HARMONIC MAASS FORMS OF WEIGHT ONE

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Abstract. The object of this paper is to initiate a study of the Fourier coefficients of a weight one harmonic Maass form and relate them to the complex Galois representation associated to a weight one newform, which is the form's image under a certain differential operator. In this paper, our focus will be on weight one dihedral newforms of prime level $p \equiv 3 \pmod 4$. In this case we give properties of the Fourier coefficients that are similar to (and sometimes reduce to) cases of Stark's conjectures on derivatives of L-functions. We also give a new modular interpretation of certain products of differences of singular moduli studied by Gross and Zagier. Finally, we provide some numerical evidence that the Fourier coefficients of a mock-modular form whose shadow is exotic are similarly related to the associated complex Galois representation.

1. Introduction

A harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ is a Maass form for $\Gamma_0(M)$ that is annihilated by the weight k Laplacian and that is allowed to have polar-type singularities in the cusps (see [9]). Associated to such a form f is the weight $2-k$ weakly holomorphic form

$$\xi_k f(z) = 2iy^k \overline{\partial_{\bar{z}} f(z)}. \tag{1.1}$$

The operator ξ_k is related to the weight k Laplacian Δ_k through the identity

$$\Delta_k = \xi_{2-k}\xi_k. \tag{1.2}$$

A special class of harmonic forms has $\xi_k f$ holomorphic in the cusps and hence has a Fourier expansion at ∞ of the shape

$$f(z) = \sum_{n \geq n_0} c^+(n)q^n - \sum_{n\geq 0} \overline{c(n)}\beta_k(n,y)q^{-n}. \tag{1.3}$$

This expansion is unique and absolutely uniformly convergent on compact subsets of $\mathcal{H}$, the upper half-plane. Here $q = e^{2\pi i z}$ with $z = x+iy \in \mathcal{H}$ and $\beta_k(n,y)$ is given for $n > 0$ by

$$\beta_k(n,y) = \int_y^\infty e^{-4\pi n t} t^{-k} dt,$$

while for $k \neq 1$ we have $\beta_k(0,y) = y^{1-k}/(k-1)$ and $\beta_1(0,y) = -\log y$. For such f the Fourier expansion of $\xi_k f$ is simply

$$\xi_k f(z) = \sum_{n\geq 0} c(n)q^n.$$

Following Zagier, the function $\sum_n c^+(n)q^n$ is said to be a mock-modular form with shadow $\sum_{n\geq 0} c(n)q^n$. It is important to observe that a mock-modular form is only determined by its shadow up to the addition of a weakly holomorphic form.

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Some (non-modular) mock-modular forms have Fourier coefficients that are well-known arithmetic functions. Let $\sigma_1(n) = \sum_{m|n} m$ and $H(n)$ be the Hurwitz class number. Then

$$-8\pi \sum_{n\geq 0} \sigma_1(n) q^n \quad \big(\sigma(0) = -\tfrac{1}{24}\big)$$

is mock-modular of weight $k = 2$ for the full modular group with shadow 1 and

$$-16\pi \sum_{n\geq 0} H(n) q^n \quad \big(H(0) = -\tfrac{1}{12}\big)$$

is mock-modular of weight $3/2$ for $\Gamma_0(4)$ with shadow the Jacobi theta series $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$. (See [50]). These two examples will also be useful in our work on the weight one case.

In general, the Fourier coefficients of mock-modular forms are not well understood. For example, in the case of weight 1/2, which includes the mock-theta functions of Ramanujan, there has been considerable progress made by Zwegers in his thesis [52], by Bringmann-Ono [6], and others (see [51] for a good exposition). The Fourier coefficients of mock-modular forms of weight 1/2 whose shadows Shimura-lift to cusp forms attached to elliptic curves have also been shown to be quite interesting by Bruinier and Ono [10] and Bruinier [14]. We remark that mock-modular forms whose shadows are only weakly holomorphic are also of interest (see [19],[20]) but in this paper we only consider those with holomorphic shadows.

The self-dual case $k = 1$ presents special features and difficulties. The Riemann-Roch theorem is without content when $k = 2 - k$, and the existence of cusp forms is a subtle issue. Furthermore, the infinite series representing the Fourier series of weight one harmonic Poincaré series are difficult to handle.

