Minimal degeneration singularities in the affine Grassmannian for PGL_2 over $\mathbb{Z}$

Master thesis by Varun Mohan Kumar
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1. Review: Prof. Dr. Timo Richarz
2. Review: Dr. Patrick Bieker
Darmstadt

TECHNISCHE UNIVERSITÄT DARMSTADT

Mathematics Department
TU Darmstadt
Algebra

Abstract

We compute minimal degeneration singularities in the affine Grassmannian for PGL_2 over $\mathbb{Z}$. This generalises results from [MOV05, Lemma 5.1] in characteristic 0, [Mül09], and [HLRed] in the case of quasi-minuscule Schubert varieties. In particular, in characteristic 2 these are not normal. We deduce these results from the GL_2 case.

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1. Introduction

To begin, we fix some notation. We let $\mathbf{Gr}_G$ denote the affine Grassmannian for a group G. Let μ be a dominant cocharacter for the diagonal torus in G. Then we let

- Λ_μ denote the corresponding point in $\mathbf{Gr}_G$, and
- $\mathbf{Gr}_{G,\leq\mu}$ denote the corresponding Schubert variety in $\mathbf{Gr}_G$.

We compute minimal degeneration singularities in Schubert varieties inside $\mathbf{Gr}_{\mathrm{PGL}_2}$ over $\mathbb{Z}$.

We identify the dominant cocharacters for PGL_2 with $\mathbb{Z}_{\geq 0}$. Then for $r \in \mathbb{Z}_{\geq 2}$, the pair $(r-2, r)$ is a minimal degeneration of dominant cocharacters for PGL_2. The corresponding minimal degeneration singularity is defined by

$$L^{--}\mathrm{PGL}_2 \cdot \Lambda_{r-2} \cap \mathbf{Gr}_{\mathrm{PGL}_2,\leq r}. \tag{1.1}$$

where

$$L^{--}\mathrm{PGL}_2 \cdot \Lambda_{r-2}$$

is a locally closed ind-subscheme of $\mathbf{Gr}_{\mathrm{PGL}_2}$.

Then (1.1) is an affine scheme, i.e. it is equal to Spec B_r where B_r is some ring. Our main result describes these rings.

Theorem 1.1. *(cf. Theorem 4.1) In the situation above, the ring B_r is isomorphic to the subring of $\mathbb{Z}[w,x,y]/(w^r + xy)$ generated by $x, y, wx, wy, 2w, w^2$.*

For the remainder of this section we denote $A_r := \mathbb{Z}[w,x,y]/(w^r + xy)$.

Note that over $\mathbb{Z}[\frac{1}{2}]$ we have that $B_r \otimes \mathbb{Z}[\frac{1}{2}] \cong A_r \otimes \mathbb{Z}[\frac{1}{2}]$; this also applies to any $\mathbb{Z}[\frac{1}{2}]$-algebra R. For $R = k$ a field with $\mathrm{char}(k) = 0$ this was first shown in [MOV05], and generalised to any field with $\mathrm{char}(k) \neq 2$ in [Mül09]. The minimal degeneration singularity in the quasi-minuscule case $r = 2$ was first discussed in [HLRed].

In order to arrive at our main result, we adapt the methodology of [Mül09, Theorem 9.2] to compute the minimal degeneration singularities in the affine Grassmannian for GL_2.

Theorem 1.2. *(cf. Theorem 3.6) Let $\mu = (m_1, m_2) \in \mathbb{Z}^2$ be a dominant cocharacter for GL_2, i.e. such that $m_1 \geq m_2$, and let $\lambda = (m_1 + 1, m_2 - 1)$. Then the pair (μ, λ) is a minimal degeneration of dominant cocharacters for GL_2. For such a pair, we set $r := m_1 - m_2 + 2$. Then there is an isomorphism of schemes*

$$L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu \cap \mathbf{Gr}_{\mathrm{GL}_2,\leq\lambda} \cong \mathrm{Spec}\ A_r.$$

To deduce the result in the PGL_2 case we make use of the fact that $\mathrm{GL}_2 \twoheadrightarrow \mathrm{PGL}_2$ induces a scheme-theoretic surjection of reduced schemes

$$L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu \cap \mathbf{Gr}_{\mathrm{GL}_2,\leq\lambda} \to L^{--}\mathrm{PGL}_2 \cdot \Lambda_{r-2} \cap \mathbf{Gr}_{\mathrm{PGL}_2,\leq r} \tag{1.2}$$

on minimal degeneration singularities. To identify the subring B_r of A_r we use the adjoint representation $\mathrm{GL}_2 \to \mathrm{Aut}(\mathfrak{gl}_2) = \mathrm{GL}_4$, generalising the argument in [HLRed, Appendix B].

Finally, we discuss presentations of the reduced rings $(B_r \otimes k)_{\mathrm{red}}$ where k is a field such that $\mathrm{char}(k) = 2$, computed using SageMath (cf. Conjecture 4.5).

2. Recollections on affine Grassmannians

We begin by presenting some results on affine Grassmannians over $\mathbb{Z}$, following [Ric20, §2–3].

2.1 The affine Grassmannian for general groups

Let R be a ring. Recall that $R(\!(z)\!)$ and $R[\![z]\!]$ are the rings of formal Laurent series and formal power series in the variable z respectively. We set $\mathbb{D}_R := \operatorname{Spec} R[\![z]\!]$ and $\mathbb{D}_R^* := \mathbb{D}_R \backslash \{z = 0\} = \operatorname{Spec} R(\!(z)\!)$.

Let G be a split reductive group over $\mathbb{Z}$.

Let $\mathcal{E}$ be a (left) fppf G-torsor over $\mathbb{D}_R$, and $\alpha \in \mathcal{E}(\mathbb{D}_R^*)$ a section. We say two pairs $(\mathcal{E}, \alpha)$ and $(\mathcal{E}', \alpha')$ are isomorphic if there exists a morphism $\pi : \mathcal{E} \to \mathcal{E}'$ such that $\pi \circ \alpha = \alpha'$. In this case, π is necessarily an isomorphism of G-torsors.

