Logarithm and Dilogarithm

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1 The logarithm

1.1. A naive sequence. Following D. Zagier, we begin with the sequence of non-zero complex numbers determined by the requirement that each element of the sequence is given as one *divided* by the product of the left-hand and right-hand neighbors of the element in question. Denoting our sequence by $a_1, a_2, a_3, a_4, a_5, \ldots$, the defining properties of the sequence give

$$a_2 = \frac{1}{a_1 a_3}\,,$$

hence the first three members of the sequence are $a_1, a_2, (a_1a_2)^{-1}$. The next member of the sequence is now determined as

$$a_3 = \frac{1}{a_2 a_4} \qquad \Longleftrightarrow \qquad a_4 = \frac{1}{a_2 a_3} = \frac{1}{a_2(a_1a_2)^{-1}} = a_1.$$

For the fifth member we thus compute

$$a_4 = \frac{1}{a_3 a_5} \qquad \Longleftrightarrow \qquad a_5 = \frac{1}{a_3 a_4} = \frac{1}{(a_1a_2)^{-1}a_1} = a_2.$$

From this we conclude that the above requirement leads to the periodic sequence

$$a_1, a_2, \frac{1}{a_1a_2}, a_1, a_2, \frac{1}{a_1a_2}, \ldots \tag{1}$$

with period 3; here a_1, a_2 are non-zero complex numbers.

1.2. The logarithm. Based on the sequence (1), we now ask for smooth, complex valued functions f on $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, which are characterized by the property that the sum of the values of f at any three consecutive elements of the sequence (1) is equal to zero, i.e., the function f satisfies the functional equation

$$f(a_1) + f(a_2) + f\left(\frac{1}{a_1a_2}\right) = 0. \tag{2}$$

From complex analysis we know that the principal logarithm $\mathrm{Log}(z)$ ($z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$) with the power series expansion

$$\mathrm{Log}(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n} \qquad (z \in \mathbb{C} :\ |z-1| < 1) \tag{3}$$

solves the functional equation (2) up to integral multiples of $2\pi i$ (see, e.g., [4]). By replacing the principal logarithm by its real part, i.e., by

$$\mathrm{Re}\big(\mathrm{Log}(z)\big) = \log(|z|),$$

we obtain a solution of the functional equation (2), which is unique up to scaling. By the normalization condition $f(\mathrm{e}) = 1$, the solution becomes unique.

1.3. Geometric interpretation. We now give a geometric interpretation of the results derived in the preceding two sections. The underlying geometry is given by the hyperbolic geometry of the hyperbolic plane (for which we refer to [2]) in the model of the upper half-plane

$$\mathbb{H}^2 := \{z \in \mathbb{C} \,|\, z = x + iy,\, y > 0\}$$

equipped with the metric

$$\mathrm{d}s_{\mathrm{hyp}} := \frac{|\mathrm{d}z|}{y}\,.$$

The isometry group of $\mathbb{H}^2$ equipped with the metric $\mathrm{d}s_{\mathrm{hyp}}$ is given by the special projective linear group $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on $\mathbb{H}^2$ as follows: For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$ and $z \in \mathbb{H}^2$, the image $\gamma(z) \in \mathbb{H}^2$ is given by

$$\gamma(z) = \frac{az + b}{cz + d}\,.$$

The hyperbolic distance between two points $z_1, z_2 \in \mathbb{H}^2$ is defined as

$$\mathrm{dist}_{\mathrm{hyp}}(z_1, z_2) := \inf_{\eta} \int_{\eta} \mathrm{d}s_{\mathrm{hyp}}\,, \tag{4}$$

where the infimum is taken over all continuously differentiable paths $\eta\colon [0,1] \to \mathbb{H}^2$ satisfying $\eta(0) = z_1$ and $\eta(1) = z_2$. In order to give an explicit formula for the hyperbolic distance (4), one shows in a first step that the hyperbolic distance from the point $i\lambda_1$ to $i\lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_2 \geq \lambda_1$ is given by $\log(\lambda_2/\lambda_1)$. In fact, this is easily seen: Given a path $\eta(t) = x(t) + iy(t)$ ($t \in [0,1]$, $\eta(0) = i\lambda_1$, $\eta(1) = i\lambda_2$), we have

$$\mathrm{dist}_{\mathrm{hyp}}(i\lambda_1, i\lambda_2) = \inf_{\eta} \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,\mathrm{d}t = \int_0^1 \frac{|y'(t)|}{y(t)}\,\mathrm{d}t = \Big[\log\big(y(t)\big)\Big]_0^1 = \log\left(\frac{\lambda_2}{\lambda_1}\right),$$

