TECHNISCHE UNIVERSITÄT DARMSTADT

FACHBEREICH MATHEMATIK

Master's Thesis

# Drinfeld's Lemma for Schemes

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# Contents

# 1 Introduction

One of the central objects of basic algebraic topology is the topological fundamental group $\pi_1(X, x)$ associated to a topological space $X$ and a point $x \in X$. It can be defined as the set of homotopy classes of loops in $X$ with base point $x$, which becomes a group under composition. A beautiful feature of $\pi_1(X, x)$ is that it classifies covering spaces of $X$ : If $X$ is sufficiently nice, there is an equivalence of categories

$$(\text{Covering spaces of } X) \cong (\pi_1(X, x)\text{-Sets}), \tag{1}$$

where the right hand side denotes the category whose objects are sets equipped with a group action by $\pi_1(X, x)$ and whose morphisms are equivariant maps with respect to that action. The equivalence is realized by the functor sending a covering space $p : Y \to X$ to the fibre $p^{-1}(x)$, on which $\pi_1(X, x)$ acts in a natural way [Hat02, p. 68-72].
In [SGA1, Exposé V], Grothendieck developed an analogous correspondence in the world of schemes. Here, finite étale morphisms play the role of (finite degree) covering spaces and the topological fundamental group is replaced by the étale fundamental group. In fact, Grothendieck gave a general framework for capturing correspondences similar to (1) by introducing Galois categories. If $\mathcal{C}$ is a category and $F : \mathcal{C} \to (\text{Finite sets})$ a functor, the automorphism group $G := \mathrm{Aut}(F)$ of $F$ becomes a profinite group when endowed with the subspace topology of $\prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Aut}(F(X))$, where the $\mathrm{Aut}(F(X))$ have the discrete topology. For any object $X$ of $\mathcal{C}$, the set $F(X)$ can be endowed with a natural continuous group action by $G$. The pair $(\mathcal{C}, F)$ is a Galois category if and only if $F$ induces an equivalence of categories

$$\mathcal{C} \cong (G\text{-FSets}),$$

where the right hand side denotes the category whose objects are finite sets $E$ equipped with a continuous group action by $G$. Grothendieck proved that if $X$ is a connected scheme and $\bar{x}$ a geometric point of $X$, i.e., a morphism $\mathrm{Spec}(k(\bar{x})) \to X$, where $k(\bar{x})$ is an algebraically closed field, the category $\mathrm{FEt}(X)$ of finite étale covers of $X$ together with the fibre functor $F_{\bar{x}} := - \times_X \mathrm{Spec}(k(\bar{x}))$ is a Galois category, see Theorem 2.22 below. The group $\pi_1(X, \bar{x}) := \mathrm{Aut}(F_{\bar{x}})$ is called the étale fundamental group of $X$, which is independent of $\bar{x}$ up to non-canonical isomorphism.
Now, let us consider the following question : For the topological fundamental group, it is straightforward to see that

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$$

for any path-connected topological spaces $X, Y$ and $x \in X$, $y \in Y$, see [Hat02, Proposition 1.12]. A natural question to ask is when the étale fundamental group of a (fibre) product of two connected schemes is isomorphic to the product of the individual étale fundamental groups. Of course, this is false in general as the fibre product of two non-empty connected schemes can be empty. A classical result due to Grothendieck is the following (slightly modified from [SGA1, Exposé X, Corollaire 1.7]) :

**Proposition 1.1.** *Let $X_1$, $X_2$ be connected qcqs schemes over an algebraically closed field $k$ such that $X_1 \to k$ is proper. Then, $X_1 \times_k X_2$ is connected and for any geometric point $\bar{z}$ of $X_1 \times_k X_2$ (and thus of $X_1$ and $X_2$), the natural map*

$$\pi_1(X_1 \times_k X_2, \bar{z}) \xrightarrow{\sim} \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z}) \tag{2}$$

*coming from base change along the two projections $X_1 \times_k X_2 \to X_1$, $X_1 \times_k X_2 \to X_2$ is an isomorphism.*

In the following, we will refer to formula (2) as the "Künneth formula" for $\pi_1$.
In their Berkeley Lectures [SW20, Lemma 16.1.1], Scholze and Weinstein claim that the Künneth formula still holds if the properness requirement is relaxed :

**Proposition 1.2.** *Let $X_1$, $X_2$ be connected qcqs schemes over an algebraically closed field $k$ such that $X_1 \to k$ is $\pi_1$-proper. Then, $X_1 \times_k X_2$ is connected and for any geometric point $\bar{z}$ of $X_1 \times_k X_2$ (and thus of $X_1$ and $X_2$), the natural map*

$$\pi_1(X_1 \times_k X_2, \bar{z}) \xrightarrow{\sim} \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z})$$

*is an isomorphism of topological groups.*

Compared to Proposition 1.1, the assumption of properness has been replaced by $\pi_1$-properness, a non-standard notion which was introduced by Kedlaya [Ked19, Definition 4.1.12]. A scheme $X$ over an algebraically closed field $k$ is called $\pi_1$-proper, if the category $\mathrm{FEt}(X)$ of finite étale covers of $X$ is invariant under base extension of algebraically closed fields $k \to k'$, see Section 3.1 below. It is a non-trivial fact that proper maps are $\pi_1$-proper [Stacks, Tag 0A49]. Moreover, any scheme over an algebraically closed field of characteristic zero is $\pi_1$-proper [SW20, Section 16.1], from which one deduces

**Corollary 1.3.** *Let $X_1$, $X_2$ be connected qcqs schemes over an algebraically closed field $k$ of characteristic $0$. Then, $X_1 \times_k X_2$ is connected and the natural map*

$$\pi_1(X_1 \times_k X_2, \bar{z}) \xrightarrow{\sim} \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z})$$

*is an isomorphism of topological groups.*

In positive characteristic, however, this fails in general, as already the example of the affine line shows, see Example 3.14 below. As a replacement can serve the following result, which we will refer to as Drinfeld's Lemma :

**Theorem 1.4.** *Let $X_1$, $X_2$ be connected qcqs schemes over $\mathbb{F}_p$. Then, $X_1 \times_{\mathbb{F}_p} X_2$ is $\varphi_1$-connected and for any geometric point $\bar{z}$ of $X_1 \times_{\mathbb{F}_p} X_2$ (and thus of $X_1$ and $X_2$), the natural map*

$$\pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z}) \xrightarrow{\sim} \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z})$$

*is an isomorphism of topological groups.*

The key innovation here is the group $\pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z})$, which is different from the classical étale fundamental group of $X_1 \times_{\mathbb{F}_p} X_2$. It belongs to the Galois category $\mathrm{FEt}(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1)$, whose objects are finite étale covers of

$X_1 \times_{\mathbb{F}_p} X_2$ equipped with an action compatible with the first partial Frobenius $\varphi_1$ on $X_1 \times_{\mathbb{F}_p} X_2$, or equivalently, an action compatible with the second partial Frobenius $\varphi_2$.

While the appearance of Frobenius morphisms might seem surprising at first glance, the example $X_1 := \mathrm{Spec}(\mathbb{F}_{p^2})$, $X_2 := \mathrm{Spec}(\overline{\mathbb{F}}_p)$ for an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ already gives some insight. Namely, by the Chinese Remainder Theorem, we have

$$\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \cong \overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p, \tag{3}$$

hence

$$X_1 \times_{\mathbb{F}_p} X_2 \cong \mathrm{Spec}(\overline{\mathbb{F}}_p) \coprod \mathrm{Spec}(\overline{\mathbb{F}}_p),$$

which is disconnected. However, one can show that, under the isomorphism (3), the map $F \otimes_{\mathbb{F}_p} \mathrm{id}$, where $F$ is the Frobenius map $a \mapsto a^p$ on $\mathbb{F}_{p^2}$, corresponds to the map on $\overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p$ which flips entries. Hence $X_1 \times_{\mathbb{F}_p} X_2$ it as least connected in the sense that $\mathrm{Spec}(F \otimes_{\mathbb{F}_p} \mathrm{id})$ acts transitively on it, in the terminology we will introduce later, we say that $X_1 \times_{\mathbb{F}_p} X_2$ is $\varphi_1$-connected.

Theorem 1.4 dates back to a result by the Ukrainian mathematician Vladimir Drinfeld [Dri80, Theorem 2.1], who treated the special case of the product of a smooth projective curve over $\mathbb{F}_p$ with itself. It attracted attention as Drinfeld used it in his celebrated proof of the global Langlands conjecture for $\mathrm{GL}_2$ over function fields, also see [BKDS$^+$03, Section 10.4.1]. Since then, different versions and generalizations have been established, see e.g. [Laf97, IV, Theorem 4], [Lau04, Lemma 8.1.1], [Laf18, Lemme 8.2] or [Ked19, Theorem 4.2.12]. The outreach of Drinfeld's Lemma even goes beyond the world of schemes : Scholze and Weinstein introduced an analogue for diamonds [SW20, Theorem 16.3.1]. In fact, also our formulation above is due to Scholze and Weinstein [SW20, Theorem 16.2.4], who did not give a complete proof, but indicated that a proof is possible by the same methods as for their diamond analogue.

## Aim and structure

The goal of this thesis is to fill these gaps and give a detailed proof of Theorem 1.4. Besides [SW20, Lecture 16], our main reference for this are Kedlaya's notes [Ked19, Sections 4.1,4.2].

Let us give a brief overview of our proof strategy. In fact, our proof will be very much related to the case of characteristic zero. As described above, Corollary 1.3 can be proven by first showing that any characteristic zero scheme over an algebraically closed field is $\pi_1$-proper and then applying Proposition 1.2. Our idea to prove Drinfeld's Lemma is quite similar : We show that any scheme over $\mathbb{F}_p$ fulfills a condition which can be viewed as a "Drinfeld version" of $\pi_1$-properness. From this, we conclude Drinfeld's Lemma by adapting the proof of Proposition 1.2. Before all this, we will of course need to introduce some formalism, such as $\varphi$-connectedness and the categories $\mathrm{FEt}(X/\varphi)$.

Unfortunately, Scholze and Weinstein neither gave a proof for Proposition 1.2. Hence, a big part of this thesis will be to establish that statement. Our strategy for this is based on Scholze's and Weinstein's diamond analogue [SW20, Proposition 16.3.3] as well as Kedlaya's notes [Ked19, Section 4.1] and resembles the original proof of Proposition 1.1. Namely, Grothendieck deduced the Künneth formula from a "homotopy exact sequence" of étale fundamental groups that can be associated to any flat proper morphism of finite presentation, see [SGA1, Exposé X.1]. The main ingredient for that exact sequence is in turn Stein factorization of proper maps, see [Stacks, Tag 03H2]. Hence, our main challenge will be to establish an analogue of Stein factorization for $\pi_1$-proper maps, at least in our very special use case. Luckily, once Proposition 1.2 is established, the Drinfeld analogue is rather straightforward.

In more detail, this thesis is structured as follows : In Chapter 2, we will present the necessary preliminaries which are mostly a recollection of Grothendieck's theory of Galois categories and the étale fundamental group, including some examples.

Then, in Chapter 3, we will prove Proposition 1.2 : First, in Section 3.1, we will shed more light on the notion of $\pi_1$-properness. Then, in Section 3.2, we will, under the assumptions of Proposition 1.2, construct an exact sequence

$$\pi_1(X_1 \times_k \bar{x}, \bar{z}) \longrightarrow \pi_1(X_1 \times_k X_2, \bar{z}) \longrightarrow \pi_1(X_2, \bar{z}) \longrightarrow 1 \tag{4}$$

of étale fundamental groups. Note that $X_1 \times_k \bar{x} \cong (X_1 \times_k X_2) \times_{X_2} \bar{x}$ is canonically isomorphic to the fibre of $X_1 \times_k X_2 \to X_2$ at the geometric point $\bar{x} \to X_2$. Hence, we may indeed think of the above sequence as a homotopy exact sequence for the projection $X_1 \times_k X_2 \to X_2$. The proof of that sequence will involve a statement similar to Stein factorization of proper maps, which Section 3.3 is dedicated to. After the proof of Proposition 1.2 is complete, Section 3.4 will review a standard example for failure of $\pi_1$-properness in positive characteristic.

In Chapter 4, we will then tackle the proof of Drinfeld's Lemma. For this, we first review different types of Frobenius morphisms on schemes over $\mathbb{F}_p$. In Section 4.1, we investigate the notion of $\varphi$-connectedness, which generalizes usual connectedness to also take into account the action of a homeomorphism, and prove that the fibre product $X_1 \times_{\mathbb{F}_p} X_2$ of any two connected schemes over $\mathbb{F}_p$ is $\varphi_1$-connected (or equivalently, $\varphi_2$-connected), if $\varphi_1$ and $\varphi_2$ are the partial Frobenii on $X_1 \times_{\mathbb{F}_p} X_2$. In Section 4.2, we introduce categories $\mathrm{FEt}(X/\varphi)$ for a scheme $X$ and a universal homeomorphism $\varphi : X \to X$, which we prove to be a Galois categories if $X$ is $\varphi$-connected. As a special case, we will introduce the abovementioned category $\mathrm{FEt}(X_1 \times_{\mathbb{F}_p} X_2/\text{p.Fr.})$ and its fundamental group.

After these preparations, we will finally start to prove Drinfeld's Lemma in Section 4.3, where we establish the abovementioned analogue of $\pi_1$-properness in the Drinfeld setting. From that on, we proceed in analogy to Chapter 3 to deduce Drinfeld's Lemma : In Section 4.4, we prove the Drinfeld analogue of the $\pi_1$-proper Stein factorization from Section 3.3. In Section 4.5, we construct the Drinfeld analogue of the homotopy exact sequence (4). And finally, in Section 4.6, we conclude Drinfeld's Lemma.

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Even though it brought me to my (co)limits, working on this thesis was a fulfilling task. I am thankful for this experience.

# 2 Conventions and preliminaries

First, we fix some conventions which hold throughout this thesis :

**Conventions 2.1.**

- A scheme $X$ is said to be connected if its underlying topological space is connected (in particular, non-empty).
- When we speak of a scheme $X$ over a field $k$, we actually mean a morphism of schemes $X \to \mathrm{Spec}(k)$. Sometimes, for ease of notation, we will denote by $X \to k$ the structure morphism $X \to \mathrm{Spec}(k)$.
- If $X_1$ and $X_2$ are schemes over a field $k$, we use the notation $X_1 \times_k X_2 := X_1 \times_{\mathrm{Spec}(k)} X_2$. If $X_2 = \mathrm{Spec}(k')$ is the spectrum of a field extension $k \to k'$, we occasionally use the notation $X_{1,k'}$ to denote $X_1 \times_k X_2$.
- Similarly, if $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ are morphisms of schemes over $k$, we use the notation $f \times_k g$ for the canonical morphism $f \times_{\mathrm{Spec}(k)} g : X_1 \times_k X_2 \to Y_1 \times_k Y_2$.
- Similarly, we abuse notation to denote by $\mathrm{id}_R$ both the identity $\mathrm{Spec}(R) \to \mathrm{Spec}(R)$ and the identity $R \to R$. We will do the same for the Frobenius morphism $F_R : a \mapsto a^p$ on a ring $R$ of characteristic $p$.

The rest of this chapter consists of a recollection of preliminaries we will need later. We will first review the theory of Galois categories and the étale fundamental group, which is mostly due to Grothendieck [SGA1, Exposé V]. Besides the original reference, [MA67], [Len85], [Sza09] or [Stacks, Tag 0BMQ] are detailed introductory texts.

## 2.1 Galois categories

We will mainly follow [Stacks, Tag 0BMQ]. Note that their axioms of a Galois category are slightly stronger than Grothendieck's, but turn out to be equivalent, cf. [Stacks, Remark below Tag 0BMY].

**Definition 2.2** ([Stacks, Tag 0BMY])**.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to$ (Finite Sets) be a functor. The pair $(\mathcal{C}, F)$ is a *Galois category* if

1. $\mathcal{C}$ has finite limits and finite colimits,
2. every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects,
3. $F$ reflects isomorphisms, i.e., a morphism $f : X \to Y$ between $X, Y \in \mathrm{Ob}(\mathcal{C})$ is an isomorphism if and only if $F(f) : F(X) \to F(Y)$ is a bijection,
4. $F$ is exact, i.e., it commutes with finite limits and finite colimits.

Here we say $X \in \mathrm{Ob}(\mathcal{C})$ is *connected* if it is not initial and for any monomorphism $Y \to X$ either $Y$ is initial or $Y \to X$ is an isomorphism. If $(\mathcal{C}, F)$ is a Galois category, we will call the functor $F$ the *fundamental functor of* $\mathcal{C}$.

We remark that by [Stacks, Tag 0BN5], a fundamental functor of $\mathcal{C}$ is uniquely determined up to (non-unique) isomorphism. Indeed, Galois categories have a lot of very special properties, a list of which is given in [Stacks, Tag 0BN0]. One of them is that if $a, b : X \to Y$ are two morphisms in $\mathcal{C}$ with connected source $X$, then they agree as soon as $F(a), F(b) : F(X) \to F(Y)$ agree on a single element of $F(X)$. In particular, since any object of $\mathcal{C}$ can be written as a coproduct of connected objects, the fundamental functor is faithful.

**Definition and Remark 2.3.** Let $(\mathcal{C}, F)$ be a Galois category and $X$ a connected object of $\mathcal{C}$. Then the group of automorphisms $\mathrm{Aut}_{\mathcal{C}}(X)$ of $X$ in $\mathcal{C}$ naturally acts on $F(X)$ by

$$\begin{aligned} \mathrm{Aut}_{\mathcal{C}}(X) \times F(X) &\to F(X) \\ (a, s) &\mapsto F(a)(s). \end{aligned}$$

Using the paragraph preceding this remark, we see that for any $s \in F(X)$, the map

$$\begin{aligned} \mathrm{Aut}_{\mathcal{C}}(X) &\to F(X) \\ a &\mapsto F(a)(s) \end{aligned}$$

is injective. If it is also surjective for all $s \in F(X)$, we call $X$ a *Galois object* of $\mathcal{C}$ and $\mathrm{Aut}_{\mathcal{C}}(X)$ the *Galois group* of $X$. Observe that the following are equivalent :

(i) $X$ is a Galois object,

(ii) $\# \mathrm{Aut}_{\mathcal{C}}(X) = \# F(X)$,

(iii) $\mathrm{Aut}_{\mathcal{C}}(X)$ acts transitively on $F(X)$,

(iv) $\mathrm{Aut}_{\mathcal{C}}(X)$ acts simply transitively $F(X)$.

We will need the following extended version of [Stacks, Tag 0BN0 (8)], which states that morphisms in a Galois category respect connected components :

**Lemma 2.4.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $X \to Y$ be a morphism in $\mathcal{C}$. Suppose $X = \coprod_{i=1}^{n} X_i$, $Y = \coprod_{j=1}^{m} Y_j$ are the decompositions of $X$ and $Y$ into connected objects, respectively. Then, for each $i \in \{1, ..., n\}$, there exists a unique $\alpha(i) \in \{1, ..., m\}$ such that $X_i \to X \to Y$ factors through the coprojection $Y_{\alpha(i)} \to Y$. Moreover, for each $j \in \{1, ..., m\}$, we have an isomorphism $X \times_Y Y_j \cong \coprod_{i \in \alpha^{-1}(\{j\})} X_i$.*

*Proof.* First of all, any of the coprojections $Y_k \to Y$ is a monomorphism, since $F$ sends it to the coprojection $F(Y_k) \to \coprod_{j=1}^{m} F(Y_j)$, as $F$ commutes with finite coproducts. In the category of sets, coprojections are injective hence monomorphisms. By [Stacks, Tag 08LR] and since $F$ preserves isomorphisms, we conclude that $Y_k \to Y$ is a monomorphism.
For the moment, fix an index $i$. For any $j$, the morphism $X_i \times_Y Y_j \to X_i$ is a monomorphism as a base change of the monomorphism $Y_j \to Y$. Therefore, since $X_i$ is a connected object, $X_i \times_Y Y_j$ is initial or $X_i \times_Y Y_j \to X_i$ is an isomorphism. The latter happens for a unique $\alpha(i) \in \{1, ..., m\}$, as we may compute

$$F(X_i) \cong F(X_i \times_Y Y) \cong F(X) \times_{F(Y)} F(Y) \cong F(X_i) \times_{F(Y)} F\left(\coprod_{j=1}^{m} Y_j\right) \cong F(X_i) \times_{F(Y)} \left(\coprod_{j=1}^{m} F(Y_j)\right) \cong \coprod_{j=1}^{m} F(X_i \times_Y Y_j),$$

where we used that $F$ is exact and that in the category of sets, fibre products commute with filtered colimits. Now $\coprod_{j=1}^{m} F(X_i \times_Y Y_j)$ is a disjoint union of sets, so $F(X_i \times_Y Y_j) \cong F(X_i)$ for a unique index $\alpha(i) := j \in \{1, ..., m\}$. As $F$ reflects isomorphisms, $\alpha(i)$ is the unique index such that $X_i \times_Y Y_j \to X_i$ is an isomorphism.

Now, if we fix $j \in \{1, ..., m\}$, a similar calculation as above shows that

$$X \times_Y Y_j \cong \coprod_{i=1}^{n} (X_i \times_Y Y_j) \cong \coprod_{i \in \alpha^{-1}(\{j\})} X_i \tag{5}$$

by definition of $\alpha$. Now, note that the statement that $X_i \to X \to Y$ factors through $Y_j$ is equivalent to the statement that $X_i \to X$ factors through $X \times_Y Y_j$, which, by (5) is possible precisely for $j = \alpha(i)$. □

For a moment, let $\mathcal{C}$ be any category and $F$ a functor from $\mathcal{C}$ to (Finite sets). Note that the automorphism group $\mathrm{Aut}(F)$ of the functor $F$ naturally acts on any set $F(X)$, where $X \in \mathrm{Ob}(\mathcal{C})$, by

$$\mathrm{Aut}(F) \times F(X) \to F(X)$$
$$((\gamma_Y)_{Y \in \mathrm{Ob}(\mathcal{C})}, s) \mapsto \gamma_X(s)$$

In fact, $\mathrm{Aut}(F)$ can be given the structure of a profinite topological group [Stacks, Tag 0BMR] such that the above action is continuous if the $F(X)$ are endowed with the discrete topology. Let us denote by ($G$-FSets) the category of finite sets (endowed with the discrete topology) together with a continuous group action by $G$. The above shows that $F$ induces a functor $\mathcal{C} \to$ ($G$-FSets). The distinctive property of Galois categories is that this functor is an equivalence :

**Theorem 2.5.** *Let $\mathcal{C}$ be a category and $F$ a functor $\mathcal{C} \to$ (Finite sets). Put $G := \mathrm{Aut}(F)$. Then, $(\mathcal{C}, F)$ is a Galois category if and only if $F$ induces an equivalence of categories $\mathcal{C} \cong$ ($G$-FSets).*

*Proof.* [Stacks, Tag 0BN4]. The converse is straightforward to check, see e.g. [SGA1, Exposé V.4]. □

In the situation of the above theorem, we call $G := \mathrm{Aut}(F)$ the *fundamental group* of $\mathcal{C}$.

**Remark 2.6.** Let $(C, F)$ be a Galois category. The equivalence from Theorem 2.5 restricts to the following bijections :

$$\{\text{Connected objects of } \mathcal{C}\}/_\sim \cong \{\text{Transitive finite } G\text{-sets}\}/_\sim \cong \{\text{Open subgroups of } G\}/_\sim$$
$$\{\text{Galois objects of } \mathcal{C}\}/_\sim \cong \{\text{Finite quotient groups of } G\}/_\sim \cong \{\text{Open normal subgroups of } G\}$$

Here, the equivalence relation "$\sim$" on the top right set means "identical up to conjugacy by an element of $G$", whereas in the other cases it denotes isomorphism in the respective categories. For the upper left bijection, see the proof of [Stacks, Tag 0BN4]. The upper right map sends a transitive finite $G$-set to the stabilizer of some element, which yields a well-defined map after taking conjugacy classes. Note that stabilizers of a continuous group action are open. The map is a bijection by the orbit-stabilizer-theorem. Note that the inverse map is indeed well-defined, since all open subgroups of a quasi-compact topological group have finite index. For the bottom left bijection, see [Stacks, Tag 03SF]. The bottom right map sends a finite quotient group to the stabilizer of its neutral element, it is a bijection again by orbit-stabilizer theorem.

We may use Galois objects and their Galois groups to describe the profinite structure of the fundamental group :

**Lemma 2.7.** *Let $(\mathcal{C}, F)$ be a Galois category and set $G := \mathrm{Aut}(F)$. Denote by $\mathcal{I}$ the set of isomorphism classes of Galois objects of $\mathcal{C}$ and let $X_i$, $i \in \mathcal{I}$, be representatives. We define a partial order on $\mathcal{I}$ by defining $i \leq j$ if and only if $X_j$ dominates $X_i$, i.e., there exists a map $X_j \to X_i$ in $\mathcal{C}$. Then, we have an isomorphism*

$$G \cong \lim_{\mathcal{I}} \mathrm{Aut}_{\mathcal{C}}(X_i)$$

*of profinite groups.*

*Proof.* This follows since the fundamental functor $F$ is in fact pro-representable by the inverse limit of the Galois objects of $\mathcal{C}$ (subject to the above relation), see [SGA1, Exposé V.4 h)]. □

Now, let $H : \mathcal{C} \to \mathcal{C}'$ be an exact functor (in the sense of Definition 2.2) between two Galois categories $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$. Set $G := \mathrm{Aut}(F)$, $G' := \mathrm{Aut}(F')$. Then, by [Stacks, Tag 0BN5], there exists an isomorphism of functors $t : F' \circ H \cong F$, which determines a continuous group homomorphism $h^t : G' \to G$ by sending

$$\varphi' = (\varphi'_{X'})_{X'} \mapsto h^t(\varphi') := (t_X \circ \varphi_{H(X)} \circ t_X^{-1})_X$$

for $X \in \mathrm{Ob}(\mathcal{C}), X' \in \mathrm{Ob}(\mathcal{C}')$. If $s : F' \circ H \cong F$ is another isomorphism, we have $h^s(\varphi') = r \circ h^t(\varphi') \circ r^{-1}$ for all $\varphi' \in G$, where $r := s \circ t^{-1} \in G$. Hence $H$ determines, up to composition with an inner automorphism of $G$, a continuous group homomorphism $h : G' \to G$. The homomorphism $h$ in turn gives a functor $(G\text{-FSets}) \to (G'\text{-FSets})$ such that the obvious diagram involving $(G\text{-FSets})$, $(G'\text{-FSets})$, $\mathcal{C}$ and $\mathcal{C}'$ is 2-commutative. Now, let $(\mathcal{C}'', F'')$ be a third Galois category and set $G'' := \mathrm{Aut}(F'')$. Let $H' : \mathcal{C}' \to \mathcal{C}''$ be an exact functor giving a map $h' : G'' \to G'$ as above. The framework presented in [Stacks, Tag 0BTQ] relates exactness of the sequence

$$G'' \xrightarrow{h'} G' \xrightarrow{h} G \longrightarrow 1$$

to properties of $H'$ and $H''$.

