$ for an X-scheme X' is the ∞-category given by the fiber product of

$$\begin{array}{ccc} & & A|_{X'} \\ & & \downarrow \\ \mathrm{D}_{\mathrm{cons}}(X', \mathcal{O}_L) & \longrightarrow & \mathrm{D}_{\mathrm{cons}}(X', L) \end{array}$$

where $\mathrm{D}_{\mathrm{cons}}(X', \mathcal{O}_L) \to \mathrm{D}_{\mathrm{cons}}(X', L)$ is the functor induced by the localization map $\mathcal{O}_L \to L$ and $A|_{X'} \to \mathrm{D}_{\mathrm{cons}}(X', L)$ is the inclusion functor. This is well-defined since Cat_∞ admits all small limits [Lur09, Section 3.3.3].

We think about this category as pairs of an $A_0 \in \mathrm{D}_{\mathrm{cons}}(X', \mathcal{O}_L)$ and an isomorphism $\alpha\colon A_0\left[\frac{1}{\ell}\right] \overset{\cong}{\to} A|_{X'}$. We think of a morphism between (A_0, α) and (B_0, β) as a morphism $f : A_0 \to B_0$ such that

$$\begin{array}{ccc} A_0 & \xrightarrow{\quad f \quad} & B_0 \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & A|_{X'} & \end{array}$$

commutes up to higher morphisms.

Before we start the proof, we cite the following lemma, which is needed quite often during the proof.

Lemma 5.9. *Let X be a w-contractible qcqs scheme and Λ a condensed ring. There is a symmetric monoidal equivalence of categories*

$$\mathrm{Perf}_{\Gamma(X,\Lambda)} \cong \mathrm{D}_{\mathrm{lis}}(X, \Lambda).$$

Proof. [HRS, Lemma 3.7. (3)]. □

Proof of Proposition 5.8. By Lemma 5.5, it suffices to prove the claim for finite extensions $L/\mathbb{Q}_\ell$. We will structure the proof into several steps.

(1) F is an arc-sheaf: For the A_0, this is due to Theorem 5.7. For the α, it suffices to show that the natural map

$$\mathrm{D}_{\mathrm{cons}}(X, L) \to \lim_{\Delta} \mathrm{D}_{\mathrm{cons}}(Y^{\bullet/X}, L)$$

is fully faithful. To see this, let $\pi\colon \mathrm{D}_{\mathrm{cons}}(X, L) \to \mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_L)$ be the forgetful functor induced by the ring map $\mathcal{O}_L \to L$. This functor is conservative, i. e. it reflects isomorphisms, and it is right adjoint to $- \otimes_{\mathcal{O}_L} L$.

Next, for $A, B \in \mathrm{D}_{\mathrm{cons}}(X, L)$ consider the map

$$\underline{\mathrm{Hom}}_L(A, B) \to \lim_{\Delta} \underline{\mathrm{Hom}}_L(A|_{Y^\bullet X}, B|_{Y^\bullet X})$$

in $\mathrm{D}_{\mathrm{cons}}(X, L)$ and look at the following diagram

$$\begin{array}{ccc} \pi\underline{\mathrm{Hom}}_L(A,B) & \longrightarrow & \pi \lim \underline{\mathrm{Hom}}_L(A|_{Y^\bullet X}, B|_{Y^\bullet X}) \\ \downarrow{=} & & \downarrow{=} \\ \underline{\mathrm{Hom}}_{\mathcal{O}_L}(A,B) & \xrightarrow{\cong} & \lim \underline{\mathrm{Hom}}_{\mathcal{O}_L}(A|_{Y^\bullet X}, B|_{Y^\bullet X}) \end{array}.$$

Since we have already proven arc descent for $\mathcal{O}_L$-coefficients, the lower morphism is an isomorphism and hence the upper is also an isomorphism. Since π is conservative, we conclude that

$$\underline{\mathrm{Hom}}_L(A,B) \to \lim_{\Delta} \underline{\mathrm{Hom}}_L(A|_{Y^\bullet X}, B|_{Y^\bullet X})$$

is already an isomorphism which proves the claim.

At last, it is clear that F sends finite disjoint unions to finite products.

