

SELF-INTERSECTION OF THE RELATIVE DUALIZING SHEAF ON MODULAR CURVES $X(N)$

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ABSTRACT. Let $N \geq 3$ be a composite, odd, and square-free integer and let Γ be the principal congruence subgroup of level N . Let $X(N)$ be the modular curve of genus g_Γ associated to Γ . In this article, we study the Arakelov invariant $e(\Gamma) = \bar{\omega}^2/\varphi(N)$, with $\bar{\omega}^2$ denoting the self-intersection of the relative dualizing sheaf for the minimal regular model of $X(N)$, equipped with the Arakelov metric, and with Euler's phi function $\varphi(N)$. Our main result is the asymptotics $e(\Gamma) = 2g_\Gamma \log(N) + o(g_\Gamma \log(N))$, as the level N tends to infinity.

1. INTRODUCTION

1.1. Relevance of the known asymptotics. Arakelov invariants, such as the Faltings's delta function and the self-intersection of the relative dualizing sheaf, play an important role in Arakelov geometry. In particular, the self-intersection of the relative dualizing sheaf on an arithmetic surface is an essential contribution to the Faltings's height of the Jacobian [24]. It is known that suitable upper bounds for this invariant lead to important applications in arithmetic geometry. For instance, an analogue of the Bogomolov–Miyaoka–Yau inequality for arithmetic surfaces implies an effective version of the Mordell conjecture [23]. Also, sufficiently good upper bounds for the self-intersection of the relative dualizing sheaf are crucial in the work of B. Edixhoven and his co-authors, when estimating the running time of his algorithm for determining Galois representations [9]. However, establishing such bounds turned out to be a complicated problem. The known bounds have been established so far by the work of Abbes, Ullmo, Michel, Edixhoven, de Jong, and others, see, e.g., [3], [7], [9], [16], [17], [22]. They all deal with the calculation of the self-intersection of the relative dualizing sheaf for regular models of certain modular curves, Fermat curves, Belyi curves, or hyperelliptic curves, either establishing bounds or computing explicit asymptotics for this numerical invariant.

In their influential work [3], A. Abbes and E. Ullmo considered the modular curves $X_0(N)/\mathbb{Q}$. Denoting by $\mathcal{X}_0(N)/\mathbb{Z}$ a minimal regular model for $X_0(N)/\mathbb{Q}$, and by $\bar{\omega}_{\mathcal{X}_0(N)/\mathbb{Z}}$ the relative dualizing sheaf on $\mathcal{X}_0(N)$, equipped with the Arakelov metric, they established the asymptotics

$$(1) \quad \bar{\omega}_{\mathcal{X}_0(N)/\mathbb{Z}}^2 \sim 3g_{\Gamma_0(N)} \log(N),$$

as $N \rightarrow \infty$, where N is assumed to be square-free with $2, 3 \nmid N$ and such that the genus $g_{\Gamma_0(N)}$ of $X_0(N)$ is greater than zero (see [3] and [22, Théorème 1.1]). As consequence of this result, P. Michel and E. Ullmo obtained in [22] an asymptotics for Zhang's admissible self-intersection number [29], and with this they proved an effective version of the Bogomolov conjecture for the curve $X_0(N)/\mathbb{Q}$. Namely, if h_{NT} denotes the Néron–Tate height on the Jacobian [4], then for all $\varepsilon > 0$ and sufficiently large N , the set

$$\{x \in X_0(N)(\overline{\mathbb{Q}}) \mid h_{\text{NT}}(x) \leq (2/3 - \varepsilon) \log(N)\}$$

is finite, whereas the set

$$\{x \in X_0(N)(\overline{\mathbb{Q}}) \mid h_{\text{NT}}(x) \leq (4/3 + \varepsilon) \log(N)\}$$

is infinite.

By its very definition, the self-intersection of the relative dualizing sheaf on modular curves is the sum of a geometric part that encodes the finite intersection of divisors coming from the cusps, and an analytic part which is given in terms of the Arakelov Green's function evaluated at these cusps. The leading term $3g_{\Gamma_0(N)} \log(N)$ in the asymptotics (1) is the sum of $g_{\Gamma_0(N)} \log(N)$ that comes from the geometric part, and $2g_{\Gamma_0(N)} \log(N)$ that comes from the analytic part. J. Kramer conjectured that this result can be generalized to arbitrary modular curves, that is, that the main term in the asymptotics of the self-intersection of the relative dualizing sheaf for a semi-stable model of modular curves of level

N and genus g is $3g \log(N)$, as N tends to infinity. Following the lines of proof in [3], Kramer's Ph.D. student H. Mayer [21] obtained a positive answer to this conjecture for the case of modular curves $X_1(N)/\mathbb{Q}$. More precisely, for squarefree N of the form $N = N' \cdot q \cdot r$, where $q, r > 4$ are different primes, he proved the asymptotics

$$\bar{\omega}_{\mathcal{X}_1(N)/\mathbb{Z}}^2 \sim 3g_{\Gamma_1(N)} \log(N),$$

as $N \rightarrow \infty$; here, $\mathcal{X}_1(N)/\mathbb{Z}$ is a minimal regular model for $X_1(N)/\mathbb{Q}$. From this asymptotics he was then able to deduce the validity of the Bogomolov conjecture in this case.

1.2. Main result. In this article, we establish asymptotics for the self-intersection of the relative dualizing sheaf on modular curves $X(N)$, as the level N tends to infinity. Following mostly the lines of proof in [3], we will first obtain a formula for this Arakelov invariant, in which the geometric and analytic parts are explicitly given. Then, we proceed to compute the asymptotics for each of these parts. Our main result is the following asymptotics (see Theorem 18)

$$\frac{1}{\varphi(N)} \bar{\omega}_{\mathcal{X}(N)/\mathbb{Z}[\zeta_N]}^2 \sim 2g_{\Gamma(N)} \log(N),$$

as $N \rightarrow \infty$, where $N \geq 3$ is a composite, odd, and square-free integer; here, $\mathbb{Z}[\zeta_N]$ denotes the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_N)$ with a primitive N -th root of unity ζ_N , and $\mathcal{X}(N)/\mathbb{Z}[\zeta_N]$ is a minimal regular model for $X(N)/\mathbb{Q}(\zeta_N)$. It turns out that the right hand side $2g_{\Gamma(N)} \log(N)$ in the asymptotics comes from the analytic part, which partially confirms Kramer's conjecture. The difference of our result with the previous cases lies in the underlying minimal regular model, where all the computations take place, namely, our model is a $\mathbb{Z}[\zeta_N]$ -scheme which is not semi-stable. The analogue computations for a semi-stable model are a theme of our future research.

1.3. Sketch of proof. To prove the main result, we follow the strategy given in the work of Abbes and Ullmo [3]. The proof starts by observing that $\bar{\omega}_{\mathcal{X}(N)/\mathbb{Z}[\zeta_N]}^2/\varphi(N)$ is the sum of an explicit geometric contribution $\mathcal{G}(N)$ and an explicit analytic contribution $\mathcal{A}(N)$ (see Proposition 13). In particular, using results of Manin/Drinfeld and Faltings/Hriljac, we show that

$$\mathcal{G}(N) = \frac{2g_{\Gamma(N)}(V_0, V_\infty)_{\text{fin}} - (V_0, V_0)_{\text{fin}} - (V_\infty, V_\infty)_{\text{fin}}}{2(g_{\Gamma(N)} - 1)}.$$

We refer the reader to section 6.1 for the definitions of the vertical divisors V_0 and V_∞ . From this, a straight-forward computation yields the asymptotics (see Proposition 14)

$$\frac{1}{\varphi(N)} \mathcal{G}(N) = o(g_{\Gamma(N)} \log(N)),$$

as $N \rightarrow \infty$. Furthermore, the analytic contribution $\mathcal{A}(N)$ is given in terms of the Arakelov (or canonical) Green's function $g_{\text{Ar}}(\cdot, \cdot)$, evaluated at the corresponding cusps, and equals

$$\mathcal{A}(N) = 4g_{\Gamma(N)}(g_{\Gamma(N)} - 1) \sum_{\sigma: \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}} g_{\text{Ar}}(0^\sigma, \infty^\sigma).$$

To derive the desired asymptotics for $\mathcal{A}(N)$, one first uses a fundamental identity by Abbes/Ullmo, which expresses the Arakelov Green's function $g_{\text{Ar}}(q_1, q_2)$ at the cusps q_1 and q_2 of $X(N)$ in terms the scattering constant $\mathcal{C}_{q_1 q_2}$ at q_1 and q_2 , the hyperbolic volume $v_{\Gamma(N)}$ of $X(N)$, the constant term in the Laurent expansion at $s = 1$ of the Rankin–Selberg transforms \mathcal{R}_{q_1} resp. \mathcal{R}_{q_2} of the Arakelov metric at q_1 and q_2 , respectively, and a contribution \mathcal{G} involving the constant term of the automorphic Green's function $G_s(z, w)$ at $s = 1$; we refer the reader to (5), (11), and (14) for precise definitions. More precisely, one has (see (13))

$$g_{\text{Ar}}(q_1, q_2) = -2\pi \mathcal{C}_{q_1 q_2} - \frac{2\pi}{v_{\Gamma(N)}} + 2\pi(\mathcal{R}_{q_1} + \mathcal{R}_{q_2}) + 2\pi \mathcal{G}.$$

The relevant scattering constants $\mathcal{C}_{q_1 q_2}$ are computed, e.g., in [13], and the asymptotics for \mathcal{G} follows from bounds proven in [18]. Furthermore, we show that to deal with the terms $\mathcal{R}_{q_1} + \mathcal{R}_{q_2}$ for the cusps in question one is reduced to only compute \mathcal{R}_∞ (see Lemma 15). To provide the explicit formula for \mathcal{R}_∞ given in Proposition 16, we apply methods from the spectral theory of automorphic functions,

namely, we represent \mathcal{R}_∞ in terms of a particular automorphic kernel. Decomposing this automorphic kernel into terms involving hyperbolic and parabolic elements, \mathcal{R}_∞ is divided into hyperbolic and parabolic contributions $\mathcal{R}_\infty^{\text{hyp}}$ and $\mathcal{R}_\infty^{\text{par}}$ (see identity (12)). The computation of $\mathcal{R}_\infty^{\text{hyp}}$ and $\mathcal{R}_\infty^{\text{par}}$ are the technical heart of the paper. Note that to determine the hyperbolic contribution $\mathcal{R}_\infty^{\text{hyp}}$, we proceed differently than Abbes–Ullmo and we reduce the calculation of the residue of the corresponding zeta function to the well-known residue of the zeta function of a suitably generalized idèle class group of the quadratic extension, determined by the corresponding hyperbolic elements. From this, it is finally shown that (see Proposition 17)

$$\frac{1}{\varphi(N)}\mathcal{A}(N) = 2g_{\bar{\Gamma}(N)} \log(N) + o(g_{\bar{\Gamma}(N)} \log(N)),$$

as $N \rightarrow \infty$. Adding up the asymptotics for the geometric and for the analytic contribution then proves the main result.

1.4. Outline of the article. The paper is organized as follows. In Section 2, we set our main notation and review basic facts on modular curves $X(N)$, non-holomorphic Eisenstein series, the spectral theory of automorphic forms, and the Arakelov metric. The core of this part is the identity (12), which states that the Rankin–Selberg constant of the Arakelov metric \mathcal{R}_∞ at the cusp ∞ is essentially the sum of two contributions $\mathcal{R}_\infty^{\text{hyp}}$ and $\mathcal{R}_\infty^{\text{par}}$, associated to the hyperbolic and parabolic elements of $\bar{\Gamma}(N)$, respectively. In Section 3, we turn our attention to the minimal regular model \mathcal{X}/S of the curve $X(N)$, where $S = \text{Spec}(\mathbb{Z}[\zeta_N])$ with ζ_N an N -th root of unity. Here, in Proposition 1, we describe the good and bad fibers of \mathcal{X}/S , and briefly recall the moduli interpretation of the cusps of $X(N)$. Further, in Proposition 2, we describe how the cusps 0 and ∞ are mapped on $X(N)$ under a given embedding $\mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$. The importance of this result will be clear in the proof of Proposition 17. In Section 4, we begin with the study of the zeta function associated to an hyperbolic element of $\bar{\Gamma}(N)$. In Proposition 4, we show that this zeta function is, in fact, equal to a partial zeta function of a suitable narrow-ray ideal class, up to a holomorphic function. Using this fact, we then proceed to compute an asymptotics for the hyperbolic contribution $\mathcal{R}_\infty^{\text{hyp}}$. This is done in Proposition 6. In Section 5, we deal with the computation of the parabolic contribution $\mathcal{R}_\infty^{\text{par}}$, and for this we proceed in the same way as in [3], adapting the results therein to our case. Finally, in Section 6, we state our main result in Theorem 18, and for the proof, we first provide a formula for the self-intersection of the relative dualizing sheaf in Proposition 13, where the geometric and analytic parts are explicitly given. Then, using the computations from previous sections, we determine the desired asymptotics.

2. BACKGROUND MATERIAL

2.1. The modular curve $X(N)$. Let $\mathbb{H} := \{z = x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}; y > 0\}$ be the hyperbolic upper half-plane endowed with the hyperbolic volume form $\mu_{\text{hyp}}(z) := dx dy / y^2$, and let $\mathbb{H}^* := \mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$ be the union of \mathbb{H} with its topological boundary. The group $\text{PSL}_2(\mathbb{R})$ acts on \mathbb{H}^* by fractional linear transformations. This action is transitive on \mathbb{H} , since $z = x + iy = n(x)a(y)i$ with

$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(y) := \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}.$$

By abuse of notation, we represent an element of $\text{PSL}_2(\mathbb{R})$ by a matrix. We set $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{PSL}_2(\mathbb{R})$. Throughout the article, let $N \geq 3$ be a composite, odd, and square-free integer and let $\Gamma := \bar{\Gamma}(N)$ denote the principal congruence subgroup of the modular group $\text{PSL}_2(\mathbb{Z})$. By $\Gamma_z := \{\gamma \in \Gamma \mid \gamma z = z\}$ we denote the stabilizer subgroup of a point $z \in \mathbb{H}^*$ with respect to Γ .

The quotient space $Y(N) := \Gamma \backslash \mathbb{H}$ resp. $X(N) := \Gamma \backslash \mathbb{H}^*$ admits the structure of a Riemann surface and a compact Riemann surface of genus g_Γ , respectively. The hyperbolic volume form $\mu_{\text{hyp}}(z)$ naturally defines a $(1, 1)$ -form on $Y(N)$, which we still denote by $\mu_{\text{hyp}}(z)$, since it is $\text{PSL}_2(\mathbb{R})$ -invariant on \mathbb{H} . Thus, the hyperbolic volume of $Y(N)$ is given by $v_\Gamma := \int_{Y(N)} \mu_{\text{hyp}}(z)$. The genus g_Γ and the hyperbolic volume v_Γ are related by the identity

$$g_\Gamma = 1 + \frac{v_\Gamma}{4\pi} \left(1 - \frac{6}{N}\right).$$

By abuse of notation, we identify $X(N)$ with a fundamental domain $\mathcal{F}_\Gamma \subset \mathbb{H}^*$; thus, at times we will identify points of $X(N)$ with their pre-images in \mathbb{H}^* .

