Abstracts

Riemann–Roch isometries in the non-compact orbifold setting

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(joint work with Gerard Freixas i Montplet)

1. Introduction

A fundamental result in intersection theory is the arithmetic Riemann–Roch theorem for arithmetic varieties by Gillet and Soulé [6]. This theorem developed from previous versions by Faltings [3] and Deligne [2], who treated the case of arithmetic surfaces. Deligne's isometry and the arithmetic Riemann–Roch theorem both require the vector bundles to be endowed with smooth hermitian metrics. However, many cases of arithmetic interest do not satisfy this assumption, for example, the case of a modular curve, when considering the trivial bundle and the dualizing sheaf endowed with the Poincaré metric. Already in this case the metric is singular at the cusps and the elliptic fixed points, and the results of Deligne and Gillet–Soulé do not apply to this setting. In presence only of cusps, hence excluding elliptic fixed points, Freixas proved a version of the arithmetic Riemann–Roch theorem for the trivial sheaf on a modular curve [4]. His method of proof has the drawback that it cannot be adapted to the presence of elliptic fixed points and does not carry over to more general bundles or to higher dimensions. Therefore, one needs to develop new ideas that are better suited to these more general settings.

2. Statement of the main theorem

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind. The quotient space $\Gamma\backslash\mathbb{H}$ admits a canonical structure of a Riemann surface. The points with non-trivial automorphisms are called elliptic fixed points. By adding a finite number of cusps, the Riemann surface $\Gamma\backslash\mathbb{H}$ can be completed into a compact Riemann surface X. We denote the set of elliptic fixed points and cusps by $p_1, \ldots, p_n$, and assign to them multiplicities $m_1, \ldots, m_n \in \mathbb{N} \cup \{\infty\}$. The multiplicity of a cusp is ∞, while for an elliptic fixed point it is the order of its automorphism group. We set $m := \prod_{m_i<\infty} m_i$. The hyperbolic metric on $\mathbb{H}$ is given by

$$ds^2_{\mathrm{hyp}} = \frac{dx^2 + dy^2}{y^2},$$

where $x + iy$ is the usual parametrization of $\mathbb{H}$. As a metric on X, it has singularities at the cusps and the elliptic fixed points.

In the recent preprint [5], we generalize the work of Deligne and Gillet–Soulé to the case of the trivial sheaf on X, equipped with the singular hyperbolic metric. Our main theorem relates the determinant of cohomology of the trivial sheaf, with an explicit Quillen type metric in terms of the Selberg zeta function of Γ, to a metrized version of the ψ line bundle of the theory of moduli spaces of pointed orbicurves, and the self-intersection bundle of a suitable twist of the canonical sheaf ω_X.

To be more precise, we consider the hermitian line bundle

$$\psi_W^{\otimes m^2} = \bigotimes_i (\omega_{X,p_i}, \|\cdot\|_{W,p_i})^{m^2(1-m_i^{-2})},$$

carrying the Wolpert metric. The underlying $\mathbb{Q}$-line bundle is denoted by ψ.

Furthermore, the singular hyperbolic metric on X induces a singular hermitian metric on the $\mathbb{Q}$-line bundle

$$\omega_X(D), \quad D := \sum_i \left(1 - \frac{1}{m_i}\right) p_i.$$

By $\omega_X(D)_{\mathrm{hyp}}$ we denote the resulting $\mathbb{Q}$-hermitian line bundle over X. It still fits the L_1^2 formalism of Bost [1], which implies that the metrized Deligne pairing

$$\langle \omega_X(D)_{\mathrm{hyp}}, \omega_X(D)_{\mathrm{hyp}} \rangle$$

is defined. This is a $\mathbb{Q}$-hermitian line bundle over $\operatorname{Spec}\mathbb{C}$.

Finally, the determinant of cohomology of $\mathcal{O}_X$ is the complex line

$$\det H^\bullet(X, \mathcal{O}_X) = \det H^0(X, \mathcal{O}_X) \otimes \det H^1(X, \mathcal{O}_X)^{-1}.$$

We define a Quillen metric on it by rescaling the L^2 metric as follows

$$\|\cdot\|_Q = (C(\Gamma) Z'(1,\Gamma))^{-1/2} \|\cdot\|_{L^2}.$$

Here, $C(\Gamma)$ is a real positive constant, which can be explicitly expressed in terms of the multiplicities m_i, special values of the Riemann zeta function $\zeta(s)$, the genus of X, and the Euler–Mascheroni constant γ (see [5]). Furthermore, $Z(s,\Gamma)$ is the Selberg zeta function of Γ; it is defined, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, by

$$Z(s,\Gamma) = \prod_\gamma \prod_{k=0}^\infty (1 - e^{-(s+k)\ell_{\mathrm{hyp}}(\gamma)})^2,$$

where γ runs over the non-oriented primitive closed geodesics in $\Gamma\backslash\mathbb{H}$, and $\ell_{\mathrm{hyp}}(\gamma)$ denotes the hyperbolic length of γ. The determinant of cohomology together with this Quillen metric will be denoted $\det H^\bullet(X, \mathcal{O}_X)_Q$.