Definition 2.1. The *affine Grassmannian for G* is the functor Rings $\to$ Sets given by

$$\mathbf{Gr}_G(R) := \{[(\mathcal{E}, \alpha)] \mid \mathcal{E} \text{ is a (left) fppf } G\text{-torsor over } \mathbb{D}_R,\ \alpha \in \mathcal{E}(\mathbb{D}_R^*) \text{ a section}\}.$$

In particular, we consider the affine Grassmannians for $G = \mathrm{GL}_n$, SL_n and PGL_n. These are all representable by ind-projective ind-schemes, as a special case of [Lev13, Theorem 3.3.11]. Then the maps

$$\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n \twoheadrightarrow \mathrm{PGL}_n$$

of $\mathbb{Z}$-group schemes induce maps

$$\mathbf{Gr}_{\mathrm{SL}_n} \hookrightarrow \mathbf{Gr}_{\mathrm{GL}_n} \twoheadrightarrow \mathbf{Gr}_{\mathrm{PGL}_n}$$

of ind-schemes.

2.2 The affine Grassmannian for GL_n

The affine Grassmannian for GL_n admits a more explicit description.

Definition 2.2. Let R be a ring. An *$R[\![z]\!]$-lattice* Λ in $R(\!(z)\!)^n$ is a finite locally free $R[\![z]\!]$-submodule of $R(\!(z)\!)^n$ such that

$$\Lambda \otimes_{R[\![z]\!]} R(\!(z)\!) = R(\!(z)\!)^n .$$

The following is essentially [Ric20, Rmk 3.3].

Lemma 2.3. *The functor $\mathbf{Gr}_{\mathrm{GL}_n}$ is isomorphic to the functor* Rings $\to$ *Sets given by:*

$$R \mapsto \{R[\![z]\!]\text{–lattices in } R(\!(z)\!)^n\}.$$

Proof. In order to see this, first note a GL_n-bundle on $\mathcal{E} \to \mathbb{D}_R$ is essentially the same as a rank n locally free $R[\![z]\!]$-module E. In this setting, α induces an isomorphism (as $R[\![z]\!]$-modules) $E[z^{-1}] \cong R(\!(z)\!)$, under which we can take the image of $E \subset E[z^{-1}]$; this gives us a lattice $\Lambda = \Lambda_{(\mathcal{E},\alpha)}$ which only depends on the class $[(\mathcal{E},\alpha)]$. The map $[(\mathcal{E},\alpha)] \mapsto \Lambda_{(\mathcal{E},\alpha)}$ is the required isomorphism. □

Proposition 2.4. *A lattice $\Lambda \in \mathbf{Gr}_{\mathrm{GL}_n}(R)$ is* special *if its highest exterior power $\bigwedge^n \Lambda = R[\![z]\!]$ as a $R[\![z]\!]$-submodule of $R(\!(z)\!)$. Then the isomorphism in Lemma 2.3 restricts to an isomorphism between $\mathbf{Gr}_{\mathrm{SL}_n}$ and the subfunctor* Rings $\to$ Sets *given by*

$$R \mapsto \{\Lambda \in \mathbf{Gr}_{\mathrm{GL}_n}(R) \mid \Lambda \text{ is special}\}.$$

Proof. See the proof of [HLRed, Lemma B.3]. □

We set $\Lambda_0 := \mathbb{Z}[\![z]\!]^n$ the base lattice, and more generally for a ring R, $\Lambda_{0,R} := R[\![z]\!]^n$.

Definition 2.5. An ordered pair $(a,b) \in \mathbb{Z}^2, a \leq b$ defines a subfunctor of $\mathbf{Gr}_{\mathrm{GL}_n}$ given by

$$\mathbf{Gr}^{[a,b]}_{\mathrm{GL}_n}(R) := \{\Lambda \in \mathbf{Gr}_{\mathrm{GL}_n}(R) \mid z^b\Lambda_{0,R} \subset \Lambda \subset z^a\Lambda_{0,R}\}.$$

We collect some important facts here.

Lemma 2.6.

1. *For any ordered pair $(a,b) \in \mathbb{Z}^2, a \leq b$, $\mathbf{Gr}^{[a,b]}_{\mathrm{GL}_n} \to$ Spec $\mathbb{Z}$ is representable by a proper scheme.*
2. *The functor $\mathbf{Gr}_{\mathrm{GL}_n}$ can be written as the filtered colimit*

$$\mathbf{Gr}_{\mathrm{GL}_n} = \mathrm{colim}_{a \leq b}\mathbf{Gr}^{[a,b]}_{\mathrm{GL}_n}.$$

Proof. See [Ric20, Theorem 2.2 and preceding discussion]. □

Definition 2.7. A *cocharacter μ for* (the diagonal torus in) GL_n is a map

$$\mu : z \mapsto \mathrm{diag}(z^{m_1}, ..., z^{m_n})$$

where $(m_1, ..., m_n) \in \mathbb{Z}^n$. We identify μ with the tuple $(m_1, ..., m_n)$. A cocharacter μ is *dominant* if $m_i \geq m_{i+1}$. We denote the set of dominant cocharacters by $\mathbb{Z}^n_+$. μ is a cocharacter for SL_n if $\sum_{i=1}^n m_i = 0$.

Definition 2.8. For $\mu \in \mathbb{Z}^n$ we define the *diagonal lattices* of $\mathbf{Gr}_{\mathrm{GL}_n}$ given by

$$\Lambda_\mu := \bigoplus_{i=1}^n z^{m_i}\mathbb{Z}[\![z]\!]$$

where the identification should be taken to mean that the matrix $z^\mu := \mathrm{diag}(z^{m_1}, ..., z^{m_n}) \in \mathrm{GL}_n(\mathbb{Z}(\!(z)\!))$ represents the lattice Λ_μ with respect to the standard basis $(e_1, ..., e_n)$. We may also use the notation $\mu(z)$ to indicate the same object.

We wish to describe cocharacters for the diagonal torus in PGL_n.