The fact that interesting non-modular mock-modular forms of weight 1 exist follows from work of Kudla, Rapoport and Yang [33]. Suppose that $M = p > 3$ is a prime with $p \equiv 3 \pmod 4$ and that $\chi_p(\cdot) = (\frac{\cdot}{p})$ is the Legendre symbol. Let

$$E_1(z) = \tfrac{1}{2}H(p) + \sum_{n\geq 1} R_p(n) q^n \tag{1.4}$$

be Hecke's Eisenstein series of weight one for $\Gamma_0(p)$ with character χ_p, where for $n > 0$

$$R_p(n) = \sum_{m|n} \chi_p(m). \tag{1.5}$$

It follows from [33] that $\tilde{E}_1(z) := \sum_{n\geq 0} R_p^+(n) q^n$ is mock-modular of weight $k = 1$ with shadow $E_1(z)$, where for $n > 0$ we have

$$R_p^+(n) = -(\log p)\mathrm{ord}_p(n) R_p(n) - \sum_{\chi_p(q)=-1} \log q(\mathrm{ord}_q(n) + 1) R_p(n/q), \tag{1.6}$$

and where $R_p^+(0)$ is a constant.[1] The associated harmonic form is constructed using the s-derivative of the non-holomorphic Hecke-Eisenstein series of weight 1. An arithmetic interpretation of its coefficients is given in [33]: it is shown that $(-2R^+(n) + 2\log p R_p(n))$ is the degree of a certain 0-cycle on an arithmetic curve parametrizing elliptic curves with CM by the ring of integers in $F = \mathbb{Q}(\sqrt{-p})$.

In this paper we will study the Fourier coefficients of mock-modular forms whose shadows are newforms. From now on, we will assume that $M = p > 3$ is a prime. The same methods

[1] Explicitly, $R_p^+(0) = -H(p)(\frac{L'(0,\chi_p)}{L(0,\chi_p)} + \frac{\zeta'(0)}{\zeta(0)} - c)$, where $c = \frac{1}{2}\gamma + \log(4\pi^{3/2})$ and γ is Euler's constant.

can be used to produce similar results when M is composite (see [34] for details). To each $\mathcal{A} \in \mathrm{Cl}(F)$, the class group of F, one can associate a theta series $\vartheta_{\mathcal{A}}(z)$ defined by

$$\vartheta_{\mathcal{A}}(z) := \tfrac{1}{2} + \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F \\ [\mathfrak{a}] \in \mathcal{A}}} q^{\mathrm{N}(\mathfrak{a})} = \sum_{n \geq 0} r_{\mathcal{A}}(n) q^n. \tag{1.7}$$

Hecke showed that $\vartheta_{\mathcal{A}}(z) \in M_1(p, \chi_p)$, the space of weight one holomorphic modular forms for $\Gamma_0(p)$ with character χ_p. Let ψ be a character of $\mathrm{Cl}(F)$ and consider $g_\psi(z) \in M_1(p, \chi_p)$ defined by

$$g_\psi(z) := \sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi(\mathcal{A}) \vartheta_{\mathcal{A}}(z) = \sum_{n \geq 0} r_\psi(n) q^n. \tag{1.8}$$

When $\psi = \psi_0$ is the trivial character, the form $g_{\psi_0}(z)$ is just $E_1(z)$ from (1.4), as a consequence of Dirichlet's fundamental formula

$$R_p(n) = \sum_{\mathcal{A} \in \mathrm{Cl}(F)} r_{\mathcal{A}}(n). \tag{1.9}$$

Otherwise, $g_\psi(z)$ is a newform in $S_1(p, \chi_p)$, the subspace of $M_1(p, \chi_p)$ consisting of cusp forms. Let H be the Hilbert class field of F with ring of integers $\mathcal{O}_H$. The following result shows that the Fourier coefficients of certain mock-modular forms of weight one with shadow $g_\psi(z)$ can be expressed in terms of logarithms of algebraic numbers in H.

Theorem 1.1. *Let $p \equiv 3 \pmod 4$ be a prime with $p > 3$. Let ψ be a non-trivial character of $\mathrm{Cl}(F)$, where $F = \mathbb{Q}(\sqrt{-p})$. Then there exists a weight one mock-modular form*

$$\tilde{g}_\psi(z) = \sum_{n \geq n_0} r_\psi^+(n) q^n$$

with shadow $g_\psi(z)$ such that the following hold.

(i) When $\chi_p(n) = 1$ or $n < -\frac{p+1}{24}$, the coefficient $r_\psi^+(n)$ equals to zero.

(ii) The coefficients $r_\psi^+(n)$ are of the form

$$r_\psi^+(n) = -\tfrac{2}{\kappa_p} \sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log |u(n, \mathcal{A})|, \tag{1.10}$$

where $\kappa_p \in \mathbb{Z}$, $u(n, \mathcal{A})$ are units in $\mathcal{O}_H$ when $n \leq 0$ and algebraic numbers in H when $n > 0$. In addition, κ_p depends only on p and $u(n, \mathcal{A})$ depends only on n and $\mathcal{A}$.