In [5], we prove the following Riemann–Roch isometry.

Theorem 1. There is a canonical isometry of $\mathbb{Q}$-hermitian line bundles

$$\det H^\bullet(X, \mathcal{O}_X)_Q^{\otimes 12} \otimes \psi_W \xrightarrow{\sim} \langle \omega_X(D)_{\mathrm{hyp}}, \omega_X(D)_{\mathrm{hyp}} \rangle. \tag{1}$$

For the proof of Theorem 1 we refer to [5]. The proof makes use of surgery techniques and Mayer–Vietoris type formulae for determinants of Laplacians, Bost's L_1^2-formalism of arithmetic intersection theory, the Selberg trace formula, and exact evaluations of determinants of Laplacians on models of cusps and cones. The explicit computations of the regularized determinants of Laplacians on models of cusps and cones for the singular hyperbolic metric are of independent interest.

3. Arithmetic applications

The advantage of the Riemann–Roch isometry (1) is that it easily leads to arithmetic versions of the Riemann–Roch formula, in the sense of Arakelov geometry. Let K be a number field and $\mathcal{X} \to \mathcal{S} = \operatorname{Spec} \mathcal{O}_K$ a flat and projective regular arithmetic surface. We suppose given sections $\sigma_1, \ldots, \sigma_n$, that are generically disjoint. We also assume that for every complex embedding $\tau \colon K \hookrightarrow \mathbb{C}$, the compact Riemann surface $\mathcal{X}_\tau(\mathbb{C})$ arises as the compactification of a quotient $\Gamma_\tau \backslash \mathbb{H}$, and that the set of elliptic fixed points and cusps is precisely given by the sections. We construct $\mathbb{Q}$-hermitian line bundles over $\mathcal{S}$, with classes in the arithmetic Picard group (up to torsion) $\widehat{\operatorname{Pic}}(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$. We use similar notations as in the complex case. A straightforward application of Theorem 1 then yields the following arithmetic Riemann–Roch formula.

Theorem 2. We have the equality

$$\begin{aligned} 12\,\widehat{\deg}\, H^\bullet(\mathcal{X}, \mathcal{O}_\mathcal{X})_Q - \delta + \widehat{\deg}\,\psi_W =& (\omega_{\mathcal{X}/\mathcal{S}}(D)_{\mathrm{hyp}}, \omega_{\mathcal{X}/\mathcal{S}}(D)_{\mathrm{hyp}}) \\ & - \sum_{i \neq j} \left(1 - \frac{1}{m_i}\right)\left(1 - \frac{1}{m_j}\right)(\sigma_i, \sigma_j)_{\mathrm{fin}}, \end{aligned}$$

where δ measures the bad reduction of $\mathcal{X} \to \mathcal{S}$, and the right most intersection numbers account for the intersections of the sections happening at finite places.

Because our results cover arbitrary Fuchsian groups, we can apply Theorem 2 to cases of arithmetic interest, for example to the case of $\mathbb{P}^1_{\mathbb{Z}}$, seen as an integral model of $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H} \cup \{\infty\}$. This leads to an explicit evaluation (carried out in [5]) of the special value $\log Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$ as a rational expression in

$$\frac{L'(0,\chi_i)}{L(0,\chi_i)}, \frac{L'(0,\chi_\rho)}{L(0,\chi_\rho)}, \frac{\zeta'(0)}{\zeta(0)}, \frac{\zeta'(-1)}{\zeta(-1)}, \gamma, \log 2, \log 3;$$

here, χ_i resp. χ_ρ (with $\rho := e^{2\pi i/3}$) is the quadratic character of $\mathbb{Q}(i)$ and $\mathbb{Q}(\rho)$, respectively. This result can be considered as the analog of the analytic class number formula for $\mathrm{PSL}_2(\mathbb{Z})$.

References

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