(2) It suffices to construct a section for A dualizable: Assume we have a section over an étale cover for all dualizable $\tilde{A}$. Since X is qcqs there exists a finite partition $X = \coprod_i X_i$ such that each $A|_{X_i}$ is dualizable. The argument is the same as in [Sta22, Tag 095E]. By assumption, there exists an étale cover $X_i' \to X_i$, a sheaf $A_{0,i} \in \mathrm{D}_{\mathrm{cons}}(X_i', \mathcal{O}_L)$ and an isomorphism $\alpha_i \colon A_{0,i}\left[\frac{1}{\ell}\right] \to (A|_{X_i})|_{X_i'} = A|_{X_i'}$ for each i. Then $X' := \coprod_i X_i' \to X$ is an étale cover. Since F is an arc-sheaf, we conclude that

$$F(X') = F(\coprod_i X_i') = \prod_i F(X_i')$$

is non-empty.

(3) F admits a section over a pro-étale cover: Assume A is dualizable, i. e. lisse, and let $X' \to X$ be a w-contractible pro-étale cover (which exists by [BS, Lemma 2.4.9]). By Lemma 5.9, $A|_{X'}$ is equivalent to a perfect complex $\tilde{A}^\bullet$ of $\mathcal{C}(\pi_0(X'), L)$-modules since L is totally disconnected and every continuous function $X' \to L$ factors canonically through $\pi_0(X')$. By [BS, Theorem 1.8, Lemma 2.4.8], $\pi_0(X')$ is extremally disconnected and we conclude from Theorem 3.7 that there exists a perfect complex $\tilde{A}_0^\bullet$ of $\mathcal{C}(\pi_0(X'), \mathcal{O}_L)$-modules with $\tilde{\alpha} \colon \tilde{A}_0^\bullet\left[\frac{1}{\ell}\right] \xrightarrow{\cong} \tilde{A}^\bullet$. By Lemma 5.9 this complex $\tilde{A}_0^\bullet$ is again equivalent to some $A_0 \in \mathrm{D}_{\mathrm{lis}}(X', \mathcal{O}_L)$ and $\tilde{\alpha}$ becomes an identification $\alpha \colon A_0\left[\frac{1}{\ell}\right] \xrightarrow{\cong} A|_{X'}$.

(4) F is finitary: By [HRS, Lemma 3.11], the property of being constructible can be checked pro-étale locally. Combining this with Lemma 5.3 and Lemma 5.4, we can assume X to be w-contractible. By (3), we can always find a section over a w-contractible pro-étale cover. Thus, we can assume $A = A_1[\ell^{-1}]$ for some sheaf

$A_1 \in \mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_l)$ that we fix. Now let $X' = \varprojlim_i X'_i$ be a cofiltered inverse limit of affine X-schemes. We need to show that the functor

$$\varinjlim_i F(X'_i) \to F(X')$$

is an equivalence. For fully faithfulness, take two elements of the left hand side and choose representatives $(A_{0,i}, \alpha_i), (B_{0,i}, \beta_i)$. Since the colimit is filtered, we can choose both representatives with the same i. A morphism $A_{0,i}|_{X'} \to B_{0,i}|_{X'}$ such that

$$\begin{array}{ccc} A_{0,i}|_{X'} & \longrightarrow & B_{0,i}|_{X'} \\ & {\scriptstyle \alpha_i|X'} \searrow \quad \swarrow {\scriptstyle \beta_i|X'} & \\ & A|_{X'} & \end{array}$$

commutes is the same as giving a morphism from the mapping cone of $A_{0,i}|_{X'} \to A|_{X'}$ to the mapping cone of $B_{0,i}|_{X'} \to A|_{X'}$. By abuse of notation, we denote these cones by $\mathrm{cone}(\alpha_i|_{X'})$ and $\mathrm{cone}(\beta_i|_{X'})$. Derived tensoring any $\tilde{A}_{0,i}$ with the short exact sequence

$$0 \to \mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0$$

yields

$$\mathrm{cone}(\tilde{A}_{0,i} \to A|_{X'_i}) \cong \tilde{A}_{0,i} \otimes^{\mathbf{L}}_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

which is a bounded complex of étale ℓ-torsion sheaves, i. e. it lies in the essential image of $\nu^*\colon \mathrm{D}(X_{\text{ét}}, \mathcal{O}_L) \to \mathrm{D}(X_{\text{proét}}, \mathcal{O}_L)$. This is due to the fact that $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is discrete and ℓ-torsion and [HRS, Proposition 3.40]. Thus, we can calculate the cohomology on the étale site [Sta22, Tag 099W]. Together with [Sta22, Tag 0GIT], we conclude