A cusp of $X(N)$ is the Γ -orbit of a parabolic fixed point of Γ in \mathbb{H}^* . By $P_\Gamma \subseteq \mathbb{P}^1(\mathbb{Q})$ we denote a complete set of representatives for the cusps of $X(N)$ and we write $p_\Gamma := \#P_\Gamma$. We will always identify a cusp of $X(N)$ with its representative in P_Γ . Hereby, identifying $\mathbb{P}^1(\mathbb{Q})$ with $\mathbb{Q} \cup \{\infty\}$, we write elements of $\mathbb{P}^1(\mathbb{Q})$ as α/β for $\alpha, \beta \in \mathbb{Z}$, not both equal to 0, and we always assume that $\gcd(\alpha, \beta) = 1$; we set $1/0 := \infty$. Given a cusp $q = \alpha/\beta \in P_\Gamma$, we choose the scaling matrix for q by $\sigma_q := g_q a(w_q) \in \mathrm{PSL}_2(\mathbb{R})$, where $g_q := \begin{pmatrix} \alpha & * \\ \beta & * \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ and $w_q := [\mathrm{PSL}_2(\mathbb{Z})_q : \Gamma_q]$. Let $U \subset \mathbb{Z}$ be a set of representatives for $(\mathbb{Z}/N\mathbb{Z})^\times / \{\pm I\}$ containing $1 \in \mathbb{Z}$. Given $\xi \in U$, the element $0_\xi := n(N-1)p(\xi)\gamma_N \in \mathbb{P}^1(\mathbb{Q})$, where $p(\xi) := \begin{pmatrix} \xi & 1 \\ rN & \xi \end{pmatrix}$ such that $\xi\tilde{\xi} - rN = 1$ with $\tilde{\xi}, r \in \mathbb{Z}$, and $\gamma_N := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ defines a cusp of Γ .

2.2. Eisenstein series, scattering constants, and the Rankin–Selberg transform. For a given cusp $q \in P_\Gamma$ with scaling matrix $\sigma_q \in \mathrm{PSL}_2(\mathbb{R})$, the non-holomorphic Eisenstein series associated to q is defined by

$$E_q(z, s) := \sum_{\gamma \in \Gamma_q \backslash \Gamma} \mathrm{Im}(\sigma_q^{-1} \gamma z)^s,$$

where $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The Eisenstein series $E_q(z, s)$ is Γ -invariant in z , and holomorphic in s for $\mathrm{Re}(s) > 1$. Furthermore, $E_q(z, s)$ admits a meromorphic continuation to the complex s -plane with a simple pole at $s = 1$ with residue equal to v_Γ^{-1} . Now, for $u \in U$, consider the subset $M(u) \subset \mathbb{Z}^2$ given by

$$M(u) := \{(m, n) \in \mathbb{Z}^2 \mid (m, n) \equiv (0, u) \pmod{N}\},$$

which is invariant under right multiplication by a matrix of Γ . Since we have $w_\infty = N$, $g_\infty = I$, and $\sigma_\infty = a(N)$, the Eisenstein series $E_\infty(z, s)$ associated to ∞ can be conveniently written as follows

$$E_\infty(z, s) = \frac{1}{N^s} \sum_{u \in U} D_u(s) \left(\sum_{(m, n) \in M(u)} \frac{y^s}{|mz + n|^{2s}} \right),$$

where $D_u(s)$ is the Dirichlet series defined by

$$(2) \quad D_u(s) := \sum_{\substack{d=1 \\ du \equiv 1 \pmod{N}}}^{\infty} \frac{\mu(d)}{d^{2s}},$$

with $\mu(d)$ denoting the Möbius function. At $s = 1$, the series $D_u(s)$ admits the following Laurent expansion

$$(3) \quad \sum_{u \in U} D_u(s) = \frac{1}{\pi} \left(\frac{v_\Gamma}{N^3} \right)^{-1} + O(s-1).$$

Let $q_1, q_2 \in P_\Gamma$ be two cusps, not necessarily distinct, with scaling matrices $\sigma_{q_1}, \sigma_{q_2} \in \mathrm{PSL}_2(\mathbb{R})$, respectively. The scattering function $\varphi_{q_1 q_2}(s)$ at the cusps q_1 and q_2 is defined by

$$\varphi_{q_1 q_2}(s) := \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c=1}^{\infty} c^{-2s} \left(\sum_{\substack{d \pmod{c} \\ \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{q_1}^{-1} \Gamma \sigma_{q_2}}} 1 \right),$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $\Gamma(s)$ denotes the Gamma function. The Eisenstein series admits the following Fourier expansion

$$(4) \quad E_{q_1}(\sigma_{q_2} z, s) = \delta_{q_1 q_2} y^s + \varphi_{q_1 q_2}(s) y^{1-s} + \sum_{n \neq 0} \varphi_{q_1 q_2}(n; s) y^{1/2} K_{s-1/2}(2\pi|n|y) e^{2\pi i n x},$$

where $\delta_{q_1 q_2}$ is the Dirac's delta function, $K_\mu(Z)$ is the modified Bessel function of the second kind, and

$$\varphi_{q_1 q_2}(n; s) := \frac{2\pi^s}{\Gamma(s)} |n|^{s-1/2} \sum_{c>0} c^{-2s} \left(\sum_{\substack{d \pmod{c} \\ \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \sigma_{q_1}^{-1} \Gamma \sigma_{q_2}}} e^{2\pi i (dm+an)/c} \right).$$

The scattering function $\varphi_{q_1 q_2}(s)$ is holomorphic for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ and admits a meromorphic continuation to the complex s -plane. At $s = 1$, there is always a simple pole of $\varphi_{q_1 q_2}(s)$ with residue equal to v_Γ^{-1} . The constant $\mathcal{C}_{q_1 q_2}$ given by

$$(5) \quad \mathcal{C}_{q_1 q_2} := \lim_{s \rightarrow 1} \left(\varphi_{q_1 q_2}(s) - \frac{v_\Gamma^{-1}}{s-1} \right)$$

is called the scattering constant at the cusps q_1 and q_2 .

Let $f : \mathbb{H} \rightarrow \mathbb{C}$ be a Γ -invariant function which is of rapid decay at a cusp $q \in P_\Gamma$, i.e., the 0-th coefficient $a_0(y, q)$ in the Fourier expansion of $f(\sigma_q z)$ satisfies $a_0(y, q) = O(y^{-C})$ for all $C > 0$, as $y \rightarrow \infty$. For $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the Rankin–Selberg transform of f at q is given by the integral

$$\mathcal{R}_q[f](s) := \int_{\mathcal{F}_\Gamma} f(z) E_q(z, s) \mu_{\text{hyp}}(z).$$

The function $\mathcal{R}_q[f](s)$ possesses a meromorphic continuation to the complex s -plane having a simple pole at $s = 1$, with residue equal to $(1/v_\Gamma) \int_{\mathcal{F}_\Gamma} f(z) \mu_{\text{hyp}}(z)$. Furthermore, we have the following useful identity

$$\mathcal{R}_q[f](s) = \int_0^\infty a_0(y, q) y^{s-2} dy.$$

2.3. The spectral expansion and automorphic kernels. Let k be either 0 or 2 and fix it. Let $z \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{PSL}_2(\mathbb{R})$. Given a function $f : \mathbb{H} \rightarrow \mathbb{C}$, we define $f[[\gamma; k]]$ by

$$(f[[\gamma; k]])(z) := j_\gamma(z; k)^{-1} f(\gamma z),$$

where $j_\gamma(z; k) := ((cz + d)/|cz + d|)^k$ is the weight- k automorphy factor. A function $f : \mathbb{H} \rightarrow \mathbb{C}$ is an automorphic function of weight k with respect to Γ if the equality $(f[[\gamma; k]])(z) = f(z)$ holds for all $\gamma \in \Gamma$. Denote by $L^2(Y(N), k)$ the Hilbert space consisting of all automorphic functions of weight k with respect to Γ that are measurable and square integrable, endowed with the inner product given by $\langle f, g \rangle := \int_{\mathcal{F}_\Gamma} f(z) \overline{g(z)} \mu_{\text{hyp}}(z)$. The hyperbolic Laplacian of weight k is defined by

$$\Delta_{\text{hyp}, k} := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + ik y \frac{\partial}{\partial x}.$$

Considering the uppering and lowering Maass operators of weight k , namely

$$(6) \quad U_k := \frac{k}{2} + iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad L_k := \frac{k}{2} + iy \frac{\partial}{\partial x} - y \frac{\partial}{\partial y},$$

respectively, the following identities hold

$$\Delta_{\text{hyp}, k} = L_{k+2} U_k - \frac{k}{2} \left(1 + \frac{k}{2} \right) = U_{k-2} L_k + \frac{k}{2} \left(1 - \frac{k}{2} \right).$$

Since the hyperbolic Laplacian $\Delta_{\text{hyp}, k}$ of weight k defines a symmetric and essentially self-adjoint operator, it extends to a unique self-adjoint operator Δ_k on a suitable domain. Consequently, there exists a countable orthonormal set $\{\psi_j\}_{j=0}^\infty$ of eigenfunctions of Δ_0 , in fact eigenfunctions of $\Delta_{\text{hyp}, 0}$, such that for all $f \in L^2(Y(N), k)$, we have the spectral expansion

$$f(z) = \sum_{j=0}^\infty \langle f, \psi_j \rangle \psi_j(z) + \frac{1}{4\pi} \sum_{q \in P_\Gamma} \int_{-\infty}^\infty \left\langle f, E_{q, k} \left(\cdot, \frac{1}{2} + ir \right) \right\rangle E_{q, k} \left(\cdot, \frac{1}{2} + ir \right) dr,$$

which converges in the norm topology; moreover, if f is smooth and bounded, then the expansion converges uniformly on compacta of \mathbb{H} . Here, $E_{q, k}(z, s)$ denotes the weight- k Eisenstein series of Γ at the cusp $q \in P_\Gamma$. For $k = 0$, this is the Eisenstein series $E_q(z, s)$ from Section 2.2, but if $k = 2$, then $E_{q, 2}(z, s)$ is determined by $E_q(z, s)$ via the uppering Maass operator U_0 given by (6), namely

$$(7) \quad U_0(E_q(z, s)) = s E_{q, 2}(z, s),$$

where s is not a pole of $E_q(z, s)$. In particular, using (4) and (7), the following Fourier expansion can be deduced

$$E_{\infty, 2}(\sigma_\infty z, s) = y^s + \varphi_{\infty, 2}(s) y^{1-s}$$

$$+ \frac{y^{1/2}}{s} \sum_{n \neq 0} \left(\left(\frac{1}{2} - 2\pi ny \right) K_{s-1/2}(2\pi|n|y) + y \frac{\partial}{\partial y} K_{s-1/2}(2\pi|n|y) \right) \varphi_{\infty\infty}(n; s) e^{2\pi i n x},$$

where we have set

$$\varphi_{\infty\infty,2}(s) := \frac{1-s}{s} \varphi_{\infty\infty}(s).$$

Next, we fix real numbers $T > 0$ and $A > 1$, once for all. Consider the holomorphic function $h_T(r)$ defined on the strip $|\operatorname{Im}(r)| < A/2$ by

$$(8) \quad h_T(r) := \exp \left(-T \left(\frac{1}{4} + r^2 \right) \right).$$

Let ϕ_k denote the inverse Selberg/Harish–Chandra transform of weight k of $h_T(r)$, namely, we have

$$\phi_k(x) := -\frac{1}{\pi} \int_{-\infty}^{\infty} Q'(x+t^2) \left(\frac{\sqrt{x+1+t^2}-t}{\sqrt{x+1+t^2}+t} \right)^{k/2} dt$$

where $x \geq 0$ and

$$Q \left(\frac{1}{4}(e^u + e^{-u} - 2) \right) = \frac{1}{2} g(u), \quad g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h_T(r) e^{-iru} dr.$$

For $z, w \in \mathbb{H}$, the weight- k point pair invariant $\pi_k(z, w)$ associated to $h_T(r)$ is given by

$$\pi_k(z, w) := \left(\frac{w - \bar{z}}{z - \bar{w}} \right)^{k/2} \phi_k(u(z, w)),$$

where $u(z, w) := |z - w|^2 / 4\operatorname{Im}(z)\operatorname{Im}(w)$. With this, we define the weight- k automorphic kernel by

$$K_k(z, w) := \sum_{\gamma \in \Gamma} j_{\gamma}(z; k) \pi_k(z, \gamma w).$$

Now, let $\mathcal{S}_2(\Gamma)$ be the \mathbb{C} -vector space of dimension g_{Γ} consisting of cusp forms of weight 2 with respect to Γ , endowed with the Petersson inner product. Once for all, we fix an orthonormal basis $\{f_1, \dots, f_{g_{\Gamma}}\}$ of $\mathcal{S}_2(\Gamma)$ and write $\lambda_j := 1/4 + r_j^2$, with $r_j \in \mathbb{C}$, for the corresponding eigenvalue of ψ_j . Then, the following spectral expansions hold

$$(9) \quad \begin{aligned} K_0(z, w) &= \sum_{j=0}^{\infty} h_T(r_j) \psi_j(z) \bar{\psi}_j(w) + S_0(z, w), \\ K_2(z, w) &= \sum_{j=1}^{g_{\Gamma}} \operatorname{Im}(z)\operatorname{Im}(w) f_j(z) \bar{f}_j(w) + \sum_{j=1}^{\infty} \frac{h_T(r_j)}{\lambda_j} (U_0 \psi_j)(z) \overline{(U_0 \psi_j)}(w) + S_2(z, w), \end{aligned}$$

where U_0 is the uppering Maass operator of weight 0 given by (6), and

$$S_k(z, w) := \frac{2-k}{2} v_{\Gamma}^{-1} + \frac{1}{4\pi} \sum_{q \in P_{\Gamma}} \int_{-\infty}^{\infty} h_T(r) E_{q,k} \left(z, \frac{1}{2} + ir \right) \overline{E_{q,k} \left(w, \frac{1}{2} + ir \right)} dr.$$

In the sequel, we write $K_k(z)$ for $K_k(z, z)$ and $S_k(z)$ for $S_k(z, z)$. Letting

$$\nu_k(\gamma; z) := j_{\gamma}(z; k) \pi_k(z, \gamma z),$$

we have $K_k(z) = K_k^{\text{hyp}}(z) + K_k^{\text{par}}(z)$ with

$$K_k^{\text{hyp}}(z) := \sum_{\substack{\gamma \in \Gamma \\ |\operatorname{tr}(\gamma)| > 2}} \nu_k(\gamma; z), \quad K_k^{\text{par}}(z) := \sum_{\substack{\gamma \in \Gamma \\ |\operatorname{tr}(\gamma)| = 2}} \nu_k(\gamma; z).$$

Note that, by abuse of notation, $\gamma = \pm I$ are included in $K_k^{\text{par}}(z)$. With this, we define

$$\mathcal{H}(z) := K_2^{\text{hyp}}(z) - K_0^{\text{hyp}}(z),$$

$$\mathcal{P}(z) := (K_2^{\text{par}}(z) - S_2(z)) - (K_0^{\text{par}}(z) - S_0(z)),$$

$$\mathcal{T}(z) := \sum_{j=1}^{\infty} \frac{h_T(r_j)}{\lambda_j} |(U_0 \psi_j)(z)|^2 - \sum_{j=0}^{\infty} h_T(r_j) |\psi_j(z)|^2.$$

2.4. The Arakelov metric and the Arakelov Green's function. For $z \in \mathbb{H}$, the Arakelov metric $F(z)$ is the Γ -invariant function defined on \mathbb{H} given by

$$F(z) := \frac{\text{Im}(z)^2}{g_\Gamma} \sum_{j=1}^{g_\Gamma} |f_j(z)|^2.$$

It can be verified that $F(z)$ is of rapid decay at all cusps $q \in P_\Gamma$, and that $\int_{\mathcal{F}_\Gamma} F(z) \mu_{\text{hyp}}(z) = 1$. Moreover, if $z \in \mathbb{H}$ and $\alpha \in \text{GL}_2^+(\mathbb{Q})$ such that $\alpha^{-1}\Gamma\alpha = \Gamma$, then the identity $F(\alpha z) = F(z)$ holds. Following the lines of Abbes–Ullmo, one obtains the following key identity for $F(z)$. First, by definition, we have $K_2(z) - K_0(z) = \mathcal{H}(z) + K_2^{\text{par}}(z) - K_0^{\text{par}}(z)$. Using the spectral decompositions given by (9), we then obtain

$$K_2(z) - K_0(z) = g_\Gamma F(z) + \mathcal{T}(z) + S_2(z) - S_0(z).$$

These two identities for the difference $K_2(z) - K_0(z)$ therefore yield the identity

$$(10) \quad F(z) = \frac{1}{g_\Gamma} \left(-\mathcal{T}(z) + \mathcal{H}(z) + \mathcal{P}(z) \right).$$

In the sequel, we refer to $\mathcal{H}(z)$ resp. $\mathcal{P}(z)$ as the hyperbolic contribution and parabolic contribution of the Arakelov metric, respectively.

