An analytic class number type formula for $\mathbf{PSL_2}(\mathbb{Z})$

Anna-Maria von Pippich
(joint work with Gerard Freixas i Montplet)

1. The Selberg zeta function

Let $\mathbb{H}$ denote the upper half plane and let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind. The quotient space $\Gamma\backslash\mathbb{H}$ admits a canonical structure of a Riemann surface. The points with non-trivial automorphisms are called elliptic fixed points. By adding a finite number of cusps, the Riemann surface $\Gamma\backslash\mathbb{H}$ can be completed into a compact Riemann surface, which we denote by X. The hyperbolic metric on $\mathbb{H}$ is given by

$$ds^2_{\mathrm{hyp}} = \frac{dx^2+dy^2}{y^2},$$

where $x+iy$ is the usual parametrization of $\mathbb{H}$. As a metric on X, it has singularities at the cusps and the elliptic fixed points.

Let now $H(\Gamma)$ denote a complete set of representatives of inconjugate, primitive, hyperbolic elements in Γ. For $\gamma \in H(\Gamma)$, we denote by $\ell_{\mathrm{hyp}}(\gamma)$ the hyperbolic length of the closed geodesic determined γ on $\Gamma\backslash\mathbb{H}$. The Selberg zeta function $Z(s,\Gamma)$ associated to Γ was introduced by Atle Selberg. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, it is defined by the absolutely and locally uniformly convergent Euler product

$$Z(s,\Gamma) = \prod_{\gamma\in H(\Gamma)} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell_{\mathrm{hyp}}(\gamma)}).$$

The Selberg zeta function is known to have a meromorphic continuation to the whole complex s-plane, and its poles and zeros can be described in terms of the spectral theory of the hyperbolic Laplacian on $\Gamma\backslash\mathbb{H}$. In particular, the Selberg zeta function has a simple zero at $s = 1$ and $Z'(1,\Gamma)$ is a positive real number.

2. The special value $Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$

In this section, we give an explicit formula for $Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$ using Arakelov theory. More precisely, we apply an arithmetic Riemann–Roch theorem, namely Theorem 10.1 of [1], in the case of the coarse moduli scheme $\mathbb{P}^1_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ of the Deligne–Mumford stack $\mathcal{M}_1 \to \mathrm{Spec}(\mathbb{Z})$ of generalized elliptic curves. We interpret $\mathbb{P}^1_{\mathbb{Z}}(\mathbb{C})$ as the Riemann surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H} \cup \{\infty\}$. The cusp at infinity and the elliptic fixed points i and $\rho = e^{2i\pi/3}$ define integral sections $\mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$, which we denote by σ_∞, σ_i, and σ_ρ, having multiplicities $m_\infty = \infty$, $m_i = 2$, and $m_\rho = 3$. In the notation of [1], we then have $D = \sigma_\infty + (1/2)\sigma_i + (2/3)\sigma_\rho$.

Let now χ_i resp. χ_ρ be the quadratic characters of $\mathbb{Q}(i)$ and $\mathbb{Q}(\rho)$, respectively, and let $L(s,\chi_i)$ and $L(s,\chi_\rho)$ denote the corresponding Dirichlet L-functions. Then, we have the following theorem.

Theorem 1. *The special value* $Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$ *is given by*

$$\log Z'(1, \mathrm{PSL}_2(\mathbb{Z})) = \frac{1}{4}\frac{L'(0,\chi_i)}{L(0,\chi_i)} + \frac{13}{27}\frac{L'(0,\chi_\rho)}{L(0,\chi_\rho)} + \frac{73}{72}\frac{\zeta'(0)}{\zeta(0)} - \frac{37}{36}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{5}{36}\gamma + \frac{5}{12}\log 3 - \frac{167}{216}\log 2 - \frac{5}{6},$$

where $\zeta(s)$ *denotes the Riemann zeta function.*

Sketch of proof. We only give a sketch of the proof; we use the notation of [1] and, for details, we refer the reader to [1]. To prove the statement, we employ Theorem 10.1 of [1], see also Theorem 2 of [2]. To compute the arithmetic degree $\widehat{\deg}\,\det H^\bullet(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}})_Q$ of the determinant of cohomology of the trivial sheaf, endowed with the Quillen metric, we first observe that $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}) = \mathbb{Z}$ and $H^1(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}) = 0$. Therefore, we get

$$12\,\widehat{\deg}\,\det H^\bullet(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}})_Q = -12\log\|1\|_{L^2} + 6\log\left(C(\mathrm{PSL}_2(\mathbb{Z}))\cdot Z'(\mathrm{PSL}_2(\mathbb{Z}),1)\right),$$

where $C(\mathrm{PSL}_2(\mathbb{Z}))$ is a real positive constant, which can be explicitly expressed in terms of the multiplicities 2, 3, and ∞, the number $c = 1$ of cusps, the number $n = 3$ of cusps and elliptic fixed points, the genus $g = 0$ of X, special values of the Riemann zeta function $\zeta(s)$, and the Euler–Mascheroni constant γ, see formula (1.2) of [1]. Theorem 10.1 of [1] thus implies the following equality of real numbers

$$\begin{aligned}\log Z'(\mathrm{PSL}_2(\mathbb{Z}),1) = {} & \frac{1}{6}(\omega_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}}(D)_{\mathrm{hyp}}, \omega_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}}(D)_{\mathrm{hyp}}) \\ & - \frac{1}{6}\sum_{\substack{j,k\in\{i,\rho,\infty\}\\ j\neq k}}\left(1-\frac{1}{m_j}\right)\left(1-\frac{1}{m_k}\right)(\sigma_j,\sigma_k)_{\mathrm{fin}} \\ & + 2\log\|1\|_{L^2} - \log C(\mathrm{PSL}_2(\mathbb{Z})) - \frac{1}{6}\widehat{\deg}\,\psi_W.\end{aligned} \tag{1}$$

It therefore remains to explicitly compute the contributions on the right-hand side of (1).