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}(X',\mathcal{O}_L)}(\mathrm{cone}\,\alpha_i|_{X'}, \mathrm{cone}\,\beta_i|_{X'}) &= H^0(X', \underline{\mathrm{Hom}}(\mathrm{cone}\,\alpha_i|_{X'}, \mathrm{cone}\,\beta_i|_{X'})) \\ &= H^0(X', \underline{\mathrm{Hom}}(\mathrm{cone}\,\alpha_i, \mathrm{cone}\,\beta_i)|_{X'}) \\ &= \mathrm{colim}_i\, H^0(X'_i, \underline{\mathrm{Hom}}(\mathrm{cone}\,\alpha_i, \mathrm{cone}\,\beta_i)|_{X'_i}) \\ &= \mathrm{colim}_i\, \mathrm{Hom}_{\mathrm{D}(X'_i,\mathcal{O}_L)}(\mathrm{cone}\,\alpha_i, \mathrm{cone}\,\beta_i). \end{aligned}$$

Hence, the functor is fully faithful.

It remains to show essential surjectivity. Let $A_0 \in \mathrm{D}_{\mathrm{cons}}(X', \mathcal{O}_\ell)$ with an isomorphism $\alpha\colon A_0[\ell^{-1}] \overset{\cong}{\to} A|_{X'} \cong A_1|_{X'}[\ell^{-1}]$ be given. We can find a finite stratification of X' into w-contractible qcqs X-schemes such that each of the three complexes becomes dualizable. By Lemma 5.9 and Theorem 3.7, we can find some power of

ℓ such that the isomorphism $A_0[\ell^{-1}] \cong A|_{X_i}$ restricted to the stratum comes from a morphism $A_0 \to A_1|_{X_i}$ multiplied by ℓ^m. Since there are finitely many strata, we can find a power of ℓ such that the isomorphism $A_0[\ell^{-1}] \cong A|_{X'}$ comes from a morphism $A_0 \to A_1|_{X_i}$ multiplied with this suitable power of ℓ. Let B be the mapping cone of this map $A_0 \to A_1|_{X_i}$. Then B is in $\mathrm{D}_{\mathrm{cons}}(X', \mathcal{O}_L)$. Tensoring the exact triangle

$$A_0 \to A_1|_{X'} \to B \to A_0[1]$$

with $\mathbb{Q}_\ell$ yields $B \otimes^{\mathbf{L}}_{Z_\ell} \mathbb{Q}_\ell \cong 0$, i. e. B is killed by some power of ℓ. We conclude that $B \in \mathrm{D}_{\mathrm{cons}}(X', \mathcal{O}_L/\ell^r)$ for some power of ℓ. Again, B is an étale sheaf, now over a finite ring. By Theorem 5.6 and [BM21, Theorem 5.6], the sheaf $X \mapsto \mathrm{D}(X_{\text{ét}}, \mathcal{O}_L/\ell^r)$ is a finitary arc-sheaf, thus $B \cong B_i|_{X'}$ for some $B_i \in \mathrm{D}_{\mathrm{cons}}(X'_i, \mathcal{O}_L)$. Since we have already shown that the functor is fully faithful, the map $A_1|_{X'} \to B$ comes from a map $A_1|_{X'_j} \to B_j$ with $B_j := B_i|_{X'_j}$ and $i \leq j$. Now, let $A_{0,j}$ be the homotopy fiber of $A_1|_{X'_j} \to B_j$. Pulling back the exact triangle

$$A_{0,j} \to A_1|_{X'_j} \to B_j \to A_{0,j}[1]$$

yields an exact triangle

$$A_{0,j}|_{X'} \to A_1|_{X'} \to B \to A_{0,j}|_{X'}[1]$$

and we conclude $A_{0,j}|_{X'} \cong A_0$.

(5) F admits a section over an étale cover: By finitaryness of F, the section constructed in step (3) is already defined over some étale cover. □

Remark 5.10. In fact, Proposition 5.8 remains true if we consider algebraic extensions of non-Archimedean local fields. The same proof works if we consider $\mathbb{F}_q((T))$ instead of $\mathbb{Q}_\ell$. In this case, the functor F would take an X-scheme X' to the ∞-category consisting of pairs (A_0, α) with $A_0 \in \mathrm{D}_{\mathrm{cons}}(X', \mathcal{O}_L)$ and an isomorphism $\alpha\colon A_0[T^{-1}] \xrightarrow{\cong} A|_{X'}$. One only needs to adjust the form of the mapping cone in the part where the finitaryness is proven. In this case, derived tensoring any $\tilde{A}_{0,i} \in \mathrm{D}_{\mathrm{cons}}(X'_i, \mathcal{O}_L)$ with the short exact sequence

$$0 \to \mathbb{F}_q[[T]] \to \mathbb{F}_q((T)) \to \mathbb{F}_q((T))/\mathbb{F}_q[[T]] \to 0$$

yields

$$\mathrm{cone}(\tilde{A}_{0,i} \to A|_{X'_i}) \cong \tilde{A}_{0,i} \otimes^{\mathbf{L}}_{\mathbb{F}_q[[T]]} \mathbb{F}_q((T))/\mathbb{F}_q[[T]].$$

The remaining parts of the proof works exactly the same.