The Rankin–Selberg constant of the Arakelov metric at q is defined by

$$(11) \quad \mathcal{R}_q := \lim_{s \rightarrow 1} \left(\mathcal{R}_q[F](s) - \frac{v_\Gamma^{-1}}{s-1} \right).$$

For $q = \infty$, taking the Rankin–Selberg transform on both sides of (10) and grouping the constant terms of the Laurent expansion at $s = 1$, we obtain the following identity

$$(12) \quad \mathcal{R}_\infty = \frac{1}{g_\Gamma} \left(-\frac{v_\Gamma^{-1}}{2} \sum_{j=1}^{\infty} \frac{h_T(r_j)}{\lambda_j} + \mathcal{R}_\infty^{\text{hyp}} + \mathcal{R}_\infty^{\text{par}} \right).$$

Here, $\mathcal{R}_\infty^{\text{hyp}}$ resp. $\mathcal{R}_\infty^{\text{par}}$ denotes the constant term in the Laurent expansion at $s = 1$ of $\mathcal{R}_\infty[\mathcal{H}](s)$ and $\mathcal{R}_\infty[\mathcal{P}](s)$, respectively, i.e., we have

$$\begin{aligned} \mathcal{R}_\infty^{\text{hyp}} &:= \lim_{s \rightarrow 1} \left(\mathcal{R}_\infty[\mathcal{H}](s) - \frac{v_\Gamma^{-1}}{s-1} \int_{\mathcal{F}_\Gamma} \mathcal{H}(z) \mu_{\text{hyp}}(z) \right), \\ \mathcal{R}_\infty^{\text{par}} &:= \lim_{s \rightarrow 1} \left(\mathcal{R}_\infty[\mathcal{P}](s) - \frac{v_\Gamma^{-1}}{s-1} \int_{\mathcal{F}_\Gamma} \mathcal{P}(z) \mu_{\text{hyp}}(z) \right). \end{aligned}$$

In the sequel, we refer to the constants $\mathcal{R}_\infty^{\text{hyp}}$ and $\mathcal{R}_\infty^{\text{par}}$ as the hyperbolic and parabolic contributions of \mathcal{R}_∞ , respectively.

Next, the canonical volume form on \mathbb{H} is given by $\mu_{\text{can}}(z) := F(z) \mu_{\text{hyp}}(z)$. Since $F(z)$ and $\mu_{\text{hyp}}(z)$ are Γ -invariant on \mathbb{H} , the canonical volume form $\mu_{\text{can}}(z)$ is naturally defined on $Y(N)$. Furthermore, $\mu_{\text{can}}(z)$ extends to a $(1, 1)$ -form on $X(N)$, since it remains smooth at the cusps $q \in P_\Gamma$. In addition, observe that $\int_{X(N)} \mu_{\text{can}}(z) = 1$.

Finally, the Arakelov Green's function g_{Ar} is the function defined on $X(N) \times X(N)$ which is smooth outside the diagonal and characterized by the following conditions

- (i) $\frac{1}{i\pi} \partial_z \partial_{\bar{z}} g_{\text{Ar}}(z, w) = \mu_{\text{can}}(z) - \delta_w(z)$,
- (ii) $\int_{X(N)} g_{\text{Ar}}(z, w) \mu_{\text{can}}(z) = 0$, for all $w \in X(N)$,

where δ_w denotes the Dirac delta distribution. Given two different cusps $q_1, q_2 \in P_\Gamma$, we have the following important identity (due to Abbes–Ullmo)

$$(13) \quad g_{\text{Ar}}(q_1, q_2) = -2\pi \mathcal{L}_{q_1 q_2} - \frac{2\pi}{v_\Gamma} + 2\pi(\mathcal{R}_{q_1} + \mathcal{R}_{q_2}) + 2\pi \mathcal{G},$$

where $\mathcal{L}_{q_1 q_2}$ is defined in (5), \mathcal{R}_{q_j} is given in (11), and

$$(14) \quad \mathcal{G} := - \int_{X(N) \times X(N)} g(z, w) \mu_{\text{can}}(z) \mu_{\text{can}}(w).$$

Here, the function $g(z, w)$ is the constant term in the Laurent expansion of the automorphic Green's function at $s = 1$. More precisely, let $G_s(z, w)$ denote the automorphic Green's function of Γ given, for $z, w \in \mathbb{H}$, $z \not\equiv w \pmod{\Gamma}$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, by the series

$$G_s(z, w) = -\frac{1}{4\pi} \sum_{\gamma \in \Gamma} Q_{s-1}(1 + 2u(z, \gamma w)),$$

where $Q_\tau(\nu)$ denotes the associated Legendre function of the second kind. The automorphic Green's function $G_s(z, w)$ admits a meromorphic continuation to the whole s -plane with a simple pole at $s = 1$. At $s = 1$, we have

$$G_s(z, w) = -\frac{v_\Gamma^{-1}}{s(s-1)} - \frac{1}{4\pi} g(z, w) + O_{z,w}(s-1),$$

where $g(z, w)$ depends only on z and w .

3. THE MINIMAL REGULAR MODEL OF THE MODULAR CURVE $X(N)$

Let K be a number field with ring of integers \mathcal{O}_K . We set $S := \text{Spec}(\mathcal{O}_K)$ and denote by η the generic point of S . An arithmetic surface \mathcal{X}/S is an integral, regular, and 2-dimensional S -scheme with a projective and flat structural morphism $\mathcal{X} \rightarrow S$, such that the generic fiber \mathcal{X}_η is a geometrically connected curve over K .

Given a smooth projective curve X/K of genus $g \geq 1$ defined over K , there exists a unique (minimal) arithmetic surface \mathcal{X}/S whose generic fiber \mathcal{X}_η is isomorphic to X/K [20, Proposition 10.1.8]. This arithmetic surface is called the minimal regular model of X/K . In particular, by the Riemann–Roch theorem, the compact Riemann surface $X(N)$ can be embedded in a projective space whose image is a smooth and projective algebraic curve $X(N)/\mathbb{C}$. Since the algebraic curve $X(N)/\mathbb{C}$ is in fact defined over the cyclotomic number field $\mathbb{Q}(\zeta_N)$, there exists a minimal regular model of $X(N)/\mathbb{Q}(\zeta_N)$.

To simplify notation, we will write \mathcal{X}/S for the minimal regular model of $X(N)/\mathbb{Q}(\zeta_N)$, where in this case $S = \text{Spec}(\mathbb{Z}[\zeta_N])$. Also, given an embedding $\sigma : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$, we write $\mathcal{X}_{\eta, \sigma}$ for the base change $\mathcal{X}_\eta \otimes_\sigma \text{Spec}(\mathbb{C})$.

Next, we will provide an explicit description of the fibers of the arithmetic surface \mathcal{X}/S . To do so, let us briefly introduce the moduli interpretation of \mathcal{X}/S .

3.1. The moduli problem of canonical structures on elliptic curves. Let $N \geq 2$ be a given integer and set $\zeta_N := e^{2\pi i/N}$. Denote by $\mu_N(\mathbb{C})$ the group of the N -th roots of unity and by $Y(N)/\mathbb{Q}(\zeta_N)$ the open subvariety of $X(N)/\mathbb{Q}(\zeta_N)$ given by the image of $\bar{\Gamma}(N) \backslash \mathbb{H}$ under the projective embedding that takes $X(N)$ to $X(N)/\mathbb{C}$.

An elliptic curve E/T over an arbitrary scheme T is a proper and smooth commutative group T -scheme with a given section, such that the geometric fibers are all connected and all have genus one. Given an elliptic curve E/T over T , the subscheme $E[N]$ of the N -torsion points of E/T is defined by $E[N] := E \times_E T$, which is obtained by base change using the N -fold morphism $[N] : E \rightarrow E$ and the given section of E/T . A canonical $\Gamma(N)$ -structure on E/T is a homomorphism of groups $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \rightarrow E[N](T)$ such that $e_N(\phi(1, 0), \phi(0, 1)) = \zeta_N$ and the following identity of Cartier divisors holds

$$\sum_{(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2} [\phi(a, b)] = E[N];$$

here, e_N denotes the Weil pairing on $E[N]$ and $[\phi(a, b)]$ is the effective Cartier divisor induced by the section $\phi(a, b) \in E[N](T)$.

Let R be a noetherian regular and excellent ring, and denote by (Sch/R) the category of R -schemes. In addition, we let (Sets) be the category of sets. Let us consider the case when $R = \mathbb{C}$. Given an elliptic curve E/\mathbb{C} over \mathbb{C} , we have $E[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$, where the isomorphism is not canonical. Then, a canonical $\Gamma(N)$ -structure on E/\mathbb{C} is an isomorphism $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \rightarrow E[N]$ which is compatible with the pairing $(\mathbb{Z}/N\mathbb{Z})^2 \times (\mathbb{Z}/N\mathbb{Z})^2 \rightarrow \mu_N(\mathbb{C})$ given by $((a, b), (c, d)) \mapsto \zeta_N^{ad-bc}$ induced by the isomorphism ϕ and the Weil pairing on $E[N]$. Then, it turns out that the variety $Y(N)/\mathbb{Q}(\zeta_N)$ represents the functor $\mathfrak{F} : (\text{Sch}/\mathbb{C}) \rightarrow (\text{Sets})$ which takes $T \in \text{Ob}(\text{Sch}/\mathbb{C})$ to the set of isomorphism classes of pairs $(E/T, \phi)$, where E/T is an elliptic curve over T and ϕ is a canonical $\Gamma(N)$ -structure on E/T .

In the general case, N. Katz and B. Mazur proved that an analogous functor $\mathfrak{F} : (\text{Sch}/\mathbb{Z}[\zeta_N]) \rightarrow (\text{Sets})$ is representable by a flat, regular, and 2-dimensional S -scheme \mathcal{Y}/S , provided that $N \geq 3$ is a composite, odd, and square-free integer. Moreover, it turns out that the arithmetic surface \mathcal{X}/S is the compactification of \mathcal{Y}/S .

Proposition 1 (Katz–Mazur). *Let $N \geq 3$ be a composite, odd, and square-free integer, and let \mathcal{X}/S be the minimal regular model of $X(N)/\mathbb{Q}(\zeta_N)$. Then, the fiber $\mathcal{X}_{\mathfrak{p}}$, for $\mathfrak{p} \in S$ with $\mathfrak{p} \nmid N$ is a smooth curve, whereas for $\mathfrak{p} \mid p$ with $p \mid N$ a prime number, the fiber $\mathcal{X}_{\mathfrak{p}}$ consists of $p + 1$ copies of smooth proper $\kappa(\mathfrak{p})$ -curves intersecting transversally at their supersingular points.*

Proof. See [19]. □

3.2. The moduli interpretation of the closed subscheme of cusps. The standard N -gon over a scheme S' is the proper curve over S' (meaning a morphism $C \rightarrow S'$ which is separated, flat, finitely presented, and of pure relative dimension 1, with non-empty fibers) obtained from $\mathbb{P}_{S'}^1 \times \mathbb{Z}/N\mathbb{Z}$ by gluing the ∞ -section of $\mathbb{P}_{S'}^1 \times \{j\}$ to the 0-section of $\mathbb{P}_{S'}^1 \times \{j + 1\}$, for all $j \in \mathbb{Z}/N\mathbb{Z}$.

In [6], B. Conrad extended the notion of canonical $\Gamma(N)$ -structures to generalized elliptic curves. Moreover, he proved that, for N sufficiently large, the scheme \mathcal{X}/S represents the functor $\mathfrak{F}' : (\text{Sch}/\mathbb{Z}[\zeta_N]) \rightarrow (\text{Sets})$ which takes $T \in \text{Ob}(\text{Sch}/\mathbb{Z}[\zeta_N])$ to the set of isomorphism classes of pairs $(E/T, \phi)$, where E/T is now a generalized elliptic curve over T and ϕ is an extended canonical $\Gamma(N)$ -structure on E/T . The advantage of this approach for \mathcal{X}/S lies in the moduli interpretation of the closed subscheme of cusps

$$\text{Cusps}(N) := (\mathcal{X} \setminus \mathcal{Y})^{\text{red}},$$

namely, the closed subscheme $\text{Cusps}(N)$ corresponds to the extended canonical $\Gamma(N)$ -structures on the Tate curve over $\mathbb{Z}[[q^{1/N}]]$. In practice, this amounts to describe an extended canonical $\Gamma(N)$ -structures on the standard N -gon.

Proposition 2. *Let $N \geq 3$ be a composite, odd, and square-free integer, and let $0, \infty \in P_{\Gamma}$ be cusps regarded as $\mathbb{Q}(\zeta_N)$ -rational points of \mathcal{X}_{η} . Given an embedding $\sigma : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$ such that $\sigma(\zeta_N) = e^{2\pi iv/N}$ with $v \in (\mathbb{Z}/N\mathbb{Z})^{\times} / \{\pm 1\}$, let $0^{\sigma}, \infty^{\sigma}$ be the corresponding points in $\mathcal{X}_{\eta, \sigma}(\mathbb{C})$ of $0, \infty$ under the embedding σ , respectively. Then, there exists an isomorphism*

$$\iota_{\sigma} : \mathcal{X}_{\eta, \sigma} \rightarrow X(N)/\mathbb{C}$$

such that, on complex points, we have $\iota_{\sigma}(0^{\sigma}) = 0_{\xi}$ for some $\xi \in U$, and $\iota_{\sigma}(\infty^{\sigma}) = \infty$.

Proof. First of all, note that, a point in $\mathcal{X}_{\eta, \sigma}(\mathbb{C})$ corresponds to a triple $(E; P, Q)$, where E is a generalized elliptic curve over \mathbb{C} and P, Q is a basis of $E^{\text{sm}}[N]$ such that $e_N(P, Q) = e^{2\pi iv/N}$. Then, ι_{σ} is given by

$$(E; P, Q) \mapsto (E, P + Q, v'Q),$$

on complex points.

Secondly, for a given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$, the map

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\mathbb{Z}/N\mathbb{Z})^2 \rightarrow \mathbb{Z}/N\mathbb{Z} \times \mu_N(\mathbb{C}),$$

defined by $(1, 0) \mapsto (a, \zeta_N^b)$ and $(0, 1) \mapsto (c, \zeta_N^d)$, describes explicitly how cusps corresponds to a $\Gamma(N)$ -structure on the Tate curve. Thus, the map $\varphi_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$ takes $(1, 0) \mapsto P_0$ and $(0, 1) \mapsto Q_0$ with $P_0 = (1, 1)$ and $Q_0 = (0, 1)$, and the moduli interpretation of the cusp ∞ is $(\text{Tate}_0/\mathbb{Q}(\zeta_N); P_0, Q_0)$. Similarly, $\varphi_{\begin{pmatrix} 1 & 0 \\ v' & 1 \end{pmatrix}}$ takes $(1, 0) \mapsto P_0$ and $(0, 1) \mapsto (v', \zeta_N)$, and the moduli interpretation of $1/v'$ is $(\text{Tate}_0/\mathbb{Q}(\zeta_N); P_0, (v', \zeta_N))$.