**Lemma 2.8.** *In the situation of the above paragraph, the following are equivalent :*

*(i) The map $h : G' \to G$ is surjective,*

*(ii) $H$ maps connected objects to connected objects, and*

*(iii) $H$ is fully faithful.*

*Proof.* See [Stacks, Tag 0BN6]. □

**Lemma 2.9.** *In the situation of the above paragraph, the following are equivalent :*

*(i) The kernel* $\ker(h)$ *is the smallest closed normal subgroup containing* $\mathrm{im}(h')$*, and*

*(ii) an object* $X'$ *of* $\mathcal{C}'$ *is in the essential image of* $H'$ *if and only if* $H(X')$ *splits as a coproduct of final objects.*

*Proof.* See [Stacks, Tag 0BS9]. □

**Lemma 2.10.** *In the situation of the above paragraph, the following are equivalent :*

*(i) The image of* $h$ *is normal, and*

*(ii) for every connected object* $X$ *of* $\mathcal{C}$ *such that there is a morphism from the final object of* $\mathcal{C}''$ *to* $H(X)$ *we have that* $H(X)$ *is isomorphic to a finite coproduct of final objects.*

*Proof.* See [Stacks, Tag 0BTS]. □

Note that we can combine Lemmas 2.9 and 2.10 in order to show exactness in the usual group-theoretic sense (i.e., $\mathrm{im}(h') = \ker(h)$). To see this, observe that $\mathrm{im}(h')$ is closed, since any continuous map between compact Hausdorff spaces is closed. This is because closed subspaces of compact spaces are quasi-compact, continuous maps preserve compactness and compact subsets of Hausdorff spaces are closed. Therefore, if $\mathrm{im}(h')$ is normal and $\ker(h)$ is the smallest closed normal subgroup containing $\mathrm{im}(h')$, it follows that $\mathrm{im}(h') = \ker(h)$.

## 2.2 Étale maps and the étale topology

**Definition 2.11.** A morphism $f : Y \to X$ of schemes is called *étale*, if it is smooth of relative dimension 0 in the sense of [Stacks, Tag 02G2].

The notion of an étale morphism plays a central role in algebraic geometry and there are many different equivalent definitions. A comprehensive list of facts about étale maps including some possible characterisations is given in [Stacks, Tag 03PC].

We explicitly point out the following statement, which Grothendieck called the "fundamental property of étale morphisms" [SGA1, Exposé I.5] :

**Proposition 2.12.** *Let* $f : Y \to X$ *be a morphism of schemes. Then,* $f$ *is an open immersion if and only if it is universally injective and étale. In particular,* $f$ *is an isomorphism if and only if it is a universal homeomorphism and étale.*

*Proof.* See [Stacks, Tag 025F]. □

Another important fact is the following :

**Lemma 2.13.** *Let* $X = \mathrm{Spec}(k)$*, where* $k$ *is a field. Then, a morphism of schemes* $Y \to X$ *is étale if and only if* $Y \cong \coprod_{i\in\mathcal{I}} \mathrm{Spec}(k_i)$ *such that for each* $i \in \mathcal{I}$*, the ring* $k_i$ *is a field which is a finite separable extension of* $k$*. In particular, if* $k$ *is separably closed,* $Y$ *is isomorphic to a coproduct of copies of* $\mathrm{Spec}(k)$*.*

*Proof.* See [Stacks, Tag 02GL]. □

While in this thesis, the property of being étale will be most important in combination with being finite, see Section 2.3 below, we will also need the étale topology of a scheme, which is constructed according to the same standard procedure as the fpqc topology for example, see [Stacks, Tag 020M].

**Definition and Lemma 2.14.** An *étale covering* of a scheme $U$ is a family $\{f_i : U_i \to U\}_i$ of étale morphisms such that $U = \bigcup_i f_i(U_i)$. Any Zariski covering is an étale covering and any étale covering is an fpqc covering. Étale coverings satisfy the conditions of [Stacks, Tag 00VH]. Hence, for a fixed scheme $X$, the category of étale morphisms $U \to X$ together with the set of étale coverings (over X) forms a site in the sense of [Stacks, Tag 00VH]. It is called the *(small) étale site* of $X$, denoted by $X_{\text{ét}}$. Here, we ignore set-theoretic difficulties and instead refer to [Stacks, Tag 00VI] on how to resolve them.

*Proof.* See [Stacks, Tags 0217, 03PH, 02GP]. □

As for all sites, there is a natural way to define sheaves and cohomology on $X_{\text{ét}}$, see e.g. [Stacks, Tags 03NK, 03NU]. Certain sheaves for the Zariski topology on X can be extended to sheaves on $X_{\text{ét}}$. Among these are quasi-coherent sheaves and also the structure sheaf $\mathcal{O}_X$, by setting $\mathcal{O}_X(U \to X) := \Gamma(U, \mathcal{O}_U)$, see [Stacks, Tags 030G, 0303, 0305].

**Definition 2.15.** Let $X$ be scheme. A *geometric point* of $X$ is a morphism $\bar{x} : \mathrm{Spec}(k(\bar{x})) \to X$ of schemes, where $k(\bar{x})$ is an algebraically closed field. Often, we will abuse notation and write $\bar{x} = \mathrm{Spec}(k(\bar{x}))$. An *étale neighborhood of* $\bar{x}$ *in* $X$ is a pair $(f, \bar{u})$, where $f : U \to X$ is étale, and $\bar{u}$ is a map $\mathrm{Spec}(k(\bar{x})) \to U$ making

$$\begin{tikzcd} & U \arrow[d, "f"] \\ \mathrm{Spec}(k(\bar{x})) \arrow[ur, "\bar{u}"] \arrow[r, "\bar{x}"] & X \end{tikzcd}$$

commute. A morphism of étale neighborhoods $(U, \bar{u}) \to (U', \bar{u}')$ is a morphism $h : U \to U'$ of $X$-schemes such that $h \circ \bar{u} = \bar{u}'$. For ease of notation, we will usually omit $\bar{u}$ and denote an étale neighborhood just by $U$.

The étale neighborhoods of a geometric point $\bar{x} \to X$ form a cofiltered category [Stacks, Tag 03PQ]. For a sheaf $\mathcal{F}$ on $X_{\text{ét}}$, one may define $\mathcal{F}_{\bar{x}} := \mathrm{colim}_{(U,\bar{u})} \mathcal{F}(U)$, where $(U, \bar{u})$ run through the étale neighborhoods of $\bar{x}$. We call $\mathcal{F}_{\bar{x}}$ *the stalk of* $\mathcal{F}$ *at* $\bar{x}$. In case of the structure sheaf, we can relate stalks in the étale topology to stalks in the Zariski topology :

**Proposition 2.16.** *Let $\bar{x} \to X$ be a geometric point of a scheme $X$ with image $x \in X$. Let $k(x) \to k(x)^{\mathrm{sep}} \to k(\bar{x})$ be the separable algebraic closure of $k(x)$ in $k(\bar{x})$, which is a separable algebraic closure of $k(x)$. Denote by $\mathcal{O}_{X,\bar{x}}$ the stalk of the structure sheaf on $X_{\text{ét}}$ at $\bar{x}$ as constructed above and by $\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at $x$ in the Zariski topology. Then, $\mathcal{O}_{X,\bar{x}}$ is isomorphic to the strict henselisation of $\mathcal{O}_{X,x}$ with respect to $k(x) \to k(x)^{\mathrm{sep}}$. In particular, the residue field of the local ring $\mathcal{O}_{X,\bar{x}}$ is isomorphic to $k(x)^{\mathrm{sep}}$.*

*Proof.* See [Stacks, Tag 04HX]. □

For an overview on henselian rings and (strict) henselisation, see [Stacks, Tags 04GE, 0BSK].

## 2.3 Finite étale maps

**Definition 2.17.** A morphism $f : Y \to X$ of schemes is called *finite*, if it is affine and for all affine opens $U \subseteq X$, the ring map $f_U^\flat : \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ makes $\mathcal{O}_Y(f^{-1}(U))$ a finite $\mathcal{O}_X(U)$-module.

For a summary of properties of finite morphisms, see [Stacks, Tag 01WG]. Useful characterizations of being finite are being of finite type and integral [Stacks, Tag 01WJ], or being proper and affine [Stacks, Tag 01WN].

**Definition 2.18.** Let $X$ be scheme. We define $\mathrm{FEt}(X)$ to be the full subcategory of $(\mathrm{Sch}/X)$ whose objects are finite étale morphisms $Y \to X$. Often, we will also call these *finite étale covers* of $X$.

In this thesis, we will think of finite étale maps as analogues of (finite degree) topological covering spaces. This is justified by the following key statement :

**Proposition 2.19.** *Let $f : Y \to X$ be a finite étale morphism of schemes. For any geometric point $\bar{x} \to X$, there exists an étale neighbourhood $\bar{x} \to U \to X$ such that $Y \times_X U$ is isomorphic to a finite coproduct $\coprod V_i$ of schemes over $U$, where each $V_i \to U$ is an isomorphism.*

*Proof.* See [Stacks, Tag 04HN]. □

Hence, we might say informally that "in the étale topology, finite étale covers of $X$ locally look like finite coproducts of copies of $X$". An important special case is :

**Lemma 2.20.** *Let $X = \mathrm{Spec}(k)$, where $k$ is a field. Then, a morphism of schemes $Y \to X$ is finite étale if and only if $Y$ is isomorphic to a finite coproduct of spectra of finite separable extensions of $k$. In particular, if $k$ is separably closed, $Y$ is isomorphic to a finite coproduct of copies of* $\mathrm{Spec}(k)$.

*Proof.* This follows from Lemma 2.13. □

**Definition 2.21.** Let $X$ be a connected scheme and $\bar{x} \to X$ a geometric point. Let $F_{\bar{x}} : \mathrm{FEt}(X) \to (\text{Finite sets})$ be the functor mapping finite étale morphisms $Y \to X$ to the underlying topological space of the fibre $Y \times_X \bar{x}$, which is finite and discrete by Lemma 2.20. We call the group $\pi_1(X, \bar{x}) := \mathrm{Aut}(F_{\bar{x}})$ the *étale fundamental group* of $X$. It comes with a natural profinite topology, see Section 2.1.

Grothendieck proved in [SGA1, Exposé V.7] under an additional mild assumption on $X$ :

**Theorem 2.22.** *Let $X$ be a connected scheme and $\bar{x} \to X$ a geometric point. Then the category $\mathrm{FEt}(X)$ together with the fibre functor $F_{\bar{x}}$, is a Galois category. In particular, $F_{\bar{x}}$ induces an equivalence of categories*

$$\mathrm{FEt}(X) \cong (\textit{Finite } \pi_1(X, \bar{x})\textit{-Sets}).$$

*Proof.* See [Stacks, Tag 0BNB]. □

We explicitly point out the following consequence :

**Lemma 2.23.** *Let $X$ be a connected scheme and $f : Y \to X$ a finite étale morphism. If there exists a geometric point $\bar{x}$ of $X$ such that $Y_{\bar{x}}$ consists of only one point, then $f$ is an isomorphism.*

*Proof.* Since $f$ is finite étale, $Y_{\bar{x}}$ is isomorphic to a finite coproduct of copies of $\bar{x}$ by Lemma 2.13. Hence the assumption implies that $f$ is a bijection after basechange to $\bar{x}$. Since $X$ is connected, $\mathrm{FEt}(X)$ is a Galois category and so the fibre functor $F_{\bar{x}}$ reflects isomorphisms. The claim follows after interpreting $f : Y \to X$ and $\mathrm{id} : X \to X$ as finite étale covers of $X$ and $f$ an $X$-morphism between $Y$ and $X$. □

It is worthwile to note :

**Lemma 2.24.** *Let $X$ be a connected scheme. The connected objects of $\mathrm{FEt}(X)$ are precisely the connected finite étale covers of $X$.*

*Proof.* See the proof of [Stacks, Tag 0BNB]. □

**Remark 2.25.** Let $X \to S$ be a morphism of connected schemes and let $\bar{s} \to S$ and $\bar{x} \to X$ be geometric points. Denote by

$$\begin{aligned} H : \mathrm{FEt}(S) &\longrightarrow \mathrm{FEt}(X) \\ T &\longmapsto T \times_S X \\ (g : T \to T') &\longmapsto (g \times_S \mathrm{id}_X : T \times_S X \to T' \times_S X) \end{aligned}$$

the base change functor along $X \to S$. As it is an exact functor, it induces a non-canonical homomorphism

$$\pi_1(X, \bar{x}) \to \pi_1(S, \bar{s})$$

as described in Section 2.1. However, if we consider $\bar{x}$ a geometric point of $S$ via the map $\bar{x} \to X \to S$, then $H$ induces a *canonical* homomorphism

$$\pi_1(X, \bar{x}) \to \pi_1(S, \bar{x}).$$

So see this, denote by $G_{\bar{x}}$ the geometric fibre functor of $\mathrm{FEt}(S)$, i.e., the base change functor along $\bar{x} \to X \to S$, and denote by $F_{\bar{x}}$ the geometric fibre functor on $\mathrm{FEt}(X)$. Then, for any finite étale cover $T \to S$, the universal property of the fibre product gives a canonical isomorphism $T \times_S \bar{x} \cong (T \times_S X) \times_X \bar{x}$, functorial in $T$. This gives an isomorphism of functors $t : G_{\bar{x}} \cong H \circ F_{\bar{x}}$, inducing a homomorphism of fundamental groups as described in Section 2.1.

## 2.4 Examples of étale fundamental groups

**Example 2.26.** Let $X := \mathrm{Spec}(k)$ be the spectrum of a field $k$. Then the étale fundamental group of $X$ is isomorphic to the absolute Galois group of $k$. More precisely, any geometric point $\bar{x} \to X$ determines a separable closure $k^{\mathrm{sep}}$ of $k$ and canonical isomorphisms

$$\pi_1(X, \bar{x}) \cong \lim_{k' \mid k \text{ finite Galois}} \mathrm{Gal}(k' \mid k) \cong \mathrm{Gal}(k^{\mathrm{sep}} \mid k) \tag{6}$$

of topological groups. To see this, one combines Lemma 2.7 with the observation that the Galois objects of $\mathrm{FEt}(X)$ are precisely the spectra of finite Galois extensions of $k$. This follows from an argument we will use later in the proof of 4.12. In particular, $\pi_1(X, \bar{x})$ is trivial if $k$ is separably closed. Also, if $k$ is a finite field, $\pi_1(X, \bar{x})$ is isomorphic to the profinite integers $\widehat{\mathbb{Z}} = \lim_n \mathbb{Z}/n\mathbb{Z} \cong \prod_{l \text{ prime}} \mathbb{Z}_l$.
Moreover, the fact that the homomorphism $\pi_1(X, \bar{x}) \to \pi_1(X, \bar{x}')$ for two geometric points $\bar{x}, \bar{x}'$ of $X$ is not canonical, see 2.25, corresponds precisely to the fact that $k$-morphisms between separable closures of $k$ are non-canonical.
Further note that in the situation $X = \mathrm{Spec}(k)$, Theorem 2.22, restricted to connected covers, recovers the fundamental theorem of infinite Galois theory for $k \to k^{\mathrm{sep}}$, at least for finite subextensions, cf. [Stacks, Tag 0BML].

**Example 2.27.** For integral normal schemes $X$, the étale fundamental group $\pi_1(X, \bar{x})$ can be expressed as the Galois group of some Galois extension of the function field $K$ of $X$, see [Stacks, Tag 0BQM]. After some additional non-trivial steps, one can deduce that $\pi_1(\mathbb{P}^1_k, \bar{x}) \cong \pi_1(\mathrm{Spec}(k))$ for any field $k$ and $\pi_1(\mathbb{A}^1_k, \bar{x}) \cong \pi_1(\mathrm{Spec}(k))$ for any field $k$ of characteristic $0$, see [Len85, 6.22,6.23].
We explicitly point out that the latter not true in characteristic $p > 0$, while there is still a surjective map $\pi_1(\mathbb{A}^1_k, \bar{x}) \twoheadrightarrow \pi_1(\mathrm{Spec}(k))$. Roughly speaking, the reason for this is the existence of Artin-Schreier coverings of $\mathbb{A}^1_k$. Those correspond to ring maps

$$R \to R[X]/(X^p - X + f),$$

where $R = k[T]$ and $f \in R$. Note that $R \to R[X]/(X^p - X + f)$ is indeed an étale ring map since $\partial(X^p - X + f)/\partial X = -1$, as $k$ has characteristic $p$. If $R = k$ is a field of characteristic $p$, these coverings correspond to Artin-Schreier field extensions of $k$. Also see [Len85, 6.23, Ex. 6.28] and Example 3.14 below.

**Example 2.28** (cf. [GM22, Corollary (10.37)]). Let $X$ be an abelian variety over an algebraically closed field $k$. Regard $0 : \mathrm{Spec}(k) \to X$ as a geometric point of $X$. Then, we have a canonical isomorphism

$$\pi_1(X, 0) \cong \lim_n X[n](k), \tag{7}$$

where $X[n]$ denotes the $n$-torsion of $X$ and $X[n](k)$ its $k$-valued points. The right hand side of (7) can be further described (at least in characteristic zero) as the product of the $l$-adic Tate modules of $X$. In particular, if $\mathrm{char}(k) = 0$ and $X$ is an elliptic curve over $k$, we get $\pi_1(X, 0) \cong \prod_{l \text{prime}} (\mathbb{Z}_l \times \mathbb{Z}_l) \cong \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$.

**Example 2.29.** Another result worth mentioning is that if $X$ is a variety over $\mathbb{C}$, its étale fundamental group is isomorphic to the profinite completion of the topological fundamental group of the analytification of $X$, see [SGA1, Exposé XII, Corollaire 5.2]. This also explains why the étale fundamental group of an elliptic curve over $\mathbb{C}$ is isomorphic to $\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$ : The latter is the profinite completion of $\mathbb{Z} \times \mathbb{Z}$, the topological fundamental group of the torus.

For more examples of étale fundamental groups, we refer to [Len85, Section 6].

## 2.5 Extending finite étale covers of geometric points

In this thesis, we will face the problem of "extending" finite étale covers of a geometric point $\bar{x}$ (hence just some finite set of copies of $\bar{x}$) to an étale neighborhood $\bar{x} \to U$ of $\bar{x}$. For this, we will use the following highly non-trivial statements :

**Theorem 2.30.** *Let $g : X \to S$ be a morphism of schemes. In the following cases, base change along $g$ induces an equivalence of categories $\mathrm{FEt}(S) \cong \mathrm{FEt}(X)$ :*

*(a) $g$ is a universal homeomorphism, or*

*(b) $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(A/I)$, where $(A, I)$ is a henselian pair, and $g$ comes from the projection $A \to A/I$.*

*Proof.* For the definition of a henselian pair, see [Stacks, 09XE]. For the proof, see [Stacks, Tags 0BQN, 09ZL]. □

Almost from the definition of the stalk of the structure sheaf at a geometric point, see Section 2.2, we get :

**Lemma 2.31.** *Let $\bar{x} \to X$ be a geometric point of a scheme $X$. Then we have an isomorphism of schemes*

$$\mathrm{Spec}(\mathcal{O}_{X,\bar{x}}) \cong \lim_{(U,\bar{u})} U,$$

*where $(U, \bar{u})$ runs through the affine étale neighborhoods of $\bar{x}$ in $X$.*

*Proof.* Since we only consider affine étale neighborhoods, the above limit exists in the category of schemes and

$$\lim_{(U,\bar{u})} U \cong \mathrm{Spec}(\mathrm{colim}_{(U,\bar{u})} \Gamma(U, \mathcal{O}_U)) \cong \mathrm{Spec}(\mathcal{O}_{X,\bar{x}}),$$

see [Stacks, Tag 01YW]. □

Note that any geometric point $\bar{x} \to X$ factors through $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}}) \to X$ and hence by the lemma above, induces a map $\bar{x} \to \lim_{(U,\bar{u})} U$. It turns out that base change along this map induces an equivalence of categories of finite étale covers.

**Lemma 2.32.** *Let $\bar{x} \to X$ be a geometric point of a scheme $X$. Base change induces an equivalence of categories*

$$\mathrm{FEt}(\lim_{(U,\bar{u})}(U)) \cong \mathrm{FEt}(\bar{x}),$$

*where on the right hand side, we use the shorthand notation $\bar{x} = \mathrm{Spec}(k(\bar{x}))$ and where $(U, \bar{u})$ runs over the affine étale neighborhoods of $\bar{x}$ in $X$.*

*Proof.* By Lemma 2.31, the category on the left hand side is equivalent to $\mathrm{FEt}(\mathrm{Spec}(A))$, where $A := \mathcal{O}_{X,\bar{x}}$. By Proposition 2.16, $A$ is a strictly henselian local ring, in particular $(A, \mathfrak{m})$ is a henselian pair, where $\mathfrak{m}$ is the unique maximal ideal of $A$, and the residue field $\kappa := A/\mathfrak{m}$ is separably closed. By Theorem 2.30(b), the category $\mathrm{FEt}(\mathrm{Spec}(A))$ is equivalent to $\mathrm{FEt}(\mathrm{Spec}(\kappa))$. Then, since $\kappa$ is separably closed, $\mathrm{FEt}(\mathrm{Spec}(\kappa))$ is equivalent to the category of finite sets, and the same holds for $\mathrm{FEt}(\bar{x})$. □

In the above situation, the category $\mathrm{FEt}(\lim_{(U,\bar{u})}(U))$ can be further written as a 2-colimit of the finite étale covers of the individual $U$. First, we review the definition of a 2-colimit given in [SW20, Remark 7.4.7] :

**Definition and Remark 2.33.** Let $\mathcal{I}$ be a filtered index category, and $F$ a functor from $\mathcal{I}$ into the category of small categories, i.e., for each object $i \in \mathcal{I}$, we have a small category $F(i) =: \mathcal{C}_i$, and for each morphism $i \to j$ in $\mathcal{I}$, a functor $F(i \to j) =: F_{ij} : \mathcal{C}_i \to \mathcal{C}_j$ such that the usual compatability conditions are satisfied. The objects of $\mathcal{C} := 2\text{-colim}\, \mathcal{C}_i$ are defined as the disjoint union of all sets of objects $\mathrm{Ob}(\mathcal{C}_i)$. For two objects $X_i, X_j$ of $\mathcal{C}$ belonging to $\mathrm{Ob}(\mathcal{C}_i)$ and $\mathrm{Ob}(\mathcal{C}_j)$, respectively, we define

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(X_i, X_j) &:= \mathrm{colim}_{i,j\to k} \mathrm{Hom}_{\mathcal{C}_k}(F_{ik}(X_i), F_{jk}(X_j)) \\ &= \{f_k \in \mathrm{Hom}_{\mathcal{C}_k}(F_{ik}(X_i), F_{jk}(X_j)) \mid k \in \mathrm{Ob}(\mathcal{I}) \text{ such that there exist morphisms } i \to k,\, j \to k\}/\sim,\end{aligned}$$

where

$$f_k \sim f_l :\Leftrightarrow \exists m \in \mathrm{Ob}(\mathcal{I}) \text{ and morphisms } k \to m,\, l \to m \text{ such that } F_{km}(f_k) = F_{lm}(f_l).$$

An identity morphism $X_i \to X_i$ is defined as the class of the respective identity morphism in $\mathcal{C}_i$ and composition of morphisms is defined as first choosing representatives, passing to a common third index object $m$ and then composing in $\mathcal{C}_m$ (this is well-defined by the definition of the sets morphisms above). Observe that with these definitions, two objects in $\mathcal{C}$ are isomorphic if and only if there exists an common third index object $m$ such that the two objects are isomorphic after passing to $\mathcal{C}_m$, cf. the description of an "ordinary" filtered colimit of sets. Therefore, the "inclusion" functors $G_i : \mathcal{C}_i \to \mathcal{C}$ form a 2-cocone from the diagram $F$ to $\mathcal{C}$, i.e., for any $i \to j$, we have isomorphisms of functors $g_{ij} : G_j \circ F_{ij} \cong G_i$. Further, it can be checked that $\mathcal{C}$ has the following universal property : For any 2-cocone $((H_i)_i, (h_{ij})_{i\to j})$ from $F$ to a category $\mathcal{T}$, there exists a functor $K : \mathcal{C} \to \mathcal{T}$, unique up to isomorphism of functors, such that for any index $i$, we have isomorphisms $H_i \cong K \circ G_i$.