We can conclude a globalized version of Theorem 3.7.

Proposition 5.11. *Let R be any topological T_1-ring and $S \subset R$ a multiplicative subset. Consider the localization $R \to S^{-1}R$ as a morphism of condensed rings. Then the functor*

$$\mathrm{D}_{\mathrm{cons}}(X, R) \to \mathrm{D}_{\mathrm{cons}}(X, S^{-1}R)$$
$$M \mapsto M \otimes_{R_X} (S^{-1}R)_X$$

is fully faithful.

Proof. The proof is the same as in Proposition 3.4 with the corresponding facts for dualizable objects in $\mathrm{D}(X, R)$ ([Sta22, Tag 08JJ], [Sta22, Tag 0FPP]) and descent for Hom-sets. □

Corollary 5.12. *For any algebraic extension L of $\mathbb{Q}_\ell$ or $\mathbb{F}_p((T))$ the functor*

$$\mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_L) \otimes_{\mathcal{O}_L} L \to \mathrm{D}_{\mathrm{cons}}(X, L)$$

is an equivalence of categories.

Proof. By Proposition 5.11, we only need to prove essential surjectivity. By [HRS, Proposition 3.20], we can assume $L/\mathbb{Q}_\ell$ to be finite.

Let $A \in \mathrm{D}_{\mathrm{cons}}(X, L)$ be any constructible sheaf. By Proposition 5.8, there exists some étale cover $X' \to X$ and some $A' \in \mathrm{D}_{\mathrm{cons}}(X', \mathcal{O}_L)$ such that $A'[\ell^{-1}] \cong A|_{X'}$. We can pass to a finite constructible stratification $X = \coprod_i X_i$ such that A is dualizable and each $X' \times_X X_i \to X_i$ is finite étale [Sta22, Tag 03S0] and since fully faithfulness is already shown, we can assume $X' \to X$ to be finite étale.

Since the functor in Proposition 5.8 is finitary, we can use relative approximation [Sta22, Tag 09MU] to reduce to the case where X and X' are of finite type over $\mathbb{Z}$ and in particular Noetherian [Sta22, Tag 01T6]. In this case, X and X' have finitely many connected components [Sta22, Tag 0BA8], which are clopen. After replacing X with a finite clopen cover, we can assume X to be connected. Likewise, we can assume X' to be connected. We can refine the stratification even further such that we can assume X and X' to be normal.

By [HRS, Corollary 3.29], we conclude that our definition of constructible sheaves coincides with the one given in [BS, Definition 6.8.8] for X Noetherian. By [Sta22, Tag 035B] and [BS, Lemma 7.3.9, Remark 7.3.10], there exists a pro-(finite étale) cover $\tilde{X} \to X' \to X$ such that $A|_{\tilde{X}}$ becomes a constant sheaf in each degree, which means that $A|_{\tilde{X}}$ is locally of the form $\Lambda \otimes_{\mathcal{O}_K} \underline{K}$ where Λ is a finitely generated

$\mathcal{O}_K$-module since A is dualizable. This stays true for all truncations, so we conclude that all truncations of A remain dualizable. Since the image of the functor $\mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_L) \to \mathrm{D}_{\mathrm{cons}}(X, L)$ is closed under cones and shifts, it suffices to find a preimage for A concentrated in degree 0.

Now, assume A is concentrated in degree 0. Since $A|_{\tilde{X}}$ is constant and dualizable, $A|_{\tilde{X}}$ is constant with value a finite dimensional L-vector space V and the descent datum to X is given by a continuous representation $\rho\colon \pi_1^{\mathrm{proét}}(X) \to \mathrm{GL}_L(V)$ [BS, Lemma 7.4.7]. Since X is normal, we can replace $\pi_1^{\mathrm{proét}}(X)$ by $\pi_1^{\mathrm{ét}}(X)$ [BS, Lemma 7.4.10]. Since $\pi_1^{\mathrm{ét}}(X)$ is compact, it is clear that ρ has an invariant $\mathcal{O}_L$-lattice[1], which then descends to an element $A_0 \in \mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_L)$ such that $A_0 \otimes_{\mathcal{O}_L} L \cong A$ (c. f. [BS, proof of Lemma 6.8.13]). □

Before we continue and prove arc-descent for constructible sheaves with coefficients in algebraic extensions of disconnected local fields, we will proof the following lemma first.