Definition 2.9. Let $\mu := (m_1, ..., m_n)$ and $\mu' := (m'_1, ..., m'_n) \in \mathbb{Z}^n$ be cocharacters for GL_n. We say $\mu \sim \mu'$ if there exists some $a \in \mathbb{Z}$ such that for all $1 \leq i \leq n$, $m'_i = m_i + a$. Then $\sim$ is an equivalence relation, the classes of which are the *cocharacters for* PGL_n. Any cocharacter for PGL_n has a unique representative given by

$$[(m_1, ..., m_n)] \mapsto (m_1 - m_n, ..., m_{n-1} - m_n, 0). \tag{2.1}$$

Hence we identify the cocharacters for PGL_n with $\mathbb{Z}^{n-1}$. A cocharacter for PGL_n is *dominant* if and only if it is the class of some $\mu \in \mathbb{Z}^n_+$, i.e. if and only if

$$m_1 - m_n \geq m_2 - m_n \geq ... \geq 0.$$

We identify the dominant cocharacters for PGL_n with

$$\mathbb{Z}^{n-1}_{+,\geq 0} := \{(m_1, ..., m_{n-1}) \in \mathbb{Z}^{n-1}_+ \text{ st. } \forall\, 1 \leq i \leq n-1,\ m_i \geq 0\}.$$

Remark 2.10. In general, $\mathbf{Gr}_{\mathrm{PGL}_n}$ does not have a similar explicit description cf. Lemma 2.3 and Proposition 2.4. However, when the Picard group $\mathrm{Pic}(R) \cong 0$, then

$$\mathrm{PGL}_n(R) = \mathrm{GL}_n(R)/R^\times.$$

If we additionally have that $\mathrm{Pic}(R(\!(z)\!)) = 0$ and that R is reduced and connected, the cocharacters for PGL_n may be treated as isomorphism classes on (matrix representations of) lattices via the identification using the standard basis outlined above. More specifically, for $\Lambda \in \mathbf{Gr}_{\mathrm{GL}_n}(R)$, let

$$[\Lambda] := \{\Lambda' \mid \exists\, a \in \mathbb{Z} \text{ st. } \Lambda = z^a \Lambda'\}.$$

Then

$$\mathbf{Gr}_{\mathrm{PGL}_n}(R) = \{[\Lambda] \mid \Lambda \in \mathbf{Gr}_{\mathrm{GL}_n}(R)\}.$$

Example 2.11. Let $n = 2$, and let $\mu = (3, -3)$ be a dominant cocharacter for SL_2 (and for GL_2). Under the map induced by (2.1), the matrix representative of $\Lambda_\mu \in \mathbf{Gr}_{\mathrm{SL}_2}$ is sent to

$$\begin{pmatrix} z^3 & 0 \\ 0 & z^{-3} \end{pmatrix} \mapsto \left[\begin{pmatrix} z^6 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

i.e. $\Lambda_{(3,-3)} \mapsto \Lambda_6$.

The following is from [Ric20, §2.3].

Definition 2.12. The determinant induces a morphism $\mathbf{Gr}_{\mathrm{GL}_n} \to \mathbf{Gr}_{\mathrm{GL}_1}$ given on R-points by

$$\Lambda \mapsto \det_{R[\![z]\!]}(\Lambda).$$

For $d \in \mathbb{Z}$, let

$$\Sigma_d(R) := \{\Lambda \in \mathbf{Gr}_{\mathrm{GL}_n}(R) \mid \det_{R[\![z]\!]}(\Lambda) = d\}.$$

Then Σ_d is a connected component and $\mathbf{Gr}_{\mathrm{GL}_n}$ can be decomposed into the disjoint union

$$\mathbf{Gr}_{\mathrm{GL}_n} = \coprod_{d \in \mathbb{Z}} \Sigma_d.$$

2.3 Loop groups

Definition 2.13. For a split reductive group G over $\mathbb{Z}$ we define the following functors from Rings $\to$ Groups:

1. $LG(R) := G(R(\!(z)\!))$, the *loop group of* G,
2. $L^+G(R) := G(R[\![z]\!])$, the *positive loop group of* G,
3. $L^{--}G(R) := \ker\left(G(R[z^{-1}]) \xrightarrow{z^{-1}\mapsto 0} G(R)\right)$, the *strictly negative loop group of* G.

In particular, we will consider these functors for $G = \mathrm{GL}_n$, SL_n, and PGL_n.

Lemma 2.14. *Let G be a split reductive group over $\mathbb{Z}$. Then*

1. *LG is representable by an ind-affine ind-scheme.*
2. *L^+G is representable by an affine scheme.*
3. *$L^{--}G$ is representable by an ind-affine ind-scheme.*

Proof. For (1) and (2) see [HR20, Lemma 3.2]. (3) follows from [HR20, Lemma 3.14] and the fact that $L^{--}G$ is closed inside L^-G. □

Proposition 2.15. *If G is a split reductive group over $\mathbb{Z}$, then the map $LG \to \mathbf{Gr}_G, g \mapsto g \cdot e = [(\mathcal{E}_0, g)]$ induces an isomorphism of étale sheaves*

$$(LG/L^+G)_{\text{ét}} \cong \mathbf{Gr}_G.$$

Proof. See [Ric20, Proposition 3.18]. □

2.4 Schubert varieties

Definition 2.16. Let $\mu \in \mathbb{Z}^2_+$. We define the *Schubert variety* $\mathbf{Gr}_{\mathrm{GL}_2,\leq\mu} \subset \mathbf{Gr}_{\mathrm{GL}_2}$ to be the scheme-theoretic image of the map

$$\begin{aligned} L^+\mathrm{GL}_2 &\to \mathbf{Gr}_{\mathrm{GL}_2} \\ g &\mapsto g \cdot \Lambda_\mu. \end{aligned}$$

The Schubert varieties for $\mathbf{Gr}_{\mathrm{SL}_2}$ are indexed by the cocharacters for SL_2 i.e. $\mu = (m, -m)$ with $m \in \mathbb{Z}_{\geq 0}$. For $\mathbf{Gr}_{\mathrm{PGL}_2}$, the dominant cocharacters are indexed by $\mathbb{Z}_{\geq 0}$. We define the Schubert variety $\mathbf{Gr}_{\mathrm{PGL}_2,\leq r}$ in an analogous manner.

Lemma 2.17. *Let $\mu = (m_1, m_2) \in \mathbb{Z}^2_+$. Then $\mathbf{Gr}_{\mathrm{GL}_2,\leq\mu}$ is a closed subscheme of $\mathbf{Gr}^{[m_2,m_1]}_{\mathrm{GL}_2}$.*

Proof. This follows from the definition, since $\Lambda_\mu \in \mathbf{Gr}^{[m_2,m_1]}_{\mathrm{GL}_2}$ which is stable under the action of L^+G. In particular, the scheme-theoretic image of a map is the smallest closed subscheme through which the map factors. □

Definition 2.18. Let $\mu = (m_1, m_2) \in \mathbb{Z}^2$. Then $\lambda = \lambda(\mu) := (m_1 + 1, m_2 - 1) \in \mathbb{Z}^2$ is the *immediate neighbour of* μ. Note that λ is dominant if and only if μ is. In this case, we call the pair (μ, λ) a *minimal degeneration of dominant cocharacters for* GL_2.