**Lemma 2.34.** *Let $S = \lim S_i$ be a limit of schemes with affine transition morphisms. Then, base change induces an equivalence of categories*

$$\mathrm{FEt}(S) \cong 2\text{-colim}_i\, \mathrm{FEt}(S_i).$$

*Proof.* Restrict [Stacks, Tag 0EYL] to finite étale covers. □

Combining this with Lemma 2.32, we get an equivalence

$$2\text{-colim}_{(U,\bar{u})}\, \mathrm{FEt}(U) \cong \mathrm{FEt}(\bar{x}).$$

This equivalence comes from the 2-cocone of base change functors along $\bar{x} \to U$. So in particular, by essential surjectivity, we have that for any finite étale $Y$ cover of $\bar{x}$, there exists an étale neighborhood $V$ of $\bar{x}$ and a finite étale cover $Z \to V$ such that $Y \cong Z \times_V \bar{x}$. Fully faithfulness says that if $Z \to U, Z' \to U'$ are two finite étale covers of affine étale neighborhoods $U$, $V$ of $\bar{x}$ such that $Z \times_U \bar{x} \cong Z' \times_{U'} \bar{x}$, there exists a third étale neighborhood $\bar{x} \to U'' \to U \times_X U'$ such that $Z$ and $Z'$ agree after base change to $U''$.

## 3 Künneth formula

Let us recall from the introduction the Künneth formula for the étale fundamental group as claimed by Scholze and Weinstein [SW20, Lemma 16.1.2] :

**Proposition 1.2.** *Let $X_1$, $X_2$ be connected qcqs schemes over an algebraically closed field $k$ such that $X_1 \to k$ is $\pi_1$-proper. Then, $X_1 \times_k X_2$ is connected and for any geometric point $\bar{z}$ of $X_1 \times_k X_2$ (and thus of $X_1$ and $X_2$), the natural map*

$$\pi_1(X_1 \times_k X_2, \bar{z}) \xrightarrow{\sim} \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z})$$

*is an isomorphism of topological groups.*

In the above Proposition, by the "natural map", we mean the map induced by the canonical maps $\pi_1(X_1 \times_k X_2, \bar{z}) \to \pi_1(X_1, \bar{z})$ and $\pi_1(X_1 \times_k X_2, \bar{z}) \to \pi_1(X_2, \bar{z})$ induced by the base change functors along $X_1 \times_k X_2 \to X_1$ and $X_1 \times_k X_2 \to X_2$ in the sense of Remark 2.25. The goal of this section is now to prove Proposition 1.2. The reason why we are doing this is that our proof will serve as a blueprint for a large part of the proof of Drinfeld's Lemma : As we will see in Section 4.3 below, any scheme over $\mathbb{F}_p$ satisfies a "Drinfeld analogue" of $\pi_1$-properness. Then, in Sections 4.4, 4.5 and 4.6, we will argue analogously as for Proposition 1.2. First, we observe that the statement about connectedness of $X_1 \times_k X_2$ already follows from well-known facts about geometrically connected schemes :

**Lemma 3.1.** *The fibre product of any two connected schemes over an algebraically closed field is connected.*

*Proof.* A connected scheme over an algebraically closed field is geometrically connected by [Stacks, Tag 0363] and the product of a geometrically connected scheme with a connected scheme is connected by [Stacks, Tag 0385]. □

The main challenge in the proof of Proposition 1.2 is of course the Künneth formula. But first, let us introduce the notion of $\pi_1$-properness.

### 3.1 $\pi_1$-properness

We adopt the following definition from [Ked19, Def. 4.1.12] :

**Definition 3.2.** Let $X \to k$ be a connected scheme over an algebraically closed field $k$. We say, $X$ is $\pi_1$*-proper* if for any extension of algebraically closed fields $k \to k'$, base change along $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ induces an equivalence of categories

$$\mathrm{FEt}(X) \cong \mathrm{FEt}(X_{k'}),$$

where $X_{k'} := X \times_k \mathrm{Spec}(k')$.

Note that, in the above Definition, $X_{k'}$ is also connected by Lemma 3.1, hence both $\mathrm{FEt}(X)$ and $\mathrm{FEt}(X_{k'})$ are Galois categories. The terminology "$\pi_1$-properness" is motivated by the highly non-trivial fact that proper maps are $\pi_1$-proper :

**Proposition 3.3.** *A connected proper scheme over an algebraically closed field is $\pi_1$-proper.*

*Proof.* See [Stacks, Tag 0A49]. □

Also, we mention the following powerful result in case of characteristic $0$ :

**Proposition 3.4.** *Any connected scheme over an algebraically closed field of characteristic $0$ is $\pi_1$-proper.*

*Proof.* See [SW20, Section 16.1] or [Ked19, Lemma 4.1.16]. □

In particular, not any $\pi_1$-proper morphism is proper.

**Remark 3.5.** The property of being $\pi_1$-proper is necessary for the Künneth formula to hold, in the following sense : Suppose that $X_1$ is a scheme over an algebraically closed field $k$, such that for any scheme $X_2$ over $k$, we have $\pi_1(X_1 \times_k X_2, \bar{z}) \cong \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z})$, then $X$ is $\pi_1$-proper. This follows by applying the assumption to $X_2 := \mathrm{Spec}(k')$, which has a trivial fundamental group and using Theorem 2.5.

The goal of the rest of this section is to show that the following statement still holds if properness is relaxed to $\pi_1$-properness :

**Lemma 3.6.** *Let $X \to S$ be a proper qcqs morphism of schemes. Then, for any geometric point $\bar{s} \to S$, base change induces an equivalence of categories*

$$\text{2-colim}_{(U,\bar{u})} \mathrm{FEt}(X \times_S U) \cong \mathrm{FEt}(X \times_S \bar{s}), \tag{8}$$

*where $(U, \bar{u})$ runs over the (affine) étale neighborhoods of $\bar{s}$ in $S$.*

*Proof.* Denote by $R := \mathrm{Spec}(\mathcal{O}_{S,\bar{s}})$ the strictly henselian local ring at $\bar{s}$ and by $\kappa$. It is a highly non-trivial statement that the equivalence from Theorem 2.30(b) remains true after proper base change, i.e., base change induces an equivalence

$$\mathrm{FEt}(X \times_S \mathrm{Spec}(R)) \cong \mathrm{FEt}(X \times_S \mathrm{Spec}(\kappa)),$$

see [Stacks, Tag 0GS2]. From this, one concludes the equivalence (8) by the same arguments as for Lemmas 2.32 and 2.34, noting that

$$X \times_S \left( \lim_{(U,\bar{u})} U \right) \cong \lim_{(U,\bar{u})} (X \times_S U),$$

as base change commutes with limits. □

Now, Lemma 3.7 below shows that Lemma 3.6 still holds if $X \to S$ is a projection $X_1 \times_k X_2 \to X_2$, where $X_1 \to k$ is $\pi_1$-proper :

**Lemma 3.7.** *Let $X_1$ be a connected $\pi_1$-proper qcqs scheme over an algebraically closed field $k$. Let $X_2$ be a second scheme over $k$ and denote by $X := X_1 \times_k X_2 \to X_2$ the base change of $X_1 \to \mathrm{Spec}(k)$ along $X_2 \to \mathrm{Spec}(k)$. Let $\bar{x} \to X_2$ be a geometric point of $X_2$. Then, base change induces an equivalence of categories*

$$\text{2-colim}_{(U,\bar{u})} \mathrm{FEt}(X \times_{X_2} U) \cong \mathrm{FEt}(X \times_{X_2} \bar{x}), \tag{9}$$

*where $(U, \bar{u})$ runs over the (affine) étale neighborhoods of $\bar{x}$ in $X_2$.*

*Proof.* First of all, by Lemma 2.34 and since base change commute with limits, we have an equivalence

$$\text{2-colim}_{(U,\bar{u})} \operatorname{FEt}(X \times_{X_2} U) \cong \operatorname{FEt}(X \times_{X_2} \lim_{(U,\bar{u})} U).$$

Hence, it suffices to show that base change along

$$X \times_{X_2} \bar{x} \to X \times_{X_2} Z$$

is an equivalence, where $Z := \lim_{(U,\bar{u})} U$. For this, we can canonically identify $X \times_{X_2} \bar{x} \cong X_1 \times_k \bar{x}$ and $X \times_{X_2} Z \cong X_1 \times_k Z$.
**Essential surjectivity :** By $\pi_1$-properness, any finite étale cover $Y \to X_1 \times_k \bar{x}$ descends to a finite étale cover of $Y' \to X_1$. Then, $Y' \times_k Z$ is a finite étale cover of $X \times_{X_2} Z$ and its base change along $X \times_{X_2} \bar{x} \to X \times_{X_2} Z$ is isomorphic to $Y$.
**Faithfulness :** Note that $X_1 \times_k Z$ and $X_1 \times_k \bar{x}$ are connected by Lemma 3.1. Now, observe that base change of finite étale covers along any map $T \to S$ of connected nonempty schemes is faithful : Choose a geometric point $\bar{t} \to T \to S$, then $\operatorname{FEt}(T)$ and $\operatorname{FEt}(S)$ together with the geometric fibre functors $G_{\bar{t}}$, $F_{\bar{t}}$, respectively, are Galois categories. Hence $G_{\bar{t}}$ and $F_{\bar{t}}$ are faithful, see Section 2.1. As $F_{\bar{t}}$ factors through base change along $T \to S$, we see that base change along $T \to S$ is faithful.
**Fullness :** The main challenge turns out to be fullness. We sketch a proof based on answers by Peter Scholze to a question the author submitted on MathOverflow.[1] First, set $R := \mathcal{O}_{X_2,\bar{x}}$ and denote by $\kappa$ its residue field. By the same arguments as in the proof of 2.32, fullness boils down to fullness of base change along $X_1 \times_k \operatorname{Spec}(\kappa) \to X_1 \times_k \operatorname{Spec}(R)$.
Now, one uses the fact that finite étale morphisms satisfy $v$-descent, which can be concluded from [HS21, Theorem 1.5] by showing that the properties "quasi-compact" and "satisfying the valuative criterion for properness" are $v$-local on the target, noting that a morphism is finite étale if and only if it is étale, separated, quasi-compact and satisfying the valuative criterion for properness. Hence, one can reduce to the case that $R$ is a valuation ring with algebraically closed fraction field $K$. In particular, $R$ is a normal integral domain [Stacks, Tag 00IC].
We further use that finite étale morphisms satisfy $h$-descent, see [Stacks, Tag 02VY] and [SGA1, Théorème IX.4.12], in order to reduce to the case that $X_1$ is normal and affine. Since $k$ is algebraically closed, $X$ is in fact geometrically normal by [Stacks, Tag 0380] and hence, both $X_1 \times_k \operatorname{Spec}(R)$ and $X_1 \times_k \operatorname{Spec}(K)$ are normal schemes by [Stacks, Tag 06DG]. Then, one can show by similar arguments as for Lemma 4.39 below that base change

$$\operatorname{FEt}(X_1 \times_k \operatorname{Spec}(R)) \to \operatorname{FEt}(X_1 \times_k \operatorname{Spec}(K))$$

is fully faithful. Now, since $X \to k$ is $\pi_1$-proper and $K$ is algebraically closed, base change induces equivalences of categories

$$\operatorname{FEt}(X_1 \times_k \operatorname{Spec}(\kappa))) \cong \operatorname{FEt}(X) \cong \operatorname{FEt}(X_1 \times_k \operatorname{Spec}(K)).$$

1. See https://mathoverflow.net/questions/432160/künneth-formula-for-pi-1-proper-morphisms (last accessed on 2022-10-20)

Hence we conclude that base change

$$\mathrm{FEt}(X_1 \times_k \mathrm{Spec}(R)) \to \mathrm{FEt}(X_1 \times_k \mathrm{Spec}(\kappa))$$

is fully faithful and in particular full. $\square$

## 3.2 Homotopy exact sequence for $X_1 \times_k X_2 \to X_2$

**Proposition 3.8.** *In the situation of Proposition 1.2, for any geometric point $\bar{x}$ of $X_2$ and any geometric point $\bar{z}$ of $X_1 \times_k \bar{x}$ (and thus also of $X_1 \times_k X_2$ and $X_2$), base change induces an exact sequence*

$$\pi_1(X_1 \times_k \bar{x}, \bar{z}) \longrightarrow \pi_1(X_1 \times_k X_2, \bar{z}) \longrightarrow \pi_1(X_2, \bar{z}) \longrightarrow 1 \tag{10}$$

*of topological groups.*

In analogy to topology, we will call the exact sequence from the above proposition a "homotopy exact sequence" with respect to $X_1 \times_k X_2 \to X_2$. Note that $X_1 \times_k \bar{x}$ is canonically isomorphic to the base change of $X_1 \times_k X_2 \to X_2$ along $\bar{x} \to X_2$.

**Lemma 3.9.** *If Proposition 3.8 holds, then also Proposition 1.2.*

*Proof.* Lemma 3.1 implies the first statement of Proposition 1.2. In particular, $X_1 \times_k X_2$ is connected, such that $\pi_1(X_1 \times_k X_2, \bar{z})$ is defined. By Proposition 3.8 and since $X_1$ is $\pi_1$-proper, we get a commutative diagram

$$\begin{tikzcd}
\pi_1(X_1 \times_k \bar{x}, \bar{z}) \arrow[r] \arrow[dr, "\cong"] & \pi_1(X_1 \times_k X_2, \bar{z}) \arrow[r] \arrow[d] & \pi_1(X_2, \bar{z}) \arrow[r] & 1 \\
& \pi_1(X_1, \bar{z}), & &
\end{tikzcd}$$

where the top row is an exact sequence of groups and $\pi_1(X_1 \times_k \bar{z}, \bar{z}) \to \pi_1(X_1, \bar{z})$ is an isomorphism. The triangle commutes, since all group homomorphisms come from base change and the projection $X_1 \times_k \bar{x} \to X_1$ factors through $X_1 \times_k X_2$ as $X_1 \times_k \bar{x}$ is meant with respect to $\bar{x} \to X_2 \to k$.
Taking the inverse of $\pi_1(X_1 \times_k \bar{x}, \bar{z}) \to \pi_1(X_1, \bar{z})$, this implies that $\pi_1(X_1, \bar{z}) \to \pi_1(X_1 \times_k \bar{x}, \bar{z}) \to \pi_1(X_1 \times_k X_2, \bar{z}) \to \pi_1(X_1, \bar{z})$ is the identity. So we get an exact sequence

$$\begin{tikzcd}
\pi_1(X_1, \bar{z}) \arrow[r] & \pi_1(X_1 \times_k X_2, \bar{z}) \arrow[r] \arrow[l, bend left] & \pi_1(X_2, \bar{z}) \arrow[r] & 1
\end{tikzcd}$$

of groups where $\pi_1(X_1, \bar{z}) \to \pi_1(X_1 \times_k X_2, \bar{z}) \to \pi_1(X_1, \bar{z})$ is the identity. Then the maps from the above sequence induce an isomorphism of groups $\pi_1(X_1 \times_k X_2, \bar{z}) \cong \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z})$. Note that this map is in fact an isomorphism of topological groups since any continuous map between quasi-compact Hausdorff spaces is closed, and hence any bijective continuous map is a homeomorphism. $\square$

To prove Proposition 3.8, we use the material presented in section 2.1. First, we show exactness at $\pi_1(X_2, \bar{z})$.

**Lemma 3.10.** *Under the assumptions of Proposition 1.2, the base change of any connected finite étale cover of $X_2$ to $X_1 \times_k X_2$ is connected. In particular, the sequence (10) is exact at $\pi_1(X_2, \bar{z})$.*

*Proof.* Let $Y$ be a connected finite étale cover of $X_2$. There is an isomorphism $Y \times_{X_2} (X_1 \times_k X_2) \cong X_1 \times_k Y$. The latter scheme is connected by Lemma 3.1. Now, by Lemmas 2.8 and 2.24, the sequence (10) is exact at $\pi_1(X_2, \bar{z})$. □

The main challenge of the proof of Proposition 3.8 is exactness at $\pi_1(X_1 \times_k X_2, \bar{z})$. For that, we will use Lemmas 2.9 and 2.10. In order to apply Lemma 2.9, we need some way to descend finite étale covers of $X_1 \times_k X_2$ to finite étale covers of $X_2$. The "right" way to do this is described in the following lemma :

**Lemma 3.11.** *Let $X \to S$ be a morphism of connected schemes with geometrically connected fibres such that the base change of any connected finite étale cover of $S$ along $X \to S$ is connected. Assume that for any finite étale $Y \to X$, there exists a finite étale morphism $T \to S$ and a morphism $Y \to T$ with geometrically connected fibres such that*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & S \end{tikzcd} \tag{11}$$

*commutes. Then for any geometric point $\bar{s} \to S$ and any geometric point $\bar{z} \to X \times_S \bar{s}$ (hence also of $X$ and $S$), the sequence*

$$\pi_1(X \times_S \bar{s}, \bar{z}) \longrightarrow \pi_1(X, \bar{z}) \longrightarrow \pi_1(S, \bar{z})$$

*induced by base change is exact.*

*Proof.* Our proof is inspired by the proofs of [Stacks, Tag 0BUM] and [MA67, Theorem 6.3.2.1]. By Lemmas 2.9 and 2.10, it suffices to show the following statement for any finite étale cover $Y \to X$ :

> *If the base change $Y \times_X (X \times_S \bar{s}) \cong Y \times_S \bar{s}$ has a connected component $Z$ isomorphic to $X \times_S \bar{s}$ (i.e., the map $Z \to Y \times_S \bar{s} \to X \times_S \bar{s}$ is an isomorphism), diagram (11) is cartesian.*

So let $Y \to X$ be a finite étale morphism. Since base change of schemes commutes with finite coproducts, and $Y$ splits as the finite coproduct of its connected components, we may assume that $Y$ is connected. Choose a finite étale cover $T \to S$ as in the assumptions of the lemma. Since $Y \to T$ has geometrically connected (and hence non-empty) fibres, it is surjective and therefore, also $T$ is connected. Set $Y' := T \times_S X$. Let $p : Y \to Y'$ be the unique morphism making

$$\begin{tikzcd} Y \arrow[dr, dashed, "p"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ & X \arrow[r] & S \end{tikzcd} \tag{12}$$

commute. We will show that $p$ is an isomorphism. First, observe that $p$ is finite étale as a morphism between finite étale covers of $X$, see [Stacks, Tags 035D and 02GW]. By assumption, base changes of connected finite étale covers of $S$ along $X \to S$ are connected. So, since $T$ is connected, also $Y'$ is connected. As a finite étale morphism, $p$ is both open and closed, hence connectedness of $Y'$ implies that $p$ is surjective. Now, in light of Lemma 2.23 and since $Y'$ is connected, it suffices to show that there exists some geometric point of $Y'$ whose fibre under $p$ consists of a single point. First, we apply the base change functor $- \times_S \bar{s}$ to diagram (12) and get

$$\begin{tikzcd} Y_{\bar{s}} \arrow[dr, dashed, "p_{\bar{s}}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Y'_{\bar{s}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T_{\bar{s}} \arrow[d] \\ & X_{\bar{s}} \arrow[r] & \bar{s}. \end{tikzcd} \tag{13}$$

It suffices to show that there is a geometric point of $Y'_{\bar{s}}$ whose fibre under $p_{\bar{s}}$ consists of a single point, since

$$\begin{tikzcd} Y_{\bar{s}} \arrow[r] \arrow[d] & Y \arrow[d] \\ Y'_{\bar{s}} \arrow[r] & Y' \end{tikzcd}$$

is cartesian. By assumption, there exists a connected component $Z$ of $Y_{\bar{s}} = Y \times_X (X \times_S \bar{s})$ such that $Z \to Y_{\bar{s}} \xrightarrow{p_{\bar{s}}} Y'_{\bar{s}} \to X_{\bar{s}}$ is an isomorphism. By Lemma 2.4, there exists a unique connected component $Z'$ of $Y'_{\bar{s}}$ such that $Z \to Y_{\bar{s}} \xrightarrow{p_{\bar{s}}} Y'_{\bar{s}}$ factors through $Z' \to Y'_{\bar{s}}$, i.e., we have a commutative square :

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y_{\bar{s}} \arrow[d, "p_{\bar{s}}"] \\ Z' \arrow[r] & Y'_{\bar{s}}. \end{tikzcd} \tag{14}$$

Next, we show that this square is cartesian. Again by Lemma 2.4, a base change of $p_{\bar{s}}$ along a connected component of $Y'_{\bar{s}}$ is isomorphic to a finite coproduct of connected components of $Y_{\bar{s}}$. Since $p_{\bar{s}}$ is surjective as a base change of the surjective map $p$, none of these coproducts is empty. We will show in the following that $Y_{\bar{s}}$ and $Y'_{\bar{s}}$ have the same number of connected components, from which we conclude that each of the coproducts consists of a single component. In particular, the square (14) is cartesian.

First observe that, since $T \to S$ is finite étale, $T_{\bar{s}} = T \times_S \bar{s}$ splits as a finite coproduct $\coprod_{i=1}^n \bar{s}$ of copies of $\bar{s}$. Now, since base change commutes with finite coproducts, the base change of a finite coproduct $\coprod_{i=1}^n A$ of copies of a base scheme $A$ along any morphism $B \to A$ of schemes is isomorphic to $\coprod_{i=1}^n B$. Therefore, $Y'_{\bar{s}}$ is isomorphic to the coproduct $\coprod_{i=1}^n X_{\bar{s}}$ of $n$ copies of $X_{\bar{s}}$. Since by assumption $X_{\bar{s}}$ is connected, this is the decomposition of $Y'_{\bar{s}}$ into connected components.

On the other hand, we have an isomorphism $Y_{\bar{s}} = Y \times_T (T \times_S \bar{s})$ and, similarly as above, $Y_{\bar{s}}$ is isomorphic to the finite coproduct $\coprod_{i=1}^n (Y \times_T \bar{s})$. Since $Y \to T$ has geometrically connected fibres, $Y \times_T \bar{s}$ is connected. Hence $\coprod_{i=1}^n (Y \times_T \bar{s})$ is the decomposition of $Y'_{\bar{s}}$ into connected components. Therefore, both $Y_{\bar{s}}$ and $Y'_{\bar{s}}$ have $n$ connected components and the square (14) is cartesian.

Moreover, by the above description of the connected components of $Y'_{\bar{s}}$, the map $Z' \to Y'_{\bar{s}} \to X_{\bar{s}}$ is in fact an isomorphism. Since also $Z \to Y_{\bar{s}} \xrightarrow{p_{\bar{s}}} Y'_{\bar{s}} \to X_{\bar{s}}$ is an isomorphism by assumption, we conclude that $Z \to Z'$ is an isomorphism. Now choose a geometric point $\bar{z} \to Z'$. Since being an isomorphism is stable under base change and since (14) is cartesian, $p_{\bar{s}}$ is an isomorphism after base change along $\bar{z} \to Z' \to Y'_{\bar{s}}$. □

In the situation of Lemma 3.11, if we additionally assume that $X \to S$ is proper, the factorization which is required in the lemma can be realized as the Stein factorization of the proper map $Y \to X \to S$, see [Stacks, Tag 03H2]. Hence, we are left to establish a similar statement under the relaxed assumption of $\pi_1$-properness (or more precisely, if $X \to S$ is a base change of a $\pi_1$-proper map.)

## 3.3 Stein factorization

**Lemma 3.12.** *Suppose the assumptions of Proposition 1.2 hold. Set $X := X_1 \times_k X_2$. For all finite étale $Y \to X$, there exists a scheme $T$ under $Y$ and over $X_2$, such that*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & X_2 \end{tikzcd} \tag{15}$$

*commutes, $Y \to T$ has geometrically connected fibres, $T \to X_2$ is finite étale and diagram (15) has the following universal property : For any $T'$ under $Y$ and finite étale over $X_2$, such that*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T' \arrow[d] \\ X \arrow[r] & X_2 \end{tikzcd} \tag{16}$$

*commutes, there is a unique morphism $T \to T'$ such that*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[rr, bend left] & T \arrow[d] \arrow[r, dotted] & T' \arrow[dl] \\ X \arrow[r] & X_2 & \end{tikzcd} \tag{17}$$

*commutes. In particular, $T$ is unique up to unique isomorphism.*

**Example 3.13.** In the situation of Lemma 3.12, assume that $X_2 = \bar{x}$ is a geometric point.

To see that diagram (15) exists, first note that $X = X_1 \times_k \bar{x}$ is connected by assumption, hence $\mathrm{FEt}(X)$ is a Galois category. In particular, $Y$ splits as the coproduct of finitely many connected components. Now, we find $T$ as the coproduct of copies of $\bar{x}$, with one copy for each connected component of $Y$. Then the map $Y \to T$ has geometrically connected fibres and $T \to X_2$ is finite étale.

Also the universal property of a diagram (15) is immediate if $X_2 = \bar{x}$ is a geometric point : Each $T'$ as in diagram

(16) splits as a finite coproduct of copies of $\bar{x}$. Since $Y \to T$ has geometrically connected fibres and is open (since $T$ is discrete), it has to induce a bijection between the connected components of $Y$ and $T$, which are the copies of $\bar{x}$. Meanwhile, being continuous, $Y \to T'$ has to map connected components to points, while not necessarily inducing a bijection. So we indeed get a unique morphism $T \to T'$ over $\bar{x}$ turning $Y \to T$ into $Y \to T'$ by mapping points to points in the right way.

Roughly, the proof idea of Lemma 3.12 is to work étale locally on $X_2$ by combining Example 3.13 with Lemmas 2.32 and 3.7. We will first show the universal property, which will enable us to glue.

*Proof of Lemma 3.12.* Our argument follows [SW20, Prop 16.3.3].

**Universal property :**

We first prove the second claim, i.e., that a diagram (15) has the desired universal property. Let

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T' \arrow[d] \\ X \arrow[r] & X_2 \end{tikzcd}$$

be a commutative square, where $T' \to X_2$ is finite étale. We want to show that there exists a unique morphism $T \to T'$ over $X_2$ and under $Y$ making diagram (17) commute.