Lemma 5.13. *Any finitely presented arc-cover $Y \to X$ can, up to universal homeomorphism, be refined by finite étale covers over a constructible stratification.*

Proof. Consider $x \in X$, a morphism $\operatorname{Spec} \kappa(x) \to X$ and some preimage $y \in Y$ of x with a morphism $\operatorname{Spec} \kappa(y) \to Y$. Since all schemes are assumed qcqs, there is some $y \in Y$ which is a closed point in the fiber Y_x. By [Sta22, Tag 01TG], the field extension $\kappa(y)/\kappa(x)$ is finite. By replacing $\kappa(x)$ with its perfection (which is a universal homeomorphism by [BS17, Lemma 3.8]), if necessary, we can assume that the field extension $\kappa(y)/\kappa(x)$ is finite separable and hence a finite étale refinement of $Y_x \to \operatorname{Spec} \kappa(x)$.

Recall that the set $\{x\} \subset X$ can be written as intersection of quasi-compact opens and their complements. Namely, choose an affine open neighborhood $\operatorname{Spec} A$ of x at first, then write the closure of x as

$$\overline{\{x\}} = \bigcap_{f \notin \mathfrak{p}_x} D(f)^C,$$

where $\mathfrak{p}_x$ is the prime ideal corresponding to x. At last, for every $y \in \overline{\{x\}}, y \neq x$, choose some $f_y \in A$ such that $x \in D(f_y)$ but $y \notin D(f_y)$. Then we have

$$\{x\} = \overline{\{x\}} \cap \bigcap_{\substack{y \in \overline{\{x\}} \\ y \neq x}} D(f_y).$$

[1] One uses that $U := \{g \in \pi_1^{\mathrm{ét}}(X) : g\Gamma \subset \Gamma\}$ is open for any lattice $\Gamma \subset V$ and that a finite sum of lattices is again a lattice.

In other words, we can write $\operatorname{Spec}\kappa(x)$ as inverse limit of open resp. closed subschemes.

Now, let $U \to \operatorname{Spec}\kappa(x)$ be the (up to universal homeomorphism) finite étale refinement constructed before. By [GVA72, Theorem 8.10.5 and Theorem 17.7.8], this morphism spreads out to a locally closed constructible subset. □

Finally, we can prove [HS, Theorem 2.2].

Theorem 5.14. *Let L be an algebraic extension of $\mathbb{Q}_\ell$ or $\mathbb{F}_p((T))$. The functor $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, L)$ is an arc-sheaf.*

Proof. By Lemma 5.5, we can again reduce to the case where L is a finite extension of $\mathbb{Q}_\ell$ resp. $\mathbb{F}_p((T))$. Let $Y \to X$ be an arc-cover. We want to show that the natural map

$$\mathrm{D}_{\mathrm{cons}}(X, L) \to \lim_{\Delta} \mathrm{D}_{\mathrm{cons}}(Y^{\bullet X}, L)$$

is an equivalence.

Since the case of $\mathcal{O}_L$-coefficients was already proven in Theorem 5.7 and we know

$$\mathrm{D}_{\mathrm{cons}}(X, \mathcal{O}_L) \otimes_{\mathcal{O}_L} L \simeq \mathrm{D}_{\mathrm{cons}}(X, L)$$

by Corollary 5.12, we can conclude that the map is fully faithful.