Finally, we have

$$\begin{aligned} \iota_\sigma(\infty^\sigma) &= (\text{Tate}_0; P_0 + (0, e^{2\pi iv/N}), v'(0, e^{2\pi iv/N})) \\ &= (\text{Tate}_0; (1, e^{2\pi iv/N}), (0, e^{2\pi i/N})) \\ &= (\text{Tate}_0; (1, 1), (0, e^{2\pi i/N})) = (\text{Tate}_0; P_0, Q_0) = \infty, \end{aligned}$$

where in the third equality we used the automorphism $(a, z) \mapsto (a, ze^{-2\pi iv/N})$ of the Tate curve. A similar reasoning can be applied for the cusp 0, using the other automorphism of the Tate curve. This concludes the proof. \square

4. THE HYPERBOLIC CONTRIBUTION OF \mathcal{R}_∞

Recall from identity (12) that $\mathcal{R}_\infty^{\text{hyp}}$ is the hyperbolic contribution of the Rankin–Selberg constant \mathcal{R}_∞ at ∞ . In this chapter, we establish an asymptotics for $\mathcal{R}_\infty^{\text{hyp}}$ as $T \rightarrow \infty$, for a given level N (see Proposition 6). To do so, in Section 4.1, we first determine the residue at $s = 1$ of certain zeta functions associated to hyperbolic elements of Γ (see Proposition 4). Next, in Section 4.2, in the proof of Proposition 6, we obtain an explicit identity for $\mathcal{R}_\infty[\mathcal{H}](s)$ in terms of the aforementioned zeta functions; thus, we derive an identity for $\mathcal{R}_\infty^{\text{hyp}}$ from the Laurent expansion of $\mathcal{R}_\infty[\mathcal{H}](s)$ at $s = 1$ in which the residues at $s = 1$ of the previous zeta functions arise.

4.1. The zeta function associated to Γ . For this section, let us consider the following settings. Let l be a fixed integer such that $|l| > 2$ and set $\Delta := l^2 - 4$. Denote by \mathcal{Q}_Δ the set of binary quadratic forms of discriminant Δ whose coefficients lie in \mathbb{Z} . Given $f(x, y) \in \mathcal{Q}_\Delta$, we will write $f \cdot (x, y) := f(y, -x)$.

Consider the sets

$$\text{sp}_l := \{\gamma \in \text{PSL}_2(\mathbb{Z}) \mid \text{tr}(\gamma) = l\}$$

$$\text{sp}_l(\Gamma) := \{\gamma \in \Gamma \mid \text{tr}(\gamma) = l\} \subset \text{sp}_l.$$

Observe that an element $\gamma \in \text{sp}_l(\Gamma)$ has the form $\gamma = \gamma_l(a, b, c)$, where

$$(15) \quad \gamma_l(a, b, c) := \begin{pmatrix} \frac{l-bN}{2} & -cN \\ aN & \frac{l+bN}{2} \end{pmatrix}$$

with $a, b, c \in \mathbb{Z}$, $a > 0$, $\gcd(a, b, c) = 1$, $(l \pm bN)/2 \equiv 1 \pmod{N}$, and $b^2 - 4ac$ is not a square. From now on, we assume that $\gamma \in \text{sp}_l(\Gamma)$ has always the form given by (15). Thus, for $\gamma \in \text{sp}_l(\Gamma)$, we set $D := b^2 - 4ac$, and also

$$L := \mathbb{Q}(\sqrt{D}), \quad \theta := -\frac{b + \sqrt{D}}{2a}, \quad \mathfrak{b} := \mathbb{Z} + \mathbb{Z}\theta.$$

Further, if \mathcal{O}_L is the ring of integers of L and $\mathfrak{f} \subset \mathcal{O}_L$ is an integral ideal, we write $U(\mathfrak{f}) := \mathcal{O}_L^\times \cap (1 + \mathfrak{f})$, and $U_+(\mathfrak{f})$ for the totally positive elements of $U(\mathfrak{f})$, i.e., we have

$$U_+(\mathfrak{f}) = \{\varepsilon \in U(\mathfrak{f}) \mid N(\varepsilon) > 0\},$$

where $N(\cdot)$ denotes the norm of an element in L . For the sequel, we set $\mathfrak{f} := N\mathcal{O}_L$.

It is well-known that the assignment

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto cx^2 + (d-a)xy - by^2$$

defines a bijection $\beta_l : \text{sp}_l \xrightarrow{\sim} \mathcal{Q}_\Delta$. Then, for convenience, we will write $\mathcal{Q}_\Delta(\Gamma) := \beta_l(\text{sp}_l(\Gamma))$, and also $f_\gamma := \beta_l(\gamma)$ and $\gamma_f := \beta_l^{-1}(f)$, for $\gamma \in \text{sp}_l(\Gamma)$ and $f(x, y) \in \mathcal{Q}_\Delta(\Gamma)$, respectively. Furthermore, the

action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on \mathcal{Q}_Δ , namely $(f \circ \gamma)(x, y) := f((xy)\gamma^t)$, descends to an action of Γ on $\mathcal{Q}_\Delta(\Gamma)$.

Let $\gamma = \gamma_l(a, b, c) \in \mathrm{sp}_l(\Gamma)$. Denote by Γ_{f_γ} the stabilizer of $f_\gamma(x, y) \in \mathcal{Q}_\Delta(\Gamma)$ under the action of Γ . Since $f_\gamma = Ng_\gamma$ with $g_\gamma(x, y) = ax^2 + bxy + cy^2$, we have $\Gamma_{f_\gamma} = \Gamma_{g_\gamma}$. Therefore, it can be verified that

$$\Gamma_{f_\gamma} = \left\{ \begin{pmatrix} \frac{t-bv}{2} & -cv \\ av & \frac{t+bv}{2} \end{pmatrix} \mid t, v \in \mathbb{Z}, t^2 - Dv^2 = 4, v \equiv 0 \pmod{N}, \frac{t-v}{2} \equiv 1 \pmod{N} \right\}.$$

Now, given $u \in U$, define

$$M_u^{f_\gamma} := \{(m, n) \in M(u) \mid f_\gamma \cdot (m, n) > 0\}.$$

Then, the group Γ_{f_γ} acts on $M_u^{f_\gamma}$ by right multiplication. Indeed, if $(m', n') = (m, n)\delta$ with $(m, n) \in M_u^{f_\gamma}$ and $\delta \in \Gamma_{f_\gamma}$, then $(m', n') \in M(u)$ holds, and since

$$f_\gamma(n', -m') = f_\gamma((n-m)(\delta^{-1})^t) = (f_\gamma \circ \delta^{-1})(n, -m) = f_\gamma(n, -m) > 0,$$

the claim follows. Let $\overline{M}_u^{f_\gamma}$ be the set of Γ_{f_γ} -orbits of $M_u^{f_\gamma}$, i.e., we have $\overline{M}_u^{f_\gamma} := M_u^{f_\gamma} / \Gamma_{f_\gamma}$.

Lemma 3. *Let $u \in U$ and $\gamma \in \mathrm{sp}_l(\Gamma)$. Suppose that $\mathfrak{b} = (u/N)\mathfrak{a}^{-1}\mathfrak{f}$, for some non-zero integral ideal $\mathfrak{a} \subset \mathcal{O}_L$. Then, the assignment $(u/N + \mathfrak{b})_+ \rightarrow M_u^{f_\gamma}$ given by $u/N + n' + m'\theta \mapsto (Nm', u + Nn')$ defines a bijection whose inverse is given by $(m, n) \mapsto (1/N)(n + m\theta)$. In particular, we have the bijection*

$$\overline{M}_u^{f_\gamma} \simeq (u/N + \mathfrak{b})_+ / U_+(\mathfrak{f}).$$

Proof. For the well-definition, note that if $\xi = u/N + n' + m'\theta$, then

$$\xi \bar{\xi} = \frac{1}{aN^3} f_\gamma(u + Nn', -Nm'),$$

which in turn implies that $f_\gamma(u + Nn', -Nm') > 0$, since $N(\xi) = \xi \bar{\xi} > 0$. Therefore, the pair $(Nm', u + Nn') \in M_u^{f_\gamma}$. The injectivity and surjectivity are immediate.

Next, for the second statement of the lemma, it turns out that the map

$$\begin{pmatrix} \frac{t-bv}{2} & -cv \\ av & \frac{t+bv}{2} \end{pmatrix} \mapsto \frac{t + v\sqrt{D}}{2}$$

establishes the bijection

$$\Gamma_{f_\gamma} \simeq U_+(\mathfrak{f})$$

(see [12, Proposition 4.2.11] for further details). This concludes the proof of the lemma. \square

Next, we proceed to define a key zeta function and study its properties. Let $\gamma \in \mathrm{sp}_l(\Gamma)$ and $u \in U$. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the zeta function associated to the pair (γ, u) is defined by

$$\zeta_{\gamma, u}(s) := \sum_{(m, n) \in \overline{M}_u^{f_\gamma}} f_\gamma(n, -m)^{-s}.$$

We claim that the function $\zeta_{\gamma, u}(s)$ possesses a meromorphic continuation to the complex plane having a pole at $s = 1$. In order to prove the claim, we will relate $\zeta_{\gamma, u}(s)$ to the zeta function associated to a narrow ideal class of L .

Let us consider the ray $\mathfrak{m} := \mathfrak{m}_\infty \mathfrak{f}$, where \mathfrak{m}_∞ denotes the ray consisting of all embeddings $L \hookrightarrow \mathbb{R}$, and denote by $Cl_{\mathfrak{m}}(L)$ the narrow-ray ideal class group of L . Given $\mathfrak{C} \in Cl_{\mathfrak{m}}(L)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the partial zeta function $\zeta(s, \mathfrak{C})$ is defined by

$$\zeta(s, \mathfrak{C}) := \sum_{\substack{\mathfrak{c} \in \mathfrak{C} \\ \mathfrak{c} \neq 0}} \frac{1}{(\mathcal{N}\mathfrak{c})^s},$$

where the sum runs over the non-zero integral ideals \mathfrak{c} contained in \mathfrak{C} and $\mathcal{N}\mathfrak{c}$ denotes the norm of the ideal \mathfrak{c} . Suppose that $\mathfrak{C} \in Cl_{\mathfrak{m}}(L)$ and $\mathfrak{a} \in \mathfrak{C}$, a nonzero integral ideal are given. Let $\tau \in L^\times$ such that $N(\tau) > 0$ and set $\mathfrak{b}' = (\tau)\mathfrak{a}^{-1}\mathfrak{f}$. Then, the partial zeta function $\zeta(s, \mathfrak{C})$ satisfies the following identity

$$\zeta(s, \mathfrak{C}) = (\mathcal{N}\mathfrak{f})^{-s} \sum_{\beta \in (\tau + \mathfrak{b}')_+ / U_+(\mathfrak{f})} \left(\frac{\mathcal{N}(\beta)}{\mathcal{N}\mathfrak{b}'} \right)^{-s}$$

(see, e.g., [26]), where $\mathcal{N}(\beta)$ denotes the norm of the principal ideal generated by the element β .

Proposition 4. *Let $u \in U$ and $\gamma \in \mathrm{sp}_l(\Gamma)$. Suppose that $\gamma = \gamma_l(a, b, c)$ is given by (15). Then, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, there exists a narrow ray ideal class $\mathfrak{C} \in Cl_{\mathfrak{m}}(L)$ such that*

$$\zeta_{\gamma, u}(s) = \left(\frac{1}{N} \right)^s \zeta(s, \mathfrak{C}).$$

Furthermore, $\zeta_{\gamma, u}(s)$ admits a meromorphic continuation to the complex plane having a pole at $s = 1$, whose pole is given by

$$\mathrm{res}_{s=1}(\zeta_{\gamma, u}(s)) = \frac{\log(\varepsilon_\gamma)}{N^3 \sqrt{D}},$$

where ε_γ denotes a generator of $U_+(\mathfrak{f})$.

Proof. We know that, if $\xi := u/N + n + m\theta \in (u/N + \mathfrak{b})_+$, then

$$f_\gamma(u + nN, -Nm) = N^3 \frac{\mathcal{N}(\xi)}{\mathcal{N}\mathfrak{b}} = N \left(\mathcal{N}\mathfrak{f} \frac{\mathcal{N}(\xi)}{\mathcal{N}\mathfrak{b}} \right).$$

Therefore, we have

$$\begin{aligned} \zeta_{\gamma, u}(s) &= \sum_{(m, n) \in \overline{M}_u^{f_\gamma}} f_\gamma(n, -m)^{-s} \\ &= \left(\frac{1}{N} \right)^s (\mathcal{N}\mathfrak{f})^{-s} \sum_{\xi \in (\frac{u}{N} + \mathfrak{b})_+ / U_+(\mathfrak{f})} \left(\frac{\mathcal{N}(\xi)}{\mathcal{N}\mathfrak{b}} \right)^{-s} \\ &= \left(\frac{1}{N} \right)^s \zeta(s, \mathfrak{C}), \end{aligned}$$

where \mathfrak{C} is the narrow-ray ideal class of the fractional ideal $\mathfrak{f}\mathfrak{b}^{-1}(u/N)$. Consequently, the function $\zeta_{\gamma, u}(s)$ inherits the analytical properties of $\zeta(s, \mathfrak{C})$, in particular, the meromorphic continuation to the complex plane. This concludes the proof. \square

4.2. Computation of the constant $\mathcal{R}_\infty^{\mathrm{hyp}}$. Given $u \in U$, let us consider the following subsets of $\mathcal{Q}_\Delta(\Gamma) \times M(u)$

$$\begin{aligned} S_\Gamma^+(u) &:= \{(f; (m, n)) \mid (m, n) \in M_u^f, f \in \mathcal{Q}_\Delta(\Gamma)\}, \\ S_\Gamma^-(u) &:= \{(f; (m, n)) \mid (m, n) \in M_u^{-f}, f \in \mathcal{Q}_\Delta(\Gamma)\}. \end{aligned}$$

It can be verified that the sets $S_\Gamma^+(u)$ and $S_\Gamma^-(u)$ remain invariant under the diagonal action of Γ . Therefore, by the ‘‘allgemeine Prinzip’’ from [28], we have

$$\begin{aligned} S_\Gamma^+(u)/\Gamma &\simeq \bigsqcup_{\gamma \in \mathrm{sp}_l(\Gamma)/\Gamma} \overline{M}_u^{f_\gamma}, \\ S_\Gamma^-(u)/\Gamma &\simeq \bigsqcup_{\gamma \in \mathrm{sp}_l(\Gamma)/\Gamma} \overline{M}_u^{(-f_\gamma)}. \end{aligned}$$

Further, we have $\mathcal{Q}_\Delta(\Gamma) \times M(u) = S_\Gamma^+(u) \sqcup S_\Gamma^-(u)$.