First, to prove uniqueness, let $f : T \to T'$ and $g : T \to T'$ be maps fitting into diagram (17). In particular, $f$ and $g$ are maps between finite étale covers of $X_2$. By Example 3.13, $f$ and $g$ agree after base change along any geometric point $\bar{x} \to X_2$. Since the fundamental functor of any Galois category is faithful by [Stacks, Tag 0BN0], $f$ and $g$ are equal.

Next, we prove existence of the morphism $T \to T'$ making diagram (17) commute. First, from Example 3.13, we also conclude existence after base change of diagram (17) along any geometric point $\bar{x} \to X$.

Now by Lemma 2.32, there exist an étale neighborhood $\bar{x} \to U \to X_2$ and a morphism $T \times_{X_2} U \to T' \times_{X_2} U$ over $U$ which, after base change to $\bar{x}$ equals $T \times_{X_2} \bar{x} \to T' \times_{X_2} \bar{x}$. As the composition of commutative diagrams,

$$\begin{tikzcd} Y \times_{X_2} U \arrow[r] \arrow[d] & T \times_{X_2} U \arrow[r] \arrow[d] & T' \times_{X_2} U \arrow[dl] \\ X \times_{X_2} U \arrow[r] & U & \end{tikzcd}$$

commutes. Also,

$$\begin{tikzcd} Y \times_{X_2} U \arrow[r] \arrow[rr, bend left] & T \times_{X_2} U \arrow[r] & T' \times_{X_2} U \end{tikzcd}$$

commutes, since, as described in the above paragraph, equality of $Y \times_{X_2} U \to T' \times_{X_2} U$ and $Y \times_{X_2} U \to T \times_{X_2} U \to T' \times_{X_2} U$ can be checked after base change to $\bar{x}$, where the maps are equal by construction.

In this way, we get an étale covering $\{U_i \to X_2\}_i$ of $X_2$ together with maps $T \times_{X_2} U_i \to T' \times_{X_2} U_i$. Since the properties of being finite étale and having geometrically connected fibres are both stable under base change and by the uniqueness above, we have a unique morphism $T \times_{X_2} (U_i \times_{X_2} U_j) \to T' \times_{X_2} (U_i \times_{X_2} U_j)$ after base change to the "intersections" $U_i \times_{X_2} U_j$.

Meanwhile, by [Stacks, Tag 040L],

$$U \mapsto \mathrm{Hom}_U(T \times_{X_2} U, T' \times_{X_2} U)$$

satisfies the sheaf condition for the fpqc topology (hence in particular for the étale topology). Therefore, we can glue the maps $T \times_{X_2} U_i \to T' \times_{X_2} U_i$ to a unique morphism $T \to T'$ over $X_2$. The obtained morphism $T \to T'$ is also a morphism under $Y$, by base change to $\bar{x}$ as before.

**Existence of $T$ is étale-local :**

Next, we claim that the existence of a diagram (15) with the claimed properties is equivalent to the following statement : For any geometric point $\bar{x}$ of $X_2$, there is an étale neighborhood $\bar{x} \to U \to X_2$ of $\bar{x}$ such that there exists a commutative diagram

$$\begin{tikzcd} Y \times_{X_2} U \arrow[r] \arrow[d] & T \arrow[d] \\ X \times_{X_2} U \arrow[r] & U \end{tikzcd} \tag{18}$$

with the properties stated in the lemma. To see this, first note that the universal property shown above implies that such $T$ is unique up to unique isomorphism. Now because being finite étale and having geometrically connected fibres are both stable under base change, for any étale covering $\{U_i \to X_2\}_i$, maps $T_i \to U_i$ as in (18) canonically give rise to a descent datum relative to $\{U_i \to X_2\}_i$ in the sense of [Stacks, Tag 023W]. (For any product $U_i \times_{X_2} U_j$, there is a unique isomorphism $T_i \times_{X_2} U_j \to U_i \times_{X_2} T_j$ making the obvious diagram commute, and because of uniqueness, these isomorphisms trivially satisfy the cocycle condition). Now, since any descent datum of schemes finite over an fpqc covering is effective by [Stacks, Tag 0245], there is a unique $T$ over $X_2$ such that base change to $U_i$ gives $T_i$. Moreover, the morphisms $Y \times_{X_2} U_i \to T_i \cong T \times_{X_2} U_i$ uniquely glue to a morphism $Y \to T$ again by [Stacks, Tag 040L]. The glued morphism $Y \to T$ has geometrically connected fibres, since a geometric point of $T$ factors through some $T_i$ and $Y \times_{X_2} U_i \to T_i$ has geometrically connected fibres. Because being finite étale is fpqc local on the target, the map $T \to X_2$ is finite étale. Now, for the rest of the proof, fix a geometric point $\bar{x} \to X_2$.

**Existence of $T$ :**

Consider the following diagram :

$$\begin{tikzcd} Y \times_{X_2} \bar{x} \arrow[r] \arrow[d] & X \times_{X_2} \bar{x} \arrow[r] \arrow[d] & \bar{x} \arrow[d] \\ Y \arrow[r] & X \arrow[r] \arrow[d] & X_2 \arrow[d] \\ & X_1 \arrow[r] & k \end{tikzcd}$$

All squares are cartesian and there is a canonical isomorphism $X \times_{X_2} \bar{x} \cong X_1 \times_k \bar{x} =: X_{1,\bar{x}}$. So $Y \times_{X_2} \bar{x} \to X_{1,\bar{x}}$ is a finite étale cover of $X_{1,\bar{x}}$. By $\pi_1$-properness of $X_1 \to k$, there exists a finite étale cover $W \to X_1$ such that $Y \times_{X_2} \bar{x}$ is isomorphic to $W \times_k \bar{x}$.

By Lemma 3.7, applied to $X \to X_2$, there exists an étale neighborhood $U$ of $\bar{x}$ in $X_2$ such that $Y \times_{X_2} U \cong W \times_k U$. Now, let $W = \coprod W_i$ be the (finite) decomposition of $W$ into connected components. Since for schemes, base change commutes with finite coproducts, we have $Y \times_{X_2} U = (\coprod W_i) \times_k U = \coprod (W_i \times_k U)$. Set $T := \coprod (U, i)$, where $(U, i)$

denotes the $i$-th copy of $U$. Consider the square

$$\begin{array}{ccc} \coprod(W_i \times_k U) & \longrightarrow & T \\ \big\downarrow & & \big\downarrow \\ X_1 \times_k U & \longrightarrow & U \end{array} \tag{19}$$

Here, the map $\coprod(W_i \times_k U) \to T = \coprod(U, i)$ denotes the one induced by the projections $W_i \times_k U \to (U, i)$ composed with the coprojections $(U, i) \to \coprod(U, i)$. The map $\coprod(W_i \times_k U) \to X_1 \times_k U$ means the canonical one induced by the finite étale covers $W_i \times_k U \to X_1 \times_k U$. The map $\coprod(U, i) \to U$ means the one induced by the identity $(U, i) \to U$. Diagram (19) commutes, since for all $i$,

$$\begin{array}{ccc} W_i \times_k U & \longrightarrow & U \\ \big\downarrow & & \big\downarrow{\scriptstyle id_U} \\ X_1 \times_k U & \longrightarrow & U \end{array}$$

commutes. Diagram (19) also fulfills the claimed properties : First, the map $\coprod(W_i \times_k U) \to T$ has geometrically connected fibres : Let $\bar{t} \to T = \coprod(U, i)$ be a geometric point. Since the underlying topological space of a coproduct of schemes is the disjoint union of the underlying topological spaces, $\bar{t}$ maps to a unique copy $(U, j)$ of $U$. Then the fibre $(\coprod(W_i \times_k U))_{\bar{t}}$ is isomorphic to $W_j \times_k \bar{t}$, since in the diagram

$$\begin{array}{ccc} W_j \times_k \bar{t} & \longrightarrow & \bar{t} \\ \big\downarrow & & \big\downarrow \\ W_j \times_k U & \longrightarrow & (U, j) \\ \big\downarrow & & \big\downarrow \\ \coprod(W_i \times_k U) & \longrightarrow & \coprod(U, i), \end{array}$$

both squares are cartesian. For the lower square, observe that $(U, j) \to \coprod(U, i)$ is an open immersion and the base change of an open subscheme along some map can be identified with the open subscheme defined by the preimage under that map, see e.g. [Stacks, Tag 01JR]. Now, $W_j \times_k \bar{t}$ is connected by Lemma 3.1. Finally, we have that $T \to U$ is finite étale, since it is a finite coproduct of isomorphisms onto $U$. $\square$

Finally, we may combine Lemmas 3.10, 3.11 and 3.12 to prove Proposition 1.2.

*Proof of Proposition 1.2.* Lemma 3.10 shows exactness at $\pi_1(X_2, \bar{z})$. Lemma 3.12 and again Lemma 3.10 show that the projection $X := X_1 \times_k X_2 \to X_2$ satisfies the assumptions of Lemma 3.11. Hence we conclude exactness at $\pi_1(X_1 \times_k X_2, \bar{z})$. $\square$

## 3.4 Failure in positive characteristic

In Example 2.27, we already stated that the étale fundamental group of the affine line over any field $k$ of characteristic $p$ behaves differently than in the case of characteristic $0$. More precisely, $\pi_1(\mathbb{A}^1_k, \bar{x})$ is "strictly larger" than $\pi_1(\operatorname{Spec}(k), \bar{x})$ in the sense that the latter is a non-trivial quotient of the former. In fact, the affine line also serves as a counterexample for $\pi_1$-properness in characteristic $p$. In particular, by Remark 3.5, it provides a counterexample for the Künneth formula. The reason for this is again related to Artin-Schreier theory.

**Example 3.14** (cf. [SW20, Example 16.1.1])**.** One way to examine the étale fundamental group of a connected scheme $X$ is to relate it to an étale cohomology group of some sheaf on $X$. More precisely, suppose $\underline{G}_X$ is the constant sheaf on the étale site $X_{\text{ét}}$ of $X$ with values in a finite abelian group $G$. By [Mil13, Example 11.3], for any geometric point $\bar{x}$ of $X$, there is an isomorphism

$$\operatorname{Hom}(\pi_1(X, \bar{x}), G) \cong H^1_{\text{ét}}(X, \underline{G}_X), \tag{20}$$

of abelian groups where the left hand side denotes the group of continuous group homomorphisms $\pi_1(X, \bar{x}) \to G$, where $G$ is given the discrete topology. Now, if $X = \operatorname{Spec}(R)$ is a connected affine scheme in characteristic $p$, we can explicitly describe $H^1_{\text{ét}}(X, \underline{G}_X)$. For this, consider the following exact sequence of sheaves of abelian groups on $X_{\text{ét}}$, which is referred to as the *Artin-Schreier sequence* of $X$, cf. [Mil13, Example 7.9 (b)] :

$$0 \longrightarrow \underline{\mathbb{Z}/p\mathbb{Z}}_X \longrightarrow \mathbb{G}_{a,X} \xrightarrow{F-\mathrm{id}} \mathbb{G}_{a,X} \longrightarrow 0.$$

Here, $\underline{\mathbb{Z}/p\mathbb{Z}}_X$ denotes the constant sheaf on $X_{\text{ét}}$ with values in $\mathbb{Z}/p\mathbb{Z}$, by $\mathbb{G}_{a,X}$ we mean the quasi-coherent sheaf sending $U \to X$ to the additive group of $\mathcal{O}_X(U)$, see [Stacks, Tag 03P4], and by $F$ the Frobenius map $a \mapsto a^p$. By [Stacks, Tags 01XB and 03P2], the cohomology groups $H^q_{\text{ét}}(X, \mathbb{G}_{a,X})$ vanish for $q \geq 2$. Therefore, the Artin-Schreier sequence induces an exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow R \xrightarrow{F-\mathrm{id}} R \longrightarrow H^1_{\text{ét}}(X, \underline{\mathbb{Z}/p\mathbb{Z}}_X) \longrightarrow 0,$$

giving us an isomorphism $R/S \cong H^1_{\text{ét}}(X, \underline{\mathbb{Z}/p\mathbb{Z}}_X)$, where $S := \operatorname{im}(F - id)$, which is a subgroup of $R$. If $R = k[T]$ for a field $k$ of characteristic $p$, we compute

$$S = \left\{ \sum_{i \geq 0} (a_i^p T^{ip} - a_i T^i) \mid a_i \in k \right\}.$$

From this explicit description, one can deduce that if $R' = k'[T]$ for a non-trivial field extension $k \to k'$, the natural map $R \to R'$ cannot induce a surjection between $R/S$ and $R'/S'$. But $X_{k'} \cong \operatorname{Spec}(k'[T])$, so (20) shows that we cannot have $\pi_1(X_{k'}, \bar{x}) \cong \pi_1(X, \bar{x})$ for any geometric point $\bar{x} \to X_{k'} \to X$.

# 4 Drinfeld's Lemma

Let us fix a prime number $p$. The goal of this chapter is to prove the main theorem of this thesis :

**Theorem 1.4** (Drinfeld's Lemma). *Let $X_1$, $X_2$ be connected qcqs schemes over $\mathbb{F}_p$. Then, $X_1 \times_{\mathbb{F}_p} X_2$ is $\varphi_1$-connected and for any geometric point $\bar{z}$ of $X_1 \times_{\mathbb{F}_p} X_2$ (and thus of $X_1$ and $X_2$), the natural map*

$$\pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z}) \xrightarrow{\sim} \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z}) \tag{21}$$

*is an isomorphism of topological groups.*

Of course, we have not yet defined what we mean by $\varphi_1$, the property "$\varphi_1$-connected" and the group $\pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z})$, respectively. These definitions will be made in the course of this chapter. First, let us review Frobenius morphisms of schemes of positive characteristic and define $\varphi_1$ :

**Definition and Remark 4.1.** Let $X$ be a scheme over $\mathbb{F}_p$. Then the identity on topological spaces together with the assignment $\mathcal{O}_X(U) \to \mathcal{O}_X(U) : a \mapsto a^p$ on structure sheaves defines a map of $\mathbb{F}_p$-schemes $X \to X$, which we call the *absolute Frobenius of $X$* and denote by $F_X$.
Note that for any map $Y \to X$ of $\mathbb{F}_p$-schemes,

$$\begin{array}{ccc} Y & \xrightarrow{F_Y} & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{F_X} & X \end{array}$$

commutes. We call the canonical factorization $Y \to F_X^* Y := Y \times_{X, F_X} X$ the *relative Frobenius of $Y$ over $X$* and denote it by $F_{Y/X}$. If $X_1$, $X_2$ are two schemes over $\mathbb{F}_p$, we denote by

$$\varphi_1 := F_{X_1} \times_{\mathbb{F}_p} \mathrm{id}_{X_2} : X_1 \times_{\mathbb{F}_p} X_2 \to X_1 \times_{\mathbb{F}_p} X_2$$

the first and by

$$\varphi_2 := \mathrm{id}_{X_1} \times_{\mathbb{F}_p} F_{X_2} : X_1 \times_{\mathbb{F}_p} X_2 \to X_1 \times_{\mathbb{F}_p} X_2$$

the second *partial Frobenius* of $X_1 \times_{\mathbb{F}_p} X_2$.

An important property of all these types of Frobenii is the following :

**Proposition 4.2.** *Absolute, partial and relative Frobenii are universal homeomorphisms.*

*Proof.* See [Stacks, 0CC8] for absolute Frobenii. Partial Frobenii are base changes of absolute Frobenii and hence also universal homeomorphisms. The relative Frobenius is a factorization of an absolute Frobenius through a partial Frobenius and hence also a universal homeomorphism. □

Now, let us turn to Drinfeld's Lemma. As indicated in the introduction, we will first introduce the notion of $\varphi$-connectedness and prove that $X_1 \times_{\mathbb{F}_p} X_2$ is $\varphi_1$-connected. Then, in Section 4.2, we will introduce the categories $\mathrm{FEt}(X/\varphi)$ for $X$

a scheme and $\varphi : X \to X$ a universal homeomorphism which we show to be a Galois category if $X$ is $\varphi$-connected. This will allow us to define fundamental groups $\pi_1(X/\varphi, \bar{z})$, a special case of which is the group $\pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z})$ occurring in Drinfeld's Lemma. In Section 4.3, we show that any scheme over $\mathbb{F}_p$ satisfies an analogue of $\pi_1$-properness, which will allow us to conclude Drinfeld's Lemma analogously to the Künneth formula given in Proposition 1.2. These steps will be carried out in Sections 4.4, 4.5 and 4.6.

## 4.1 $\varphi$-connectedness

**Definition 4.3.** Let $X$ be a topological space and $\varphi : X \to X$ be a homeomorphism. A subset $U \subseteq X$ is called *$\varphi$-stable* if $\varphi(U) = U$. We say $X$ is *$\varphi$-connected*, if it is non-empty and it has no clopen and $\varphi$-stable subsets except $\emptyset$ and $X$. A $\varphi$-stable subset $U \subseteq X$ is called *$\varphi$-connected* if it is *$\varphi|_U$-connected* with respect to the subspace topology of $U$. A *$\varphi$-connected component* is a maximal $\varphi$-connected subset. If $X$ is a scheme and $\varphi$ a map of schemes, we say $X$ is $\varphi$-connected, if its underlying topological space is $\varphi$-connected.

The notion of $\varphi$-connectedness behaves the same as the usual connectedness :

**Remark 4.4.** A topological space $X$ is $\varphi$-connected if and only if there are no non-empty open (or, equivalently, closed) and $\varphi$-stable subsets $U, V$ of $X$ such that $X = U \coprod V$. Indeed, since $\varphi$ is bijective, we have $\varphi(X \setminus U) = X \setminus \varphi(U) = X \setminus U$ for any $\varphi$-stable $U$, i.e., the complement of a $\varphi$-stable subset is $\varphi$-stable. Moreover, $\varphi$-connected components of $X$ are closed, since the closure of a $\varphi$-connected subset $U$ is $\varphi$-connected : As $\varphi$ and $\varphi^{-1}$ are continuous, we have $\varphi(\overline{U}) \subseteq \overline{\varphi(U)} = \overline{U}$ and $\varphi^{-1}(\overline{U}) \subseteq \overline{\varphi^{-1}(U)} \subseteq \overline{U}$, hence $\varphi(\overline{U}) = \overline{U}$, i.e., $\overline{U}$ is $\varphi$-stable. Moreover, if $\overline{U} = U_1 \coprod U_2$, where $U_1$ and $U_2$ are closed in $\overline{U}$ and $\varphi$-stable, then $U \cap U_i$ are closed in $U$ and $\varphi$-stable and $U = (U \cap U_1) \coprod (U \cap U_2)$.

**Example 4.5.** Let $X$ be a set endowed with the discrete topology and $\varphi : X \to X$ a bijection. Then $X$ is $\varphi$-connected if and only if the group $\varphi^{\mathbb{Z}} := \{\varphi^i \mid i \in \mathbb{Z}\}$ acts transitively on $X$. This is because the orbit $\{\varphi^i(x) : i \in \mathbb{Z}\}$ of any $x \in X$ is a non-empty $\varphi$-stable subset of $X$. More precisely, one can check that the $\varphi$-connected components of $X$ are precisely given by the set $X/\varphi^{\mathbb{Z}}$ of orbits of the action of $\varphi^{\mathbb{Z}}$ on $X$. Note that if $X$ is finite with $n$ elements, then $\varphi^{\mathbb{Z}}$ acting transitively on $X$ is further equivalent to $\varphi$ being an $n$-cycle in the symmetric group of $X$ in the sense of [Rot12, Ch. 1].

An important tool will be the following lemma, which slightly generalizes the topological fact that an open map with connected fibres induces a bijection on connected components, see [Stacks, Tag 0378].

**Lemma 4.6.** *Let $X, Y$ be topological spaces and $\varphi : X \to X$ a homeomorphism. Let $f : X \to Y$ be an open continuous map. If all fibres $f^{-1}(y)$ are $\varphi$-stable and $\varphi$-connected, we have a bijection*

$$\begin{aligned} \{\varphi\text{-connected components of } X\} &\cong \{\text{connected components of } Y\} \\ U &\mapsto f(U) \\ f^{-1}(V) &\leftarrow\!\shortmid V \end{aligned}$$

*Proof.* First note that the $\varphi$-connectedness condition on the fibres of $f$ implies that they are non-empty, i.e., $f$ is surjective. Since $f$ is also open, we have that the topology on $Y$ is the quotient topology with respect to $f$. So a subset of $V \subseteq Y$ is clopen if and only if $f^{-1}(V) \subseteq X$ is clopen.

- **Images of $\varphi$-connected components are connected :** It suffices to show that $Y$ is connected if $X$ is $\varphi$-connected. For this, suppose $V \subseteq Y$ is a non-trivial clopen subset of $Y$. Then $f^{-1}(V)$ is clopen in $X$ and also $\varphi$-stable, because all fibres of $f$ are $\varphi$-stable and hence also their union $f^{-1}(V)$. But since $f$ is surjective, $f^{-1}(V)$ is non-empty. Then, since $X$ is $\varphi$-connected, $f^{-1}(V) = X$ and since $f$ is surjective, $V = f(f^{-1}(V)) = f(X) = Y$, contradiction.
- **Preimages of connected components are $\varphi$-connected :** Now, let $V$ be a connected component of $Y$. In particular, $V$ and thus $f^{-1}(V)$ are closed. Suppose $f^{-1}(V)$ is not $\varphi$-connected, then $f^{-1}(V) = U_1 \coprod U_2$ for non-empty clopen and $\varphi$-stable subsets $U_1$, $U_2$ of $f^{-1}(V)$. Since $f^{-1}(V)$ is closed, the $U_i$ are also closed in $X$. For any $v \in V$, we have $f^{-1}(v) = (U_1 \cap f^{-1}(v)) \coprod (U_2 \cap f^{-1}(v))$. The subsets $f^{-1}(v) \cap U_i$ are $\varphi$-stable and closed in $f^{-1}(v)$. So $f^{-1}(v) \cap U_i = f^{-1}(v)$ or $f^{-1}(v) \cap U_i = \emptyset$ and since the $U_i$ are disjoint, $f^{-1}(v)$ is contained in either $U_1$ or $U_2$. In other words, for $V_i := \{v \in V : f^{-1}(v) \subseteq U_i\}$ it holds that $V = V_1 \coprod V_2$. Since $f^{-1}(V) = U_1 \coprod U_2$, we actually have $f^{-1}(V_i) = U_i$ and thus the $f^{-1}(V_i)$ are closed in $X$. Therefore, we apply that the topology on $Y$ is the quotient topology with respect to $f$ and conclude that the $V_i$ are closed in $Y$ and therefore also in the subspace topology of $V$. This is a contradiction to $V$ being connected.
- **Images of $\varphi$-connected components are connected components :** Let $U$ be a $\varphi$-connected component of $X$. By the above, $f(U)$ is connected. So there exists a connected component $V$ of $Y$ with $f(U) \subseteq V$, i.e., $U \subseteq f^{-1}(V)$. Since $f^{-1}(V)$ is $\varphi$-connected by the above and by maximality of $U$, $U = f^{-1}(V)$ and by surjectivity of $f$, $f(U) = V$.
- **Preimages of connected components are $\varphi$-connected components :** If $V$ is a component of $Y$, then $f^{-1}(V)$ is $\varphi$-connected by the above and thus contained in a component $U$ of $X$. By the surjectivity of $f$, $V = f(f^{-1}(V)) \subseteq f(U)$. But $f(U)$ is connected and by maximality of $V$, $V = f(U)$. Then, $f^{-1}(V) \subseteq U$ implies $U = f^{-1}(V)$, so $f^{-1}(V)$ is in fact a connected component.
- **Bijectivity :** Let $U$ be a component of $X$. $U$ is contained in $f^{-1}(f(U))$, which, by the above, is $\varphi$-connected. By maximality, $U = f^{-1}(f(U))$. Surjectivity of $f$ implies $V = f(f^{-1}(V))$ for any $V \subseteq Y$.

$\square$

**Remark 4.7.** Let $X$ be a topological space and $\varphi : X \to X$ a homeomorphism. Let $X/\varphi^{\mathbb{Z}}$ be the set of orbits of the action of the group $\varphi^{\mathbb{Z}}$ on $X$. For any $x \in X$, denote by $[x]$ its orbit. We may endow $X/\varphi^{\mathbb{Z}}$ with the quotient topology. Then, in fact, the quotient map $p : X \to X/\varphi^{\mathbb{Z}}$ is open : Let $U \subseteq X$ be open, then

$$p^{-1}(p(U)) = p^{-1}(\{[u] \mid u \in U\}) = \{x \in X \mid \exists u \in U : [x] = [u]\} = \bigcup_{u \in U} [u] = \bigcup_{i \in \mathbb{Z}} \varphi^i(U)$$

is open, as $\varphi$ is open. So, applying Lemma 4.6 to the open map $X \to X/\varphi^{\mathbb{Z}}$ gives us that the $\varphi$-connected components of $X$ are precisely the connected components of the topological space $X/\varphi^{\mathbb{Z}}$. Observe that this generalizes the situation when $X$ has the discrete topology, see Example 4.5.