For the essential surjectivity, consider some $A \in \mathrm{D}_{\mathrm{cons}}(Y, L)$ equipped with a descent datum. Let $\tilde{Y}$ be the finitary arc-sheaf considered in Proposition 5.8 with Y as base scheme, i. e. $\tilde{Y}$ is a functor from the category of schemes over Y. We equip $\mathrm{PSh}(Y_{\text{proét}})$, resp. $\mathrm{PSh}(X_{\text{proét}})$ with the topology described in Remark 4.7 and identify X and Y with their images via the Yoneda embedding. Since the pullback functor $\mathrm{D}_{\mathrm{cons}}(X, L) \to \mathrm{D}_{\mathrm{cons}}(Y, L)$ is exact, the descent datum on A induces a descent datum on $\tilde{Y}$ by construction of $\tilde{Y}$. This descent datum is effective, thus it descends to a sheaf $\tilde{X}$ (c. f. [Lur09, §6.2.4]). Since $\tilde{Y} \to Y$ is an arc-cover (Y is the terminal object in $\mathrm{PSh}(Y_{\text{proét}})$ as Y is the constant presheaf with a singleton value) and the arc-cover $Y \to X$ lifts to an arc-cover in the presheaf category, the induced map $\tilde{X} \to X$ is an arc-cover as well. At last, we consider that $\tilde{Y}$ is a representable sheaf in $\mathrm{Sh}_{\mathrm{can}}(\mathrm{Sh}(Y_{\text{proét}}))$, hence there exists a universal $A_0 \in \mathrm{D}_{\mathrm{cons}}(-, \mathcal{O}_L)$ over $\tilde{Y}$. By Theorem 5.7, this universal A_0 descends to $\tilde{X}$. In conclusion, it is enough to prove descent along $\tilde{X} \to X$.

To reduce further, note that by relative approximation [Sta22, Tag 09MV] any qcqs scheme over X can be written as a limit of a directed system of finitely presented X-schemes X_i with affine transition maps over X. Since $\tilde{X}$ is a finitary arc sheaf and an arc-cover of X (because $\tilde{Y}$ is one), there exists a finitely presented X-scheme X' such that $\tilde{X}(X')$ has a section over X and $X' \to X$ is an arc-cover. Hence, it is enough to prove descent along finitely presented arc-covers.

Now, assume $Y \to X$ to be a finitely presented arc-cover. Since descent for morphisms is already proven, we can restrict to a constructible stratification. By Lemma 5.13 we can, up to universal homeomorphism, refine $Y \to X$ by finite étale covers. In other words, we reduced to the case, where $Y \to X$ is finite étale. But by [HRS, Lemma 3.11, Corollary 3.13], the functor $U \mapsto \mathrm{D}_{\mathrm{cons}}(U, \Lambda)$ satisfies descent on $X_{\mathrm{pro\acute{e}t}}$ and we are done.

□

5.2 Coefficients in the adele ring

In this section, let K denote a global field and $\mathcal{O}_K$ its ring of integers. For each place ν, we denote the completion of K with respect to ν by K_ν and its ring of integers by $\mathcal{O}_{K_\nu}$. By $\hat{\mathcal{O}}_K$ we denote the profinite completion of $\mathcal{O}_K$.

Recall that any finite place in a number field, resp. any place in a function field over a finite field corresponds one-to-one to a non-zero prime ideal. For any place ν with corresponding prime ideal $\mathfrak{p}_\nu$, one has

$$\hat{\mathcal{O}}_{K_\nu} \cong \varprojlim_{n \geq 1} \mathcal{O}_K / \mathfrak{p}_\nu^n. \tag{5.2}$$

By the Chinese remainder theorem one can conclude

$$\hat{\mathcal{O}}_K \cong \prod_{\nu < \infty} \mathcal{O}_{K_\nu}$$

for number fields K, respectively

$$\hat{\mathcal{O}}_K \cong \prod_{\nu \leq \infty} \mathcal{O}_{K_\nu}$$

for function fields K over some finite field.

Theorem 5.15. *The functor $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \hat{\mathcal{O}}_K)$ is an arc-sheaf.*

Proof. The basic idea is to apply [HRS, Proposition 3.19] to the diagram in (5.2) and perform the same proof as in Theorem 5.7. To do so, we need to reformulate (5.2) in such a way that the limit diagram becomes sequential.

The number of prime ideals in $\mathcal{O}_K$ is countable. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$ be an enumeration of non-zero prime ideals in $\mathcal{O}_K$. For every $m \geq 1$, we set

$$R_m := \mathcal{O}_K / \mathfrak{p}_1^m \cdot \ldots \cdot \mathfrak{p}_m^m.$$

The canonical transition maps $R_{m+1} \to R_m$ are surjective with nilpotent kernel. The collection $\{R_m\}_{m\geq 1}$ is a cofinal subset in the set of finite quotient rings of $\mathcal{O}_K$ and hence, their limits coincide, i. e.

$$\hat{\mathcal{O}}_K \cong \varprojlim_{m\geq 1} R_m.$$

For every m, the ring R_m is finite and discrete, so we use the same approach as in Theorem 5.7. For every m the functor

$$X \mapsto \mathrm{D}_{\mathrm{cons}}(X, R_m) \cong \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, R_m)$$

is an arc-sheaf by Theorem 5.6. Using [HRS, Proposition 3.19], we conclude that

$$X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \hat{\mathcal{O}}_K) = \lim_{m\geq 1} \mathrm{D}_{\mathrm{cons}}(X, R_m)$$

is an arc-sheaf. □

The proof also works considering only a subset of all places. Hence, in fact we did prove the following:

Corollary 5.16. *Let S be a subset of places such that for all $\nu \in S$ the completion K_ν is non-Archimedean. Then the functor*

$$X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \prod_{\nu\in S} \mathcal{O}_{K_\nu})$$

is an arc-sheaf.