Now, we conveniently define the integrals

$$J_k^+ := \int_{\mathcal{F}_\Gamma} \left(\sum_{(f;(m,n)) \in S_\Gamma^+(u)} \nu_k(\gamma_f; z) \frac{\text{Im}(z)^s}{|mz + n|^{2s}} \right) \mu_{\text{hyp}}(z),$$

$$J_k^- := \int_{\mathcal{F}_\Gamma} \left(\sum_{(f;(m,n)) \in S_\Gamma^-(u)} \nu_k(\gamma_f; z) \frac{\text{Im}(z)^s}{|mz + n|^{2s}} \right) \mu_{\text{hyp}}(z),$$

and the matrix $\lambda(l)$ by

$$\lambda(l) := \begin{pmatrix} l/2 & l^2/4 - 1 \\ 1 & l/2 \end{pmatrix}.$$

Lemma 5. *Let $u \in U$, $l \in \mathbb{Z}$ such that $|l| > 2$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. Then, the following identity holds*

$$J_k^+ = \left(\int_{\mathbb{H}} \nu_k(\lambda(l); z) \text{Im}(z)^s \mu_{\text{hyp}}(z) \right) \sum_{\gamma \in \text{sp}_l(\Gamma)/\Gamma} \zeta_{\gamma, u}(s),$$

$$J_k^- = \left(\int_{\mathbb{H}} \nu_k(\lambda(-l); z) \text{Im}(z)^s \mu_{\text{hyp}}(z) \right) \sum_{\gamma \in \text{sp}_l(\Gamma)/\Gamma} \zeta_{\gamma, u}(s),$$

where $\nu_k(\gamma; z)$ is given in Section 2.4.

Proof. For the proof of the lemma, observe that it suffices to prove the statement for the integral J_k^+ , since we have $-f_\gamma = f_{-\gamma} = f_{\gamma^{-1}}$. Let $(f; (m, n)) \in S_\Gamma^+(u)$ with $f(x, y) = cx^2 + (d-a)xy - by^2$, therefore $\gamma_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Put $(m_\delta, n_\delta) = (m, n)\delta^t$, for $\delta \in \Gamma$, and set

$$M := \frac{1}{f(n, -m)} \begin{pmatrix} n & -bm - n(d-a)/2 \\ -m & cn - m(d-a)/2 \end{pmatrix}.$$

Using the identities

$$\nu_k(\delta^{-1}\gamma\delta; z) = \nu_k(\gamma; \delta z), \quad \frac{\text{Im}(z)}{|m_\delta z + n_\delta|^2} = \frac{\text{Im}(\delta z)}{|m(\delta z) + n|^2},$$

which hold for $\delta \in \text{PSL}_2(\mathbb{R})$, we obtain

$$\begin{aligned} \sum_{(f;(m,n)) \in S_\Gamma^+(u)} \nu_k(\gamma_f; z) \frac{\text{Im}(z)^s}{|mz + n|^{2s}} &= \sum_{(f;(m,n)) \in S_\Gamma^+(u)/\Gamma} \sum_{\delta \in \Gamma} \nu_k(\delta^{-1}\gamma_f\delta; z) \frac{\text{Im}(z)^s}{|m_\delta z + n_\delta|^{2s}} \\ &= \sum_{(f;(m,n)) \in S_\Gamma^+(u)/\Gamma} \sum_{\delta \in \Gamma} \nu_k(\gamma_f; \delta z) \frac{\text{Im}(\delta z)^s}{|m(\delta z) + n|^{2s}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} J_k^+ &= \int_{\mathcal{F}_\Gamma} \left(\sum_{(f;(m,n)) \in S_\Gamma^+(u)} \nu_k(\gamma_f; z) \frac{\text{Im}(z)^s}{|mz + n|^{2s}} \right) \mu_{\text{hyp}}(z) \\ &= \sum_{(f;(m,n)) \in S_\Gamma^+(u)/\Gamma} \left(\int_{\mathcal{F}_\Gamma} \sum_{\delta \in \Gamma} \nu_k(\gamma_f; \delta z) \frac{\text{Im}(\delta z)^s}{|m(\delta z) + n|^{2s}} \mu_{\text{hyp}}(z) \right) \\ &= \sum_{(f;(m,n)) \in S_\Gamma^+(u)/\Gamma} \left(\int_{\mathbb{H}} \nu_k(\gamma_f; z) \frac{\text{Im}(z)^s}{|mz + n|^{2s}} \mu_{\text{hyp}}(z) \right). \end{aligned}$$

Finally, applying the change of variables $z \mapsto Mz$ in the last integral, and using the identities

$$\frac{\text{Im}(Mz)}{|m(Mz) + n|^2} = \frac{\text{Im}(z)}{|mz + n|^2}, \quad M^{-1}\gamma_f M = \lambda(l),$$

we deduce

$$J_k^+ = \left(\int_{\mathbb{H}} \nu_k(M^{-1}\gamma_f M; z) \text{Im}(z)^s \mu_{\text{hyp}}(z) \right) \sum_{(f;(m,n)) \in S_\Gamma^+(u)/\Gamma} f(n, -m)^{-s}.$$

This concludes the proof of the lemma. □

Next, let us set

$$I_k(s; l) := \int_{\mathbb{H}} \left(\nu_k(\lambda(l); z) + \nu_k(\lambda(-l); z) \right) \text{Im}(z)^s \mu_{\text{hyp}}(z).$$

Proposition 6. *Let $N \geq 3$ be an odd square-free integer and $s \in \mathbb{C}$ with $1 < \text{Re}(s) < \inf\{A, 3/2\}$. Then the following identity holds*

$$\mathcal{R}_{\infty}[\mathcal{H}](s) = \frac{1}{N^s} \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} (I_2(s; l) - I_0(s; l)) \sum_{u \in U} \left(D_u(s) \sum_{\gamma \in \text{sp}_l(\Gamma)/\Gamma} \zeta_{\gamma, u}(s) \right),$$

where $D_u(s)$ is the Dirichlet series given by (2). Furthermore, we have

$$\mathcal{R}_{\infty}^{\text{hyp}} = \frac{v_{\Gamma}^{-1}}{2} \left(\lim_{s \rightarrow 1} \left(\frac{Z'_{\Gamma}(s)}{Z_{\Gamma}(s)} - \frac{1}{s-1} \right) - T + 1 + o(1) \right),$$

as $T \rightarrow \infty$; here $Z_{\Gamma}(s)$ stands for the Selberg zeta function of Γ .

Proof. For the proof, we write $K_k^{\text{hyp}}(z)$ as follows

$$K_k^{\text{hyp}}(z) = \sum_{\substack{\gamma \in \Gamma \\ |\text{tr}(\gamma)| > 2}} \nu_k(\gamma; z) = \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{\gamma \in \text{sp}_l(\Gamma)} \nu_k(\gamma; z).$$

Then, note that

$$K_k^{\text{hyp}}(z) E_{\infty}(z, s) = \frac{1}{N^s} \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{u \in U} D_u(s) \sum_{(\gamma; (m, n))} \nu_k(\gamma; z) \frac{\text{Im}(z)^s}{|mz + n|^{2s}},$$

where the innermost sum runs over the elements of $\text{sp}_l(\Gamma) \times M(u)$, and since $\text{sp}_l(\Gamma) \simeq \mathcal{Q}_{\Delta}(\Gamma)$, we have

$$K_k^{\text{hyp}}(z) E_{\infty}(z, s) = \frac{1}{N^s} \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{u \in U} D_u(s) \sum_{(f; (m, n))} \nu_k(\gamma_f; z) \frac{\text{Im}(z)^s}{|mz + n|^{2s}}.$$

Thus, the previous identity yields

$$\begin{aligned} \mathcal{R}_{\infty}[K_k^{\text{hyp}}](s) &= \int_{\mathcal{F}_{\Gamma}} K_k^{\text{hyp}}(z) E_{\infty}(z, s) \mu_{\text{hyp}}(z) \\ &= \frac{1}{N^s} \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{u \in U} D_u(s) \int_{\mathcal{F}_{\Gamma}} \left(\sum_{(f; (m, n))} \nu_k(\gamma_f; z) \frac{y^s}{|mz + n|^{2s}} \right) \mu_{\text{hyp}}(z). \end{aligned}$$

Using the decomposition $\mathcal{Q}_{\Delta}(\Gamma) \times M(u) \simeq S_{\Gamma}^{+}(u) \sqcup S_{\Gamma}^{-}(u)$ and Lemma 5, we get

$$\begin{aligned} \mathcal{R}_{\infty}[K_k^{\text{hyp}}](s) &= \frac{1}{N^s} \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{u \in U} D_u(s) (J_k^{+} + J_k^{-}) \\ &= \frac{1}{N^s} \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} I_k(s; l) \sum_{u \in U} \left(D_u(s) \sum_{\gamma \in \text{sp}_l(\Gamma)/\Gamma} \zeta_{\gamma, u}(s) \right). \end{aligned}$$

This proves the first assertion.

For the second part of the proposition, observe that

$$\sum_{u \in U} D_u(s) \zeta_{\gamma, u}(s) = \left(\sum_{u \in U} D_u(s) \right) \left(\frac{\log(\varepsilon_{\Gamma})}{N^3 \sqrt{D}} \frac{1}{s-1} + C + O(s-1) \right).$$

Then, by [3, Proposition 3.3.2], namely

$$I_2(s; l) - I_0(s; l) = \pi A_l(T)(s-1) + O((s-1)^2),$$

where

$$A_l(T) := -\frac{1}{2\eta_l} + \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h_T(r)}{\frac{1}{4} + r^2} e^{-2ir \log(\eta_l)} dr$$

with $l > 2$ and $\eta_l := (l + \sqrt{l^2 - 4})/2$, and by virtue of identity (3), we deduce the following expansion at $s = 1$

$$\begin{aligned} \mathcal{R}_{\infty}[\mathcal{H}](s) &= \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{u \in U} \sum_{\gamma \in \text{sp}_l(\Gamma)/\Gamma} \frac{1}{N^s} (I_2(s; l) - I_0(s; l)) D_u(s) \zeta_{\gamma, u}(s) \\ &= \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{\gamma \in \text{sp}_l(\Gamma)/\Gamma} \frac{\log(\varepsilon_{\gamma})}{\sqrt{l^2 - 4}} \frac{A_l(T)}{v_{\Gamma}} + O(s - 1). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathcal{R}_{\infty}^{\text{hyp}} &= \mathcal{R}_{\infty}[\mathcal{H}](1) \\ &= \frac{1}{v_{\Gamma}} \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{\gamma \in \text{sp}_l(\Gamma)/\Gamma} \frac{\log(\varepsilon_{\gamma})}{\sqrt{l^2 - 4}} A_l(T). \end{aligned}$$

Now, by [3, Proposition 3.3.3], namely

$$A_l(T) = -\frac{1}{2} \int_0^T g(t, 2 \log(\eta_l)) dt$$

for $l > 2$ with

$$g(t, u) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{t}{4} - \frac{u^2}{4t}},$$

we get

$$\begin{aligned} \mathcal{R}_{\infty}^{\text{hyp}} &= -\frac{v_{\Gamma}^{-1}}{2} \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{\gamma \in \text{sp}_l(\Gamma)/\Gamma} \frac{\log(\varepsilon_{\gamma})}{\sqrt{l^2 - 4}} \int_0^T g(t, 2 \log(\eta_l)) dt \\ &= -\frac{v_{\Gamma}^{-1}}{2} \int_0^T \Theta_{\Gamma}(t) dt, \end{aligned}$$

where

$$\Theta_{\Gamma}(t) := \sum_{\substack{l \in \mathbb{Z} \\ |l| > 2}} \sum_{\gamma \in \text{sp}_l(\Gamma)/\Gamma} \frac{\log(\varepsilon_{\gamma})}{\sqrt{l^2 - 4}} g(t, 2 \log(\eta_l)).$$

Finally, since

$$\int_0^T \Theta_{\Gamma}(t) dt = T - \lim_{s \rightarrow 1} \left(\frac{Z'_{\Gamma}(s)}{Z_{\Gamma}(s)} - \frac{1}{s - 1} \right) - 1 + o(1)$$

holds as $T \rightarrow \infty$, the assertion follows. \square

5. THE PARABOLIC CONTRIBUTION OF \mathcal{R}_{∞}

Recall from identity (12) that $\mathcal{R}_{\infty}^{\text{par}}$ is the parabolic contribution of the Rankin–Selberg constant \mathcal{R}_{∞} at ∞ . In this chapter, we compute $\mathcal{R}_{\infty}^{\text{par}}$ explicitly (see Proposition 11). To do so, we first note that

$$\mathcal{R}_{\infty}[\mathcal{P}](s) = \int_0^{\infty} p(y) y^{s-2} dy,$$

where $p(y)$ is the 0-th coefficient in the Fourier expansion of $\mathcal{P}(\sigma_{\infty} z)$. Further, since

$$p(y) = \sum_{j=1}^4 (p_j(y; 2) - p_j(y; 0)),$$

where $p_j(y; k)$ is given by (18), it suffices to compute the integrals

$$\mathcal{M}_j(s) := \int_0^\infty (p_j(y; 2) - p_j(y; 0))y^{s-2}dy,$$

which is done in Section 5.1. Then, in Section 5.2, we gather the identities for $\mathcal{M}_j(s)$ and conclude the computation of $\mathcal{R}_\infty^{\text{par}}$ in Proposition 11.

In this section, we will consider the following settings. We define $\lambda(2) := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\lambda(-2) := \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, and let $I_k(s; 2)$ be the integral given by

$$(16) \quad I_k(s; 2) := \int_{\mathbb{H}} \left(\nu_k(\lambda(2); z) + \nu_k(\lambda(-2); z) \right) \text{Im}(z)^s \mu_{\text{hyp}}(z).$$

From [3, Proposition 3.3.4], we have the Laurent expansion at $s = 1$ of the difference $I_2(s; 2) - I_0(s; 2)$, namely

$$(17) \quad I_2(s; 2) - I_0(s; 2) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \left(A_2(T)(s - 1) + C_1(T)(s - 1)^2 + O((s - 1)^3) \right),$$

where

$$A_2(T) := -\frac{1}{2} + \frac{1}{4\pi} \int_{-\infty}^\infty \frac{h_T(r)}{\frac{1}{4} + r^2} dr.$$

This expansion will be useful later on. Next, recall that ϕ_k stands for the inverse Selberg/Harish-Chandra transform of weight k of the function $h_T(r)$ given by (8). Then, we set

$$\Psi_k^\pm(y) := \Psi_k(y) + \Psi_k(-y),$$

where $\Psi_k(y)$ is the function given by

$$\Psi_k(y) := \int_{-\infty}^\infty \phi_k(u^2) \left(\frac{1 - iu}{1 + iu} \right)^{k/2} e^{2i\pi uy} du.$$

We will write

$$\tilde{E}_{\infty, k}(\sigma_\infty z, s) := E_{\infty, k}(\sigma_\infty z, s) - y^s - \varphi_{\infty, k}(s),$$

where $E_{\infty, 0}(z, s)$ and $\varphi_{\infty, 0}(s)$ stands for $E_\infty(z, s)$ and $\varphi_{\infty}(s)$, respectively. For convenience, we define the following integrals

$$(18) \quad \begin{aligned} p_1(y; k) &:= \int_{-1/2}^{1/2} \left(\sum_{\substack{\gamma \in \Gamma \setminus \Gamma_\infty \\ |\text{tr}(\gamma)|=2}} \nu_k(\gamma; \sigma_\infty z) \right) dx; \\ p_2(y; k) &:= \left(\sum_{\gamma \in \Gamma_\infty} \nu_k(\gamma; \sigma_\infty z) \right) - \frac{y}{2\pi} \int_{-\infty}^\infty h_T(r) dr; \\ p_3(y; k) &:= -\frac{y}{2\pi} \int_{-\infty}^\infty h_T(r) \left(\frac{\frac{1}{2} + ir}{\frac{1}{2} - ir} \right)^{k/2} \varphi_{\infty} \left(\frac{1}{2} - ir \right) y^{2ir} dr - \frac{2-k}{2} v_\Gamma^{-1}; \\ p_4(y; k) &:= -\int_{-1/2}^{1/2} \left(\frac{1}{4\pi} \int_{-\infty}^\infty h_T(r) \sum_{q \in P_\Gamma} \left| \tilde{E}_{\infty, k} \left(\sigma_\infty z, \frac{1}{2} + ir \right) \right|^2 dr \right) dx; \\ p_4^*(y) &:= -\frac{1}{4\pi} \sum_{q \in P_\Gamma} \int_{-\infty}^\infty \left(\frac{h_T(r)}{\frac{1}{4} + r^2} \mathcal{R}_\infty \left[\left| \tilde{E}_{\infty, k} \left(\sigma_\infty z, \frac{1}{2} + ir \right) \right|^2 \right] (s) \right) dr; \end{aligned}$$

where the sum in the integrand of the function $p_1(y; k)$ runs over all elements of Γ that are not in Γ_∞ .