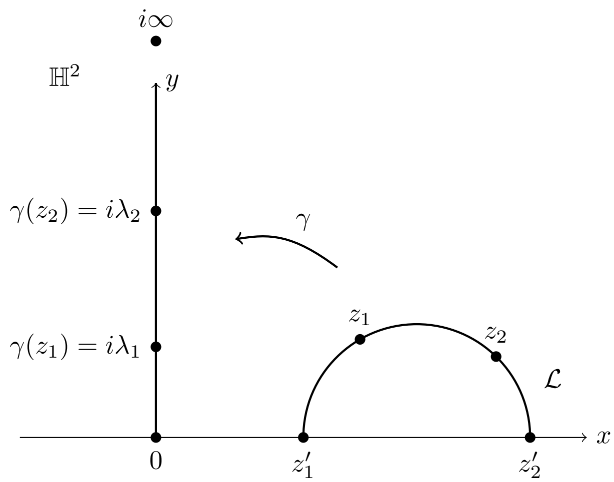
since the infimum, in fact the minimum, is achieved for the path satisfying $x'(t) = 0$ for all $t \in [0,1]$, i.e., for the path $\eta(t) = iy(t)$. In a second step one then proves the existence of an element $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ with the property $\gamma(z_1) = i\lambda_1$ and $\gamma(z_2) = i\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_2 \geq \lambda_1$ satisfy

$$\frac{\lambda_2}{\lambda_1} = \frac{1 + \left|\frac{z_1 - z_2}{z_1 - \bar{z}_2}\right|}{1 - \left|\frac{z_1 - z_2}{z_1 - \bar{z}_2}\right|}\,. \tag{5}$$

Using the invariance of the hyperbolic distance by the action of $\mathrm{PSL}_2(\mathbb{R})$, we find that the hyperbolic distance between z_1 and z_2 equals the *logarithm* of the ratio λ_2/λ_1 given by formula (5), i.e., we have the relation

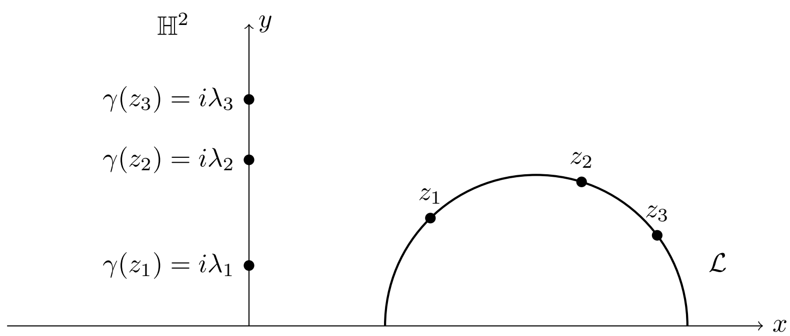
$$\mathrm{dist}_{\mathrm{hyp}}(z_1, z_2) = \log(\lambda_2/\lambda_1).$$

After having "rediscovered" the logarithm as the hyperbolic distance between two points in the upper half-plane $\mathbb{H}^2$, we next provide a hyperbolic geometric interpretation characterizing the naive sequence (1), which underlies the functional equation of the logarithm.



To this end, we recall that the boundary of the upper half-plane $\mathbb{H}^2$ is given by the real line $\mathbb{R}$, the x-axis, together with the point $i\infty$ at infinity, i.e., the real projective line $\mathbb{P}^1(\mathbb{R})$. Furthermore, the geodesics in the upper half-plane are given by Euclidean semicircles in $\mathbb{H}^2$, which are orthogonal to the x-axis, or straight lines in $\mathbb{H}^2$, which are parallel to the y-axis. Therefore, the unique geodesic passing through z_1 and z_2 is given by the semicircle $\mathcal{L}$ from z_1' to z_2' as depicted in the preceding figure. Using an appropriate $\gamma \in \mathrm{PSL}_2(\mathbb{R})$, the geodesic $\mathcal{L}$ can be mapped onto the geodesic from 0 to $i\infty$ such that $\gamma(z_1) = i\lambda_1$ and $\gamma(z_2) = i\lambda_2$.

Let us now connect our geometric insight to the naive sequence (1). For this we give ourselves three complex numbers $z_1, z_2, z_3 \in \mathbb{H}^2$ lying on a hyperbolic line $\mathcal{L}$ as depicted below:



Denoting by $[z_j, z_k]$ ($j, k = 1, 2, 3$ with $j \neq k$) the closed segment of $\mathcal{L}$ consisting of all the points of $\mathcal{L}$ lying between z_j and z_k, we may assume without loss of generality that $z_2 \in [z_1, z_3]$ as shown in the above figure. By means of formula (5) we are now able to determine real numbers

$$\lambda_1,\, \lambda_2,\, \lambda_3 \in \mathbb{R} \quad \text{satisfying} \quad \lambda_1 \leq \lambda_2 \leq \lambda_3$$

such that $\gamma(z_j) = i\lambda_j$ ($j = 1, 2, 3$) for a suitable $\gamma \in \mathrm{PSL}_2(\mathbb{R})$. Now, we introduce the positive real numbers $a_j := \lambda_{j+1}/\lambda_j$ ($j = 1, 2, 3$), where the indices are to be read modulo 3, that is,

$$a_1 = \frac{\lambda_2}{\lambda_1}, \quad a_2 = \frac{\lambda_3}{\lambda_2}, \quad a_3 = \frac{\lambda_1}{\lambda_3}\,.$$