To prove arc-descent for constructible sheaves with finite adelic coefficients, we will use the following proposition which shows that constructible sheaves are well-behaved with finite products in the coefficient ring.

Proposition 5.17. *Let X be a scheme and let Λ_1, Λ_2 be condensed rings. There is an equivalence of categories*

$$\mathrm{D}_{\mathrm{cons}}(X, \Lambda_1) \times \mathrm{D}_{\mathrm{cons}}(X, \Lambda_2) \simeq \mathrm{D}_{\mathrm{cons}}(X, \Lambda_1 \times \Lambda_2).$$

Proof. Given $A \in \mathrm{D}(X, \Lambda_1), B \in \mathrm{D}(X, \Lambda_2)$, the complex $A \times B$ is a complex of $(\Lambda_1 \times \Lambda_2)$-modules with componentwise scalar multiplication. In particular, we can view A and B as $(\Lambda_1 \times \Lambda_2)$-modules via the obvious scalar multiplication.

We can find a constructible stratification $X \coprod_i X_i$ such that both A and B become dualizable on each stratum. By Lemma 5.9 and the fact that the tensor product commutes with finite products (which are finite direct sums), we conclude that $A|_{X_i} \times B|_{X_i}$ is dualizable for each i. Hence, $A \times B$ is constructible. This construction is functorial, so we get a functor

$$\mathrm{D}_{\mathrm{cons}}(X, \Lambda_1) \times \mathrm{D}_{\mathrm{cons}}(X, \Lambda_2) \to \mathrm{D}_{\mathrm{cons}}(X, \Lambda_1 \times \Lambda_2).$$

This functor is fully faithful by the universal property of the product, resp. coproduct. For essential surjectivity, let $A \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda_1 \times \Lambda_2)$. This sheaf splits as $A = A_1 \oplus A_2$ with a Λ_1-module A_1 and a Λ_2-module A_2. Both A_1 and A_2 are dualizable on a constructible stratification with respect to the inherited monoidal structure, i. e. $A_1 \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda_1)$ and $A_2 \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda_2)$. □

Now, we can prove arc-descent for constuctible sheaves with (finite) adelic coefficients.

Theorem 5.18. *Let K be a global field. Then the functor $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \mathbb{A}_{K,\mathrm{fin}})$ is an arc-sheaf. If K is a global field of characteristic $p > 0$, then the functor $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \mathbb{A}_K)$ is an arc-sheaf as well.*

Proof. Let P be the set of finite places of K. For the second case, let P simply be the set of places of K. For any finite subset $S \subset P$, we define

$$\mathbb{A}_{K,S} := \prod_{\nu \in S} K_\nu \times \prod_{\substack{\nu \notin S \\ \nu < \infty}} \mathcal{O}_{K_\nu}.$$

By Proposition 5.17 and Theorem 5.14, we conclude that

$$X \mapsto \mathrm{D}_{\mathrm{cons}}\left(X, \prod_{\nu \in S} K_\nu\right) = \prod_{\nu \in S} \mathrm{D}_{\mathrm{cons}}(X, K_\nu)$$

is an arc-sheaf. By Corollary 5.16 and using Proposition 5.17 again, we get that

$$X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \mathbb{A}_{K,S})$$

is an arc-sheaf.

Since we have

$$\mathbb{A}_{K,\mathrm{fin}} = \varinjlim_{\substack{S \subset P \\ \text{finite}}} \mathbb{A}_{K,S}$$

as topological rings, we are done by Lemma 5.5. □

At last, we want to prove Corollary 5.12 for adelic coefficients.