5.1. Preparatory lemmas.

Lemma 7. *Let $N \geq 3$ be an odd square-free integer and $s \in \mathbb{C}$ with $1 < \operatorname{Re}(s) < A$. Then, the following identity holds*

$$\mathcal{M}_1(s) = \frac{2\zeta(s)\zeta(2s-1)}{\zeta(2s)N^{2s-1}}(I_2(s; 2) - I_0(s; 2)),$$

where $\zeta(s)$ denotes the Riemann zeta function and $I_k(s; 2)$ is given by (16). Furthermore, the Laurent expansion of $\mathcal{M}_1(s)$ at $s = 1$ is given by

$$\frac{12A_2(T)}{\pi N} \frac{1}{s-1} + \frac{12}{\pi N} \left(C_1(T) + A_2(T)(2\mathcal{C} + \gamma_{\text{EM}} - 2\log(N)) \right) + O(s-1),$$

where γ_{EM} denotes the Euler–Mascheroni constant and $C_1(T)$ is a constant that depends only on the fixed positive real T .

Proof. By definition, we have

$$\mathcal{M}_1(s) = \int_0^\infty (p_1(y; 2) - p_1(y; 0))y^{s-2}dy.$$

In order to compute $\mathcal{M}_1(s)$, let us consider the following integral

$$\int_0^\infty p_1(y; k)y^{s-2}dy = \int_0^\infty \int_{-1/2}^{1/2} \left(\sum_{\substack{\gamma \in \Gamma \setminus \Gamma_\infty \\ |\operatorname{tr}(\gamma)|=2}} \nu_k(\gamma; \sigma_\infty z) \right) y^{s-2} dx dy.$$

Put $B := \{n(b) \mid b \in \mathbb{Z}\}$. Then, note that we have bijections

$$N\mathbb{Z} \setminus \{0\} \times (B \setminus \operatorname{PSL}_2(\mathbb{Z})) \xrightarrow{\sim} \{\gamma \in \Gamma \setminus \Gamma_\infty \mid \operatorname{tr}(\gamma) = 2\},$$

$$N\mathbb{Z} \setminus \{0\} \times -(B \setminus \operatorname{PSL}_2(\mathbb{Z})) \xrightarrow{\sim} \{\gamma \in \Gamma \setminus \Gamma_\infty \mid \operatorname{tr}(\gamma) = -2\},$$

given by

$$\begin{aligned} \left(m, B \begin{pmatrix} * & * \\ c & d \end{pmatrix} \right) &\mapsto \gamma_+(m; c, d) := \begin{pmatrix} 1 + mcd & md^2 \\ -mc^2 & 1 - mcd \end{pmatrix}, \\ \left(m, B \begin{pmatrix} * & * \\ c & d \end{pmatrix} \right) &\mapsto \gamma_-(m; c, d) := \begin{pmatrix} -1 + mcd & md^2 \\ -mc^2 & -1 - mcd \end{pmatrix}, \end{aligned}$$

respectively. With these bijections, we obtain

$$\begin{aligned} \int_0^\infty p_1(y; k)y^{s-2}dy &= \int_0^\infty \int_{-1/2}^{1/2} \left(\sum_{(m, B \begin{pmatrix} * & * \\ c & d \end{pmatrix})}^+ \nu_k(\gamma_+(m; c, d); \sigma_\infty z) \right. \\ &\quad \left. + \sum_{(m, B \begin{pmatrix} * & * \\ c & d \end{pmatrix})}^- \nu_k(\gamma_-(m; c, d); \sigma_\infty z) \right) y^{s-2} dx dy, \end{aligned}$$

where the first sum runs over all pairs $(m, B \begin{pmatrix} * & * \\ c & d \end{pmatrix})$ in $N\mathbb{Z} \setminus \{0\} \times (B \setminus \operatorname{PSL}_2(\mathbb{Z}))$, and the second sum runs over all pairs $(m, B \begin{pmatrix} * & * \\ c & d \end{pmatrix})$ in $N\mathbb{Z} \setminus \{0\} \times -(B \setminus \operatorname{PSL}_2(\mathbb{Z}))$. Then it suffices to compute

$$P_k^+ := \int_0^\infty \int_{-1/2}^{1/2} \left(\sum_{(m, B \begin{pmatrix} * & * \\ c & d \end{pmatrix})}^+ \nu_k(\gamma_+(m; c, d); \sigma_\infty z) \right) y^{s-2} dx dy.$$

Observe that,

$$P_k^+ = \int_0^\infty \int_{-1/2}^{1/2} \left(\sum_{(m, B \begin{pmatrix} * & * \\ c & d \end{pmatrix})}^+ \sum_{n \in \mathbb{Z}} \nu_k(\gamma_+(m; c, d); \sigma_\infty z) \right) y^{s-2} dx dy,$$

where now, the first sum runs over all pairs $(m, B(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}))$ representing a class in $N\mathbb{Z} \setminus \{0\} \times (B \backslash \mathrm{PSL}_2(\mathbb{Z}) / \Gamma_\infty)$; further, by convergence, we can rewrite P_k^+ as follows

$$P_k^+ = \sum'_{(m, B(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}))} \int_0^\infty \left(\sum_{n \in \mathbb{Z}} \int_{-1/2}^{1/2} \nu_k(\gamma_+(m; c, d); N(x+n) + iNy) dx \right) y^{s-2} dy.$$

Now, by a suitable change of variables, we obtain

$$\begin{aligned} P_k^+ &= \frac{1}{N} \sum'_{(m, B(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}))} \int_0^\infty \int_{-\infty}^\infty \nu_k(\gamma_+(m; c, d); x + iNy) y^{s-2} dx dy \\ &= \frac{1}{N^s} \sum'_{(m, B(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}))} \int_{\mathbb{H}} \nu_k(\gamma_+(m; c, d); z) \mathrm{Im}(z)^s \mu_{\mathrm{hyp}}(z). \end{aligned}$$

Then, by [3, Lemme 3.2.12], namely

$$\int_{\mathbb{H}} \nu_k(\gamma_+(m; c, d); z) \mathrm{Im}(z)^s \mu_{\mathrm{hyp}}(z) = \frac{1}{|mc^2|^s} \int_{\mathbb{H}} \nu_k(\lambda(2); z) \mathrm{Im}(z)^s \mu_{\mathrm{hyp}}(z),$$

we get

$$P_k^+ = \frac{1}{N^s} \left(\int_{\mathbb{H}} \nu_k(\lambda(2); z) \mathrm{Im}(z)^s \mu_{\mathrm{hyp}}(z) \right) \sum'_{(m, B(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}))} \frac{1}{|mc^2|^s},$$

Similarly, if we set

$$P_k^- := \int_0^\infty \int_{-1/2}^{1/2} \left(\sum_{(m, B(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}))} \nu_k(\gamma_+(m; c, d); \sigma_\infty z) \right) y^{s-2} dx dy,$$

we will obtain

$$P_k^- = \frac{1}{N^s} \left(\int_{\mathbb{H}} \nu_k(\lambda(-2); z) \mathrm{Im}(z)^s \mu_{\mathrm{hyp}}(z) \right) \sum'_{(m, B(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}))} \frac{1}{|mc^2|^s}.$$

Therefore,

$$\int_0^\infty p_1(y; k) y^{s-2} dy = \frac{1}{N^s} I_k(s; 2) \sum'_{(m, B(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}))} \frac{1}{|mc^2|^s},$$

and since

$$\begin{aligned} \sum'_{(m, B(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}))} \frac{1}{|mc^2|^s} &= \sum_{m \in N\mathbb{Z} \setminus \{0\}} \frac{1}{|m|^s} \left(\sum_{c=1}^\infty \frac{1}{c^{2s}} \sum_{\substack{d \bmod c \\ (\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}) \in B \backslash \mathrm{PSL}_2(\mathbb{Z}) / \Gamma_\infty}} 1 \right) \\ &= \frac{2\zeta(s)\zeta(2s-1)}{\zeta(2s)N^{s-1}}, \end{aligned}$$

it follows that

$$\mathcal{M}_1(s) = 2N^{1-2s} \frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)} (I_2(s; 2) - I_0(s; 2)).$$

This proves the first assertion of the lemma. For the second part of the lemma, use (17) together with the well-known Laurent expansion

$$\sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} = \frac{3/\pi}{s-1} + \frac{6}{\pi} \mathcal{C} + O(s-1)$$

at $s = 1$ with $\mathcal{C} := 1 - \log(4\pi) + \zeta'(-1)/\zeta(-1)$. This concludes the proof. \square

Lemma 8. *Let $N \geq 3$ be an odd square-free integer and $s \in \mathbb{C}$ with $1 < \operatorname{Re}(s) < A$. Then, the following identity holds*

$$\mathcal{M}_2(s) = 2\zeta(s) \left(\int_0^\infty \Psi_2^\pm(2y)y^{s-1}dy - \int_0^\infty \Psi_0^\pm(2y)y^{s-1}dy \right).$$

Furthermore, the Laurent expansion of $\mathcal{M}_2(s)$ at $s = 1$ is given by

$$\frac{C_2(T) + (4\pi)^{-1}}{s-1} + \left(\frac{1 - \log(4\pi)}{4\pi} + \gamma_{\text{EM}} C_2(T) + C_3(T) \right) + O(s-1),$$

where $C_2(T)$ and $C_3(T)$ are constants that depend only on the fixed positive real T .

Proof. By definition, we have

$$\mathcal{M}_2(s) = \int_0^\infty (p_2(y; 2) - p_2(y; 0))y^{s-2}dy.$$

In order to compute $\mathcal{M}_2(s)$, let us consider the integral

$$\int_0^\infty p_2(y; k)y^{s-2}dy = \int_0^\infty \left(\sum_{\gamma \in \Gamma_\infty} \nu_k(\gamma, \sigma_\infty z) - \frac{y}{2\pi} \int_{-\infty}^\infty h_T(r)dr \right) y^{s-2}dy.$$

We claim that

$$\sum_{\gamma \in \Gamma_\infty} \nu_k(\gamma; \sigma_\infty z) = \frac{y}{2\pi} \int_{-\infty}^\infty h_T(r)dr + 2y \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \Psi_k(2ny).$$

Indeed, since $\gamma \in \Gamma_\infty$ has the form $\gamma = \begin{pmatrix} 1 & nN \\ 0 & 1 \end{pmatrix}$, for some $n \in \mathbb{Z}$, we have $j_\gamma(z; k) = 1$. This implies

$$\begin{aligned} \nu_k(\gamma; \sigma_\infty z) &= \pi_k(Nz, N(z+b)) \\ &= \left(\frac{1 - i(n/2y)}{1 + i(n/2y)} \right)^{k/2} \phi_k \left(\frac{n^2}{4y^2} \right), \end{aligned}$$

where for the second equality, we used the definition of $\pi_k(z, w)$ and $u(z, w)$ given in Section 2.3. Therefore, we get the identities

$$\begin{aligned} \sum_{\gamma \in \Gamma_\infty} \nu_k(\gamma; \sigma_\infty z) &= \sum_{n \in \mathbb{Z}} \left(\frac{1 - i(n/2y)}{1 + i(n/2y)} \right)^{k/2} \phi_k \left(\frac{n^2}{4y^2} \right) \\ &= \sum_{n \in \mathbb{Z}} \int_{-\infty}^\infty \phi_k \left(\frac{v^2}{4y^2} \right) \left(\frac{1 - i(v/2y)}{1 + i(v/2y)} \right)^{k/2} e^{2\pi inv} dv, \end{aligned}$$

where in the last equality we applied the Poisson summation formula. The claim follows by noting that

$$\int_{-\infty}^\infty \phi_k(v^2) \left(\frac{1 + iv}{1 - iv} \right)^{k/2} dv = \frac{1}{4\pi} \int_{-\infty}^\infty h_T(r)dr.$$

Consequently, we have

$$\begin{aligned} \int_0^\infty p_2(y; k)y^{s-2}dy &= 2 \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^\infty \Psi_k(2ny)y^{s-1}dy \\ &= 2 \sum_{n=1}^\infty \int_0^\infty (\Psi_k(2ny) + \Psi_k(-2ny))y^{s-1}dy \\ &= 2\zeta(s) \int_0^\infty \Psi_k^\pm(2y)y^{s-1}dy, \end{aligned}$$

and the first assertion of the lemma follows. For the second assertion, note that $\Psi_k^\pm(2y)$ is one-half the function $\psi(y) + \psi(-y)$ given by [3, p. 42] when $k = 2$, and given by [27, p. 326–329] when $k = 0$. Finally, the assertion of the lemma follows using [27, p. 389] and [3, Proposition 3.2.6]. This concludes the proof. \square

Lemma 9. *Let $N \geq 3$ be an odd square-free integer and $s \in \mathbb{C}$ with $1 < \operatorname{Re}(s) < A$. Then, the following identity holds*

$$\mathcal{M}_3(s) = \left(\frac{s}{1+s} \right) h_T \left(\frac{is}{2} \right) \varphi_{\infty\infty} \left(\frac{1+s}{2} \right).$$

Furthermore, the Laurent expansion of $\mathcal{M}_3(s)$ at $s = 1$ is given by

$$\frac{v_\Gamma^{-1}}{s-1} + \left(\frac{\mathcal{C}_{\infty\infty}}{2} + \frac{v_\Gamma^{-1}}{2}(T+1) \right) + O(s-1).$$

Proof. The proof of the first identity follows from an immediate extension of [3, Lemme 3.2.17, p. 44] and [21, Lemma 5.1.1., p. 137] to subgroups of the modular group of finite index. This is possible since it only uses general analytic properties of the scattering function. Next, for the Laurent expansion at $s = 1$, we just multiply the following Laurent expansions at $s = 1$

$$\frac{1}{2} h_T \left(\frac{is}{2} \right) = \frac{1}{2} + \frac{T}{4}(s-1) + O((s-1)^2);$$

$$\varphi_{\infty\infty} \left(\frac{1+s}{2} \right) = \frac{2v_\Gamma^{-1}}{s-1} + \mathcal{C}_{\infty\infty} + O(s-1);$$

$$\frac{2s}{s+1} = 1 + \frac{1}{2}(s-1) + O((s-1)^2).$$

This concludes the proof. \square

Lemma 10. *Let $N \geq 3$ be an odd square-free integer and $s \in \mathbb{C}$ with $1 < \operatorname{Re}(s) < A$. Then, the following identity holds*

$$\mathcal{M}_4(s) = s \left(\frac{s-1}{2} \right) \int_0^\infty p_4^*(y) y^{s-2} dy,$$

where $p_4^*(y)$ is given by (18). Furthermore, the Laurent expansion of $\mathcal{M}_4(s)$ at $s = 1$ is given by

$$C_4(T) + O(s-1),$$

where $C_4(T)$ is a constant that depends only on the fixed positive real T .