**Lemma 4.8.** *Let $X, Y$ be topological spaces and $\varphi_X : X \to X$, $\varphi_Y : Y \to Y$ homeomorphisms. Let $f : X \to Y$ be an open continuous map such that $f \circ \varphi_X = \varphi_Y \circ f$. If all fibres of $f$ are connected, we have a bijection*

$$\{\varphi_X\text{-connected components of } X\} \cong \{\varphi_Y\text{-connected components of } Y\}$$
$$U \mapsto f(U)$$
$$f^{-1}(V) \leftarrow V$$

*Proof.* The compatability condition $f \circ \varphi_X = \varphi_Y \circ f$ implies that $f$ induces an (open) map $X/\varphi_X^{\mathbb{Z}} \to Y/\varphi_Y^{\mathbb{Z}}$. To that map we may apply the already mentioned classical fact from topology that an open map with connected fibres induces a bijection of connected components, see [Stacks, Tag 0378]. Then Remark 4.7 implies the claim. □

The following is the analogue of [Stacks, Tag 07VB] :

**Lemma 4.9.** *Let $X, Y$ be nonempty topological spaces and $\varphi_X : X \to X$, $\varphi_Y : Y \to Y$ homeomorphisms. Let $f : X \to Y$ be a clopen continuous map such that $f \circ \varphi_X = \varphi_Y \circ f$. Suppose $Y$ is $\varphi_Y$-connected and there exists $y \in Y$ such that $\#f^{-1}(y)$ is finite. Then $X$ has at most $f^{-1}(y)$ many $\varphi_X$-connected components.*

*Proof.* Denote by $n \in \mathbb{N} \cup \{\infty\}$ the number of $\varphi_X$-connected components of $X$. Choose $y$ as in the assumptions of the lemma. We show that for any $N \in \mathbb{N}$ with $N \leq n$, we have $N \leq \#f^{-1}(y)$. Namely, given such $N$, by induction, we can write $X$ as the disjoint union of $N$ clopen and $\varphi_X$-stable subsets $X_i$. Since $f$ is clopen and $f \circ \varphi_X = \varphi_Y \circ f$, each $f(X_i)$ is clopen and $\varphi_Y$-stable. Since $Y$ is $\varphi_Y$-connected, $f(X_i) = Y$ for all $i = 1, ..., N$. In particular, $f^{-1}(y)$ meets any $X_i$, hence $N \leq \#f^{-1}(y)$. □

In the following, we will use the notion of $\varphi$-connectedness in the situation when $X$ is a scheme over $\mathbb{F}_p$ and $\varphi$ the absolute Frobenius of $X$ or, if $X = X_1 \times_{\mathbb{F}_p} X_2$, a partial Frobenius. Note that the formalism indeed applies because of Proposition 4.2. The goal of the rest of this section is to give a proof of the following claim by Kedlaya [Ked19, Lemma 4.2.11] :

**Proposition 4.10.** *If $X_1$ and $X_2$ are connected schemes over $\mathbb{F}_p$, then $X_1 \times_{\mathbb{F}_p} X_2$ is $\varphi_1$- and $\varphi_2$-connected.*

First, let us remark :

**Remark 4.11.** For any product of schemes over $\mathbb{F}_p$, connectedness with respect to one partial Frobenius is equivalent to connectedness with respect to the other. This is because their composition is the absolute Frobenius, which is the identity on topological spaces.

The key algebraic input for Proposition 4.10 is the lemma below. To simplify notation, for the rest of this section, we set $k := \mathbb{F}_p$ and fix an algebraic closure $k \to \bar{k}$. Also, we denote by $F_{\bar{k}} : a \mapsto a^p$ the Frobenius map on $\bar{k}$.

**Lemma 4.12.** *Let $k \to \widetilde{k}$ be an algebraically closed extension of $k$. Then, there exists a (non-canonical) homeomorphism between $\operatorname{Spec}(\widetilde{k} \otimes_k \bar{k})$ and the absolute Galois group $\operatorname{Gal}(\bar{k}|k)$ of $k$. Under that homeomorphism, the second partial Frobenius $\varphi_2 = \mathrm{id}_{\widetilde{k}} \times_k F_{\bar{k}} : \operatorname{Spec}(\widetilde{k} \otimes_k \bar{k}) \to \operatorname{Spec}(\widetilde{k} \otimes_k \bar{k})$ corresponds to the map $F_{\bar{k}} \circ - : \operatorname{Gal}(\bar{k}|k) \to \operatorname{Gal}(\bar{k}|k)$.*

*Proof.* We have that $\bar{k} \cong \mathrm{colim}_{k'|k \text{ finite}} k'$ and so

$$\mathrm{Spec}(\widetilde{k} \otimes_k \bar{k}) \cong \lim_{k'|k \text{ finite}} \mathrm{Spec}(\widetilde{k} \otimes_k k'). \tag{22}$$

as topological spaces by [Stacks, Tags 00DD, 01YW and 0CUF]. On the other hand,

$$\mathrm{Gal}(\bar{k}|k) \cong \lim_{k'|k \text{ finite}} \mathrm{Gal}(k'|k) \tag{23}$$

as topological groups, see e.g. [Stacks, Tag 0BU2]. Therefore, it suffices to show the above proposition after replacing $\bar{k}$ with some finite extension $k'$ of $k$. So, fix some finite extension $k' = \mathbb{F}_{p^n}$ of $k$. We can write $k' = k[T]/(f)$, where $f \in k[T]$ is an irreducible polynomial of degree $n$. Since $\widetilde{k}$ is algebraically closed, we can choose an embedding $\bar{k} \to \widetilde{k}$. Since the field extension $k \to k'$ is separable, $f$ splits over $\bar{k}$, and hence over $\widetilde{k}$, into $n$ distinct linear factors, corresponding to the roots $a_1, ..., a_n$ of $f$ in $\bar{k}$. By the Chinese Remainder Theorem, we have

$$\widetilde{k} \otimes_k k' \cong \widetilde{k}[T]/(f) \cong \widetilde{k}[T]/(T - a_1) \times \cdots \times \widetilde{k}[T]/(T - a_n) \cong \widetilde{k}^n, \tag{24}$$

where the isomorphism $\widetilde{k} \otimes_k k' \cong \widetilde{k}^n$ is given by

$$b \otimes \bar{g} \mapsto (bg(a_1), \cdots, bg(a_n)), \tag{25}$$

where we identified $k' \cong k[T]/(f)$. Hence $\mathrm{Spec}(\widetilde{k} \otimes_k k') \cong \mathrm{Spec}(\widetilde{k}^n) \cong \coprod \mathrm{Spec}(\widetilde{k})$ is a discrete topological space whose points are in bijection with the set of roots $\{a_1, ..., a_n\}$.

Now, by Galois theory, the group $\mathrm{Gal}(k'|k)$ acts simply transitively on the set of roots $\{a_1, ..., a_n\}$. More precisely, after suitably reordering the $a_i$, the Frobenius $F_{k'}$, which generates $\mathrm{Gal}(k'|k)$, maps $a_i$ to $a_{(i+1) \bmod n}$. Put differently, $\mathrm{Gal}(k'|k)$ is in bijection with $\{a_1, ..., a_n\}$ and under that bijection, the map $F'_{\bar{k}} \circ -$ on $\mathrm{Gal}(k'|k)$ corresponds to the map $a_i \mapsto a_{(i+1) \bmod n}$.

Combined with the above, this implies that we get a bijection $\mathrm{Spec}(\widetilde{k} \otimes_k k') \cong \mathrm{Gal}(k'|k)$. Moreover, under the isomorphism (25), $\mathrm{id}_{\widetilde{k}} \otimes F_{k'}$ corresponds to the map on $\widetilde{k}^n$ which shifts entries by one position, since $F_{k'}(g(a_i)) = g(F_{k'}(a_i)) = g(a_{(i+1) \bmod n})$, as $g \in k[T]$. On $\{a_1, ..., a_n\}$, this corresponds to $a_i \mapsto a_{(i+1) \bmod n}$, which is the same map as for $F_{\bar{k}} \circ - : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(k'|k)$. □

**Corollary 4.13.** *In the situation of Lemma 4.12, the scheme* $\mathrm{Spec}(\widetilde{k} \otimes_k \bar{k})$ *is* $\varphi_2$*-connected.*

*Proof.* Let $U$ be a non-empty clopen and $\mathrm{id}_{\widetilde{k}} \times_k F_{\bar{k}}$-stable subset of $\mathrm{Spec}(\widetilde{k} \otimes_k \bar{k})$. Fix a homeomorphism $\mathrm{Spec}(\widetilde{k} \otimes_k \bar{k}) \cong \mathrm{Gal}(\bar{k}|k)$ as obtained from Lemma 4.12. Under this homeomorphism, $U$ corresponds to a subset $V$ of $\mathrm{Gal}(\bar{k}|k)$ with $F_{\bar{k}} \circ V = V$. In particular, $F_{\bar{k}}^m \circ V = V$, where $F_{\bar{k}}^m$ denotes the $m$-fold composition of $F_{\bar{k}}$ with itself, and so $V$ must contain a coset of the subgroup generated by $F_{\bar{k}}$.

Now, a fact from Galois theory is that $\mathrm{Gal}(\bar{k}|k)$ is topologically generated by $F_{\bar{k}}$, i.e., the subgroup generated by $F_{\bar{k}}$ and thus all its cosets are dense in $\mathrm{Gal}(\bar{k}|k)$. Since $U$ is closed, it therefore contains all of $\mathrm{Spec}(\widetilde{k} \otimes_k \bar{k})$. □

The rest of Proposition 4.10 can be concluded by merely topological arguments.

**Lemma 4.14.** *For any connected scheme $X$ over $k$, the base change $X_{\bar{k}} = X \times_k \operatorname{Spec}(\bar{k})$ is $\varphi_2$-connected.*

*Proof.* Let $X$ be a connected scheme over $k$. Let $x \in X$ be a point. Choose an algebraic closure $k(\bar{x})$ of the residue field $k(x)$ and consider the following diagram :

$$\begin{tikzcd}
\operatorname{Spec}(k(\bar{x}) \otimes_k \bar{k}) \arrow[r, "j \times_k \mathrm{id}_{\bar{k}}"] \arrow[d] & \operatorname{Spec}(k(x) \otimes_k \bar{k}) \arrow[r, "i \times_k \mathrm{id}_{\bar{k}}"] \arrow[d] & X_{\bar{k}} \arrow[r] \arrow[d, "p"] & \operatorname{Spec}(\bar{k}) \arrow[d] \\
\operatorname{Spec}(k(\bar{x})) \arrow[r, "j"] & \operatorname{Spec}(k(x)) \arrow[r, "i"] & X \arrow[r] & \operatorname{Spec}(k)
\end{tikzcd}$$

Note that each square is cartesian, see [GW10, Prop. 4.16]. On each scheme in the top row, we have the action of the respective base change of $F_{\bar{k}}$, which are all compatible along the maps in the top row. We check compatability by the example of the top-left map $j \times_k \mathrm{id}_{\bar{k}}$ : Here we indeed have

$$(j \times_k \mathrm{id}_{\bar{k}}) \circ (\mathrm{id}_{k(\bar{x})} \times_k F_{\bar{k}}) = j \times_k F_{\bar{k}} = (\mathrm{id}_{k(\bar{x})} \times_k F_{\bar{k}}) \circ (j \times_k \mathrm{id}_{\bar{k}}).$$

Now, by Corollary 4.13, $\operatorname{Spec}(k(\bar{x}) \times_k \bar{k})$ is $\mathrm{id}_{k(\bar{x})} \times_k F_{\bar{k}}$-connected. The idea is now to conclude the $\mathrm{id}_X \times_k F_{\bar{k}}$-connectedness of $X_{\bar{k}}$ from Lemma 4.6. Therefore, we first show that $\operatorname{Spec}(k(x) \otimes_k \bar{k})$ is $\mathrm{id}_{k(\bar{x})} \times_k F_{\bar{k}}$-connected. For this, observe that the map $j \times_k \mathrm{id}_{\bar{k}} : \operatorname{Spec}(k(\bar{x}) \otimes_k \bar{k})) \to \operatorname{Spec}(k(x) \otimes_k \bar{k})$ is surjective as the base change of a surjective map. Moreover, $j \times_k \mathrm{id}_{\bar{k}}$ is compatible with the respective base changes of the Frobenius $F_{\bar{k}}$, see above. But then, $\mathrm{id}_{k(\bar{x})} \times_k F_{\bar{k}}$-connectedness of $\operatorname{Spec}(k(\bar{x}) \times_k \bar{k})$ implies $\mathrm{id}_{k(x)} \times_k F_{\bar{k}}$-connectedness of $\operatorname{Spec}(k(x) \otimes_k \bar{k})$.
Next, we know that $\operatorname{Spec}(k(x) \otimes_k \bar{k})$ is canonically isomorphic to the scheme-theoretic fibre of the projection $X_{\bar{k}} \to X$ at $x$ (which is homemorphic to the set-theoretic fibre) and $i \times_{F_p} \mathrm{id}_{\bar{k}}$ is a homeomorphism on the set-theoretic fibre $p^{-1}(x)$. Moreover, the projection $p : X_{\bar{k}} \to X$ is open and surjective by [Stacks, Tags 0383 and 01S1]. As above, we see that the morphism $i \times_k \mathrm{id}_{\bar{k}}$ is compatible with the respective base changes of $F_{\bar{k}}$. Note that $x$ was an arbitrary point of $X$, so indeed all fibres $X_{k(x)}$ of $p$ are $\mathrm{id}_{k(x)} \times_k F_{\bar{k}}$-connected. So we may apply Lemma 4.6 to $p$ with $\varphi_{X_{\bar{k}}} = \mathrm{id}_X \times_k F_{\bar{k}}$, get a bijection

$$\{(\mathrm{id}_X \times_k F_{\bar{k}})\text{-connected components of } X_{\bar{k}}\} \cong \{\text{connected components of X}\}$$

Since $X$ is connected by assumption, the claim follows. $\square$

Now, in order to conclude Proposition 4.10, we proceed in analogy to the following "classical" statement :

**Lemma 4.15.** *Let $X_1$, $X_2$ be schemes over some field $l$. Suppose that $X_2$ is connected and the base change of $X_1$ to a separable algebraic closure $\bar{l}$ of $l$ is connected. Then $X_1 \times_l X_2$ is connected.*

This of course implies Lemma 3.1. A proof of Lemma 4.15 goes as follows :

1. Any connected scheme $X$ over a separably closed field $\widetilde{l}$ is geometrically connected, i.e., for any field extension $\widetilde{l} \to l'$, the base change $X_{l'}$ is connected. This is shown in [Stacks, Tag 0363]. Thus, in the situation of the Lemma, $X_{1,\bar{l}}$ is geometrically connected over $\bar{l}$.

2. If $X_{1,\bar{l}}$ is geometrically connected over $\bar{l}$, then also $X_1$ is geometrically connected over $l$ since for any extension $l \to l'$, there exists a common extension $\bar{l}'$ of $\bar{l}$ and $l'$. Then $X_{1,\bar{l}'}$ is connected by hypothesis and since $X_{1,\bar{l}'} \to X_{1,l'}$ is surjective, also $X_{1,l'}$ is connected, cf. [Stacks, Tag 0387].
3. For any field $l$, the fibre product of a geometrically connected scheme $X_1$ over $l$ with a connected $X_2$ scheme over $l$ is connected, see [Stacks, Tag 0385].

In the following, we will promote these steps to the setting of $\varphi$-connectedness.

**Definition 4.16.** Let $X$ be a scheme over a field $l$ and $\varphi$ an $l$-linear universal homeomorphism of $X$. We say $X$ is *geometrically $\varphi$-connected*, if for all field extensions $l \to l'$, $X_{l'}$ is $\varphi \times_l \mathrm{id}_{l'}$-connected.

Now, the first proof step translates to :

**Lemma 4.17.** *Let $X$ be a scheme over a separably closed field $\widetilde{l}$ and $\varphi : X \to X$ a $\widetilde{l}$-linear universal homeomorphism of $X$. Then if $X$ is $\varphi$-connected, it is geometrically $\varphi$-connected.*

*Proof.* Let $\widetilde{l} \to l'$ be any field extension of $\widetilde{l}$ and let $x \in X$ be a point. Consider the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(k(x) \otimes_{\widetilde{l}} l') & \xrightarrow{i \times_{\widetilde{l}} \mathrm{id}_{l'}} & X_{l'} & \longrightarrow & \mathrm{Spec}(l') \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ \mathrm{Spec}(k(x)) & \xrightarrow{\quad i \quad} & X & \longrightarrow & \mathrm{Spec}(\widetilde{l}), \end{array}$$

where both squares are cartesian. As in the proof of Lemma 4.14, we see that $i \times_{\widetilde{l}} \mathrm{id}_{l'}$ is a homeomorphism between $\mathrm{Spec}(k(x) \otimes_{\widetilde{l}} l')$ and the set-theoretic fibre $p^{-1}(x)$. Now, since $\widetilde{l}$ is separably closed, the map $\mathrm{Spec}(l') \to \mathrm{Spec}(\widetilde{l})$ is geometrically connected, see [Stacks, Tags 037U and 0386]. Therefore, $\mathrm{Spec}(k(x) \otimes_{\widetilde{l}} l')$ is connected. Then we apply Corollary 4.8 to $p$ with $\varphi_{X_{l'}} := \varphi \times_{\widetilde{l}} \mathrm{id}_{l'}$ and $\varphi_X := \varphi$ and get that $X_{l'}$ is $\varphi_{X_{l'}}$-connected. □

The second step translates to :

**Lemma 4.18.** *Let $X$ be a scheme over a field $l$, $\varphi : X \to X$ a $l$-linear universal homeomorphism and $l \to \widetilde{l}$ a field extension. If $X_{\widetilde{l}}$ is geometrically $(\varphi \times_l \mathrm{id}_{\widetilde{l}})$-connected, then $X$ is geometrically $\varphi$-connected.*

*Proof.* Let $l \to l'$ be any field extension. There exists a field extension $l \to \widetilde{l}'$ containing both $\widetilde{l}$ and $l'$. Note that $p : X_{\widetilde{l}'} \to X$ factors through both $X_{l'}$ and $X_{\widetilde{l}}$. Since $X_{\widetilde{l}}$ is geometrically $\varphi \times_l \mathrm{id}_{\widetilde{l}}$-connected, $X_{\widetilde{l}'}$ is $\varphi \times_l \mathrm{id}_{\widetilde{l}'}$-connected. On the other hand, the map $X_{\widetilde{l}'} \to X_{l'}$ is surjective and compatible with the respective base changes of $\varphi$. Therefore, $X_{l'}$ is $\varphi \times_l \mathrm{id}_{l'}$-connected. □

The third step is just a special case of Lemma 4.6 :

**Lemma 4.19.** *Let $X_1$, $X_2$ be schemes over some field $l$, $\varphi$ be a $l$-linear universal homeomorphism of $X_1$ and let $X_1$ be geometrically $\varphi$-connected. Then, the projection $X_1 \times_l X_2 \to X_2$ induces a bijection*

$$\{\varphi \times_l \mathrm{id}_{X_2}\text{-connected components of } X_1 \times_l X_2\} \cong \{\text{connected components of } X_2\}.$$

*Proof.* Denote by $p : X_1 \times_l X_2 \to X_2$ the projection to $X_2$. The fibres of $p$ are canonically isomorphic to base changes $X_{1,l'}$, which by assumption are $\varphi \times_l \mathrm{id}_{l'}$-connected. Under the mentioned isomorphism, $\varphi \times_l \mathrm{id}_{l'}$ agrees with $\varphi \times_l \mathrm{id}_{X_2}$, which means that the fibres of $p$ are $\varphi \times_l \mathrm{id}_{X_2}$-connected. By [Stacks, Tag 0383], $p$ is open. So applying Lemma 4.6 again to $p$ with $\varphi_{X_1 \times_l X_2} := \varphi \times_l \mathrm{id}_{X_2}$ yields the claim. □

Now, finally, we are able to prove Proposition 4.10 :

*Proof of Proposition 4.10.* Let $X_1$, $X_2$ be connected schemes over $k = \mathbb{F}_p$. By Lemma 4.14, $X_1 \times_k \bar{k}$ is $\varphi_2$-connected. Then it is also $\varphi_1$-connected by Remark 4.11. Now, by Lemmas 4.17 and 4.18, $X_1$ is geometrically $F_{X_1}$-connected. Then, since $X_2$ is connected, $X_1 \times_k X_2$ is $\varphi_1$-connected by Lemma 4.19. □

## 4.2 The Galois category $\mathrm{FEt}(X/\varphi)$

Our next step towards Drinfeld's Lemma is to introduce a fundamental group $\pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z})$. For this, we construct a category $\mathrm{FEt}(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1)$, which we prove to be a Galois category. In turns out that this is just a special case of a more general construction which works for any scheme $X$ and universal homeomorphism $\varphi : X \to X$ such that $X$ is $\varphi$-connected. The reason for this is that universal homeomorphisms interplay well with étale morphisms, as we saw in Proposition 2.12.

**Remark 4.20.** Suppose $X$ is a scheme, $\varphi : X \to X$ is a universal homeomorphism and $f : Y \to X$ is finite étale. Consider a commutative square

$$\begin{tikzcd} Y \arrow[r, "\varphi_Y"] \arrow[d, "f"] & Y \arrow[d, "f"] \\ X \arrow[r, "\varphi"] & X, \end{tikzcd}$$

where $\varphi_Y : Y \to Y$ is a universal homeomorphism. We claim that any such square is already cartesian. To see this, denote by $\varphi^* X := Y \times_{X,\varphi} X$ the base change of $Y \to X$ along $\varphi$ and consider the following commutative diagram :

$$\begin{tikzcd} Y \arrow[dr, "\beta"] \arrow[drr, bend left, "f^*\varphi\circ\beta"] \arrow[ddr, bend right, "f"'] & & \\ & \varphi^*Y \arrow[r, "f^*\varphi"] \arrow[d, "\varphi^*f"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ & X \arrow[r, "\varphi"'] & X. \end{tikzcd}$$

The canonical factorization $\beta : Y \to \varphi^* Y$ coming from the universal property of the fibre product is itself a universal homeomorphism as $\varphi_Y = f^*\varphi \circ p$ and both $\varphi_Y$ and $f^*\varphi$ are universal homeomorphisms. On the other hand, $\beta$ is étale as a map between étale $X$-schemes. But an étale universal homeomorphism is an isomorphism by Proposition 2.12. Hence, it is equivalent to give the following data :

- A pair $(f : Y \to X, \varphi_Y)$, where $f$ is finite étale and $\varphi_Y : Y \to Y$ is a universal homeomorphism with $\varphi \circ f = f \circ \varphi_Y$,

- a pair $(f : Y \to X, \varphi_Y)$, where $f$ is finite étale and $\varphi_Y : Y \to Y$ is some morphism of schemes such that the square

$$\begin{tikzcd} Y \arrow[r, "\varphi_Y"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ X \arrow[r, "\varphi"] & X \end{tikzcd}$$

is cartesian,
- a pair $(f : Y \to X, \beta)$, where $f$ is finite étale and $\beta : Y \to \varphi^* Y$ is an isomorphism.

So, let us make the following definition :

**Definition 4.21.** Let $X$ be a scheme and $\varphi : X \to X$ a universal homeomorphism. We define a category $\mathrm{FEt}(X/\varphi)$ with objects

$$\{(f : Y \to X, \varphi_Y) : f \text{ is finite étale, } \varphi_Y \text{ is a universal homeomorphism with } \varphi \circ f = f \circ \varphi_Y\}$$

and morphisms

$$\mathrm{Mor}((Y, \varphi_Y), (Y', \varphi_{Y'})) := \{X\text{-morphisms } g : Y \to Y' \text{ with } g \circ \varphi_Y = \varphi_{Y'} \circ g\}.$$

In light of Remark 4.20, we have another two equivalent descriptions of objects of $\mathrm{FEt}(X/\varphi)$ : Firstly, as pairs $(Y, \varphi_Y)$ such that the square

$$\begin{tikzcd} Y \arrow[r, "\varphi_Y"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ X \arrow[r, "\varphi"] & X, \end{tikzcd}$$

is cartesian, or secondly, as pairs $(Y, \beta)$, where $\beta : Y \to \varphi^* Y$ is the unique isomorphism fitting into the obvious diagram. We will also call an $X$-morphism $g : Y \to Y'$ with $g \circ \varphi_Y = \varphi_{Y'} \circ g$ *equivariant*.