From the definition of the arithmetic self-intersection number of $\omega_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}}(D)_{\mathrm{hyp}}$, endowed with the hyperbolic metric, we derive using the relation $\|\cdot\|_{\mathrm{hyp}} = \frac{8}{(4\pi)^6}\|\cdot\|_{\mathrm{Pet}}$ the equality

$$(\omega_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}}(D)_{\mathrm{hyp}}, \omega_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}}(D)_{\mathrm{hyp}}) = \frac{1}{36}(\mathcal{M}_{12}(\Gamma(1))_{\mathrm{Pet}}, \mathcal{M}_{12}(\Gamma(1))_{\mathrm{Pet}}) + \frac{1}{3}\log(2\pi) + \frac{1}{6}\log 2.$$

Hence, employing the identity

$$(\mathcal{M}_{12}(\Gamma(1))_{\mathrm{Pet}}, \mathcal{M}_{12}(\Gamma(1))_{\mathrm{Pet}}) = -12\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right),$$

proven by Bost and Kühn, we conclude that

$$(\omega_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}}(D)_{\mathrm{hyp}}, \omega_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}}(D)_{\mathrm{hyp}}) = -\frac{1}{3}\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right) + \frac{1}{3}\log(2\pi) + \frac{1}{6}\log 2. \tag{2}$$

In the next step, we prove that one has the following finite intersection numbers

$$\begin{aligned} (\sigma_\infty, \sigma_i)_{\mathrm{fin}} &= 0, \\ (\sigma_\infty, \sigma_\rho)_{\mathrm{fin}} &= 0, \\ (\sigma_i, \sigma_\rho)_{\mathrm{fin}} &= \log(1728) = 6\log 2 + 3\log 3. \end{aligned} \tag{3}$$

Furthermore, the square-norm of 1 for the L^2 metric is given by the volume

$$\|1\|^2_{L^2} = \frac{1}{2\pi} \int\limits_{\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{dx \wedge dy}{y^2} = \frac{1}{2\pi}\frac{\pi}{3} = \frac{1}{6},$$

hence, we obtain

$$2\log\|1\|_{L^2} = -\log 2 - \log 3. \tag{4}$$

It remains to compute the arithmetic degree $\widehat{\deg}\,\psi_W$ of the ψ-bundle, endowed with the Wolpert metric. To this end, let E_i resp. E_ρ be the elliptic curves, defined over $\mathbb{Q}$, having complex multiplication by $\mathbb{Q}(i)$ and $\mathbb{Q}(\rho)$, respectively. We denote by $h_F(E_i)$ and $h_F(E_\rho)$ their stable Faltings height. Then, one can prove that

$$\widehat{\deg}\,\psi_W = 3h_F(E_i) + \frac{16}{3}h_F(E_\rho) - \frac{43}{18}(\sigma_i, \sigma_\rho)_{\mathrm{fin}} + \frac{25}{6}\log(4\pi).$$

Consequently, by the Chowla–Selberg formula and (3), we get

$$\widehat{\deg}\,\psi_W = -\frac{3}{2}\frac{L'(0,\chi_i)}{L(0,\chi_i)} - \frac{8}{3}\frac{L'(0,\chi_\rho)}{L(0,\chi_\rho)} + \frac{25}{6}\frac{\zeta'(0)}{\zeta(0)} - \frac{17}{2}\log 3 - \frac{15}{2}\log 2.$$

Inserting the explicit formula for $C(\Gamma)$ together with (2), (3), and (4) into (1), finally yields the claimed formula for $Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$. □

Since there is a formal resemblance between the equality in Theorem 10.1 of [1] to the analytic class number formula of Dedekind zeta functions, we call the explicit expression for $\log Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$ the analytic class number formula for $\mathrm{PSL}_2(\mathbb{Z})$.

We finally remark that it would be interesting to have a direct "analytic number theoretic" evaluation of $Z'(\mathrm{PSL}_2(\mathbb{Z}), 1)$, and differently see how the special values of Dirichlet L-functions above arise. The advantage of the Arakelov theoretic strategy is that the result has a geometric interpretation.

References

[1] G. Freixas i Montplet, A.-M. von Pippich, *Riemann–Roch isometries in the non-compact orbifold setting*, `arXiv:1604.00284 [math.NT]`, 2016.

[2] A.-M. von Pippich, *Riemann–Roch isometries in the non-compact orbifold setting*, Oberwolfach reports from the workshop on moduli spaces and modular forms held April 24 – April 30, 2016, Organized by J.H. Bruinier, G. van der Geer, and V. Gritenko. Report No. 23/2016. DOI: 10.4171/OWR/2016/23, 2016.