Proof. The lemma follows from an immediate extension of [21, Proposition 5.2.3, p. 141] to subgroups of the modular group of finite index. \square

5.2. Computation of the constant $\mathcal{R}_\infty^{\text{par}}$.

Proposition 11. *Let $N \geq 3$ be an odd square-free integer and $s \in \mathbb{C}$ with $1 < \operatorname{Re}(s) < \inf\{A, 3/2\}$. Then the following identity holds*

$$\begin{aligned} \mathcal{R}_\infty[\mathcal{P}](s) &= \frac{2\zeta(s)\zeta(2s-1)}{\zeta(2s)N^{2s-1}} (I_2(s; 2) - I_0(s; 2)) + \frac{s}{s+1} h_T \left(\frac{is}{2} \right) \varphi_{\infty\infty} \left(\frac{1+s}{2} \right) \\ &\quad + \zeta(s) \left(\int_0^\infty \Psi_2^\pm(y) y^{s-1} dy - \int_0^\infty \Psi_0^\pm(y) y^{s-1} dy \right) + s \left(\frac{s-1}{2} \right) \int_0^\infty p_4^*(y) y^{s-2} dy. \end{aligned}$$

Furthermore, the constant in the Laurent expansion at $s = 1$ of the previous expression is

$$\begin{aligned} \mathcal{R}_\infty^{\text{par}} &= \frac{12}{\pi N} \left(C_1(T) + A_2(T) \left(2\mathcal{C} + \gamma_{\text{EM}} - 2 \log(N) \right) \right) + \frac{1 - \log(4\pi)}{4\pi} \\ &\quad + \gamma_{\text{EM}} C_2(T) + C_3(T) + \frac{\mathcal{C}_{\infty\infty}}{2} + \frac{v_\Gamma^{-1}}{2}(T+1) + C_4(T). \end{aligned}$$

Proof. By summing up the Laurent expansions for each $\mathcal{M}_j(s)$ at $s = 1$ proved in the previous lemmas, we can immediately deduce the constant term $\mathcal{R}_\infty^{\text{par}}$, namely we have

$$\mathcal{R}_\infty^{\text{par}} = \frac{12}{\pi N} \left(C_1(T) + A_2(T) \left(2\mathcal{C} + \gamma_{\text{EM}} - 2 \log(N) \right) \right) + \frac{1 - \log(4\pi)}{4\pi}$$

$$+ \gamma_{\text{EM}} C_2(T) + C_3(T) + \frac{\mathcal{C}_{\infty\infty}}{2} + \frac{v_{\Gamma}^{-1}}{2}(T+1) + C_4(T).$$

The result follows after noting that

$$\frac{N\varphi(N)}{v_{\Gamma}} \prod_{p|N} \left(1 + \frac{1}{p}\right) = \frac{6}{\pi N}$$

and reordering terms. This concludes the proof. \square

6. THE SELF-INTERSECTION OF THE RELATIVE DUALIZING SHEAF

Let $\omega := \omega_{\mathcal{X}/S}$ be the relative dualizing sheaf of \mathcal{X}/S , write $\bar{\omega}$ for the relative dualizing sheaf ω , equipped with the Arakelov metric, and let $\bar{\omega}^2 := \bar{\omega}_{\mathcal{X}/S}^2$ denote its self-intersection. In this chapter, we establish the main result of our article, namely an asymptotics for the invariant

$$e(\Gamma) := \frac{1}{\varphi(N)} \bar{\omega}^2,$$

as $N \rightarrow \infty$. To prove this asymptotics, we proceed in two steps. In section 6.1, we first obtain an explicit formula for $\bar{\omega}^2$ in terms of an geometric contribution $\mathcal{G}(N)$ and an analytic contribution $\mathcal{A}(N)$ (see Proposition 13). In section 6.2, we then establish asymptotics for the geometric and analytic contribution (see Propositions 14 and 17), and we conclude with the main result in Theorem 18.

6.1. Explicit formula for the self-intersection. Let H_0 resp. H_{∞} be the horizontal divisor on \mathcal{X}/S defined by the cusps 0 and ∞ , respectively. For a prime ideal $\mathfrak{p} \in S$ such that $\mathfrak{p}|p$, where $p|N$ is a prime number, we set

$$\begin{aligned} r_{\mathfrak{p}} &:= p + 1, \\ s_{\mathfrak{p}} &:= \frac{p-1}{24} [\text{PSL}_2(\mathbb{Z}) : \bar{\Gamma}(N/p)]. \end{aligned}$$

Note that $s_{\mathfrak{p}}$ is the number of supersingular points on the fiber $\mathcal{X}_{\mathfrak{p}}$. Let us write $C_{1,\mathfrak{p}}, \dots, C_{r_{\mathfrak{p}},\mathfrak{p}}$ for the $r_{\mathfrak{p}}$ irreducible components of the fiber $\mathcal{X}_{\mathfrak{p}}$ stated in Proposition 1, and $C_{0,\mathfrak{p}}$ resp. $C_{\infty,\mathfrak{p}}$ for the irreducible component of $\mathcal{X}_{\mathfrak{p}}$ intersected by H_0 and H_{∞} , respectively.

Consider the following divisors on \mathcal{X} with rational coefficients

$$V_0 := - \sum_{\mathfrak{p}|N} \frac{2(g_{\Gamma} - 1)}{r_{\mathfrak{p}} s_{\mathfrak{p}}} C_{0,\mathfrak{p}}, \quad V_{\infty} := - \sum_{\mathfrak{p}|N} \frac{2(g_{\Gamma} - 1)}{r_{\mathfrak{p}} s_{\mathfrak{p}}} C_{\infty,\mathfrak{p}},$$

where both sums run over all prime ideals $\mathfrak{p} \in S$ satisfying $\mathfrak{p}|p$ for some prime number $p|N$. Now, for $q \in \{0, \infty\}$, we define the admissible line bundle $\bar{\mathcal{L}}_q$ on \mathcal{X} as follows

$$\bar{\mathcal{L}}_q := \bar{\omega} \otimes \overline{\mathcal{O}(H_q)}^{\otimes -(2g_{\Gamma}-2)} \otimes \overline{\mathcal{O}(V_q)}.$$

In addition, we define the divisor \mathcal{M} on \mathcal{X} by

$$\mathcal{M} := H_{\infty} - H_0 + \frac{1}{2g_{\Gamma} - 2} (V_0 - V_{\infty}).$$

Lemma 12. *Let $N \geq 3$ be a composite, odd, and square-free integer. Then, the identities*

$$(19) \quad \begin{aligned} (\bar{\mathcal{L}}_q, \overline{\mathcal{O}(V)})_{\text{Ar}} &= 0, \\ (\mathcal{M}, V)_{\text{Ar}} &= 0, \end{aligned}$$

hold for all vertical divisors V of \mathcal{X} and $q \in \{0, \infty\}$. Furthermore, we have

$$\begin{aligned} (V_{\infty}, V_{\infty})_{\text{fin}} &= (V_0, V_0)_{\text{fin}} = -4(g_{\Gamma} - 1)\varphi(N) \left(1 - \frac{6}{N}\right) \sum_{p|N} \frac{p^2 \log(p)}{p^2 - 1}, \\ (V_0, V_{\infty})_{\text{fin}} &= 4(g_{\Gamma} - 1)\varphi(N) \left(1 - \frac{6}{N}\right) \sum_{p|N} \frac{p \log(p)}{p^2 - 1}. \end{aligned}$$

Proof. Throughout the proof, we let $\mathfrak{p}, \tilde{\mathfrak{p}} \in S$ be prime ideals satisfying $\mathfrak{p}|N$ and $\tilde{\mathfrak{p}} \nmid N$, respectively.

First of all, we claim that the identity

$$(\overline{\mathcal{L}_q}, \overline{\mathcal{O}(V)})_{\text{Ar}} = 0$$

holds for $V = \mathcal{X}_{\tilde{\mathfrak{p}}}$ and for $V = C_{q,\mathfrak{p}}$. Indeed, note that, from the definition of $\overline{\mathcal{L}_q}$, we have

$$(\overline{\mathcal{L}_q}, \overline{\mathcal{O}(V)})_{\text{Ar}} = (\overline{\omega}, \overline{\mathcal{O}(V)})_{\text{fin}} - (2g_{\Gamma} - 2)(\overline{\mathcal{O}(H_q)}, \overline{\mathcal{O}(V)})_{\text{fin}} + (\overline{\mathcal{O}(V_q)}, \overline{\mathcal{O}(V)})_{\text{fin}}.$$

Then, the claim follows by a straightforward computation using the following identities

$$(\overline{\omega}, \overline{\mathcal{O}(V)})_{\text{fin}} = \begin{cases} (2g_{\Gamma} - 2) \log(\#k(\tilde{\mathfrak{p}})), & V = \mathcal{X}_{\tilde{\mathfrak{p}}}; \\ \frac{2g_{\Gamma} - 2}{r_{\mathfrak{p}}} \log(\#k(\mathfrak{p})), & V = C_{q,\mathfrak{p}}; \end{cases}$$

$$(\overline{\mathcal{O}(H_q)}, \overline{\mathcal{O}(V)})_{\text{fin}} = \begin{cases} \log(\#k(\tilde{\mathfrak{p}})), & V = \mathcal{X}_{\tilde{\mathfrak{p}}}; \\ \log(\#k(\mathfrak{p})), & V = C_{q,\mathfrak{p}}; \end{cases}$$

$$(\overline{\mathcal{O}(V_q)}, \overline{\mathcal{O}(V)})_{\text{fin}} = \begin{cases} 0, & V = \mathcal{X}_{\tilde{\mathfrak{p}}}; \\ \frac{2(g_{\Gamma} - 1)(r_{\mathfrak{p}} - 1)}{r_{\mathfrak{p}}} \log(\#k(\mathfrak{p})), & V = C_{q,\mathfrak{p}}; \end{cases}$$

(see, e.g., [20, Chapter 9]). Now, since every vertical divisor V of \mathcal{X} can be written as a linear combination of $\mathcal{X}_{\tilde{\mathfrak{p}}}$ and $C_{q,\mathfrak{p}}$, with coefficients in \mathbb{Q} , the first identity of (19) follows. Similarly, note that $(\mathcal{M}, V)_{\text{Ar}} = 0$ holds for $V = \mathcal{X}_{\tilde{\mathfrak{p}}}$ and $V = C_{q,\mathfrak{p}}$. Furthermore, if $V = C_{q',\mathfrak{p}}$ with $q' \in \{0, \infty\}$ and $q \neq q'$, then we have

$$(\overline{\mathcal{O}(H_q)}, \overline{\mathcal{O}(V)})_{\text{fin}} = 0,$$

$$(\overline{\mathcal{O}(V_q)}, \overline{\mathcal{O}(V)})_{\text{fin}} = -\frac{2(g_{\Gamma} - 1)}{r_{\mathfrak{p}}} \log(\#k(\mathfrak{p})).$$

This yield the first assertion of the lemma.

Next, to prove the second part of the lemma, observe that

$$(V_0, V_{\infty})_{\text{fin}} = \sum_{\mathfrak{p}|N} \sum_{\mathfrak{p}'|N} \frac{4(g_{\Gamma} - 1)^2}{r_{\mathfrak{p}} r_{\mathfrak{p}'} s_{\mathfrak{p}} s_{\mathfrak{p}'}} (C_{0,\mathfrak{p}}, C_{\infty,\mathfrak{p}'})_{\text{fin}}.$$

Since we have $(C_{0,\mathfrak{p}}, C_{\infty,\mathfrak{p}'})_{\text{fin}} = 0$ provided that $\mathfrak{p} \neq \mathfrak{p}'$, and

$$(C_{q,\mathfrak{p}}, C_{q',\mathfrak{p}})_{\text{fin}} = \begin{cases} s_{\mathfrak{p}} \log(\#k(\mathfrak{p})), & q \neq q'; \\ -(r_{\mathfrak{p}} - 1) s_{\mathfrak{p}} \log(\#k(\mathfrak{p})), & q = q'; \end{cases}$$

where $q' \in \{0, \infty\}$ with $q \neq q'$, we get

$$(V_0, V_{\infty})_{\text{fin}} = \sum_{\mathfrak{p}|N} \frac{4(g_{\Gamma} - 1)^2}{r_{\mathfrak{p}}^2 s_{\mathfrak{p}}} \log(\#k(\mathfrak{p})).$$

Further, if $\mathfrak{p}|p$ for some prime number $p|N$, then we have

$$s_{\mathfrak{p}} = \frac{N}{24p(p+1)} \prod_{p'|N} ((p')^2 - 1);$$

therefore, we obtain

$$\begin{aligned} (V_0, V_{\infty})_{\text{fin}} &= \frac{4(g_{\Gamma} - 1)}{N} \frac{24(g_{\Gamma} - 1)}{\prod_{p'|N} ((p')^2 - 1)} \sum_{p|N} \left(\frac{p}{p+1} \sum_{\mathfrak{p}|p} \log(\#k(\mathfrak{p})) \right) \\ &= 4(g_{\Gamma} - 1) \varphi(N) \left(1 - \frac{6}{N} \right) \sum_{p|N} \frac{p \log(p)}{p^2 - 1}. \end{aligned}$$

The other identities can be proved in a similar way. This concludes the proof. \square

Proposition 13. *Let $N \geq 3$ be a composite, odd, and square-free integer. Then, the following identity holds*

$$\bar{\omega}^2 = \mathcal{G}(N) + \mathcal{A}(N),$$

where

$$\begin{aligned} \mathcal{G}(N) &= \frac{2g_\Gamma(V_0, V_\infty)_{\text{fin}} - (V_0, V_0)_{\text{fin}} - (V_\infty, V_\infty)_{\text{fin}}}{2(g_\Gamma - 1)}, \\ \mathcal{A}(N) &= 4g_\Gamma(g_\Gamma - 1) \sum_{\sigma: \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}} g_{\text{Ar}}(0^\sigma, \infty^\sigma). \end{aligned}$$

Here, the sum runs over all embeddings of $\mathbb{Q}(\zeta_N)$ into \mathbb{C} , and 0^σ resp. ∞^σ denote the image of $0, \infty$ in $\mathcal{X}_\eta(\mathbb{Q}(\zeta_N))_\sigma$ under the embedding σ , respectively.

Proof. Throughout the proof, we will write $\mathcal{E}^2 := (\mathcal{E}, \mathcal{E})_{\text{Ar}}$. Let $\mathcal{L}_q := \omega \otimes \mathcal{O}(H_q)^{\otimes -(2g_\Gamma - 2)} \otimes \mathcal{O}(V_q)$.

First of all, note that the pullback $L_q := \mathcal{L}_q \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Q}(\zeta_N)$ of \mathcal{L}_q to the generic fiber \mathcal{X}_η defines a line bundle on $X(N)/\mathbb{Q}(\zeta_N)$, that is supported on the cusps (see [3, Lemme 4.1.1]). By the theorem of Manin–Drinfeld, L_q defines a torsion point of $\text{Jac}(X(N))(\mathbb{Q}(\zeta_N))$. Furthermore, the theorem of Faltings–Hriljac implies

$$\bar{\mathcal{L}}_q^2 = -2\varphi(N)h_{\text{NT}}(L_q),$$

where $h_{\text{NT}}(\cdot)$ denotes the Néron–Tate height. Since the latter vanishes on torsion points of the Jacobian, we obtain that

$$\bar{\mathcal{L}}_q^2 = 0.$$

Similarly, the restriction of the divisor \mathcal{M} to the generic fiber \mathcal{X}_η is supported on the cusps of $X(N)/\mathbb{Q}(\zeta_N)$; therefore, by using the same arguments of the previous paragraph, we obtain $\mathcal{M}^2 = 0$.