Since we obviously have $a_3 = (a_1 a_2)^{-1}$, the sequence $a_1, a_2, a_3, \ldots$ gives rise to the naive sequence (1). The functional equation (2), i.e.,

$$\log(a_1) + \log(a_2) + \log(a_3) = 0 \quad \Longleftrightarrow \quad \log\left(\frac{\lambda_2}{\lambda_1}\right) + \log\left(\frac{\lambda_3}{\lambda_2}\right) = \log\left(\frac{\lambda_3}{\lambda_1}\right)$$

now becomes equivalent to the obvious additive relation among the hyperbolic distances between the points z_1, z_2, and z_3 lying on the geodesic $\mathcal{L}$, namely

$$\mathrm{dist}_{\mathrm{hyp}}(z_1, z_2) + \mathrm{dist}_{\mathrm{hyp}}(z_2, z_3) = \mathrm{dist}_{\mathrm{hyp}}(z_1, z_3).$$

2 The dilogarithm

2.1. A more sophisticated sequence. Next, we slightly vary the construction of the sequence studied in subsection 1.1 by switching from the multiplicative to the additive viewpoint. More specifically, again following D. Zagier, we now ask for a sequence of non-zero complex numbers characterized by the property that each element of the sequence is given as one *minus* the product of the left-hand and right-hand neighbors of the element in question. Denoting our sequence by $a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots$, the defining properties of the sequence give

$$a_2 = 1 - a_1 a_3 \quad \Longleftrightarrow \quad a_3 = \frac{1-a_2}{a_1},$$

hence the first three members of the sequence are $a_1, a_2, (1-a_2)/a_1$. The next member of the sequence is now determined as

$$a_3 = 1 - a_2 a_4 \quad \Longleftrightarrow \quad a_4 = \frac{1-a_3}{a_2} = \frac{a_1+a_2-1}{a_1 a_2}.$$

For the fifth member we thus compute

$$a_4 = 1 - a_3 a_5 \quad \Longleftrightarrow \quad a_5 = \frac{1-a_4}{a_3} = \frac{1-\frac{a_1+a_2-1}{a_1a_2}}{\frac{1-a_2}{a_1}} = \frac{1-a_1}{a_2}.$$

Turning to the sixth member of the sequence, we compute

$$a_5 = 1 - a_4 a_6 \quad \Longleftrightarrow \quad a_6 = \frac{1-a_5}{a_4} = \frac{1-\frac{1-a_1}{a_2}}{\frac{a_1+a_2-1}{a_1a_2}} = a_1.$$

It is now not a surprise that the seventh member in sequence is given by a_2, namely we have

$$a_6 = 1 - a_5 a_7 \quad \Longleftrightarrow \quad a_7 = \frac{1-a_6}{a_5} = \frac{1-a_1}{\frac{1-a_1}{a_2}} = a_2.$$

From this we conclude that the above requirement leads to the periodic sequence

$$a_1, a_2, \frac{1-a_2}{a_1}, \frac{a_1+a_2-1}{a_1a_2}, \frac{1-a_1}{a_2}, a_1, a_2, \frac{1-a_2}{a_1}, \frac{a_1+a_2-1}{a_1a_2}, \frac{1-a_1}{a_2}, \ldots \tag{6}$$

with period 5; here a_1, a_2 are non-zero complex numbers.

2.2. The dilogarithm. In analogy to the situation of the first section relating the sequence (1) to the logarithm, we now ask for smooth, complex valued functions f on $\mathbb{C}^\times$, which are characterized by the property that the sum of the values of f at any five consecutive elements of the sequence (6) is equal to zero, i.e., the function f satisfies the functional equation

$$f(a_1) + f(a_2) + f\left(\frac{1-a_2}{a_1}\right) + f\left(\frac{a_1+a_2-1}{a_1a_2}\right) + f\left(\frac{1-a_1}{a_2}\right) = 0. \tag{7}$$

Inspired by the solution in the case of the previous section, namely the power series expansion (3) with z replaced by $1-z$, i.e.,

$$-\mathrm{Log}(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad (z \in \mathbb{C} : \, |z| < 1),$$

we now consider the dilogarithm $\mathrm{Li}_2(z)$ defined by the power series expansion