**Lemma 4.22.** *The connected objects of* $\mathrm{FEt}(X/\varphi)$ *are precisely those pairs* $(Y, \varphi_Y)$*, where* $Y$ *is* $\varphi_Y$*-connected in the sense of Definition 4.3.*

*Proof.* The proof easily carries over from the classical setting, cf. [Stacks, Tag 0BNB]. We have the following chain of equivalences :

$(Y, \varphi_Y)$ is a connected object of $\mathrm{FEt}(X/\varphi)$

$\Leftrightarrow$ For any monomorphism $g : (Y', \varphi_{Y'}) \to (Y, \varphi_Y)$ in $\mathrm{FEt}(X/\varphi)$, $(Y', \varphi_{Y'})$ is initial or $g$ is an isomorphism

$\overset{(1)}{\Leftrightarrow}$ For any equivariant closed-open immersion $g : Y' \to Y$ over $X$, either $Y' = \emptyset$ or $g$ is an isomorphism

$\overset{(2)}{\Leftrightarrow}$ For any clopen and $\varphi_Y$-stable subset $U$ of $Y$, either $U = Y$ or $U = \emptyset$

$\Leftrightarrow$ $Y$ is $\varphi_Y$-connected

For (1) : Monomorphisms in the category $\mathrm{FEt}(X/\varphi)$ are monomorphisms of schemes $Y' \to Y$ over $X$ which are equivariant in the sense of Definition 4.21. Any map $Y' \to Y$ of schemes étale over $X$ is étale. The fundamental property

of étale maps ([Stacks, Tag 025G]) implies that the monomorphisms of schemes between $Y' \to Y$ over $X$ are precisely the open immersions $Y' \to Y$ over $X$. Also any $Y' \to Y$ over $X$ is finite, hence closed. Hence the monomorphisms in $\mathrm{FEt}(X/\varphi)$ are precisely the equivariant clopen immersions.
For (2) : Any clopen $\varphi_Y$-stable subset is the image of a clopen equivariant immersion, hence (LHS) $\Rightarrow$ (RHS). An open immersion is an isomorphism if and only if it is surjective, hence (RHS) $\Rightarrow$ (LHS). $\square$

**Lemma 4.23.** *Let $X, Y$ be schemes and $\varphi : X \to X$, $\varphi_Y : Y \to Y$ universal homeomorphisms. Let $f : Y \to X$ be a finite étale morphism such that $f \circ \varphi_Y = \varphi \circ f$. Suppose $X$ is $\varphi$-connected. Then $f$ is an isomorphism if and only if there exists a geometric point $\bar{x} \to X$ such that $\#|Y_{\bar{x}}| = 1$.*

*Proof.* If $f$ is an isomorphism, then also after base change along $\bar{x} \to X$. For the other implication, let $x \in X$ be the image of $\bar{x}$ in $X$. By surjectivity of the map $Y_{\bar{x}} \to Y_x$, the set-theoretic fibre $f^{-1}(x)$ consists of a single point. The condition $f \circ \varphi_Y = \varphi \circ f$ implies that $f^{-1}(\varphi^n(x))$ consists of a single point for all $n \in \mathbb{Z}$. Therefore, on topological spaces, $f$ is bijective and open (being étale), hence a homeomorphism. We are left to show that it is an isomorphism on structure sheaves. For this, we show that the degree of $f$ (in the sense of [GW10, Section (12.6)]) is constant on $X$ and equals 1. First observe that $\bar{x} : \mathrm{Spec}(k(\bar{x})) \to X$ factors through the spectrum of the residue field $k(x)$, and hence we may compute

$$\deg(f)(x) = \dim_{k(x)}((f_*\mathcal{O}_Y)_x \otimes_{\mathcal{O}_{X,x}} k(x)) = (\dim_{k(\bar{x})}(f_*\mathcal{O}_Y)_x \otimes_{\mathcal{O}_{X,x}} k(\bar{x})) = \dim_{k(\bar{x})}\left(\prod_{i=1}^{m} k(\bar{x})\right) = m,$$

where $m$ equals the number of copies of $\mathrm{Spec}(k(\bar{x}))$ in the base change of $f$ to $\bar{x}$, i.e., $m = \#|Y_{\bar{x}}|$. In the above calculation, we used that the rank of a module is preserved under base change and that $(f_*\mathcal{O}_Y)_x$ is a finite étale $\mathcal{O}_{X,x}$-algebra and hence after base change to the algebraically closed field $k(\bar{x})$ splits into a finite product of copies of $k(\bar{x})$. So, by assumption, $\deg(f)(x) = \#|Y_{\bar{x}}| = 1$. As the degree is locally constant, $\deg(f)$ equals 1 on the connected component of $x$. We are left to show that it equals 1 everywhere. By Remark 4.20,

$$\begin{tikzcd} Y \arrow[r, "\varphi_Y"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ X \arrow[r, "\varphi"] & X \end{tikzcd}$$

is cartesian. Now note that the degree of a finite locally free morphism is preserved under base change (as locally on the target, such a morphism corresponds to an algebra over a ring $A$ which is finite as an $A$-module, and the rank of an $A$-module is stable under change of base rings). Applied to the above square, we conclude that $\deg(f)(\varphi^n(x)) = \deg(f)(x) = 1$ for all $n \in \mathbb{Z}$. But since by assumption, $X$ is $\varphi$-connected, for each connected component $T$ of $X$, there exists $n_T \in \mathbb{Z}$ such that $\varphi^{n_T}(x) \in T$. Hence, $\deg(f)$ is equal to 1 on each connected component of $X$, which implies that $f$ is an isomorphism. $\square$

**Proposition 4.24.** *If $X$ is $\varphi$-connected, then $\mathrm{FEt}(X/\varphi)$, together with a fibre functor $F_{\bar{x}}$ to a geometric point, is a Galois category.*

*Proof.* In this proof, we will use the alternative description of objects of $\mathrm{FEt}(X/\varphi)$ as pairs $(Y,\beta)$, where $\beta : Y \to \varphi^*Y$ is an isomorphism, see Definition 4.21. We show that $\mathrm{FEt}(X/\varphi)$ satisfies the axioms of Definition 4.21. For that, we proceed in analogy to the proof in the classical setting, cf. [Stacks, Tags 0BN9,0BNB]. First, we note that the functor $F_{\bar{x}}$ maps into the category of finite sets by 2.20.

- **Existence of finite limits :** We show the existence of a final object and fibre products. First observe that

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\square"] & X \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "\varphi"] & X \end{tikzcd}$$

  is cartesian, hence there exists a canonical isomorphism $\beta : X \to \varphi^*X$, which makes $(X,\beta)$ a final object by definition of objects of $\mathrm{FEt}(X/\varphi)$. Moreover, let $(Y,\beta)$, $(Y',\beta')$ be objects of $\mathrm{FEt}(X/\varphi)$, then $Y \times_X Y' \to Y \to X$ is again finite étale over $X$ as the composition of the base change of a finite étale morphism with a finite étale morphism. Moreover, since base change (here, along $\varphi : X \to X$) preserves fibre products, we have a canonical isomorphism $(\varphi^*Y \times_X \varphi^*Y') \cong \varphi^*(Y \times_X Y')$. Therefore, the isomorphism $\beta \times_X \beta' : Y \times_X Y' \to \varphi^*Y \times_X \varphi^*Y'$ gives rise to an isomorphism $Y \times_X Y' \to \varphi^*(Y \times_X Y')$.

- **Existence of finite colimits :** We show the existence of finite coproducts and coequalizers. Both exist in $\mathrm{FEt}(X)$ since it is a Galois category. As used in the proof of [Stacks, 0BN9], base change of schemes (here, along $\varphi : X \to X$), commutes with (finite) coproducts and coequalizers. Therefore, similar to above, isomorphisms $Y_i \to \varphi^*Y_i$ for each summand $Y_i$ give rise to an isomorphism between the respective colimit and its $\varphi$-pullback. By the example of the coproduct, isomorphisms $\beta_i : Y_i \to \varphi^*Y_i$ induce an isomorphism $\coprod Y_i \to \coprod \varphi^*Y_i \to \varphi^*(\coprod Y_i)$.

- **Any object splits into a finite coproduct of connected objects :** Let $(f : Y \to X, \beta)$ be an object of $\mathrm{FEt}(X/\varphi)$. Since $f$ is finite étale, it is clopen and has finite fibres [Stacks, Tag 02NH]. Hence we may apply Lemma 4.9 to $f$, $\varphi_Y := f^*\varphi \circ \beta$ and $\varphi_X := \varphi$ to get that the underlying topological space of $Y = \bigcup Y_i$ is a finite disjoint union of $\varphi_Y$-connected components $Y_i$. Since the $Y_i$ are closed by Remark 4.4 and there are only finitely many of them, they are also open. Hence the $Y_i$ define clopen subschemes and since $\mathcal{O}_Y$ is a sheaf, $Y$ is isomorphic to $\coprod Y_i$ as schemes. Proposition 4.22 shows that the $Y_i$, together with the respective restrictions of $\beta$, are in fact connected objects of $\mathrm{FEt}(X/\varphi)$.

- **Fibre functor is conservative :** Suppose there exists a morphism $g$ between objects $(Y,\beta), (Y',\beta')$ in $\mathrm{FEt}(X/\varphi)$ such that $g \times_X \mathrm{id}_{\bar{x}} : Y_{\bar{x}} \to Y'_{\bar{x}}$ is a bijection. Since both $Y_{\bar{x}}$ and $Y'_{\bar{x}}$ are isomorphic to coproducts of copies of $\bar{x}$, this implies that $g \times_X \mathrm{id}_{\bar{x}}$ is in fact an isomorphism, in particular it is an isomorphism after base change along a geometric point $\bar{y}' \to Y'_{\bar{x}}$. Therefore, $g$ is an isomorphism after base change along $\bar{y}' \to Y'_{\bar{x}} \to Y'$. Then Lemma 4.23 shows that $g$ is an isomorphism.

- **Exactness of $F_{\bar{x}}$ :** By the above, the underlying schemes of finite limits and colimits in $\mathrm{FEt}(X/\varphi)$ are the same as the finite limits and colimits of the underlying schemes. Since $F_{\bar{x}}$ is only applied to the underlying schemes (and the isomorphism $\beta$ is ignored), we conclude exactness of $F_{\bar{x}}$ from the fact that $\mathrm{FEt}(X)$ together with $F_{\bar{x}}$ is a Galois category. □

**Definition 4.25.** In the situation of Proposition 4.24, we define

$$\pi_1(X/\varphi, \bar{x}) := \mathrm{Aut}(F_{\bar{x}})$$

and conclude from Theorem 2.5 an equivalence of categories between $\mathrm{FEt}(X/\varphi)$ and the category of finite $\pi_1(X/\varphi, \bar{x})$-sets.

**Definition 4.26.** Let $X \to S$ be a morphism of schemes and $\varphi_S : S \to S$ a universal homeomorphism. Let $\varphi_X : X \to X$ be a morphism of schemes fitting into a cartesian diagram

$$\begin{tikzcd} X \arrow[r, "\varphi_X"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ S \arrow[r, "\varphi_S"] & S, \end{tikzcd}$$

in particular, $\varphi_X$ is a universal homeomorphism. We call the following functor the *base change functor* from $\mathrm{FEt}(S/\varphi_S)$ to $\mathrm{FEt}(X/\varphi_X)$ :

$$\begin{aligned} \mathrm{FEt}(S/\varphi_S) &\to \mathrm{FEt}(X/\varphi_X) \\ (T, \varphi_T) &\mapsto (T \times_S X, \varphi_T \times_S \mathrm{id}_X) \\ (f : T \to T') &\mapsto (f \times_S \mathrm{id}_X : T \times_S X \to T' \times_S X) \end{aligned}$$

This is a well-defined functor since if

$$\begin{tikzcd} T \arrow[r, "\varphi_T"] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ S \arrow[r, "\varphi_S"] & S, \end{tikzcd}$$

is cartesian, this remains true after applying the base change functor $- \times_S X$. Similarly, $f \times_S \mathrm{id}_X$ is equivariant with respect to $\varphi_T \times_S \mathrm{id}_X$, $\varphi_{T'} \times_S \mathrm{id}_X$ if $f$ is equivariant with respect to $\varphi_T$ and $\varphi_{T'}$.

**Remark 4.27.** In the situation of Definition 4.26, let $\bar{x} \to X$ be a geometric point of $X$, which is a geometric point of $S$ via $\bar{x} \to X \to S$. Then, base change as defined above, induces a *canonical* homomorphism

$$\pi_1(X/\varphi_X, \bar{x}) \to \pi_1(S/\varphi_S, \bar{x})$$

of fundamental groups. The construction is analogous to Remark 2.25.

**Lemma 4.28.** *In the situation of Definition 4.26, suppose that base change along $X \to S$ induces an equivalence of categories $\mathrm{FEt}(S) \cong \mathrm{FEt}(X)$. Then base change in the sense of Definition 4.26 induces an equivalence $\mathrm{FEt}(S/\varphi_S) \cong \mathrm{FEt}(X/\varphi_X)$.*

*Proof.* Denote by $F$ the base change functor $\mathrm{FEt}(S) \to \mathrm{FEt}(X)$ and by $G$ the base change functor $\mathrm{FEt}(S/\varphi_S) \to \mathrm{FEt}(X/\varphi_X)$ from Definition 4.26. The fact that $G$ is faithful directly follows from faithfulness of $F$. To see that $G$ is full, let $(T, \varphi_T), (T', \varphi_{T'})$ be objects in $\mathrm{FEt}(S/\varphi_S)$. Then, since $F$ is full, any morphism $g : Y := T \times_S X \to Y' := T' \times_S X$

in $\mathrm{FEt}(X/\varphi_X)$ compatible with $\varphi_Y$, $\varphi_{Y'}$ descends to some morphism $f : T \to T'$ in $\mathrm{FEt}(S)$. That morphism has to be compatible with $\varphi_T$, $\varphi_{T'}$, since the morphisms $f \circ \varphi_T$ and $\varphi_{T'} \circ f$ are equal after base change to $X$, since $\varphi_Y$, $\varphi_{Y'}$ are base changes of $\varphi_T, \varphi_{T'}$, respectively, and $F$ is faithful. Essential surjectivity follows from essential surjectivity of $F$ since base change commutes with fibre products, i.e., if $Y$ descends to $T$, then $\varphi_X^* Y$ descends to $\varphi_S^* T$. □

For the rest of this chapter, we will consider the situation $X = X_1 \times_{\mathbb{F}_p} X_2$ for two connected schemes $X_1$, $X_2$ over $\mathbb{F}_p$ and $\varphi = \varphi_1$ or $\varphi_2$, the first or second partial Frobenius on $X$.

**Definition and Remark 4.29.** Let $X_1$, $X_2$ be connected schemes over $\mathbb{F}_p$ and set $X := X_1 \times_{\mathbb{F}_p} X_2$. By Proposition 4.10, $X$ is $\varphi_1$- and $\varphi_2$-connected. Hence $\mathrm{FEt}(X/\varphi_1)$ and $\mathrm{FEt}(X/\varphi_2)$ are Galois categories. Scholze-Weinstein and Kedlaya introduce a third category $\mathrm{FEt}(X/\mathrm{pFr.})$, see [SW20, Definition 16.2.1], [Ked19, Definition 4.2.12]. Its objects are given by triples $(Y, \beta_1, \beta_2)$, where $Y \to X$ is a finite étale cover and $\beta_1 : Y \to \varphi_1^* Y$, $\beta_2 : Y \to \varphi_2^* Y$ are isomorphisms such that $\varphi_2^*\beta_1 \circ \beta_2 = \varphi_1^*\beta_2 \circ \beta_1 = F_{Y/X}$, the relative Frobenius of $Y$ over $X$. As usual, by Remark 4.20, we could equivalently describe the objects of $\mathrm{FEt}(X/\mathrm{pFr.})$ as triples $(Y, \varphi_{Y,1}, \varphi_{Y,2})$, where $\varphi_{Y,1}$ and $\varphi_{Y,2}$ are cartesian over $\varphi_1$ and $\varphi_2$, respectively, and $\varphi_{Y,1} \circ \varphi_{Y,2} = \varphi_{Y,2} \circ \varphi_{Y,1} = F_Y$, the absolute Frobenius of $Y$. Since by the product relation, one of $\beta_1$, $\beta_2$ is determined by the other and we have natural equivalences of categories

$$\mathrm{FEt}(X/\varphi_1) \cong \mathrm{FEt}(X/\mathrm{pFr.}) \cong \mathrm{FEt}(X/\varphi_2),$$

also see [SW20, Remark 16.2.3].

## 4.3 $\pi_1$-properness modulo a partial Frobenius

In this section, we will apply the formalism introduced in Section 4.2 and prove :

**Proposition 4.30.** *Let $X$ be a scheme over $\mathbb{F}_p$ and $\mathbb{F}_p \to k$ an algebraically closed field extension. Denote $X_k := X \times_{\mathbb{F}_p} \mathrm{Spec}(k)$. Then, base change*

$$\begin{aligned} \mathrm{FEt}(X) &\to \mathrm{FEt}(X_k/\varphi_2) \\ Y &\mapsto (Y_k, \mathrm{id}_Y \times_{\mathbb{F}_p} F_k) \\ (g : Y \to Y') &\mapsto (f \times_{\mathbb{F}_p} \mathrm{id}_k : Y_k \to Y'_k) \end{aligned}$$

*is an equivalence of categories.*

Observe how the property described in Proposition 4.30 is the Drinfeld analogue of $\pi_1$-properness as introduced in Definition 3.2.

Our proof of Proposition 4.30 closely follows the proof sketches provided by Scholze-Weinstein and Kedlaya, see [SW20, Lemma 16.2.6] and [Ked19, Lemmas 4.2.2, 4.2.5 and 4.2.6].

**Definition 4.31.** Let $k$ be a field of characteristic $p > 0$ and $V$ a $k$-vector space. A map $\varphi : V \to V$ is called $p$-linear if it is additive and for all $a \in k, v \in V$, we have $\varphi(av) = a^p \varphi(v)$. If $V$ is a $k$-algebra, we require that $\varphi$ is also multiplicative.

A key algebraic input is the following result proved in [SGA7, Exposé XXII, Proposition 1.1].

**Proposition 4.32.** *Let $\mathbb{F}_p \to k$ a separably closed field extension, $V$ a finite dimensional $k$-vector space and $\varphi : V \to V$ a $p$-linear map such that $\varphi(V)$ generates $V$. Then $V$ has a $k$-basis consisting of fixed points of $\varphi$.*

*Proof.* For the following, we choose a basis of $V$ and an isomorphism $V \cong k^n$ mapping that basis to the standard basis $E = (e_1|...|e_n)$ of $k^n$. Accordingly, we consider $\varphi$ as a $p$-linear map $k^n \to k^n$. Set $A := \varphi(E)$, i.e., $\varphi$ applied to each column of $E$. For any $v \in k^n$, we have

$$\varphi(v) = AF_k(v), \tag{26}$$

where $F_k(v)$ denotes the application of the Frobenius endomorphism $F_k : k \to k$ to each entry of $v$. Since by assumption, the image of $\varphi$ generates $k^n$, the matrix $A$ has to be invertible. Our strategy is to find an invertible matrix $M \in k^{n\times n}$, such that

$$M = A \cdot F_k(M), \tag{27}$$

since then its columns form a basis of fixed points of $\varphi$. For this, we need to solve $n^2 + 1$ polynomial equations in $n^2 + 1$ variables. Namely, there is a bijection between $\mathrm{GL}(n, k)$ and the set of $k$-rational points of the affine scheme $\mathrm{Spec}(R)$, where

$$R := k[\{U_{ij}\}_{i,j=1,...,n}][T]/(d \cdot T - 1)$$

and $d \in k[\{U_{ij}\}_{i,j}]$ denotes the determinant of the matrix $(U_{ij})_{i,j}$. The condition (27) translates to $n^2$ equations

$$f_{ij} := U_{ij} - \sum_{l=1,...,n} a_{il} U_{lj}^p = 0$$

where the $a_{rs}$ denote the coefficients of the matrix $A$. The subset of those $M \in \mathrm{GL}(n, k)$ satisfying equation (27) is in bijection with the $k$-rational points of $\mathrm{Spec}(S)$, where

$$S := R/(\{f_{ij}\}_{i,j}) \cong k[\{U_{ij}\}_{i,j}, T]/(\{f_{ij}\}_{i,j}, d \cdot T - 1).$$

Now, since $k$ has characteristic $p$, we calculate

$$\frac{\partial f_{ij}}{\partial U_{rs}} = \begin{cases} 1 \text{ if } (i, j) = (r, s) \\ 0 \text{ else} \end{cases}, \frac{\partial f_{ij}}{\partial T} = 0.$$

and

$$\frac{\partial(d \cdot T - 1)}{\partial U_{rs}} = 0, \frac{\partial(d \cdot T - 1)}{\partial T} = d,$$

where $r, s$ run through $1, ..., n$. Therefore, the determinant of the Jacobian w.r.t. to the $f_{ij}$ and $d \cdot T - 1$ is equal to $d$, which is invertible in $S$. Hence, the ring map $k \to S$ is standard smooth of relative dimension 0 in the sense of [Stacks, Tag 00T6], and thus étale by [Stacks, Tag 02GU]. Then, since $k$ is separably closed, $S$ is isomorphic to a finite product of copies of $k$ by [Stacks, Tag 00U3]. In particular, there exists a ring map $S \to k$, i.e., a $k$-rational point of $\mathrm{Spec}(S)$. $\square$

In the following, assume that $k$ is algebraically closed. Note that in this case and with notation as in 4.32, the condition that $\varphi(V)$ generates $V$ is equivalent to $\varphi$ being bijective. Injectivity follows from equation (26) and surjectivity follows from the fact that $k$ contains the $p$-th roots of all its elements. The following corollary promotes Proposition 4.32 to the case that $V$ is an algebra over $k$.

**Corollary 4.33.** *Let $\mathbb{F}_{\mathrm{p}} \to k$ be an algebraically closed field extension, $\psi : k \to B$ a $k$-algebra which is finite dimensional as a $k$-vector space, and $\varphi : B \to B$ a $p$-linear ring isomorphism. Denote by $B^\varphi \hookrightarrow B$ the subring of fixed points of $\varphi$. Then the restriction of $\psi$ to $\mathbb{F}_{\mathrm{p}}$ factors through $B^\varphi$, turning it into an $\mathbb{F}_{\mathrm{p}}$-algebra, and both squares of the diagram*

$$\begin{tikzcd}
B & B \arrow[l, "\varphi"'] & B^\varphi \arrow[l, hook', "\iota"'] \\
k \arrow[u, "\psi"] & k \arrow[l, "F_k"] \arrow[u, "\psi"] & \mathbb{F}_{\mathrm{p}} \arrow[l, hook', "\tau"] \arrow[u]
\end{tikzcd} \tag{28}$$

*are cocartesian. In particular, $\varphi = \beta \circ (\mathrm{id}_{B^\varphi} \otimes_{\mathbb{F}_{\mathrm{p}}} F_k) \circ \beta^{-1}$, where $\beta$ is the unique ring isomorphism making*

$$\begin{tikzcd}
B & & \\
 & B^\varphi \otimes_{\mathbb{F}_{\mathrm{p}}} k \arrow[ul, "\beta"] & B^\varphi \arrow[l] \arrow[ull, bend right, "\iota"'] \\
 & k \arrow[u] \arrow[uul, bend left, "\psi"] & \mathbb{F}_{\mathrm{p}}\,. \arrow[l, hook', "\tau"] \arrow[u]
\end{tikzcd}$$

*commute.*

*Proof.* The condition that $\varphi$ is $p$-linear means that the left square of diagram (28) commutes. This implies that for any $a \in \mathbb{F}_{\mathrm{p}}$, we have $\varphi(\psi(a)) = \psi(a^p) = \psi(a)$, i.e., $\psi(a) \in B^\varphi$. Hence $\psi \circ \tau$ factors through $B^\varphi$ and the right square of (28) commutes. Therefore, the abovementioned ring map $\beta$ exists by the universal property of the tensor product. By Proposition 4.32, $B$ has a $k$-basis consisting of elements $b_1, ..., b_n \in B^\varphi$. By construction, $\beta(b_i \otimes 1) = b_i$, in particular, $\beta$ maps a $k$-basis of $B^\varphi \otimes_{\mathbb{F}_{\mathrm{p}}} k$ to a $k$-basis of $B$. Also, $\beta$ is $k$-linear by construction, hence $\beta$ is an isomorphism of $k$-vector spaces, and since it is a ring homomorphism, also of $k$-algebras. Note that $\varphi \circ \iota = \iota$ by construction of $B^\varphi$ and $F_k \circ \tau = \tau$, hence also the composed diagram (28) is cocartesian with the same unique isomorphism $\beta$. Therefore also the left square of diagram (28) is cocartesian, see [GW10, Prop. 4.16], and the claimed equality holds. □

**Remark 4.34.** Let $k, B$ be as in Corollary 4.33. By Corollary 4.33, for any $p$-linear isomorphism $\varphi_B : B \to B$, the square

$$\begin{tikzcd}
B & B \arrow[l, "\varphi_B"'] \\
k \arrow[u, "\psi"] & k \arrow[l, "F_k"] \arrow[u, "\psi"]
\end{tikzcd}$$

is cocartesian. Let us define a category (F. d. $k$-algebras$/F_k$) with objects $(B, \varphi_B)$, where $B$ is a $k$-algebra finite-dimensional as a $k$-vector space and $\varphi_B : B \to B$ is a ring homomorphism fitting into a cocartesian square over $F_k$ as above. Morphisms between $(B, \varphi_B)$ and $(B', \varphi_{B'})$ are given by morphisms $B \to B'$ of $k$-algebras which are compatible with the corresponding $\varphi_B$ and $\varphi_{B'}$. As usual, we may equivalently describe and object $(B, \varphi_B)$ as a pair $(B, \beta)$, where $\beta : F_k^* B \to B$ an isomorphism for $F_k^* B := B \otimes_{k,F_k} k$. There is a natural "base change" functor

$$\begin{aligned}(\text{F. d. } \mathbb{F}_p\text{-algebras}) &\to (\text{F. d. } k\text{-algebras}/F_k)\\ A &\mapsto (A \otimes_{\mathbb{F}_p} k, \mathrm{id}_A \otimes_{\mathbb{F}_p} F_k),\\ (\alpha : A \to A') &\mapsto \alpha \otimes_{\mathbb{F}_p} \mathrm{id}_k \,.\end{aligned}$$

By Corollary 4.33, this functor is essentially surjective (note that the corollary also states that the isomorphism $\beta : B^{\varphi_B} \otimes_{\mathbb{F}_p} k \to B$ is compatible with $\mathrm{id}_{B^{\varphi_B}} \otimes_{\mathbb{F}_p} F_k$ and $\varphi_B$, hence indeed an isomorphism in the category (F. d. $k$-algebras$/F_k$)). In fact, it is an equivalence of categories. We postpone fully faithfulness to the more general Lemma 4.36 below.

The next step is to promote Corollary 4.33 to coherent algebras over projective schemes over $\mathbb{F}_p$.