Secondly, if we expand the left hand side of $\bar{\mathcal{L}}_q^2 = 0$ using the definition of $\bar{\mathcal{L}}_q$, and apply the identities (19), then we obtain

$$\bar{\omega}^2 = 2(2g_\Gamma - 2)(\bar{\omega}, \overline{\mathcal{O}(H_q)})_{\text{Ar}} - (2g_\Gamma - 2)^2 \overline{\mathcal{O}(H_q)}^2 - (\bar{\omega} \otimes \overline{\mathcal{O}(H_q)}^{\otimes -(2g_\Gamma - 2)}, \overline{\mathcal{O}(V_q)})_{\text{Ar}},$$

where $q \in \{0, \infty\}$. By virtue of the equalities

$$\begin{aligned} (\bar{\omega} \otimes \overline{\mathcal{O}(H_q)}^{\otimes -(2g_\Gamma - 2)}, \overline{\mathcal{O}(V_q)})_{\text{Ar}} &= -\overline{\mathcal{O}(V_q)}^2, \\ (\bar{\omega} \otimes \overline{\mathcal{O}(H_q)}, \overline{\mathcal{O}(H_q)})_{\text{Ar}} &= 0, \end{aligned}$$

we now have

$$\bar{\omega}^2 = -4g_\Gamma(g_\Gamma - 1)\overline{\mathcal{O}(H_q)}^2 + \overline{\mathcal{O}(V_q)}^2,$$

for each $q \in \{0, \infty\}$. Thus, by adding the resulting identities for $q = 0$ and $q = \infty$, we get

$$\bar{\omega}^2 = -2g_\Gamma(g_\Gamma - 1)(\overline{\mathcal{O}(H_0)}^2 + \overline{\mathcal{O}(H_\infty)}^2) + \frac{1}{2}(\overline{\mathcal{O}(V_0)}^2 + \overline{\mathcal{O}(V_\infty)}^2).$$

Thirdly, putting $D := H_\infty - H_0$ and $E := (1/(2g_\Gamma - 2))(V_0 - V_\infty)$, we have

$$D^2 = \mathcal{M}^2 - 2(\mathcal{M}, E)_{\text{Ar}} + E^2 = E^2,$$

which in turn implies

$$\overline{\mathcal{O}(H_\infty)}^2 + \overline{\mathcal{O}(H_0)}^2 = \frac{1}{(2g_\Gamma - 2)^2} \left(\overline{\mathcal{O}(V_0)}^2 - 2(\overline{\mathcal{O}(V_0)}, \overline{\mathcal{O}(V_\infty)})_{\text{Ar}} + \overline{\mathcal{O}(V_\infty)}^2 \right) + 2(\overline{\mathcal{O}(H_\infty)}, \overline{\mathcal{O}(H_0)})_{\text{Ar}}.$$

Therefore, we have

$$\bar{\omega}^2 = -4g_\Gamma(g_\Gamma - 1)(\overline{\mathcal{O}(H_\infty)}, \overline{\mathcal{O}(H_0)})_{\text{Ar}} + \frac{2g_\Gamma(V_0, V_\infty)_{\text{fin}} - (V_0, V_0)_{\text{fin}} - (V_\infty, V_\infty)_{\text{fin}}}{2(g_\Gamma - 1)}.$$

Finally, since

$$\overline{(\mathcal{O}(H_\infty), \mathcal{O}(H_0))}_{\text{Ar}} = - \sum_{\sigma: \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}} g_{\text{Ar}}(0^\sigma, \infty^\sigma),$$

the result follows. This concludes the proof of the proposition. \square

6.2. Asymptotics for the self-intersection.

Proposition 14. *Let $N \geq 3$ be a composite, odd, and square-free integer. Then, the following asymptotics holds*

$$\frac{1}{\varphi(N)} \mathcal{G}(N) = o(g_\Gamma \log(N)),$$

as $N \rightarrow \infty$.

Proof. Indeed, using Lemma 12 and Proposition 13, we have

$$\frac{1}{\varphi(N)} \mathcal{G}(N) = 4 \left(1 - \frac{6}{N}\right) \left(\log(N) + g_\Gamma \sum_{p|N} \frac{p \log(p)}{p^2 - 1} + \sum_{p|N} \frac{\log(p)}{p^2 - 1} \right).$$

Using the fact that

$$\sum_{p|N} \frac{\log(p)}{p} = O(\log \log(N)),$$

as $N \rightarrow \infty$ (see, e.g., [5]), we can deduce the following asymptotics

$$\sum_{p|N} \frac{p \log(p)}{p^2 - 1} = O(\log \log(N)), \quad \sum_{p|N} \frac{\log(p)}{p^2 - 1} = O(\log \log(N)),$$

as $N \rightarrow \infty$. This yields

$$\frac{1}{\varphi(N)} \frac{\mathcal{G}(N)}{g_\Gamma \log(N)} = \frac{4}{g_\Gamma \log(N)} \left(1 - \frac{6}{N}\right) \left(\log(N) + O(g_\Gamma \log \log(N)) \right),$$

and the result follows, since the right hand side of the previous identity converges to zero as $N \rightarrow \infty$. This concludes the proof. \square

Lemma 15. *Let $N \geq 3$ be a composite, odd, and square-free integer. Then, the following identity holds*

$$\mathcal{R}_{0_\xi} = \mathcal{R}_\infty,$$

for all $\xi \in U$.

Proof. By taking $\alpha = \begin{pmatrix} -mN & \xi \\ -\xi & 1 \end{pmatrix}$ with $\det(\alpha) = 1$, one can verify that $0_\xi = \alpha^{-1} \infty$ and $\alpha^{-1} \Gamma \alpha = \Gamma$; therefore, from Section 2.4, we have that $F(\alpha z) = F(z)$ with $z \in \mathbb{H}$. Thus, since the identity $E_\infty(\alpha z, s) = E_{0_\xi}(z, s)$ holds, we obtain $\mathcal{R}_{0_\xi}[F](s) = \mathcal{R}_\infty[F](s)$. Consequently, by comparing coefficients in the Laurent expansions of $\mathcal{R}_{0_\xi}[F](s)$ and $\mathcal{R}_\infty[F](s)$ at $s = 1$, we get $\mathcal{R}_{0_\xi} = \mathcal{R}_\infty$. This concludes the proof. \square

Proposition 16. *Let $N \geq 3$ be a composite, odd, and square-free integer. Then, the following identity holds*

$$\begin{aligned} \mathcal{R}_\infty &= \frac{1}{g_\Gamma} \left(\frac{v_\Gamma^{-1}}{2} \lim_{s \rightarrow 1} \left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} - \frac{1}{s-1} \right) + \frac{12}{\pi N} \left(C_1 - \mathcal{C} - \frac{\gamma_{\text{EM}}}{2} + \log(N) \right) \right) \\ &\quad + v_\Gamma^{-1} \left(\mathcal{C} + 1 - 2 \log(N) - \sum_{p|N} \frac{\log(p)}{p^2 - 1} \right) + \frac{1 - \log(4\pi)}{4\pi}, \end{aligned}$$

where $C_1 = \lim_{T \rightarrow \infty} C_1(T)$.

Proof. From identity (12), we know that

$$\mathcal{R}_\infty = \frac{1}{g_\Gamma} \left(-\frac{v_\Gamma^{-1}}{2} \sum_{j=1}^{\infty} \frac{h_T(r_j)}{\lambda_j} + \mathcal{R}_\infty^{\text{hyp}} + \mathcal{R}_\infty^{\text{par}} \right),$$

where now, by virtue of Proposition 6 and Proposition 11, we have

$$\mathcal{R}_\infty^{\text{hyp}} = \frac{v_\Gamma^{-1}}{2} \left(\lim_{s \rightarrow 1} \left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} - \frac{1}{s-1} \right) - T + 1 + o(1) \right),$$

as $T \rightarrow \infty$, and

$$\begin{aligned} \mathcal{R}_\infty^{\text{par}} &= \frac{12}{\pi N} \left(C_1(T) + A_2(T) \left(2\mathcal{C} + \gamma_{\text{EM}} - 2 \log(N) \right) \right) + \frac{1 - \log(4\pi)}{4\pi} \\ &+ \frac{v_\Gamma^{-1}}{2} (T + 1) + \frac{\mathcal{C}_{\infty\infty}}{2} + \gamma_{\text{EM}} C_2(T) + C_3(T) + C_4(T). \end{aligned}$$

Since

$$\mathcal{C}_{\infty\infty} = 2v_\Gamma^{-1} \left(\mathcal{C} - 2 \log(N) - \sum_{p|N} \frac{\log(p)}{p^2 - 1} \right)$$

(see, e.g., [13]), one gets

$$\begin{aligned} \mathcal{R}_\infty &= \frac{1}{g_\Gamma} \left(\frac{v_\Gamma^{-1}}{2} \lim_{s \rightarrow 1} \left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} - \frac{1}{s-1} \right) + \frac{12}{\pi N} \left(C_1(T) + A_2(T) (2\mathcal{C} + \gamma_{\text{EM}} - 2 \log(N)) \right) \right) \\ &+ v_\Gamma^{-1} \left(\mathcal{C} + 1 - 2 \log(N) - \sum_{p|N} \frac{\log(p)}{p^2 - 1} \right) + \frac{1 - \log(4\pi)}{4\pi} + \vartheta(T), \end{aligned}$$

where

$$\vartheta(T) := -\frac{v_\Gamma^{-1}}{2} \sum_{j=1}^{\infty} \frac{h_T(r_j)}{\lambda_j} + \gamma_{\text{EM}} C_2(T) + C_3(T) + C_4(T) + \frac{v_\Gamma^{-1}}{2} o_T(1),$$

as $T \rightarrow \infty$.

Now, since $h_T(r_j)$, $C_2(T)$, $C_3(T)$, and $C_4(T)$ tend to zero as $T \rightarrow \infty$ (see [3]), we have that $\vartheta(T) \rightarrow 0$. Consequently, we get

$$\begin{aligned} \mathcal{R}_\infty &= \frac{1}{g_\Gamma} \left(\frac{v_\Gamma^{-1}}{2} \lim_{s \rightarrow 1} \left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} - \frac{1}{s-1} \right) + \frac{12}{\pi N} \left(C_1 - \mathcal{C} - \frac{\gamma_{\text{EM}}}{2} + \log(N) \right) \right) \\ &+ v_\Gamma^{-1} \left(\mathcal{C} + 1 - 2 \log(N) - \sum_{p|N} \frac{\log(p)}{p^2 - 1} \right) + \frac{1 - \log(4\pi)}{4\pi}. \end{aligned}$$

This concludes the proof. \square

Proposition 17. *Let $N \geq 3$ be a composite, odd, and square-free integer. Then, the following asymptotics holds*

$$\frac{1}{\varphi(N)} \mathcal{A}(N) = 2g_\Gamma \log(N) + o(g_\Gamma \log(N)),$$

as $N \rightarrow \infty$.

Proof. By Proposition 13, we have

$$\begin{aligned} \frac{1}{\varphi(N)} \mathcal{A}(N) &= \frac{4g_\Gamma(g_\Gamma - 1)}{\varphi(N)} \sum_{\sigma: \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}} g_{\text{Ar}}(0^\sigma, \infty^\sigma) \\ &= \frac{4g_\Gamma(g_\Gamma - 1)}{\varphi(N)} \sum_{\xi \in U} g_{\text{Ar}}(0_\xi, \infty), \end{aligned}$$

where in the second identity we identified each embedding $\sigma: \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$ with a unique element $\xi \in U$ via the assignment $\sigma(\zeta_N) = e^{2\pi i \xi / N}$, and used the isomorphism ι_σ from Proposition 2.

Next, from the identity (13) and Lemma 15, we write $g_{\text{Ar}}(0_\xi, \infty)$ as follows

$$g_{\text{Ar}}(0_\xi, \infty) = -2\pi\mathcal{C}_{0_\xi\infty} - \frac{2\pi}{v_\Gamma} + 4\pi\mathcal{R}_\infty + \mathcal{G},$$

where \mathcal{G} is given by (14), and $\mathcal{C}_{0_\xi\infty}$ is given by

$$\mathcal{C}_{0_\xi\infty} = 2v_\Gamma^{-1} \left(\mathcal{C} - \log(N) + \sum_{p|N} \frac{p \log(p)}{p^2 - 1} \right) + \kappa(\tilde{\xi}; 1, N);$$

we refer to [13] for a proof of the latter identity and for the definition of $\kappa(\tilde{\xi}; 1, N)$. Letting $C := 4\pi(g_\Gamma - 1)v_\Gamma^{-1}$, we obtain

$$\begin{aligned} \frac{1}{\varphi(N)} \mathcal{A}(N) &= 2Cg_\Gamma \log(N) - 2g_\Gamma C \left(\mathcal{C} + \frac{1}{2} + \sum_{p|N} \frac{p \log(p)}{p^2 - 1} \right) - \frac{8\pi g_\Gamma (g_\Gamma - 1)}{\varphi(N)} \sum_{\xi \in U} \kappa(\bar{\xi}; 1, N) \\ &\quad + 8\pi g_\Gamma (g_\Gamma - 1) \mathcal{R}_\infty + 2\mathcal{G} g_\Gamma (g_\Gamma - 1). \end{aligned}$$

Observe that, from the definition of $\kappa(\bar{\xi}; 1, N)$ and the orthogonality relations of Dirichlet characters, we obtain

$$\sum_{\xi \in U} \kappa(\bar{\xi}; 1, N) = 0.$$

Therefore, since

$$\sum_{p|N} \frac{p \log(p)}{p^2 - 1} = O(\log \log(N)),$$

as $N \rightarrow \infty$ (see the proof of Proposition 14), we get the following asymptotics

$$\frac{1}{\varphi(N)} \mathcal{A}(N) = 2Cg_\Gamma \log(N) + 8\pi g_\Gamma (g_\Gamma - 1) \mathcal{R}_\infty + 2\mathcal{G} g_\Gamma (g_\Gamma - 1) + O(g_\Gamma \log \log(N)),$$

as $N \rightarrow \infty$.

Now, we claim that

$$8\pi g_\Gamma (g_\Gamma - 1) \mathcal{R}_\infty + 2\mathcal{G} g_\Gamma (g_\Gamma - 1) = o(g_\Gamma \log(N)),$$

as $N \rightarrow \infty$. Indeed, using Proposition 16, we have

$$\begin{aligned} 8\pi g_\Gamma (g_\Gamma - 1) \mathcal{R}_\infty &= C \lim_{s \rightarrow 1} \left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} - \frac{1}{s-1} \right) + 2C \left(\mathcal{C} + 1 - \log(N) - \sum_{p|N} \frac{\log(p)}{p^2 - 1} \right) \\ &\quad + \frac{96(g_\Gamma - 1)}{N} \left(C_1 - \left(\mathcal{C} + \frac{\gamma_{\text{EM}}}{2} - \log(N) \right) \right) + 2(g_\Gamma - 1)(1 - \log(4\pi)), \end{aligned}$$

and by virtue of [18, p. 27], namely

$$\lim_{s \rightarrow 1} \left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} - \frac{1}{s-1} \right) = O(N^\varepsilon),$$

for $\varepsilon > 0$, as $N \rightarrow \infty$, the claim follows for a sufficiently small ε .

Finally, note that

$$2Cg_\Gamma \log(N) = 2g_\Gamma \log(N) + o(g_\Gamma \log(N)).$$

This concludes the proof of the proposition. \square

Finally, we can prove the main result of this paper.

Theorem 18. *Let $N \geq 3$ be a composite, odd, and square-free integer. Then, the following asymptotics holds*

$$e(\Gamma) = 2g_\Gamma \log(N) + o(g_\Gamma \log(N)),$$

as $N \rightarrow \infty$.

Proof. Recalling that $e(\Gamma) = \bar{\omega}^2/\varphi(N)$ and using Proposition 13, we get

$$e(\Gamma) = \frac{1}{\varphi(N)} (\mathcal{G}(N) + \mathcal{A}(N)).$$

Substituting therein the asymptotics proven in Propositions 14 and 17, and adding up, yields the asserted asymptotics. \square

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