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (z \in \mathbb{C} : \, |z| < 1); \tag{8}$$

the definition of $\mathrm{Li}_2(z)$ on the whole complex plane then follows by analytic continuation. As in the case of the logarithm, this function is multi-valued and it turns out not to satisfy the functional equation (7). However, a suitable combination of the dilogarithm $\mathrm{Li}_2(z)$ and the solution of the functional equation (2), i.e., $\log(|z|)$, leads to a positive answer of our question. Consider the so-called Bloch–Wigner dilogarithm $D(z)$, defined as

$$D(z) := \mathrm{Im}\big(\mathrm{Li}_2(z)\big) + \arg(1-z)\log(|z|)$$

for $z \in \mathbb{C}\backslash\mathbb{R}$; for $z \in \mathbb{R}$, we put $D(z) = 0$. It can then be shown that the Bloch–Wigner dilogarithm $D(z)$ provides a solution of the functional equation

$$D(a_1) + D(a_2) + D(a_3) + D(a_4) + D(a_5) = 0 \tag{9}$$

for the generating members a_1, a_2, a_3, a_4, a_5 of the 5-periodic sequence (6), which is unique up to scaling. For later purposes, we note that the Bloch–Wigner dilogarithm $D(z)$ satisfies several other functional equations, such as

$$D\left(\frac{1}{z}\right) = -D(z), \quad D\left(1-\frac{1}{z}\right) = D(z), \tag{10}$$

which are formal consequenes of (9). For more details, we refer the reader to [9], pp. 8–11.

2.3. Geometric interpretation. In analogy to subsection 1.3, we now give a geometric interpretation of the results derived in the preceding two sections. The underlying geometry is now given by the hyperbolic geometry of the hyperbolic 3-space (for which we refer to [3]) in the model of the upper half-space

$$\mathbb{H}^3 := \{(z,w) \in \mathbb{C}\times\mathbb{R} \,|\, z = x+iy,\, w > 0\}$$

equipped with the metric

$$\mathrm{d}s_{\mathrm{hyp}} := \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}w^2}}{w}\,. \tag{11}$$

The line element (11) gives rise to the volume element

$$\mathrm{d}\mu_{\mathrm{hyp}} := \frac{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}w}{w^3}\,,$$

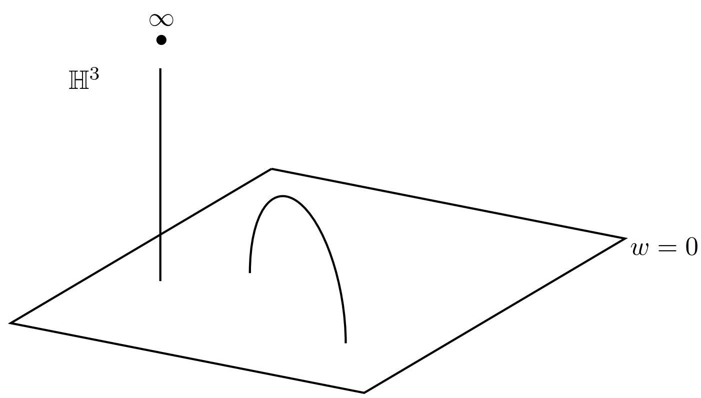
which allows to compute volumes with respect to the hyperbolic metric.

The isometry group of $\mathbb{H}^3$ equipped with the metric $\mathrm{d}s_{\mathrm{hyp}}$ is given by the special projective linear group $\mathrm{PSL}_2(\mathbb{C})$ acting by fractional linear transformations on $\mathbb{H}^3$ as follows: For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{C})$ and $(z,w) \in \mathbb{H}^3$, the image $\gamma(z,w) \in \mathbb{H}^3$ is given by

$$\gamma(z,w) = \left(\frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}w^2}{|cz+d|^2 + |c|^2w^2}\,, \frac{w}{|cz+d|^2 + |c|^2w^2}\right).$$

In analogy to the 1-dimensional case, we will now see that the Bloch–Wigner dilogarithm $D(z)$ computes the hyperbolic volumes of certain bodies in $\mathbb{H}^3$.

To this end, we recall that the boundary of the upper half-space $\mathbb{H}^3$ consists of the complex plane $\mathbb{C}$ given by $\{(z,w) \in \mathbb{H}^3 \,|\, w = 0\}$ and the point ∞ at infinity, i.e., it is given by the complex projective line $\mathbb{P}^1(\mathbb{C})$; to simplify our notation, we will write z instead of $(z,0)$ for points on the boundary $\mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$. Furthermore, the geodesics in the upper half-space are given by Euclidean (possibly degenerate) semicircles in $\mathbb{H}^3$, which are orthogonal to the boundary, as depicted below:



An ideal hyperbolic tetrahedron is a geodesic 3-simplex all whose vertices z_1, z_2, z_3, z_4 lie on the boundary $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{H}^3$. We denote it by $T(z_1, z_2, z_3, z_4)$, which keeps track of the ordering of the vertices and thus of the underlying orientation of the ideal hyperbolic tetrahedron. We note that for a given $\gamma \in \mathrm{PSL}_2(\mathbb{C})$, the 3-simplex $T\big(\gamma(z_1), \gamma(z_2), \gamma(z_3), \gamma(z_4)\big)$ is again an ideal hyperbolic tetrahedron of the same hyperbolic volume as $T(z_1, z_2, z_3, z_4)$. Introducing the cross-ratio

$$[z_1 : z_2 : z_3 : z_4] := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in \mathbb{P}^1(\mathbb{C}),$$