Note that for such a pair (μ, λ), $|\lambda| = |\mu|$, so the corresponding Schubert varieties are contained in the same connected component $\Sigma_{|\mu|}$. In particular, $\mathbf{Gr}_{\mathrm{GL}_2, \leq\mu} \subset \mathbf{Gr}_{\mathrm{GL}_2, \leq\lambda}$ and this inclusion is minimal, i.e. there is no λ' such that $\mathbf{Gr}_{\mathrm{GL}_2, \leq\mu} \subsetneq \mathbf{Gr}_{\leq\lambda'} \subsetneq \mathbf{Gr}_{\mathrm{GL}_2, \leq\lambda}$.

Definition 2.19. For μ a cocharacter of GL_2, let $z^\mu := \mathrm{diag}(z^{m_1}, z^{m_2}) \in \mathrm{GL}_2(\mathbb{Z}(\!(z)\!))$. For a ring R, we denote $\Lambda^-_{\mu,R} := z^\mu \cdot (z^{-1}R[z^{-1}]^2)$. Then we define

$$\mathcal{U}_\mu(R) := \{\Lambda \in \mathbf{Gr}_{\mathrm{GL}_2}(R) \mid \Lambda^-_{\mu,R} \oplus \Lambda \cong R(\!(z)\!)^2 \text{ as } R\text{-modules}\}.$$

In particular, $\mathcal{U}_0$ contains Λ_0.

Note that for any $\mu \in \mathbb{Z}^2$, $\mathcal{U}_\mu$ is the z^μ-translate of $\mathcal{U}_0$ under the action of $\mathrm{GL}_2(R(\!(z)\!))$ on $\mathbf{Gr}_{\mathrm{GL}_2}$.

Proposition 2.20.

$$\mathbf{Gr}_{\mathrm{GL}_2} = \bigcup_{\mu\in\mathbb{Z}^2} \mathcal{U}_\mu \tag{2.2}$$

is an ind-affine open covering. In particular, any $\mathcal{U}_\mu$ is an ind-affine ind-scheme, and the inclusion into $\mathbf{Gr}_{\mathrm{GL}_2}$ is representable by a quasi-compact open immersion.

Proof. Follows from the discussion in §2.5 and §2.6 below (see [Ric20, Proposition 2.4]). □

Proposition 2.21. *Under the natural transformation $L^{--}\mathrm{GL}_n \to \mathbf{Gr}_{\mathrm{GL}_n}, g \mapsto g \cdot \Lambda_0$ we have*

$$L^{--}\mathrm{GL}_2 \cong \mathcal{U}_0.$$

Proof. See [Ric20, Lem 2.7]. □

Identifying $L^{--}\mathrm{GL}_2$ with its image gives

$$L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu \cong \mathcal{U}_\mu. \tag{2.3}$$

2.5 The classical Grassmannian

Definition 2.22. Let M be a finite locally free $\mathbb{Z}$-module. The classical Grassmannian of M is the functor Rings $\to$ Sets given by

$$\mathrm{Grass}(M)(R) := \{N \subset M \otimes_{\mathbb{Z}} R \mid (M \otimes_{\mathbb{Z}} R)/N \text{ finite locally free } R\text{-module}\}.$$

Lemma 2.23. $\mathrm{Grass}(M)$ *is representable by a projective scheme.*

Proof. See [GW10, Proposition 8.14] □

Let $a \leq b \in \mathbb{Z}$. We define $M_{[a,b]} := z^a\Lambda_0/z^b\Lambda_0 \cong \mathbb{Z}^{2(b-a)}$. Then $M_{[a,b]}$ has the natural ordered $\mathbb{Z}$-basis

$$B = (z^a e_1, ..., z^{b-1}e_1, z^a e_2, ..., z^{b-1}e_2).$$

Definition 2.24. The *subfunctor of z-stable subspaces* is given by

$$\mathrm{Grass}^z(M_{[a,b]})(R) := \{N \in \mathrm{Grass}(M_{[a,b]})(R) \mid z \cdot N \subset N\}.$$

This is a closed subscheme of $\mathrm{Grass}(M_{[a,b]})$ [Ric20, Proof of Theorem 2.2].

Remark 2.25. Here we consider z as a $\mathbb{Z}$-linear nilpotent operator, thus the characteristic polynomial of z as an endomorphism is traceless.

Lemma 2.26. *Let $a \leq b \in \mathbb{Z}$. There is an isomorphism*

$$\begin{aligned} \mathbf{Gr}_{\mathrm{GL}_2,[a,b]} &\xrightarrow{\sim} \mathrm{Grass}^z(M_{[a,b]}) \\ \Lambda &\longmapsto \Lambda/z^b\Lambda_0. \end{aligned}$$

Proof. See the proof of Theorem 2.2 in [Ric20]. □

2.6 The standard open cover

Definition 2.27. Let $J \subset B$, then we denote by $\langle J \rangle$ the free $\mathbb{Z}$-submodule generated by the basis vectors in J. Note that J inherits a natural order from B. Then the subfunctor

$$V_J := \{N \in \mathrm{Grass}(M_{[a,b]}) \mid N \oplus \langle J \rangle = M_{[a,b]}\}$$

defines an affine open subscheme of a connected component of $\mathrm{Grass}(M_{[a,b]})$.

Lemma 2.28. *Let $s := |B|$ and $r := |J_c|$ where $J_c := B \backslash J$. Then*

$$V_J \cong \mathbb{A}^{r(s-r)}.$$

In particular, it is representable.

Proof. See [GW10, Lemma 8.13(2)]. □

In particular, let $\mu \in \mathbb{Z}^2$ such that $a \leq m_1, m_2 \leq b$, then we let

$$J(\mu) := (z^a e_1, ..., z^{m_1-1}e_1, z^a e_2, ..., z^{m_2-1}e_2).$$

Then the isomorphism in Lemma 2.26 restricts to

$$\mathcal{U}_\mu \cap \mathbf{Gr}_{\mathrm{GL}_2}^{[a,b]} \xrightarrow{\sim} V_{J(\mu)} \cap \mathrm{Grass}^z(M_{[a,b]}). \tag{2.4}$$

Consider a module $N \in V_J(R)$ in the target of the above map. Locally this is prescribed by the vectors in J_c. As above, we let $s := |B|$ and $r := |J_c|$. As B is an R-basis of $\mathrm{Grass}^z(M_{[a,b]})(R)$, N is represented by an R-matrix M_N of dimension $(s \times r)$, whose columns represent the basis vectors in J_c.