**Definition and Remark 4.35.** Let $\mathbb{F}_p \to k$ be an algebraically closed field extension and $X$ a noetherian scheme over $\mathbb{F}_p$. Denote by $X_k := X \times_{\mathbb{F}_p} \mathrm{Spec}(k)$ its base change to $k$. For any coherent $\mathcal{O}_{X_k}$-algebra $\mathcal{E}$, denote by $\varphi_2^* \mathcal{E}$ the pullback of $\mathcal{E}$ along $\varphi_2 : X_k \to X_k$, i.e., $\varphi_2^* \mathcal{E}$ is given as the tensor product $\varphi_2^{-1} \mathcal{E} \otimes_{\varphi_2^{-1}\mathcal{O}_{X_k}} \mathcal{O}_{X_k}$ in the category of $\varphi_2^{-1}\mathcal{O}_{X_k}$-algebras. Similar as in Remark 4.34, we may define a category $\mathrm{CohAlg}(\mathcal{O}_{X_k}/\varphi_2)$ with objects $(\mathcal{E}, \varphi_{\mathcal{E}})$, where $\mathcal{E}$ is a coherent $\mathcal{O}_{X_k}$-algebra and $\varphi_{\mathcal{E}} : \varphi_2^{-1} \mathcal{E} \to \varphi_2^* \mathcal{E}$ a map of sheaves of rings on $X_k$ fitting into a cocartesian diagram

$$\begin{tikzcd} \varphi_2^* \mathcal{E} & \varphi_2^{-1} \mathcal{E} \arrow[l, "\varphi_{\mathcal{E}}"'] \\ \mathcal{O}_{X_k} \arrow[u] & \varphi_2^{-1}\mathcal{O}_{X_k}. \arrow[l, "\varphi_2^{\#}"'] \arrow[u] \end{tikzcd} \tag{29}$$

Morphisms between $(\mathcal{E}, \beta)$ and $(\mathcal{E}', \beta')$ are given by morphisms of $\mathcal{O}_{X_k}$-algebras which are compatible with the corresponding $\varphi_{\mathcal{E}}$ and $\varphi_{\mathcal{E}'}$. Further, if $q : X_k \to X$ denotes the canonical projection, there is a natural functor

$$\begin{aligned}\mathrm{CohAlg}(\mathcal{O}_X) &\to \mathrm{CohAlg}(\mathcal{O}_{X_k}/\varphi_2)\\ \mathcal{F} &\mapsto (q^* \mathcal{F},\ \text{the structure map } \varphi_2^{-1} q^* \mathcal{F} \to \varphi_2^* q^* \mathcal{F})\\ (\psi : \mathcal{F} \to \mathcal{F}') &\mapsto q^* \psi,\end{aligned}$$

which can be viewed as "base change" from $\mathrm{CohAlg}(\mathcal{O}_X)$ to $\mathrm{CohAlg}(\mathcal{O}_{X_k}/\varphi_2)$.

Now, let us further generalize Corollary 4.33. In fact, the following statement is already referred to as "Drinfeld's Lemma" by some authors, cf. [Lau04, Lemma 8.1.1], [Laf18, Lemme 8.2].

**Lemma 4.36.** *Let $\mathbb{F}_p \to k$ be algebraically closed and $X$ a projective scheme over $\mathbb{F}_p$. The natural functor*

$$\mathrm{CohAlg}(\mathcal{O}_X) \to \mathrm{CohAlg}(\mathcal{O}_{X_k}/\varphi_2)$$

*from Remark 4.35 is an equivalence of categories.*

*Proof.* Before proving the above equivalence, we note some general observations in the situation of the lemma.

The first important ingredient is the fact that the global sections (in fact, all cohomology groups) of a coherent sheaf $\mathcal{M}$ on a projective scheme $S$ over a field $l$ form a finite dimensional $l$-vector space. This is a consequence of the fundamental fact that (under mild additional assumptions), higher direct images of a coherent sheaf along a proper morphism are again coherent, see [Stacks, Tag 0205]. More precisely, as closed immersions are proper and $\mathbb{P}^n_l \to l$ is proper, also $t : S \hookrightarrow \mathbb{P}^n_l \to l$ is proper, hence $t_*\mathcal{M}$ is coherent on $\mathrm{Spec}(l)$, hence $\Gamma(\mathrm{Spec}(l), t_*\mathcal{M})$ is a finite dimensional $l$-vector space. On the other hand, we have an isomorphism $\Gamma(S, \mathcal{M}) \cong \Gamma(\mathrm{Spec}(l), t_*\mathcal{M})$ by [Stacks, Tag 01XK]. Also see [Stacks, Tag 0206]. In particular, in the situation of the lemma above, for any coherent $\mathcal{O}_{X_k}$-module $\mathcal{M}$, the global sections $\Gamma(X_k, \mathcal{M})$ form a finite dimensional $k$-vector space.

Secondly, observe that since the projection $q : X_k \to X$ is flat as a base change of the flat morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{F}_p)$ and $X \to \mathbb{F}_p$ is qcqs as it is projective, for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$, we have an isomorphism

$$\Gamma(X_k, q^*\mathcal{H}) \cong \Gamma(X, \mathcal{H}) \otimes_{\mathbb{F}_p} k \tag{30}$$

by flat base change [Stacks, Tag 02KH].

Thirdly, we may express $X$ (resp. $X_k$) as the projective spectrum of a graded $\mathbb{F}_p$-algebra (resp. $k$-algebra). For this, fix a projective embedding $X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_p}$ and the corresponding projective embedding $X_k \hookrightarrow \mathbb{P}^n_k$. By [GW10, Summary 13.71], there exists a graded $\mathbb{F}_p$-algebra $R := \bigoplus_n R_n$ such that $X \cong \mathrm{Proj}(R)$ over $\mathbb{F}_p$, where $R$ is finitely generated in degree 1, i.e., $R$ is generated as an $R_0$-algebra by finitely many elements of $R_1$. Similarly, $X_k \cong \mathrm{Proj}(S)$, for a graded $k$-algebra $S := \bigoplus_n S_n$ which is finitely generated in degree 1. Note that the $R_n$ (resp. $S_n$) arise as global sections of the $n$-th twist of a very ample invertible $\mathcal{O}_X$-module (resp. $\mathcal{O}_{X_k}$-module) by [GW10, Corollary 13.75], so we may apply (30) to see that $S \cong R \otimes_{\mathbb{F}_p} k$ as graded $k$-algebras. Note that as projective, hence finite type schemes over a noetherian base, $X$ and $X_k$ are noetherian, hence $R$ and $S$ are noetherian rings. In particular, we may apply [Stacks, Tag 0BXD] to relate coherent $\mathcal{O}_X$-algebras (resp. $\mathcal{O}_{X_k}$-algebras) to finite graded $R$-algebras (resp. $S$-algebras).

For a summary of definitions related to graded algebras and the $\mathrm{Proj}$ functor, we refer to [GW10, Sections (13.1), (13.2), (13.7)]. We now turn to the proof of the lemma.

- **Essential surjectivity :**

  Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an object of $\mathrm{CohAlg}(X_k/\varphi_2)$. Set $B := \bigoplus_n B_n$, where $B_n := \Gamma(X_k, \mathcal{E}(n))$. Here, the $\mathcal{E}(n)$ are meant with respect to the projective embedding fixed above. By the abovementioned description of coherent modules on projective schemes [Stacks, Tag 0BXD], $\mathcal{E}$ is isomorphic to $\widetilde{B}$ and $B$ is a finite graded $S$-algebra. Moreover, $\varphi_{\mathcal{E}}$ comes from a $p$-linear map $\varphi : B \to B$ of graded rings (i.e., $\varphi$ is a $p$-linear ring map with $\varphi(B_n) \subseteq B_n$ for all $n$). We need to show that $B$ is the base change of a finite graded $R$-algebra $A$ and $\varphi$ corresponds to $\mathrm{id}_A \otimes_R F_S$.

  As described above, each $B_n$ is a finite dimensional $k$-vector space via the identification

  $$\Gamma(X_k, \mathcal{E}(n)) \cong \Gamma(\mathrm{Spec}(k), t_*\, \mathcal{E}(n))$$

  from [Stacks, Tag 01XK], where $t : X_k \hookrightarrow \mathbb{P}^n_k \to k$. Therefore, setting $A_n := B_n^{\varphi}$, we get isomorphisms of $k$-vector spaces $B_n \cong A_n \otimes_{\mathbb{F}_p} k$ by Proposition 4.32. Set $A := \bigoplus A_n$. Since $\varphi$ is a map of graded rings, i.e., $\varphi(B_n) \subseteq B_n$, we have the equality of sets $B^\varphi = \bigoplus(B_n^\varphi) = A$. Note that $A$ is in fact a subring of $B$, as $\varphi$ is a ring homomorphism. Since $\varphi$ is even a map of graded rings, the grading $A = \bigoplus_n A_n$ makes $A$ a graded ring. In fact, $B$ is a $k$-algebra (via $k \to R \otimes_{\mathbb{F}_p} k \to B$), $A$ is an $\mathbb{F}_p$-algebra (via $\mathbb{F}_p \to R \to A$) and $B \cong A \otimes_{\mathbb{F}_p} k$ as $k$-algebras by Corollary 4.33. Hence we get the following cocartesian diagram of (graded) rings

$$\begin{tikzcd}
B & A \arrow[l] \\
R \otimes_{\mathbb{F}_p} k \arrow[u] & R \arrow[l] \arrow[u] \\
k \arrow[u] & \mathbb{F}_p \arrow[l] \arrow[u]
\end{tikzcd}$$

In particular, the upper square is cocartesian. Now note that the map $R \to R \otimes_{\mathbb{F}_p} k$ is an fpqc morphism as a base change of the fpqc morphism $\mathbb{F}_p \to k$. Now, by assumption, $B$ is a finite $R \otimes_{\mathbb{F}_p} k$-module. Since being finite is fpqc local on the target, see [Stacks, Tag 02LA], this implies that $A$ is finite as an $R$-module. In summary, we see that $A$ is a graded $R$-algebra which is finitely generated as an $R$-module with $A \otimes_R (R \otimes_{\mathbb{F}_p} k) \cong B$ as $R \otimes_{\mathbb{F}_p} k$-algebras. Hence, $A$ gives rise to a coherent $\mathcal{O}_X$-algebra $\widetilde{A}$ whose image under the functor from the lemma is isomorphic to $(\mathcal{E}, \varphi_{\mathcal{E}})$.

- **Fully faithfulness :** Fully faithfulness is a direct consequence of (30). Namely, for any two coherent $\mathcal{O}_X$-algebras $\mathcal{F}, \mathcal{F}'$, consider the internal Hom $\mathcal{G} := \mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}')$, which is itself a coherent $\mathcal{O}_X$-algebra, cf. [Stacks, Tag 01CM]. Applying (30) above to $\mathcal{H} := \mathcal{G}$, we get

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}') \otimes_{\mathbb{F}_p} k = \Gamma(X, \mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}')) \otimes_{\mathbb{F}_p} k \cong \Gamma(X_k, q^*(\mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}'))) \cong \mathrm{Hom}_{\mathcal{O}_{X_k}}(q^*\mathcal{F}, q^*\mathcal{F}'),$$

see [Stacks, Tag 0C6I]. Since the category $\mathrm{CohAlg}(X_k/\varphi_2)$ precisely contains those $\mathcal{O}_{X_k}$-homomorphisms between $q^*\mathcal{F}$ and $q^*\mathcal{F}'$ which are compatible with $\mathrm{id}_X \otimes_{\mathbb{F}_p} F_k$, we conclude from this that the functor from the lemma is fully faithful. □

Lemma 4.36 now allows us to conclude Proposition 4.30. In rough terms, the idea is to embed $X$ into a projective scheme $X'$, and correspondingly $X_k$ into $X'_k$. Due to some reductions, we will be able make use of relative normalization in order to switch between finite étale covers of $X$ and certain normal finite covers of $X'$ and similarly for $X_k \hookrightarrow X'_k$. Over the projective schemes $X'$ (resp. $X'_k$), we will interpret finite covers as coherent $\mathcal{O}_{X'}$-algebras (resp. $\mathcal{O}_{X'_k}$-algebras). Then, we apply Lemma 4.36. But first, let us carry out some reduction steps :

**Lemma 4.37.** *If Proposition 4.30 holds under the additional assumptions that $X$ is affine, normal, connected and of finite type over $\mathbb{F}_p$, then it holds as stated.*

*Proof.* • **Reduction to $X$ affine :** Suppose we have shown the equivalence claimed in Proposition 4.30 for all affine schemes over $\mathbb{F}_p$. Then, if $X$ is an arbitrary scheme over $\mathbb{F}_p$, we might choose an affine open covering $X = \bigcup U_i$ of $X$ giving also an affine open covering $X_k = \bigcup U_{i,k}$, where $U_{i,k} := U_i \times_{\mathbb{F}_p} \mathrm{Spec}(k)$. Note that base change along the $U_i \to X$ induces an equivalence between the category of schemes over $X$ and the category of descent data relative to the covering $\{U_i \to X\}_i$ by [Stacks, Tag 02VY]. This equivalence restricts to finite étale covers of $X$, since being finite étale is stable under base change and Zariski local on the target and finite étale covers of $X$ form a full subcategory of $(\mathrm{Sch}/X)$. The analogous equivalence holds relative to the affine open covering $\{U_i \times_X X_k \to X_k\}_i$ and after quotienting by Frobenius, which can be shown using the same arguments as for Lemma 4.28.

Therefore it suffices to show the equivalence of descent data relative to $\{U_i \to X\}_i$ and $\{U_i \times_X X_k \to X_k\}_i$. For this however, it suffices to show $\mathrm{FEt}(U_i) \cong FEt(U_{i,k}/\varphi_2)$, $\mathrm{FEt}(U_{ij})(U_{ij,k}/\varphi_2)$ and $\mathrm{FEt}(U_{ijl})(U_{ijl,k}/\varphi_2)$ for all "intersections" $U_{ij} := U_i \times_X U_j$ and $U_{ijl} := U_i \times_X U_j \times_X U_l$. Here, the notation $U_{ij,k}$ and $U_{ijl,k}$ means the base changes of $U_{ij}$, $U_{ijl}$ along $X_k \to X$, respectively. Since the $U_i$ are affine, $\mathrm{FEt}(U_i) \cong FEt(U_{i,k}/\varphi_2)$ holds by assumption. For the $U_{ij}$ and $U_{ijl}$, observe that they embed as open subschemes into in the affine scheme $U_i$, hence $U_{ij}$ and $U_{ijl}$ are separated. Therefore, intersections of affine opens of $U_{ij}$ (resp. $U_{ijl}$) are affine schemes. Therefore, in order to prove $\mathrm{FEt}(U_{ij})(U_{ij,k}/\varphi_2)$ and $\mathrm{FEt}(U_{ijl})(U_{ijl,k}/\varphi_2)$ for all $i, j, l$, we may repeat the same argument once again, i.e., we may cover the $U_{ij}$ and $U_{ijl}$ by affine opens, respectively, whose intersections are now affine. So, here we may apply the hypothesis that the claim is shown for all affine schemes. From this, we conclude the equivalences $\mathrm{FEt}(U_{ij}) \cong FEt(U_{ij,k}/\varphi_2)$ and $\mathrm{FEt}(U_{ijl})(U_{ijl,k}/\varphi_2)$. See also the proof of [Stacks, Tag 0BQB].

- **Reduction to $X$ of finite type :** Suppose we have shown the claim for all affine schemes of finite type over $\mathbb{F}_p$. By the above paragraph, it suffices to show the claim for all affine schemes over $\mathbb{F}_p$. Now, if $X = \mathrm{Spec}(A)$ is affine over $\mathbb{F}_p$, we may write $A \cong \mathrm{colim}_i A_i$ as a colimit of finite type $\mathbb{F}_p$-algebras by [Stacks, Tag 00QN]. By [Stacks, Tag 0EYL] restricted to finite étale covers as described in [Ked19, Definition 4.1.1(a)], we have an equivalence of categories $2\text{-}\mathrm{colim}\,\mathrm{FEt}(\mathrm{Spec}(A_i)) \cong \mathrm{FEt}(\mathrm{Spec}(A))$. By the same arguments as for Lemma 4.28, we have an equivalence $2\text{-}\mathrm{colim}\,\mathrm{FEt}(\mathrm{Spec}(A_{i,k}/\varphi_2) \cong \mathrm{FEt}(\mathrm{Spec}(A_k))/\varphi_2)$. Hence it suffices to show the equivalences $\mathrm{FEt}(\mathrm{Spec}(A_i)) \cong \mathrm{FEt}(\mathrm{Spec}(A_{i,k})/\varphi_2)$, which follow from our assumption.

- **Reduction to $X$ normal :** Suppose we have shown the claim for all normal and affine schemes of finite type over $\mathbb{F}_p$. By the above reductions, it suffices to show the claim for all affine schemes of finite type over $\mathbb{F}_p$. So let $X$ be such a scheme. Since $X$ is of finite type over a field, it is noetherian, hence the absolute normalization $\nu : X' \to X$ is defined, see [Stacks, Tag 035N]. The morphism $\nu$ is surjective by [Stacks, Tag 035Q] and finite by [Stacks, Tags 035B, 035S], hence in particular proper. Since $X$ is noetherian, $\nu$ is also of finite presentation [Stacks, Tag 01TX]. Now, proper surjective morphisms of finite presentation are coverings in the $h$-topology and finite étale morphisms satisfy $h$-descent, i.e., $U \mapsto \mathrm{FEt}(U)$ is a 2-sheaf on the $h$-site of $X$, see [Ryd10, Sections 5,8]. In particular, we may apply the 2-sheaf axioms to the $h$-cover $\nu : X' \to X$. Note that we can also cover the "intersections" $X' \times_X X'$ and $X' \times_X X' \times_X X'$ by their normalization. From this, one can deduce that if the claim is shown under the assumptions from the beginning of this paragraph, it also holds for $X$.

- **Reduction to $X$ connected :** Now, finally, let $X$ be a normal affine scheme of finite type over $\mathbb{F}_p$. In particular, $X$

is noetherian. As noetherian normal schemes split as coproducts of their connected components by [Stacks, Tag 035Q], it suffices to show the claim for each connected component of $X$. Hence, the claim on $X$ follows if we have shown it for all connected, normal, affine schemes of finite type over $\mathbb{F}_p$. $\square$

The next step is to embed $X$ (and, correspondingly $X_k$) into a projective scheme over $\mathbb{F}_p$ (resp. $k$).

**Lemma 4.38.** *After the reductions from Lemma 4.37, there exists a quasi-compact open immersion $X \hookrightarrow X'$ into a projective normal $\mathbb{F}_p$-scheme.*

*Proof.* Being affine and of finite type over $\mathbb{F}_p$, the scheme $X$ is isomorphic to a closed subscheme of $\mathbb{A}^n_{\mathbb{F}_p}$ for some $n$. By composing with an open immersion $\mathbb{A}^n_{\mathbb{F}_p} \hookrightarrow \mathbb{P}^n_{\mathbb{F}_p}$, we get an immersion $X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_p}$, i.e., $X$ is quasi-projective. This immersion is in fact quasi-compact, as $X$ is noetherian and any subspace of a noetherian topological space is quasi-compact. By [GW10, Remark 10.31], or [Stacks, Tag 01RG], the immersion $X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_p}$ factors as an open immersion followed by a closed immersion, i.e., there exists an open immersion $X \hookrightarrow X''$, where $X''$ is a projective $\mathbb{F}_p$-scheme. Now let $X'$ be the absolute normalization of $X''$, which is still projective over $\mathbb{F}_p$ by [Stacks, Tags 01WC, 0GK4].
We are left to construct an open immersion $X \hookrightarrow X'$. Let $X' \times_{X''} X \to X$ be the base change of the normalization morphism $X' \to X''$ along $X \hookrightarrow X''$. Since $X \hookrightarrow X''$ is an open immersion, we conclude from [Stacks, Tag 035K] that $X' \times_{X''} X$ is isomorphic to the normalization of $X$. Since $X$ is already normal, $X' \times_{X''} X \to X$ is an isomorphism. Therefore composing its inverse with the open immersion $X' \times_{X''} X \hookrightarrow X'$ gives an open immersion $X \hookrightarrow X'$. $\square$

Next, we relate finite étale covers of $X$ to certain covers of $X'$ :

**Lemma 4.39.** *After the reductions from Lemma 4.37 and with $X'$ as in Lemma 4.38, we have an equivalence of categories*

$$\mathrm{FEt}(X) \cong (\textit{finite morphisms } Y' \to X', \textit{ where } Y' \textit{ is normal and } Y' \times_{X'} X \to X \textit{ is étale})$$
$$\Phi : Y \mapsto \textit{normalization of } X' \textit{ in } Y$$
$$Y' \times_{X'} X \leftarrow\!\shortmid Y' : \Psi$$

*Here, the right hand side is meant as a full subcategory of $(Sch/X')$.*

*Proof.*
- **$\Phi$ is well-defined :** First note that relative normalization is functorial by [Stacks, Tag 035J], i.e., the above assignment also yields a map between sets of morphisms. Now, let $Y$ be finite étale over $X$. Denote by $Y'$ the normalization of $X'$ in $Y$ with respect to the morphism $Y \to X \hookrightarrow X'$. We first show that the normalization morphism $Y' \to X'$ is finite. This follows from [Stacks, Tag 03GR]. Indeed, since $X'$ is a projective scheme over a field, it is Nagata by [Stacks, Tag 035B]. Moreover, the morphism $Y \to X \hookrightarrow X'$ is of finite type as the composition of the finite morphism $Y \to X$ with $X \hookrightarrow X'$, which is a quasi-compact immersion by [Stacks, Tag 01OX] and hence of finite type by [Stacks, Tag 01T5]. Furthermore, $Y$ is normal, since it is étale over a normal scheme [Stacks, Tag 033C]. In particular, $Y$ is reduced. To see that $Y'$ is indeed normal, note that $Y$ is noetherian, being of finite type over a field. Since $Y$ is also normal, $Y'$ is normal by [Stacks, Tag 035L]. Below, we will show that $Y' \times_{X'} X$ is isomorphic to $Y$, therefore, $Y' \times_{X'} X$ is étale over $X$.

- $\Psi$ **is well-defined :** Clear.
- $\Psi \circ \Phi \cong \mathrm{id}$ : Let $Y$ be finite étale over $X$. Denote by $Y'$ the normalization of $X'$ in $Y$, with respect to the morphism $Y \to X \hookrightarrow X'$. By [Stacks, Tag 035K], $Y' \times_{X'} X$ is isomorphic to the normalization $Y^0$ of $X$ in $Y$. Since $Y \to X$ is finite, hence integral, $Y^0$ is further isomorphic to $Y$ by [Stacks, Tag 03GP].
- $\Phi \circ \Psi \cong \mathrm{id}$ : Let $Y'$ be a normal scheme which is finite over $X'$ such that $Y' \times_{X'} X$ is étale over $X$.

  Since both $\Psi$ and $\Phi$ commute with finite coproducts (for the latter see [Stacks, Tag 03GO]), we may assume that $Y'$ is connected.

  Denote by $Y''$ the normalization of $X'$ in $Y' \times_{X'} X$. Since $Y' \to X'$ is finite, hence integral, the map $Y' \times_{X'} X \to Y'$ factors through an integral morphism $h : Y'' \to Y'$ by [Stacks, Tag 035I]. Its base change $Y'' \times_{X'} X \to Y' \times_{X'} X$ is an isomorphism, since $Y'' \times_{X'} X$ is isomorphic to the normalization of $X$ in $Y' \times_{X'} X$, which is isomorphic to $Y' \times_{X'} X$, since $Y' \times_{X'} X \to X$ is integral. Further, $Y'$ is normal, connected and noetherian being finite over the noetherian scheme $X'$, therefore it is integral and in particular irreducible. Then by the above, and since $Y' \times_{X'} X \to Y'$ is an open immersion, we conclude that $h : Y'' \to Y'$ is birational. Since $h$ is also integral, we may conclude from [Stacks, Tag 0AB1] that $h$ is even an isomorphism. □

We can establish an analogue of 4.39 after base change to $k$ and quotienting by $\varphi_2$ :

**Lemma 4.40.** *After the reductions from Lemma 4.37 and with $X'$ as in Lemma 4.38, we have an equivalence of categories*

$$\begin{aligned}
\mathrm{FEt}(X_k/\varphi_2) \;&\cong\; \textit{(Finite morphisms } Y' \to X'_k\textit{, where } Y' \textit{ is normal and } Y' \times_{X'_k} X_k \to X_k \textit{ is étale,}\\
&\qquad \textit{equipped with morphisms } \varphi_{Y'} : Y' \to Y' \textit{ cartesian over } \varphi_2\textit{)}\\
(Y, \varphi_Y) \;&\mapsto\; (Y^0, \varphi_Y^0)\\
(Y' \times_{X'_k} X_k, \varphi_{Y'} \times_{X'_k} X_k) \;&\leftarrow\!\shortmid\; (Y', \varphi_{Y'})
\end{aligned}$$

*Here, the right hand side is meant as a subcategory of (Sch$/X'_k$), where morphisms should be equivariant in the sense of Definition 4.21. By $Y^0$, we mean the normalization of $X'_k$ in $Y$ and by $\varphi_Y^0$, we mean the map $Y^0 \to Y^0$ obtained from $\varphi_Y$ by [Stacks, Tag 035J].*

*Proof.* The base change $X_k$ is still an affine scheme of finite type over $k$. Moreover, since $\mathbb{F}_p$ is perfect, we may apply [Stacks, Tag 0380] to see that $X_k$ is again normal. Similarly, $X'_k$ is again normal and projective and $X_k \hookrightarrow X'_k$ is still a quasi-compact open immersion. Note that since $X_k$ is in particular noetherian, it is a (finite) coproduct of connected normal schemes [Stacks, Tag 0357] and similarly for $X'_k$. Therefore, we may apply Lemma 4.39 to each connected component of $X_k$ to conclude the equivalence

$$(\text{finite étale } Y \to X_k) \cong (\text{finite normal schemes } Y' \to X'_k \text{ such that } Y' \times_{X'_k} X_k \to X_k \text{ is étale}).$$

Then, note that $X'_k \cong X' \times_X (X \times_{\mathbb{F}_p} \mathrm{Spec}(k))$, so we have $\mathrm{id}_{X'} \times_X (\mathrm{id}_X \times_{\mathbb{F}_p} F_k) = \mathrm{id}_{X'} \times_{\mathbb{F}_p} F_k$ (up to conjugation with a unique isomorphism), i.e., the partial Frobenius on $X'_k$ is a base change of the partial Frobenius on $X_k$. Therefore, we conclude the claim from Lemma 4.28. □

The heart of the proof of Proposition 4.30 is the following step, where Lemma 4.36 comes in.