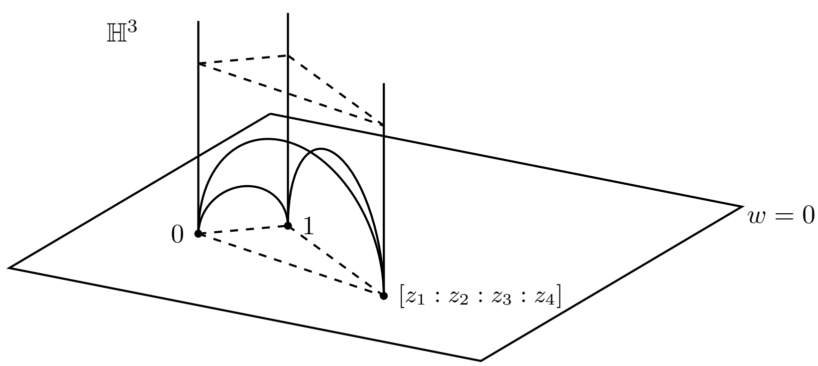
and choosing $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ such that

$$\gamma(z_1) = \infty,\ \gamma(z_2) = 0,\ \gamma(z_3) = 1,$$

a short calculation yields

$$T\big(\gamma(z_1), \gamma(z_2), \gamma(z_3), \gamma(z_4)\big) = T\big(\infty, 0, 1, [z_1 : z_2 : z_3 : z_4]\big).$$

This shows that in order to compute the hyperbolic volume of an ideal hyperbolic tetrahedron $T(z_1, z_2, z_3, z_4)$, one is reduced to calculate the hyperbolic volume of $T\big(\infty, 0, 1, [z_1 : z_2 : z_3 : z_4]\big)$, which looks as follows:



As predicted by the preceding consideration, the hyperbolic volume of $T(z_1, z_2, z_3, z_4)$ depends only on the cross-ratio $[z_1 : z_2 : z_3 : z_4]$. A theorem going back to Lobachevsky now states that this volume equals the *Bloch–Wigner dilogarithm* of the corresponding cross-ratio, i.e., we have the relation

$$\mathrm{vol}_{\mathrm{hyp}}\big(T(z_1, z_2, z_3, z_4)\big) = D\big([z_1 : z_2 : z_3 : z_4]\big),$$

taking into account the orientation, namely, the volume is positive if the orientation is and negative otherwise. From this and the definition of the dilogarithm we see that the ideal hyperbolic tetrahedron $T(z_1, z_2, z_3, z_4)$ is flat if and only if the cross-ratio is real. Furthermore, we see that the orientation of $T(z_1, z_2, z_3, z_4)$ given by the ordering of its vertices agrees with the orientation induced from $\mathbb{H}^3$ if and only if the cross-ratio has positive imaginary part.

After having "rediscovered" the Bloch–Wigner dilogarithm as the hyperbolic volume of an ideal hyperbolic tetrahedron in the upper half-space $\mathbb{H}^3$, we next provide a hyperbolic geometric interpretation characterizing the naive sequence (6), which underlies the functional equation of the Bloch–Wigner dilogarithm.

For this we give ourselves five complex numbers z_1, z_2, z_3, z_4, z_5 lying on the boundary $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{H}^3$. We then introduce the complex numbers $b_j := [z_1 : \ldots : \hat{z}_j : \ldots : z_5]$ $(j = 1, \ldots, 5)$, where the entry marked with the hat has to be omitted, i.e., we consider the cross-ratios

$$\begin{aligned} &b_1 = [z_2 : z_3 : z_4 : z_5], \quad b_2 = [z_1 : z_3 : z_4 : z_5], \quad b_3 = [z_1 : z_2 : z_4 : z_5], \\ &b_4 = [z_1 : z_2 : z_3 : z_5], \quad b_5 = [z_1 : z_2 : z_3 : z_4]. \end{aligned}$$

Now, letting

$$a_1 := \frac{1}{b_1}, \quad a_2 := 1 - \frac{1}{b_2}, \quad a_3 := \frac{1}{b_3}, \quad a_4 := 1 - \frac{1}{b_4}, \quad a_5 := \frac{1}{b_5}, \tag{12}$$

and reading indices modulo 5, it can easily be shown that the recursion formula

$$a_j = 1 - a_{j-1}a_{j+1} \quad (j = 1, \ldots, 5)$$

holds. For example, if $j = 2$, one has to show that $a_2 = 1 - a_1 a_3$, which, using (12), amounts to show that $b_2 = b_1 b_3$. The latter equality can be immediately verified, since we have

$$\begin{aligned} b_1 \cdot b_3 &= [z_2 : z_3 : z_4 : z_5] \cdot [z_1 : z_2 : z_4 : z_5] \\ &= \frac{(z_2 - z_4)(z_3 - z_5)}{(z_2 - z_5)(z_3 - z_4)} \cdot \frac{(z_1 - z_4)(z_2 - z_5)}{(z_1 - z_5)(z_2 - z_4)} \\ &= \frac{(z_1 - z_4)(z_3 - z_5)}{(z_1 - z_5)(z_3 - z_4)} = [z_1 : z_3 : z_4 : z_5] = b_2. \end{aligned}$$

Consequently, the sequence a_1, a_2, a_3, a_4, a_5 above gives rise to the 5-periodic sequence (6).