There is a natural order on J_c, inherited from the one on B. Let ρ_i be the index in B of the ith element of J_c, which is represented by the column κ_i. Via the ordering, we know that $\rho_1 < ... < \rho_r$. Then the ρ_ith row of M_N is given by $(0, ..., 0, 1, 0, ..., 0)$, with the 1 in the ith place.

Lemma 2.29. *Let (μ, λ) be a minimal degeneration of dominant cocharacters for* GL_2*, with $\mu = (m_1, m_2)$, and let $r := m_1 - m_2 + 2$. Then $N \in V_{J(\mu)} \cap \mathsf{Grass}^z(M_{[m_2-1, m_1+1]})(R)$ corresponds to an R-matrix of the form*

$$\left(\begin{array}{ccccc} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,r} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{r-1,1} & a_{r-1,2} & a_{r-1,3} & \dots & a_{r-1,r} \\ 1 & 0 & 0 & \dots & 0 \\ \hline c_1 & c_2 & c_3 & \dots & c_r \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{array}\right). \tag{2.5}$$

Proof. Here we have

$$\begin{aligned} J(\mu) &= (z^{m_2-1}e_1, ..., z^{m_1-1}e_1, z^{m_2-1}e_2) \\ \text{and } J_c(\mu) &= (z^{m_1}e_1, z^{m_2}e_2, ..., z^{m_1}e_2). \end{aligned}$$

Since $|J(\mu)| = |J_c(\mu)| = m_1 - m_2 + 2 := r$, we have $|B| = 2r$. Following the above discussion, a module in the target of the restricted isomorphism corresponds to a $(2r \times r)$ matrix of the form in the equation whose entries are in R. □

Remark 2.30. In (2.5), the horizontal line separates the $(r \times r)$-block corresponding to the basis vector e_1 above, from the block corresponding to e_2 below.

Remark 2.31. In particular, Λ_μ is represented by the matrix of the form in (2.5) with $a_{i,j} = c_j = 0$ for all $1 \le i \le r-1$, $1 \le j \le r$.

3. Minimal degeneration singularities over GL_2

3.1 Loop rotation over $\mathbb{Z}$

Denote by $\mathbb{G}_m$ the multiplicative group, which we consider as a group scheme over $\mathbb{Z}$. Note that $\mathbb{G}_m \cong \mathbb{A}^1\backslash\{0\}$.

Definition 3.1. Let $s \in \mathbb{G}_m(R)$. We define the map

$$\begin{aligned} s^{-1} : R(\!(z)\!) &\to R(\!(z)\!) \\ z &\mapsto s^{-1}z. \end{aligned}$$

This map induces a $\mathbb{G}_m$-action on $L\mathrm{GL}_2$, which descends to a $\mathbb{G}_m$-action on $\mathbf{Gr}_{\mathrm{GL}_2}$ via the quotient description in Proposition 2.15.

Definition 3.2. For a ring R and a point $\Lambda \in \mathbf{Gr}_{\mathrm{GL}_2}(R)$, the *orbit map* is given by

$$\begin{aligned} f_\Lambda : \mathbb{G}_m \otimes_{\mathbb{Z}} R &\to \mathbf{Gr}_{\mathrm{GL}_2} \\ s &\mapsto s^{-1}\Lambda. \end{aligned}$$

We say $\lim_{s\to 0} s^{-1}\Lambda$ *exists* if there exists a (necessarily unique) map $\tilde{f}_\Lambda : \mathbb{A}^1_R \to \mathbf{Gr}_{\mathrm{GL}_2}$ such that $\tilde{f}_\Lambda\Big|_{\mathbb{G}_m\otimes_{\mathbb{Z}} R} = f_\Lambda$. Then we set

$$\lim_{s\to 0} s^{-1}\Lambda := \tilde{f}_\Lambda(0) \in \mathbf{Gr}_{\mathrm{GL}_2}(R).$$

We will apply this to $\Lambda \in L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu$, where μ is a cocharacter for GL_2.

Proposition 3.3. *Let R be a ring, and let μ be a cocharacter for GL_2. The orbit map is constant on $\Lambda_\mu \in \mathbf{Gr}_{\mathrm{GL}_2}(R)$.*

Proof. Recall that $\mathbf{Gr}_{\mathrm{GL}_2} \cong (L\mathrm{GL}_2/L^+\mathrm{GL}_2)_{\text{ét}}$, and let $s \in \mathbb{G}_m(R)$. Now $s^{-1}\Lambda_\mu$ is represented by $\mu(zs^{-1}) \in L\mathrm{GL}_2(R)$. But

$$\mu(zs^{-1}) = \mu(z)\mu(s^{-1})$$

and $\mu(s^{-1}) \in L^+\mathrm{GL}_2(R)$.
Hence the orbit map is constant on Λ_μ, and so $\lim_{s\to 0} s^{-1}\Lambda_\mu = \Lambda_\mu$. □

Proposition 3.4. *Let R be a ring and let $\Lambda \in \mathcal{U}_0(R)$. Then*

$$\lim_{s\to 0} s^{-1}\Lambda = \Lambda_0.$$

Proof. Such a lattice Λ corresponds to a matrix $g \in \mathrm{GL}_2(R[z^{-1}])$ given by

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

where $a_i = \sum_{j=0} a_{i,j} z^{-j} \in R[z^{-1}]$ such that under the reduction map $z^{-1} \mapsto 0$, $g \mapsto 1_{2\times 2}$. This corresponds to the conditions

$$a_{1,0} = a_{4,0} = 1 \text{ and } a_{2,0} = a_{3,0} = 0.$$

Under the orbit map, $s^{-1}\Lambda$ is thus represented by the matrix with entries $a_i' = \sum_{j=0} a_{i,j} z^{-j} s^j$. Since all powers of s are nonnegative, we may thus naively take the limit from which we recover $1_{2\times 2}$. □

Lemma 3.5. *Let R be a ring and let $\Lambda \in L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu(R)$. Then*

$$\lim_{s\to 0} s^{-1}\Lambda = \Lambda_\mu.$$

Proof. We have that $\Lambda = g \cdot \Lambda_\mu$, where $g \in L^{--}\mathrm{GL}_2(R)$. Then

$$\lim_{s\to 0} s^{-1}\Lambda = \lim_{s\to 0} s^{-1} g \cdot \lim_{s\to 0} s^{-1}\Lambda_\mu = 1_{2\times 2} \cdot \Lambda_\mu = \Lambda_\mu$$

as required. □

3.2 Minimal degeneration singularities over GL_2

We adapt the argument of [Mül09, Theorem 9.2] to compute minimal degeneration singularities in the affine Grassmannian for GL_2.