Theorem 5.19. *Let K be a number field. Then the functor*

$$\mathrm{D}_{\mathrm{cons}}(X, \hat{\mathcal{O}}_K) \otimes_{\mathcal{O}_K} K \to \mathrm{D}_{\mathrm{cons}}(X, \mathbb{A}_{K,\mathrm{fin}})$$

is an equivalence. If K is a function field over a finite field, then

$$\mathrm{D}_{\mathrm{cons}}(X, \hat{\mathcal{O}}_K) \otimes_{\mathcal{O}_K} K \to \mathrm{D}_{\mathrm{cons}}(X, \mathbb{A}_K)$$

is an equivalence.

Proof. First, we recall some facts about the adeles. For every place ν of K let $\mathfrak{p}_\nu \subset \mathcal{O}_K$ be the corresponding prime ideal. For every $x \in \mathcal{O}_K$, we have

$$\mathcal{O}_K[x^{-1}] \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\nu} = \begin{cases} K_\nu & \text{if } x \in \mathfrak{p}_\nu, \\ \mathcal{O}_{K_\nu} & \text{if } x \notin \mathfrak{p}_\nu. \end{cases}$$

We conclude

$$\mathcal{O}_K[x^{-1}] \otimes_{\mathcal{O}_K} \hat{\mathcal{O}}_K = \prod_{x \in \mathfrak{p}_\nu} K_\nu \times \prod_{x \notin \mathfrak{p}_\nu} \mathcal{O}_{K_\nu}.$$

Hence, we have

$$\begin{aligned} \mathbb{A}_{K,\mathrm{fin}} &= K \otimes_{\mathcal{O}_K} \hat{\mathcal{O}}_K \\ &= \mathrm{colim}_{x \in \mathcal{O}_K} \mathcal{O}_K[x^{-1}] \otimes_{\mathcal{O}_K} \hat{\mathcal{O}}_K \\ &= \mathrm{colim}_{x \in \mathcal{O}_K} \left(\prod_{x \in \mathfrak{p}_\nu} K_\nu \times \prod_{x \notin \mathfrak{p}_\nu} \mathcal{O}_{K_\nu} \right) \end{aligned}$$

as topological rings.

With Corollary 5.12, Proposition 5.17 and [HRS, Proposition 3.20], we calculate

$$\begin{aligned}
\mathrm{D}_{\mathrm{cons}}(X, \mathbb{A}_{K,\mathrm{fin}}) &= \mathrm{D}_{\mathrm{cons}}\left(X, \operatorname{colim}\left(\prod_{x\in\mathfrak{p}_\nu} K_\nu \times \prod_{x\notin\mathfrak{p}_\nu} \mathcal{O}_{K_\nu}\right)\right) \\
&= \operatorname{colim} \mathrm{D}_{\mathrm{cons}}\left(X, \prod_{x\in\mathfrak{p}_\nu} K_\nu \times \prod_{x\notin\mathfrak{p}_\nu} \mathcal{O}_{K_\nu}\right) \\
&= \operatorname{colim}\left(\left(\prod_{x\in\mathfrak{p}_\nu} \mathrm{D}_{\mathrm{cons}}\left(X, K_\nu\right)\right) \times \mathrm{D}_{\mathrm{cons}}\left(X, \prod_{x\notin\mathfrak{p}_\nu} \mathcal{O}_{K_\nu}\right)\right) \\
&= \operatorname{colim}\left(\left(\prod_{x\in\mathfrak{p}_\nu} \mathrm{D}_{\mathrm{cons}}\left(X, \mathcal{O}_{K_\nu}\right) \otimes_{\mathcal{O}_K} \mathcal{O}_K[x^{-1}]\right) \times \mathrm{D}_{\mathrm{cons}}\left(X, \prod_{x\notin\mathfrak{p}_\nu} \mathcal{O}_{K_\nu}\right)\right) \\
&= \operatorname{colim}\left(\left(\prod_{x\in\mathfrak{p}_\nu} \mathrm{D}_{\mathrm{cons}}\left(X, \mathcal{O}_{K_\nu}\right)\right) \times \mathrm{D}_{\mathrm{cons}}\left(X, \prod_{x\notin\mathfrak{p}_\nu} \mathcal{O}_{K_\nu}\right)\right) \otimes_{\mathcal{O}_K} \mathcal{O}_K[x^{-1}] \\
&= \operatorname{colim} \mathrm{D}_{\mathrm{cons}}(X, \hat{\mathcal{O}}_K) \otimes_{\mathcal{O}_K} \mathcal{O}_K[x^{-1}] \\
&= \mathrm{D}_{\mathrm{cons}}(X, \hat{\mathcal{O}}_K) \otimes_{\mathcal{O}_K} K,
\end{aligned}$$

where we used that the derived tensor product commutes with finite products. □

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