After these preliminary computations, we now find using the functional equations (10) that

$$\begin{aligned} &D(b_1) - D(b_2) + D(b_3) - D(b_4) + D(b_5) \\ &\quad = -D\left(\frac{1}{b_1}\right) - D\left(1 - \frac{1}{b_2}\right) - D\left(\frac{1}{b_3}\right) - D\left(1 - \frac{1}{b_4}\right) - D\left(\frac{1}{b_5}\right) \\ &\quad = -D(a_1) - D(a_2) - D(a_3) - D(a_4) - D(a_5) = 0, \end{aligned}$$

taking into account the functional equation (9). In other words, we have proven that the functional equation (9) is equivalent to the following additive relation among the hyperbolic volumes of the ideal hyperbolic tetrahedra $T(z_1, \ldots, \hat{z}_j, \ldots, z_5)$ $(j = 1, \ldots, 5)$, namely

$$\sum_{j=1}^{5} (-1)^j \operatorname{vol}_{\mathrm{hyp}}\big(T(z_1, \ldots, \hat{z}_j, \ldots, z_5)\big) = 0. \tag{13}$$

The relation (13) can also be checked directly geometrically by taking five complex numbers z_1, z_2, z_3, z_4, z_5 lying on the boundary $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{H}^3$ and then verifying that the alternating sum of the hyperbolic volumes of the five ideal hyperbolic tetrahedra $T(z_1, \ldots, \hat{z}_j, \ldots, z_5)$ vanishes. This fact can be *seen* quite easily, if we choose

$$z_1 = 0, z_2 = z', z_3 = 1, z_4 = \infty, z_5 = z,$$

where we can pick z' without loss of generality in such a way that it lies inside of the Euclidean triangle determined by z_1, z_3, z_5. We then find that the union of the two ideal hyperbolic tetrahedra

$$T(z_1, \hat{z}_2, z_3, z_4, z_5), \, T(z_1, z_2, z_3, \hat{z}_4, z_5)$$

equals the union of the three ideal hyperbolic tetrahedra

$$T(\hat{z}_1, z_2, z_3, z_4, z_5), \, T(z_1, z_2, \hat{z}_3, z_4, z_5), \, T(z_1, z_2, z_3, z_4, \hat{z}_5),$$

from which the claimed relation immediately follows.

3 Special values of zeta functions

In this section we illustrate how the logarithm and dilogarithm together with their geometric interpretations presented in the previous sections as well as their generalizations, called polylogarithms, can be used to compute special values of zeta functions attached to number fields. However, in general, one is still far from having complete results and thus these special values are to be viewed as mysterious quantities.

For this section, we assume that the reader has some basic knowledge of algebraic number theory as one can find it for example in the book [6] of S. Lang.

3.1. Zeta functions of number fields. The Riemann zeta function $\zeta(s)$ is defined by the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. This function is known to have a meromorphic continuation to the whole complex s-plane with a simple pole at $s = 1$ with residue 1, which can be restated as

$$\mathrm{res}_{s=1}\zeta(s) = \lim_{s\to 1}(s-1)\zeta(s) = 1.$$

The Riemann zeta function encodes basic arithmetic properties of the integers such as the fundamental theorem of arithmetic by means of its Euler product expansion and the infinitude of the number of primes through its pole at $s = 1$.

The special values of $\zeta(s)$ at positive even integers have been computed already by L. Euler as

$$\zeta(2m) = \frac{(-1)^{m-1}(2\pi)^{2m}B_{2m}}{2(2m)!} \qquad (m \in \mathbb{N}_{>0}),$$

where the B_{2m}'s are the so-called Bernoulli numbers, which are rational numbers. In other words, the special values $\zeta(2m)$ are rational multiples of even powers of π, i.e., $\zeta(2m) \in \mathbb{Q}[\pi^2]$. In contrast to the special values of $\zeta(s)$ at positive even integers, their values at positive odd integers are still quite mysterious. For example, it was only in the 80's of the last century that R. Apéry was able to prove that the special value $\zeta(3)$ is irrational (see [1]). Although alternative proofs of this result have meanwhile been found, none of them seems to generalize to other odd zeta values.