Theorem 3.6. *Let (μ, λ) be a minimal degeneration of dominant cocharacters for GL_2 with $\mu = (m_1, m_2)$, and let $r := m_1 - m_2 + 2$. Then*

$$L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu \cap \mathbf{Gr}_{\mathrm{GL}_2, \leq \lambda} \cong \mathrm{Spec}\ \mathbb{Z}[w, x, y]/(w^r + xy).$$

Proof. Let $\Lambda \in (L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu \cap \mathbf{Gr}_{\mathrm{GL}_2,\leq\lambda})(R)$. We identify Λ with its image in (2.5):

$$\left(\begin{array}{ccccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,r} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{r-1,1} & a_{r-1,2} & a_{r-1,3} & \cdots & a_{r-1,r} \\ 1 & 0 & 0 & \cdots & 0 \\ \hline c_1 & c_2 & c_3 & \cdots & c_r \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{array}\right).$$

We know from Lemma 3.5 that

$$\Lambda \in L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu(R) \implies \lim_{s\to 0} s^{-1}\Lambda = \Lambda_\mu.$$

Now

$$\lim_{s\to 0} s^{-1}\Lambda = \lim_{s\to 0}\left(\begin{array}{ccccc} s^{1-m_2}a_{1,1} & s^{1-m_2}a_{1,2} & s^{1-m_2}a_{1,3} & \dots & s^{1-m_2}a_{1,r} \\ s^{-m_2}a_{2,1} & s^{-m_2}a_{2,2} & s^{-m_2}a_{2,3} & \dots & s^{-m_2}a_{2,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s^{1-m_1}a_{r-1,1} & s^{1-m_1}a_{r-1,2} & s^{1-m_1}a_{r-1,3} & \dots & s^{1-m_1}a_{r-1,r} \\ s^{-m_1} & 0 & 0 & \dots & 0 \\ \hline s^{1-m_2}c_1 & s^{1-m_2}c_2 & s^{1-m_2}c_3 & \dots & s^{1-m_2}c_r \\ 0 & s^{-m_2} & 0 & \dots & 0 \\ 0 & 0 & s^{-m_2-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & s^{-m_1} \end{array}\right)$$

$$\equiv \lim_{s\to 0}\left(\begin{array}{ccccc} s^{r-1}a_{1,1} & sa_{1,2} & s^2a_{1,3} & \dots & s^{r-1}a_{1,r} \\ s^{r-2}a_{2,1} & a_{2,2} & sa_{2,3} & \dots & s^{r-2}a_{2,r} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ sa_{r-1,1} & s^{3-r}a_{r-1,2} & \dots & a_{r-1,r-1} & sa_{r-1,r} \\ 1 & 0 & 0 & \dots & 0 \\ \hline s^{r-1}c_1 & sc_2 & s^2c_3 & \dots & s^{r-1}c_r \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{array}\right).$$

Since this must evaluate to Λ_μ (cf. Remark 2.31), all entries with s^α such that $\alpha \leq 0$ must be 0. Thus Λ is of the form:

$$\left(\begin{array}{ccccc} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,r} \\ a_{2,1} & 0 & a_{2,3} & \dots & a_{2,r} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{r-1,1} & 0 & \dots & 0 & a_{r-1,r} \\ 1 & 0 & \dots & \dots & 0 \\ \hline c_1 & c_2 & c_3 & \dots & c_r \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{array}\right).$$

Under our identification in Lemma 2.26, this represents a z-stable subspace. This corresponds to the condition

$z \cdot \Lambda = \Lambda \cdot X \subset \Lambda$ where $X \in \mathrm{GL}_r(R)$. More precisely:

$$\left(\begin{array}{ccccc} 0 & \dots & \dots & \dots & 0 \\ a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,r} \\ a_{2,1} & 0 & a_{2,3} & \dots & a_{2,r} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{r-1,1} & 0 & \dots & 0 & a_{r-1,r} \\ \hline 0 & \dots & \dots & \dots & 0 \\ c_1 & c_2 & c_3 & \dots & c_r \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccccc} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,r} \\ a_{2,1} & 0 & a_{2,3} & \dots & a_{2,r} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{r-1,1} & 0 & \dots & 0 & a_{r-1,r} \\ 1 & 0 & \dots & \dots & 0 \\ \hline c_1 & c_2 & c_3 & \dots & c_r \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{array}\right) \cdot X \tag{3.1}$$

so we may use the rows in the submatrix given by deleting all rows containing entries labelled $a_{i,j}$ or c_j to recover

$$X = \begin{pmatrix} a_{r-1,1} & 0 & \dots & 0 & a_{r-1,r} \\ c_1 & c_2 & c_3 & \dots & c_r \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Comparing rows on the left and right hand sides of (3.1), we have a recursive relation: for $2 \leq i \leq r-1$

$$\begin{aligned} (a_{i-1,1}, ..., a_{i-1,r}) &= (a_{i,1}, ..., a_{i,r}) \cdot X \\ &= (a_{r-1,1}, ..., a_{r-1,r}) \cdot X^{i-2}. \end{aligned}$$

Thus all $a_{i,j}$ are entirely determined by $a_{r-1,1}$ and $a_{r-1,r}$. We set $w := a_{r-1,1}$ and $x := a_{r-1,r}$. Then the $(r-1)$-th row is $(w, 0, ..., 0, x)$. At the same time, we have

$$(c_1, ..., c_r) \cdot X = (0, ..., 0),$$

i.e. for $2 \leq i \leq r$

$$c_i = c_2(-c_2)^{i-2}.$$

We set $y := c_1$ and $t := c_2$. Thus X becomes

$$\begin{pmatrix} w & 0 & \dots & 0 & x \\ y & t & -t^2 & \dots & t(-t)^{r-2} \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Since X represents z as an endomorphism, its trace is zero, as noted in Remark 2.25, hence

$$t = -w.$$

Thus

$$\Lambda = \left(\begin{array}{ccccc} w^{r-1} & x & wx & \dots & w^{r-2}x \\ w^{r-2} & 0 & x & \dots & w^{r-3}x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w & 0 & 0 & \dots & x \\ 1 & 0 & 0 & \dots & 0 \\ \hline y & -w & -w^2 & \dots & -w^{r-1} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{array}\right). \tag{3.2}$$