**Lemma 4.41.** *After the reductions from Lemma 4.37 and with $X'$ as in Lemma 4.38, base change induces an equivalence*

$$\begin{aligned}&(\textit{Finite morphisms } Y' \to X', \textit{ where } Y' \textit{ is normal and } Y' \times_{X'} X \to X \textit{ is étale})\\ \cong\ &(\textit{Finite morphisms } Y' \to X'_k, \textit{ where } Y' \textit{ is normal and } Y' \times_{X'_k} X_k \to X_k \textit{ is étale,}\\ &\textit{equipped with morphisms } \varphi_{Y'} : Y' \to Y' \textit{ cartesian over } \varphi_2).\end{aligned}$$

*Proof.* Note that for noetherian schemes $S$, formation of the relative spectrum gives an equivalence between coherent $\mathcal{O}_S$-algebras and finite schemes over $S$, cf. [GW10, Remark 12.10]. Since projective schemes over a field are noetherian, and by Lemma 4.36, we thus have the following chain of equivalences :

$$\begin{aligned}&(\text{Finite morphisms } Y' \to X')\\ &\cong \mathrm{CohAlg}(\mathcal{O}_{X'})\\ &\overset{4.36}{\cong} \mathrm{CohAlg}(\mathcal{O}_{X'_k}/\varphi_2)\\ &\cong (\text{Finite morphisms } Y' \to X'_k, \text{ equipped with morphisms } \varphi_{Y'} : Y' \to Y' \text{ cartesian over } \varphi_2)\end{aligned}$$

That equivalence restricts to the claimed equivalence. □

Finally, we conclude Proposition 4.30 :

*Proof of Proposition 4.30.* Do the reductions of Lemma 4.37 and choose an open immersion $X \hookrightarrow X'$ as in Lemma 4.38. Then we have equivalences

$$\begin{aligned}\mathrm{FEt}(X) &\overset{4.39}{=} (\text{finite morphisms } Y' \to X', \text{ where } Y' \text{ is normal and } Y' \times_{X'} X \to X \text{ is étale})\\ &\overset{4.41}{=} (\text{finite morphisms } Y' \to X'_k, \text{ where } Y' \text{ is normal and } Y' \times_{X'_k} X_k \to X_k \text{ is étale,}\\ &\qquad \text{equipped with morphisms } Y' \to Y' \text{ cartesian over } \varphi_2)\\ &\overset{4.40}{=} \mathrm{FEt}(X_k/\varphi_2).\end{aligned}$$

Note that all equivalences are induced by base change. □

**Remark 4.42.** Observe that by Remark 4.29, we also conclude from Proposition 4.30 that $\mathrm{FEt}(X) \cong \mathrm{FEt}(X_k/\varphi_1)$ by the "base change" functor

$$\begin{aligned}\mathrm{FEt}(X) &\to \mathrm{FEt}(X_k/\varphi_1)\\ Y &\mapsto (Y_k, F_Y \times_{\mathbb{F}_p} \mathrm{id}_k)\\ (g : Y \to Y') &\mapsto (g \times_{\mathbb{F}_p} \mathrm{id}_k : Y_k \to Y'_k).\end{aligned}$$

Note that this is indeed a well-defined functor, since any morphism of schemes in characteristic $p$ commutes with absolute Frobenii. In particular,

$$\begin{tikzcd} Y \arrow[r, "F_Y"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "F_X"] & X \end{tikzcd}$$

commutes and for any map $g : Y \to Y'$ of finite étale covers,

$$\begin{tikzcd} Y \arrow[r, "F_Y"] \arrow[d, "g"] & Y \arrow[d, "g"] \\ Y' \arrow[r, "F_{Y'}"] & Y' \end{tikzcd}$$

commutes. Of course, both squares still commute after applying the base change functor $- \times_{\mathbb{F}_p} \mathrm{Spec}(k)$. Also see Definition 4.26.

## 4.4 Stein factorization in the Drinfeld setting

Proposition 4.30 showed that any scheme $X$ over $\mathbb{F}_p$ satisfies a "Drinfeld analogue" of $\pi_1$-properness as introduced in Definition 3.2. So from now on, we proceed as for the proof of Proposition 1.2. First, we establish an analogue of Lemma 3.12 in the Drinfeld setting.

**Lemma 4.43.** *Let $X_1$ and $X_2$ be connected qcqs schemes over $\mathbb{F}_p$. Set $X := X_1 \times_{\mathbb{F}_p} X_2$. Then, for any object $(Y, \varphi_Y)$ of $\mathrm{FEt}(X/\varphi_1)$, there exists a scheme $T$ under $Y$ and over $X_2$, such that*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & X_2 \end{tikzcd} \tag{31}$$

*commutes, $T \to X_2$ is finite étale, $\varphi_Y : Y \to Y$ is a $T$-morphism, the geometric fibres $Y_{\bar{t}}$ of $Y \to T$ are $\varphi_{Y,\bar{t}} := (\varphi_Y \times_T id_{\bar{t}})$-connected and such that diagram (31) has the following universal property : For any $T'$ under $Y$ and finite étale over $X_2$, such that*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T' \arrow[d] \\ X \arrow[r] & X_2 \end{tikzcd} \tag{32}$$

*commutes, there is a unique morphism $T \to T'$ such that*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[rr, bend left] & T \arrow[d] \arrow[r, dotted] & T' \arrow[dl] \\ X \arrow[r] & X_2 & \end{tikzcd}$$

*commutes. In particular, $T$ is unique up to unique isomorphism.*

Here comes the analogue of Example 3.13 :

**Example 4.44.** In the situation of Lemma 4.43, assume that $X_2 = \bar{x}$ is a geometric point. Let $(Y, \varphi_Y)$ be an object of $\mathrm{FEt}(X/\varphi_1)$. By Proposition 4.30 and after switching to the first partial Frobenius, $\mathrm{FEt}(X_1 \times_{\mathbb{F}_p} \bar{x}/\varphi_1) \cong \mathrm{FEt}(X_1)$, so there exists a finite étale cover $W \to X_1$ such that $Y \cong W \times_{\mathbb{F}_p} \bar{x}$ and $\varphi_Y = F_W \times_{\mathbb{F}_p} \mathrm{id}_{\bar{x}}$, up to composition with the canonical isomorphism $Y \cong W \times_{\mathbb{F}_p} \bar{x}$.
If $W = \coprod W_i$ is the decomposition of $W$ into finitely many connected components, $Y \cong (\coprod W_i) \times_{\mathbb{F}_p} \bar{x} \cong \coprod (W_i \times_{\mathbb{F}_p} \bar{x})$ is the decomposition of $Y$ into $\varphi_Y$-connected components. Now, we find $T$ as a coproduct of copies of $\bar{x}$, one for each $i$. As the map $Y \to T$, we take the coproduct of the projection maps $W_i \times_{\mathbb{F}_p} \bar{x} \to \bar{x}$. Then each geometric fibre $Y_{\bar{t}}$ of $Y \to T$ is isomorphic to $W_j \times_{\mathbb{F}_p} \bar{t}$ for the index $j$ corresponding to the copy of $\bar{x}$ met by $\bar{t}$. Note that we can write $F_W = \coprod F_{W_i}$, by definition of the absolute Frobenius of a scheme and because of the trivial fact that the Frobenius on a product of rings is the product of the Frobenii on its factors. Therefore, $\varphi_Y = F_W \times_{\mathbb{F}_p} \mathrm{id}_{\bar{x}} = \coprod (F_{W_i} \times_{\mathbb{F}_p} \mathrm{id}_{\bar{x}})$. So $\varphi_Y$ is indeed a $T$-morphism and each fibre $W_i \times_{\mathbb{F}_p} \bar{t}$ is $\varphi_{Y,\bar{t}} = F_{W_i} \times_{\mathbb{F}_p} \mathrm{id}_{\bar{t}}$-connected by Proposition 4.10.
Also the universal property of a diagram (31) is immediate, if $X_2 = \bar{x}$ is a geometric point : Each $T'$ as in diagram (32) splits as a finite coproduct of copies of $\bar{x}$. Since $Y \to T$ has $\varphi_{Y,\bar{t}}$-connected geometric fibres,

$$Y \cong Y \times_T T \cong Y \times_T \left(\coprod \bar{x}\right) \cong \coprod (Y \times_T \bar{x})$$

is the decomposition of $Y$ into $\varphi_Y$-connected components, hence $Y \to T$ induces a bijection between the $\varphi_Y$-connected components of $Y$ and the connected components of $T$, which are the copies of $\bar{x}$. Meanwhile, $Y \to T'$ has to map $\varphi_Y$-connected components to points, while not necessarily inducing a bijection. So we indeed get a unique morphism $T \to T'$ over $\bar{x}$ turning $Y \to T$ into $Y \to T'$ by mapping points to points in the right way.

As for Lemma 3.12, the proof strategy for Proposition 4.43 is to globalize Example 4.44. For that, we will use the following analogue of Lemma 3.7 :

**Lemma 4.45.** *Let $X_1, X_2 \to \mathbb{F}_p$ connected qcqs schemes over $\mathbb{F}_p$. Denote $X := X_1 \times_{\mathbb{F}_p} X_2$. Let $\bar{x} \to X_2$ be a geometric point of $X_2$. Then, base change induces an equivalence of categories*

$$2\text{-colim}_{(U,\bar{u})} \mathrm{FEt}(X \times_{X_2} U/\varphi_1) \xrightarrow{\sim} \mathrm{FEt}(X \times_{X_2} \bar{x}/\varphi_1) \tag{33}$$

*where $(U, \bar{u})$ runs through the (affine) étale neighborhoods of $\bar{x}$ in $X_2$.*

*Proof.* The proof is analogous to Lemma 3.7. Note that fullness again relies on the suggestions by Scholze [2]. We sketch why they also apply in the Drinfeld situation. Let us denote $Z := \lim_{(U,\bar{u})} U$. It suffices to show that base change induces an equivalence

$$\mathrm{FEt}(X \times_{X_2} \bar{x}/\varphi_1) \to \mathrm{FEt}(X \times_{X_2} Z/\varphi_1).$$

Further, observe the canonical isomorphisms $X \times_{X_2} Z \cong X_1 \times_{\mathbb{F}_p} Z$ and $X \times_{X_2} \bar{x} \cong X_1 \times_{\mathbb{F}_p} \bar{x}$.

2. See https://mathoverflow.net/questions/432160/künneth-formula-for-pi-1-proper-morphisms (last accessed on 2022-10-20)

- **Essential surjectivity :** By Proposition 4.30, base change is an equivalence $\mathrm{FEt}(X) \cong \mathrm{FEt}(X \times_{\mathbb{F}_p} \bar{x}/\varphi_1)$. This functor factors through base change $\mathrm{FEt}(X) \to \mathrm{FEt}(X \times_{\mathbb{F}_p} Z/\varphi_1)$. So, any object of $\mathrm{FEt}(X \times_{\mathbb{F}_p} \bar{x}/\varphi_1)$ descends to an object of $\mathrm{FEt}(X)$, which we can base change to $\mathrm{FEt}(X) \to \mathrm{FEt}(X \times_{\mathbb{F}_p} Z/\varphi_1)$ in order to get a preimage.
- **Faithfulness :** This follows from the same arguments as for Lemma 3.7, since $\mathrm{FEt}(X \times_{\mathbb{F}_p} Z/\varphi_1)$ and $\mathrm{FEt}(X \times_{X_2} \bar{x}/\varphi_1)$ together with fibre functors to a common geometric point are Galois categories by Proposition 4.10.
- **Fullness :** Set $R := \mathcal{O}_{X_2,\bar{x}}$ and denote by $\kappa$ its residue field. By $v$-descent and $h$-descent, we can do the same reductions as for Lemma 3.7, i.e. reduce to the case that $R$ is a normal integral domain with algebraically closed fraction field $K$ and to the case that $X_1$ is normal. As $\mathbb{F}_p$ is perfect, $X_1$ is geometrically normal, hence $X_1 \times_{\mathbb{F}_p} \mathrm{Spec}(R)$ and $X_1 \times_{\mathbb{F}_p} \mathrm{Spec}(K)$ are normal. Then we can argue as for Lemma 3.7.

□

*Proof of Lemma 4.43.* The proof is completely analogous to Lemma 3.12. Hence this proof will be kept brief.

- **Universal property :** The universal property of diagram (31) follows by the same arguments as in the proof of Lemma 3.12 : Uniqueness of the map $T \to T'$ follows from faithfulness of the fibre functor on $\mathrm{FEt}(X_2)$ and Example 4.44. Existence of the map $T \to T'$ is shown in an étale neighborhood of any geometric point of $X_2$ by first applying Example 4.44 and then Lemma 2.32. By uniqueness and the sheaf property of morphisms over $X_2$, we can glue to get the factorization $T \to T'$ globally.
- **Existence of $T$ is étale-local :** As in the proof of Lemma 4.43, it suffices to show existence of $T$ in an étale neighborhood of any geometric point $\bar{x}$ of $X_2$ since by the universal property shown before, any family of locally existent covers gives rise to a descent datum and finite étale morphisms satisfy fpqc (in particular, étale) descent.
- **Existence of $T$ :** Fix a geometric point $\bar{x} \to X_2$ and base change the composed morphism $Y \to X \to X_2$ as well as $\varphi_1$ and $\varphi_Y$ to $\bar{x}$. We have a canonical isomorphism $X \times_{X_2} \bar{x} \cong X_1 \times_{\mathbb{F}_p} \bar{x}$ under which $\varphi_1 \times_{X_2} \mathrm{id}_{\bar{x}}$ corresponds to $F_{X_1} \times_{\mathbb{F}_p} \mathrm{id}_{\bar{x}}$. Proposition 4.30 tells us (after switching to the first partial Frobenius) that there exists a finite étale cover $W \to X_1$ and an isomorphism $Y \times_{X_2} \bar{x} \cong W \times_{\mathbb{F}_p} \bar{x}$ under which $\varphi_Y \times_{X_2} \mathrm{id}_{\bar{x}}$ corresponds to $F_W \times_{\mathbb{F}_p} \mathrm{id}_{\bar{x}}$. Lemma 4.45 implies the existence of an étale neighborhood $\bar{x} \to U \to X_2$ such that $Y \times_{X_2} U \cong W \times_{\mathbb{F}_p} U$ and $\varphi_Y \times_{X_2} \mathrm{id}_U$ corresponds to $F_W \times_{\mathbb{F}_p} \mathrm{id}_U$. Then we might argue as in the proof of Lemma 3.12 by writing $W = \coprod_{i=1,..,n} W_i$ as the finite coproduct of its connected components and defining $T$ as the $n$-fold coproduct of copies of $U$.

□

## 4.5 Homotopy exact sequence in the Drinfeld setting

The goal of this section is to show

**Proposition 4.46.** *Let $X_1$, $X_2$ be connected qcqs schemes over $\mathbb{F}_p$ and set $X := X_1 \times_{\mathbb{F}_p} X_2$. Then, for any geometric point $\bar{x}$ of $X_2$ and any geometric point $\bar{z}$ of $X_1 \times_{\mathbb{F}_p} \bar{x}$ (and thus also of $X_1 \times_{\mathbb{F}_p} X_2$ and $X_2$), base change induces an exact sequence of topological groups*

$$\pi_1(X_1 \times_{\mathbb{F}_p} \bar{x}/\varphi_1, \bar{z}) \longrightarrow \pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z}) \longrightarrow \pi_1(X_2, \bar{z}) \longrightarrow 1. \tag{34}$$

In the above Proposition, we abused notation and denoted by $\varphi_1$ both the first partial Frobenius on $X_1 \times_{\mathbb{F}_p} \bar{x}$ and on $X_1 \times_{\mathbb{F}_p} X_2$, respectively. By the base change functor $\mathrm{FEt}(X_1 \times_{\mathbb{F}_p} \bar{x}/\varphi_1) \to \mathrm{FEt}(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1)$, we mean the one from Definition 4.26. By the base change functor $\mathrm{FEt}(X_2) \to \mathrm{FEt}(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1)$, we mean a functor as in Remark 4.42.
Here comes the analogue of Lemma 3.10 :

**Lemma 4.47.** *Under the assumptions of Proposition 4.46, the base change of any connected finite étale cover of $X_2$ to $X_1 \times_{\mathbb{F}_p} X_2$ is $\varphi_1$-connected. In particular, the sequence (34) is exact at $\pi_1(X_2, \bar{z})$.*

*Proof.* Let $Y_2 \to X_2$ be finite étale with $Y_2$ connected. Base change sends $Y_2$ to $(X_1 \times_{\mathbb{F}_p} Y_2, F_{X_1} \times_{\mathbb{F}_p} \mathrm{id}_{Y_2})$ and $X_1 \times_{\mathbb{F}_p} Y_2$ is $(F_{X_1} \times_{\mathbb{F}_p} \mathrm{id}_{Y_2})$-connected by Proposition 4.10. Then by Lemma 4.22, $(X_1 \times_{\mathbb{F}_p} Y_2, F_{X_1} \times_{\mathbb{F}_p} \mathrm{id}_{Y_2})$ is a connected object of $\mathrm{FEt}(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1)$. Hence we have shown that base change sends connected objects to connected objects, which implies that the sequence (34) is exact at $\pi_1(X_2, \bar{z})$ by Lemma 2.8. □

This is the analogue of Lemma 3.11 :

**Lemma 4.48.** *Let $X \to S$ be a morphism of schemes. Let $\varphi : X \to X$ be a universal homeomorphism over $S$ such that the geometric fibres of $X \to S$ are $(\varphi \times_S \mathrm{id}_{\bar{s}})$-connected. Suppose that for any finite étale morphism $T \to S$ the base change $X \times_S T \to X$ is $(\varphi \times_S \mathrm{id}_T)$-connected. Further assume that for any object $(Y, \varphi_Y)$ of $\mathrm{FEt}(X/\varphi)$, there exists a finite étale morphism $T \to S$ and a morphism $Y \to T$ with $(\varphi_Y \times_T \bar{t})$-connected geometric fibres such that*

$$\begin{array}{ccc} Y & \longrightarrow & T \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & S \end{array} \tag{35}$$

*commutes. Then, for any geometric point $\bar{s} \to S$ and any geometric point $\bar{z} \to X \times_S \bar{s}$ (hence also of $X$ and $S$), the sequence*

$$\pi_1(X \times_S \bar{s}, \bar{z}) \longrightarrow \pi_1(X, \bar{z}) \longrightarrow \pi_1(S, \bar{z})$$

*induced by base change is exact.*

*Proof.* The proof works in total analogy to Lemma 3.11. So, as before, we will be brief and refer the reader to the proof of Lemma 3.11. By Lemmas 2.9 and 2.10, it suffices to show the following statement :

*If the base change $Y \times_X (X \times_S \bar{s}) \cong Y \times_S \bar{s}$ has a $(\varphi_Y \times_S \bar{s})$-connected component $Z$ isomorphic to $X \times_S \bar{s}$ (i.e., the map $Z \to Y \times_S \bar{s} \to X \times_S \bar{s}$ is an isomorphism), then diagram (35) is cartesian.*

As in the proof of Lemma 3.11, we first may assume that $Y$ is $\varphi_Y$-connected. Choose a finite étale cover $T \to S$ as in the assumptions of the lemma. Since $Y \to T$ has geometrically $(\varphi_Y \times_T \bar{t})$-connected (and hence non-empty) fibres, it is surjective and therefore, $T$ is connected. Set $Y' := T \times_S X$ and $\varphi_{Y'} := id_T \times_S \varphi$. Let $p : Y \to Y'$ be the unique morphism making

$$\begin{tikzcd} Y \arrow[dr, dashed, "p"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ & X \arrow[r] & S \end{tikzcd} \tag{36}$$

commute. We show that $p$ is an isomorphism. First, observe that $p$ is finite étale as a morphism between finite étale covers of $X$, see [Stacks, Tags 035D and 02GW]. Since $T$ is connected, $Y'$ is $\varphi_{Y'}$-connected by assumption. Observe that, since $\varphi_Y$ and $\varphi$ are compatible along $Y \to X$, also $\varphi_Y$ and $\varphi_{Y'}$ are compatible along $p$. Therefore, $p$ indeed defines a morphism in $\mathrm{FEt}(X/\varphi)$.

Now, by Lemma 4.23, it suffices to show that there exists some geometric point of $Y'$ whose fibre under $p$ consists of a single point. For this, as for Lemma 3.11, we apply the base change functor $- \times_S \bar{s}$ to diagram (36) and consider the base change $p_{\bar{s}} := p \times_S \mathrm{id}_{\bar{s}} : X \times_S \bar{s} \to Y' \times_S \bar{s}$. By assumption, there exists a $\varphi_Y \times_S \bar{s}$-connected component $Z$ of $Y_{\bar{s}} = Y \times_X (X \times_S \bar{s})$ such that $Z \to Y_{\bar{s}} \xrightarrow{p_{\bar{s}}} Y'_{\bar{s}} \to X_{\bar{s}}$ is an isomorphism.

By Lemma 2.4 (and the description of connected objects in $\mathrm{FEt}(X_{\bar{s}}/\varphi \times_S \mathrm{id}_{\bar{s}})$), there exists a unique $(\varphi_{Y'} \times_S \mathrm{id}_{\bar{s}})$-connected component $Z'$ of $Y'_{\bar{s}}$ such that $Z \to Y_{\bar{s}} \xrightarrow{p_{\bar{s}}} Y'_{\bar{s}}$ factors through $Z' \to Y'_{\bar{s}}$. Moreover, $Z \to Z'$ is an isomorphism by the assumption on $Z$. Now, in analogy to the proof of Lemma 3.11, we observe that $Y_{\bar{s}}$ and $Y'_{\bar{s}}$ have the same number of components, which is where the assumption that $Y \to T$ has $(\varphi_Y \times_T \bar{t})$-connected geometric fibres comes in. Then, by Lemma 2.4, the square

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\cong"] & Y_{\bar{s}} \arrow[d, "p_{\bar{s}}"] \\ Z' \arrow[r] & Y'_{\bar{s}} \end{tikzcd}$$

is cartesian. Since $Z \to Z'$ is an isomorphism, $p_{\bar{s}}$ and hence also $p$ is an isomorphism after base change to a geometric point. $\square$

Hence, as in the classical setting, we conclude Proposition 4.46 from Lemmas 4.47, 4.48 and 4.43.

## 4.6 Conclusion of Drinfeld's Lemma

*Proof of Theorem 1.4.* Propositions 4.30 and 4.46 give us a commutative diagram

$$\begin{tikzcd} \pi_1(X_1 \times_{\mathbb{F}_p} \bar{x}/\varphi_1, \bar{z}) \arrow[r] \arrow[dr, "\cong"] & \pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z}) \arrow[r] \arrow[d] & \pi_1(X_2, \bar{z}) \arrow[r] & 1 \\ & \pi_1(X_1, \bar{z}), & & \end{tikzcd}$$

where the top row is exact and $\pi_1(X_1 \times_{\mathbb{F}_p} \bar{x}/\varphi_1, \bar{z}) \to \pi_1(X_1, \bar{z})$ is an isomorphism. From that on, the same arguments as for Lemma 3.9 show the claim : We get an exact sequence

$$\begin{tikzcd} \pi_1(X_1, \bar{z}) \arrow[r] & \pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z}) \arrow[r] \arrow[l, bend left] & \pi_1(X_2, \bar{z}) \arrow[r] & 1 \end{tikzcd}$$

of groups where $\pi_1(X_1, \bar{z}) \to \pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z}) \to \pi_1(X_1, \bar{z})$ is the identity, i.e., the sequence splits. Hence the maps from the above sequence induce an isomorphism of groups $\pi_1(X_1 \times_{\mathbb{F}_p} X_2/\varphi_1, \bar{z}) \cong \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z})$. This map is in fact an isomorphism of topological groups since any continuous map between compact Hausdorff spaces is closed and hence any bijective continuous map is a homeomorphism. $\square$

**Remark 4.49.** By Remark 4.29, Theorem 1.4 also implies

$$\pi_1(X/\mathrm{pFr}, \bar{z}) \cong \pi_1(X/\varphi_2, \bar{z}) \cong \pi_1(X_1, \bar{z}) \times \pi_1(X_2, \bar{z}).$$

**Remark 4.50.** By induction, one can generalize Theorem 1.4 to higher products with higher partial Frobenii. More precisely, if $X := X_1 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$ is the fibre product of $n$ connected qcqs schemes over $\mathbb{F}_p$, we may denote by

$$\varphi_i := \mathrm{id}_{X_1} \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} F_{X_i} \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} \mathrm{id}_{X_n},$$

the $i$-th partial Frobenius of $X$. Then, we may define a category $\mathrm{FEt}(X/\mathrm{pFr.})$ analogously to Remark 4.29 and Theorem 1.4 generalizes to

$$\pi_1(X/\mathrm{pFr}, \bar{z}) \cong \prod_i \pi_1(X_i, \bar{z}),$$

for any geometric point $\bar{z}$ of $X$ (and hence of all $X_i$). See also [Ked19, 4.2.10, 4.2.12].

**Example 4.51.** Let $X$ be a connected scheme over $\mathbb{F}_p$ and $F_X$ its absolute Frobenius. Drinfeld's Lemma states that for any connected scheme $X$ over $\mathbb{F}_p$, the group $\pi_1(X/F_X)$ is related to the "usual" étale fundamental group of $X$ as follows :

$$\pi_1(X/F_X, \bar{z}) \cong \pi_1(X \times_{\mathbb{F}_p} \mathrm{Spec}(\overline{\mathbb{F}}_p)/\varphi_1, \bar{z}) \cong \pi_1(X, \bar{z}) \times \pi_1(\mathrm{Spec}(\overline{\mathbb{F}}_p), \bar{z}) \cong \pi_1(X, \bar{z}) \times \widehat{\mathbb{Z}}.$$

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