More generally, let now $F/\mathbb{Q}$ be a number field. The Dedekind zeta function $\zeta_F(s)$ attached to F is defined as

$$\zeta_F(s) := \sum_{\mathfrak{a}\subseteq O_F} \frac{1}{N(\mathfrak{a})^s},$$

where again $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and the sum is taken over all non-trivial integral ideals $\mathfrak{a}$ contained in the ring of integers O_F of F; here, $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$, which is given by the quantity $|O_F/\mathfrak{a}|$. One easily checks that $\zeta(s) = \zeta_{\mathbb{Q}}(s)$. As in the case of the Riemann zeta function it can be shown that the Dedekind zeta function $\zeta_F(s)$ has a meromorphic continuation to the whole complex s-plane with a simple pole at $s = 1$. Also in this more general case, the Dedekind zeta function encodes important arithmetic information about the ring of integers O_F, such as the unique factorization of integral ideals into powers of prime ideals and the infinitude of the number of prime ideals. As will be explained below, it is expected that the logarithm, dilogarithm and, more generally, the polylogarithms will allow to "compute" special values of Dedekind zeta functions of number fields at positive integers.

3.2. The logarithm and the residue of $\zeta_F(s)$ at $s = 1$. In this paragraph we will show how the logarithm and its geometric characterization by means of hyperbolic lengths of geodesics in the upper half-plane $\mathbb{H}^2$ presented in the first section is encoded in the residue of the Dedekind zeta function $\zeta_F(s)$ at $s = 1$. To illustrate this phenomenon, it is best to start with the case of a real quadratic extension $F = \mathbb{Q}(\sqrt{D})$, where $D \in \mathbb{N}$ is a positive discriminant, i.e., $D \equiv 0$ or $1 \bmod 4$. The group of units $O_F^{\times}$ of the ring of integers O_F of F is generated, modulo sign, by the so-called fundamental unit $\varepsilon_D > 1$. By the basic theory of integral binary quadratic forms it can then be shown that there exists $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$\delta := \gamma \begin{pmatrix} \varepsilon_D & 0 \\ 0 & 1/\varepsilon_D \end{pmatrix} \gamma^{-1} \in \mathrm{PSL}_2(\mathbb{Z}),$$

and we see that the hyperbolic distance between the point $z := \gamma(i) \in \mathbb{H}^2$ and $\delta(z) \in \mathbb{H}^2$ is given as

$$\mathrm{dist}_{\mathrm{hyp}}(z, \delta(z)) = \mathrm{dist}_{\mathrm{hyp}}(i, \varepsilon_D^2 i) = 2\log(\varepsilon_D).$$

Projecting the geodesic line from z to $\delta(z)$ to the quotient space $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$, it thus turns out that we obtain a primitive closed geodesic of hyperbolic length $2\log(\varepsilon_D)$. The detailed analysis in [7] in fact shows that *all* the primitive closed geodesics on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$ are obtained in this way, where a fixed real quadratic extension $F = \mathbb{Q}(\sqrt{D})$ together with its fundamental unit ε_D gives rise to exactly h_F different primitive closed geodesics, with h_F denoting the class number of F. The residue of $\zeta_F(s)$ at $s = 1$ now counts the lengths of all the h_F primitive closed geodesics weighted by the square-root of the discriminant, i.e., we have the formula

$$\mathrm{res}_{s=1}\zeta_F(s) = \lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2\,h_F \log(\varepsilon_D)}{\sqrt{D}}\,. \tag{14}$$

Turning now to the general case, we let $F/\mathbb{Q}$ be any number field with ring of integers O_F and discriminant D_F. It is known that F comes with r_1 real embeddings and r_2 pairs of complex embeddings, where $[F : \mathbb{Q}] = r_1 + 2r_2$. Dirichlet's unit theorem now states that the group $O_F^\times$ of units modulo the roots of unity in $O_F^\times$ is a free abelian group generated by $r := r_1 + r_2 - 1$ units $\varepsilon_1, \ldots, \varepsilon_r$. Denoting the real and half of the complex embeddings by $\sigma_1, \ldots, \sigma_{r_1+r_2}$, the residue of $\zeta_F(s)$ at $s = 1$ is finally given by the so-called class number formula, which reads

$$\mathrm{res}_{s=1}\zeta_F(s) = \lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \det\big((\log|\sigma_j(\varepsilon_k)|^{e_j})_{j,k=1,\ldots,r}\big)}{\omega_F\,\sqrt{|D_F|}}\,, \tag{15}$$

where ω_F is the number of roots of unity in $O_F^\times$ and $e_j = 1$, if σ_j is a real embedding, and $e_j = 2$ otherwise. One easily checks that formula (14) is a special case of the general class number formula (15), and we also recognize that the special value of $\zeta_F(s)$, or rather its residue, at $s = 1$ can be expressed in terms of logarithms evaluated at distinguished elements of F.