Finally, since

$$(w^{r-1}, x, wx, \dots, w^{r-2}x) \cdot X = (0, ..., 0),$$

the only remaining relation is

$$w^r + xy = 0.$$

Thus, (3.2) gives a point in the intersection $L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu \cap \mathbf{Gr}_{\mathrm{GL}_2, \leq \lambda}$, from which we see that

$$L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu \cap \mathbf{Gr}_{\mathrm{GL}_2, \leq \lambda} = \operatorname{Spec} \mathbb{Z}[w, x, y]/(w^r + xy).$$

□

We consider the preimage of (3.2) under (2.4). Let κ_i be the ith column of (3.2). It is easy to see that, for $i \geq 2$,

$$\kappa_{i+1} = (z + w)\kappa_i.$$

Then we only have two linearly independent columns, κ_1 and κ_2, and we restrict our attention to these. This gives us our preimage in the following way: recall that the horizontal line separates the e_1-entries from the e_2-entries. Each entry of the 2×2 matrix is given by summing all entries of a fixed basis vector inside a fixed column. In particular, the preimage of the above matrix is

$$M := z^{m_2 - 1} \begin{pmatrix} \sum_{k=0}^{r-1} w^{r-1-k} z^k & x \\ y & z - w \end{pmatrix} \tag{3.3}$$

4. Minimal degeneration singularities over PGL_2

We recall that the stated goal of [Mül09, Theorem 9.2] was to investigate minimal degeneration singularities in the affine Grassmannian for PGL_2 over a field k.

To summarise, the proof makes use of the morphism $\mathbf{Gr}_{\mathrm{SL}_2} \to \mathbf{Gr}_{\mathrm{PGL}_2}$ in order to take advantage of the concrete description of lattices in terms of the classical Grassmannian for SL_2. The problem is that the morphism

$$\mathbf{Gr}_{\mathrm{SL}_2, \leq (m,-m)} \to \mathbf{Gr}_{\mathrm{PGL}_2, \leq 2m}$$

is not étale over characteristic 2.

In order to compute minimal degeneration singularities in the affine Grassmannian for PGL_2, we instead consider the scheme-theoretic surjection

$$\text{Spec } A_r := L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu \cap \mathbf{Gr}_{\mathrm{GL}_2, \leq \lambda} \longrightarrow L^{--}\mathrm{PGL}_2 \cdot \Lambda_{r-2} \cap \mathbf{Gr}_{\mathrm{PGL}_2, \leq r} =: \text{Spec } B_r \tag{4.1}$$

induced by $\mathrm{GL}_2 \twoheadrightarrow \mathrm{PGL}_2$, where (μ, λ) and r are as in the setting of Theorem 3.6, and generalise the argument in [HLRed, Appendix B] to identify B_r as a subring of A_r.

4.1 The adjoint representation of GL_2

As groups, we have

$$\begin{array}{ccc} \mathrm{GL}_2 & \xrightarrow{\quad \mathrm{Ad} \quad} & \mathrm{Aut}(\mathfrak{gl}_2) = \mathrm{GL}_4 \\ & \searrow\!\!\!\!\searrow \quad \nearrow & \\ & \mathrm{PGL}_2 & \end{array}$$

Under the ordered basis $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of $\mathfrak{gl}_2$ the map Ad is represented by the matrix

$$\mathrm{ad} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 & 0 & 0 \\ 0 & ad+bc & -ac & bd \\ 0 & -2ab & a^2 & -b^2 \\ 0 & 2cd & -c^2 & d^2 \end{pmatrix}. \tag{4.2}$$

This induces a closed immersion $\mathrm{PGL}_2 \hookrightarrow \mathrm{GL}_4$ which further induces a closed immersion $\mathbf{Gr}_{\mathrm{PGL}_2} \hookrightarrow \mathbf{Gr}_{\mathrm{GL}_4}$ [Ric20, Proposition 3.6]. Via the categorical antiequivalence between rings and affine schemes [GW10, Theorem 2.35], we see that $L^{--}\mathrm{PGL}_2 \cdot \Lambda_{r-2} \cap \mathbf{Gr}_{\mathrm{PGL}_2, \leq r}$ is the spectrum of some subalgebra of A_r. This leads to our main result.

Theorem 4.1. *Let (μ, λ) and r be as in the statement of Theorem 3.6. Then the minimal degeneration singularity over PGL_2 is an affine scheme*

$$L^{--}\mathrm{PGL}_2 \cdot \Lambda_{r-2} \cap \mathbf{Gr}_{\mathrm{PGL}_2, \leq r} \cong \text{Spec } B_r$$

where B_r is the subalgebra of $A_r = \mathbb{Z}[w,x,y]/(w^r + xy)$ generated by $x, y, wx, wy, w^2, 2w$.

Proof. Let B_r be the subalgebra corresponding to $L^{--}\mathrm{PGL}_2 \cdot \Lambda_{r-2} \cap \mathbf{Gr}_{\mathrm{PGL}_2, \leq r}$. In order to describe B_r as a subring of A_r, we take the image of M in (3.3) under the map in (4.2), then investigate the entries as polynomials in z, z^{-1} with coefficients in $\mathbb{Z}[w,x,y]$. Having identified the minimal generating set Γ of these coefficients, we have that $B_r = \mathbb{Z}[\Gamma]$ as a subalgebra of A_r.

Cancelling out the common factor of z^{2m-2}, the image of M under the adjoint map is

$$\frac{1}{z^r}\begin{pmatrix} z^r & 0 & 0 & 0 \\ 0 & z^r + 2xy & -y\left(\sum_{k=0}^{j} w^{j-k} z^k\right) & x(z-w) \\ 0 & -2x\left(\sum_{k=0}^{j} w^{j-k} z^k\right) & \sum_{k=0}^{j} w^{2(j-k)} z^{2k} + 2\sum_{0\leq k<l\leq j} w^{2j-k-l} z^{k+l} & -x^2 \\ 0 & 2y(z-w) & -y^2 & z^2 - 2zw + w^2 \end{pmatrix}$$

where we let $j := r - 1$ for convenience. By inspection it is clear that B_r is generated by the elements $x, y, wx, wy, w^2, 2w$. □

Remark 4.2. In the case where $r = 2$, we have that $w^2 = xy$, and we recover [HLRed, Proposition B.1]; namely, B_2 is the subalgebra of $A_2 = \mathbb{Z}[w,x,y]/(w^2 + xy)$ generated by $x, y, wx, wy, 2w$.

We also generalise [HLRed, Corollary B.2].

Corollary 4.3. *1. The ring $B_r \otimes \mathbb{F}_2$ is not reduced. Its reduction $(B_r \otimes \mathbb{F}_2)_{red}$ is isomorphic to the subring of $A_r \otimes \mathbb{F}_2 \cong \mathbb{F}_2[w,x,y]/(w^r + xy)$ generated by x, y, wx, wy, and w^2.*

2. The ring $(B_r \otimes \mathbb{F}_2)_{\mathrm{red}}$ is neither normal nor seminormal. Further, $A_r \otimes \mathbb{F}_2$ is both its normalisation and its seminormalisation.

Proof. 1. We repeat the proof here, now for all r.

First, note that $w \notin B_r$, so $2w \neq 0$ in $B_r \otimes \mathbb{F}_2$. On the other hand, $w^2 \in B_r$ (via $xy = w^2$ for the case $r = 2$ or otherwise directly) so $(2w)^2 = 4w^2 = 0$. Hence $B_r \otimes \mathbb{F}_2$ is not reduced.

The image of $(B_r \otimes \mathbb{F}_2)_{\mathrm{red}}$ in $A_r \otimes \mathbb{F}_2 \cong \mathbb{F}_2[w,x,y]/(w^r + xy)$ is the subring generated by the elements listed above. The kernel of the map is nilpotent as the spectra of both rings are irreducible of Krull dimension 2. Hence $(B_r \otimes \mathbb{F}_2)_{\mathrm{red}}$ identifies with the stated subring.

2. That $A_r \otimes \mathbb{F}_2$ is normal follows from [Mül09, §6.6–6.8]. In particular, the Schubert variety $\mathbf{Gr}_{\mathrm{GL}_2, \leq\lambda}$ is normal, hence so is $L^{--}\mathrm{GL}_2 \cdot \Lambda_\mu \cap \mathbf{Gr}_{\mathrm{GL}_2, \leq\lambda}$.

That $(B_r \otimes \mathbb{F}_2)_{\mathrm{red}}$ is not seminormal (hence not normal) follows from [LV81, Lemma 1.4(3)]. Let $r \in \mathbb{Z}_{\geq 2}$. Then $w^r = xy$, $w^{r+1} = (wx)y \in (B_r \otimes \mathbb{F}_2)_{\mathrm{red}}$ but $w \notin (B_r \otimes \mathbb{F}_2)_{\mathrm{red}}$.

The same argument shows that the seminormalisation C of $(B_r \otimes \mathbb{F}_2)_{\mathrm{red}}$ must contain w hence $C \supset A_r \otimes \mathbb{F}_2$, and since $A_r \otimes \mathbb{F}_2$ is seminormal and normal, $C = A_r \otimes \mathbb{F}_2$. Hence $A_r \otimes \mathbb{F}_2$ is both the normalisation and the seminormalisation of $(B_r \otimes \mathbb{F}_2)_{\mathrm{red}}$.

□

4.2 Presentations of the subrings B_r over a field of characteristic 2

Here we fix a field k of characteristic 2. Following Corollary 4.3, we wish to compute a presentation of $(B_r \otimes k)_{\mathrm{red}}$. The essential idea is to compute a minimal generating set of the kernel of the map $k[x,y,a,b,c] \to A_r \otimes k$

given by:

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y \\ a &\mapsto wx \\ b &\mapsto wy \\ c &\mapsto w^2 \end{aligned}$$

Our starting point is the following lemma.

Lemma 4.4. *The k-algebra $(B_2 \otimes \mathbb{F}_2)_{\mathrm{red}}$ has the following presentation*

$$k[x, y, a, b]/(ya + xb, xy^3 + b^2, x^2y^2 + ab, x^3y + a^2).$$

Proof. See [HLRed, §1]. □

For general $r > 2$, we work with SageMath using the interface to Singular to compute a Gröbner basis of the kernel, from which we extract a minimal generating set (see Appendix A for the code). Testing numerous cases has led us to the following conjecture.

Conjecture 4.5. Let $r > 2$. The k-algebra $(B_r \otimes \mathbb{F}_2)_{\mathrm{red}}$ has the following presentation:

1. If r is even, then:

$$(B_r \otimes \mathbb{F}_2)_{\mathrm{red}} \cong k[x, y, a, b, c]/(ya + xb, y^2c + b^2, xyc + ab, x^2c + a^2, c^{\frac{r}{2}} + xy).$$

2. If r is odd, then:

$$(B_r \otimes \mathbb{F}_2)_{\mathrm{red}} \cong k[x, y, a, b, c]/(ya + xb, y^2c + b^2, xyc + ab, x^2c + a^2, c^{\frac{r+1}{2}} + xb, bc^{\frac{r-1}{2}} + xy^2, ac^{\frac{r-1}{2}} + x^2y).$$

A.

```
F = GF(2)
var("m1", "m2", "r")
def Basis(m1, m2):
    r = m1 - m2 + 2
    if r < 2:
        return "Invalid input"
    #Special case: if r = 2, we only need 4 variables
    elif r == 2:
        ret = singular.eval('ring R = 2, (x,y,a,b), dp')
        ret = singular.eval('ring S = 2, (x,y,w), dp')
        ret = singular.eval('ideal I = w^('+str(r)+') + x*y')
        ret = singular.eval('qring Q = std(I);')
        ret = singular.eval('map phi = R, x,y,w*x,w*y;')
        ret = singular.eval('setring R;')
        B = singular.eval('mstd(kernel(Q, phi))[2];')
    else:
        ret = singular.eval('ring R = 2, (x,y,a,b,c), dp')
        ret = singular.eval('ring S = 2, (x,y,w), dp')
        ret = singular.eval('ideal I = w^('+str(r)+') + x*y')
        ret = singular.eval('qring Q = std(I);')
        ret = singular.eval('map phi = R, x,y,w*x,w*y,w^2;')
        ret = singular.eval('setring R;')
        B = singular.eval('mstd(kernel(Q, phi))[2];')
    print("r is "+str(r))
    print("The basis is")
    print(B)
```

This is also available as a SageMath cell here.

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