3.3. The dilogarithm and the value of $\zeta_F(s)$ at $s = 2$. In this paragraph we will show how the dilogarithm and its geometric characterization by means of hyperbolic volumes of ideal hyperbolic tetrahedra in the upper half-space $\mathbb{H}^3$ presented in the second section is encoded in the value of the Dedekind zeta function $\zeta_F(s)$ at $s = 2$. To illustrate this fact, it is best to start with the case of an imaginary quadratic extension $F = \mathbb{Q}(\sqrt{-D})$, where $-D \in \mathbb{Z}$ is a negative discriminant, i.e., $D \equiv 0$ or $3 \bmod 4$. It is then known that the Bianchi group $\mathrm{PSL}_2(O_F)$, with O_F denoting again the ring of integers of F, acts strongly discontinuously on $\mathbb{H}^3$ and the resulting quotient space $\mathrm{PSL}_2(O_F)\backslash\mathbb{H}^3$ is an oriented hyperbolic 3-orbifold (i.e., a hyperbolic 3-manifold with the exception of finitely many singular points due to the torsion elements of $\mathrm{PSL}_2(O_F)$) of finite volume. This hyperbolic volume has been computed by M. G. Humbert, who found the formula

$$\mathrm{vol}_{\mathrm{hyp}}\big(\mathrm{PSL}_2(O_F)\backslash\mathbb{H}^3\big) = \frac{D\sqrt{D}}{4\pi^2}\,\zeta_F(2).$$

Since, on the other hand, every hyperbolic 3-manifold or 3-orbifold can be triangulated into ideal hyperbolic tetrahedra, the volume $\mathrm{vol}_{\mathrm{hyp}}\big(\mathrm{PSL}_2(O_F)\backslash\mathbb{H}^3\big)$ can also be expressed in terms of special values of dilogarithms. For example, for $D = 3$, one is thus led to the formula

$$\begin{aligned}\zeta_{\mathbb{Q}(\sqrt{-3})}(2) &= \frac{2}{27}\,\pi^2\sqrt{3}\,D\Big(\frac{1+\sqrt{-3}}{2}\Big)\\ &\underset{\mathbb{Q}^\times}{\sim}\ \pi^2\sqrt{3}\,D\Big(\frac{1+\sqrt{-3}}{2}\Big),\end{aligned}$$

where the latter notation means that equality holds up to a non-zero rational number. Turning now to the general case, where $F/\mathbb{Q}$ is any number field with ring of integers O_F and discriminant D_F, and letting $\sigma_1, \dots, \sigma_{r_2}$ denote the r_2 complex embeddings, the value of $\zeta_F(s)$ at $s = 2$ is given as

$$\zeta_F(2) \underset{\mathbb{Q}^\times}{\sim} \pi^{2(r_1+r_2)} \sqrt{|D_F|} \det \big(D(\sigma_j(\xi_k))_{j,k=1,\dots,r_2} \big), \tag{16}$$

where $\xi_1, \dots, \xi_{r_2}$ are suitable elements of F. We thus recognize that the special value of $\zeta_F(s)$ at $s = 2$ can indeed be expressed in terms of dilogarithms evaluated at distinguished elements of F.

3.4. Zagier's conjecture. In order to state this conjecture about the values of the Dedekind zeta function $\zeta_F(s)$ of number fields F at positive integers $s = m$, we introduce the m-th polylogarithm by generalizing the series expansion (8) of the dilogarithm to

$$\mathrm{Li}_m(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^m} \quad (z \in \mathbb{C} : \ |z| < 1),$$

and – in analogy to the Bloch–Wigner dilogarithm – the corrected function

$$\mathcal{L}_m(z) := \mathrm{Re}_m \Bigg(\sum_{j=0}^{m-1} \frac{2^j B_j}{j!} \log(|z|)^j \, \mathrm{Li}_{m-j}(z) \Bigg),$$

where Re_m equals the real or the imaginary part, when m is odd or even, respectively, and B_j refer again to the Bernoulli numbers. One immediately verifies that

$$\mathcal{L}_1(z) = -\log(|1-z|) \quad \text{and} \quad \mathcal{L}_2(z) = D(z).$$

Letting $n_+ := r_1 + r_2$ and $n_- := r_2$, as well as $\pm = (-1)^{m-1}$ (and thus $\mp = (-1)^m$), Zagier's conjecture (see [8]) then states that

$$\zeta_F(m) \underset{\mathbb{Q}^\times}{\sim} \pi^{mn_\mp} \sqrt{|D_F|} \det \big(\mathcal{L}_m(\sigma_j(\xi_k))_{j,k=1,\dots,n_\pm} \big); \tag{17}$$

here D_F is the discriminant of F, σ_j denote the r_1 real and the r_2 complex embeddings of F, and $\xi_1, \dots, \xi_{n_\pm}$ are suitable elements of F. It is straightforward to check that Zagier's conjectured formula (17) coincides with formulas (15) and (16), if $m = 1$ and $m = 2$, respectively. This conjecture has been proven for $m = 3$ by A. B. Goncharov (see [5]). The general case remains open to this date and is the subject of current research.

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