(iii) Let $\sigma_{\mathcal{C}} \in \mathrm{Gal}(H/F)$ be the element associated to the class $\mathcal{C} \in \mathrm{Cl}(F)$ via Artin's isomorphism. Then it acts on $u(n, \mathcal{A})$ by

$$\sigma_{\mathcal{C}}(u(n, \mathcal{A})) = u(n, \mathcal{A}\mathcal{C}^{-1}).$$

(iv) $\mathrm{N}_{H/\mathbb{Q}}(u(n, \mathcal{A}))$ is an integer and satisfies

$$-\tfrac{1}{\kappa_p} \log(|\mathrm{N}_{H/\mathbb{Q}}(u(n, \mathcal{A}))|) = R_p^+(n)$$

for all non-zero integers n.

Observe that parts (ii) and (iii) are very similar in form to Stark's Conjectures for special values of derivatives of L-functions (see [16, 44]). In fact, they are consequences of known cases of his conjectures when $n \leq 0$. Part (iv) can be viewed as a mock-modular version of Dirichlet's identity (1.9). It is important to observe that $\tilde{g}_\psi(z)$ is not necessarily unique.

The proof of this result relies heavily on the Rankin-Selberg method for computing heights of Heegner divisors as developed in [25], but entails the use of weight one harmonic forms with polar singularities in cusps in place of weight one Eisenstein series and hence requires regularized inner products. In order to get to the individual mock-modular coefficients, it is necessary to consider modular curves of large prime level N and their Heegner divisors of height zero. The relation of this part of the proof to [25] has some independent interest.

Zagier observed that his identity with Gross for the norms of differences of singular values of the modular j-function can be neatly expressed in terms of the coefficients $R_p^+(n)$ given in (1.6) (see Eq. (0.21) in [33]). For simplicity, let $-d < 0$ be a fundamental discriminant not equal to $-p$. and set $F' = \mathbb{Q}(\sqrt{-d})$. As is well-known, the modular j-function is well-defined on ideal classes of F and F' and takes values in the rings of integers of their respective Hilbert class fields. Also, values of the j-function at different ideal classes are Galois conjugates of each other. For any $\mathcal{A} \in \mathrm{Cl}(F)$ define the quantity

$$a_{d,\mathcal{A}} := \prod_{\mathcal{A}' \in \mathrm{Cl}(F')} (j(\mathcal{A}) - j(\mathcal{A}')). \tag{1.11}$$

The norm of $a_{d,\mathcal{A}}$ to F is thus $\prod_{\mathcal{A} \in \mathrm{Cl}(F)} a_{d,\mathcal{A}}$ and is an ordinary integer. The result of Gross and Zagier [24, Theorem 1.3] is that this integer can be expressed in terms of $R_p^+(n)$ as follows:

$$\log \prod_{\mathcal{A} \in \mathrm{Cl}(F)} |a_{d,\mathcal{A}}|^{2/w_d} = -\tfrac{1}{4} \sum_{k \in \mathbb{Z}} \delta(k)\, R_p^+(\tfrac{pd-k^2}{4}), \tag{1.12}$$

where w_d is the number of roots of unity in F' and $\delta(k) = 2$ if $p|k$ and 1 otherwise.

There are two proofs of this factorization in [24]: one analytic and one algebraic. In fact, the algebraic approach, which is based on Deuring's theory of supersingular elliptic curves over finite fields, gives the factorization of the ideal $(a_{d,\mathcal{A}})$ in $\mathcal{O}_H$ for each class $\mathcal{A} \in \mathrm{Cl}(F)$. To state it, suppose ℓ is a rational prime such that $\chi_p(\ell) \neq 1$. Then the ideal (ℓ) factors in $\mathcal{O}_H$ as

$$\ell = \prod_{\mathcal{A} \in \mathrm{Cl}(F)} \mathfrak{l}_{\mathcal{A}}^{\delta(\ell)}. \tag{1.13}$$

The $\mathfrak{l}_{\mathcal{A}}$'s are primes in H above ℓ uniquely labeled so that $\sigma_{\mathcal{C}}(\mathfrak{l}_{\mathcal{A}}) = \mathfrak{l}_{\mathcal{A}\mathcal{C}^{-1}}$ for all $\mathcal{C} \in \mathrm{Cl}(F)$ and complex conjugation sends $\mathfrak{l}_{\mathcal{A}}$ to $\mathfrak{l}_{\mathcal{A}^{-1}}$. Let $\mathcal{A}_0$ be the principal class. It is shown in [24] that the order of $a_{d,\mathcal{A}_0}$ at the place associated to the prime $\mathfrak{l}_{\mathcal{A}}$ is given by

$$\mathrm{ord}_{\mathfrak{l}_{\mathcal{A}}}(a_{d,\mathcal{A}_0}) = \tfrac{1}{2} \sum_{k \in \mathbb{Z}} \delta(k) \sum_{m \geq 1} r_{\mathcal{A}^2}\left(\tfrac{pd-k^2}{4\ell^m}\right). \tag{1.14}$$

The Galois action then yields the prime factorization of the ideal $(a_{d,\mathcal{A}})$ for any $\mathcal{A}$.

Our second main result gives a mock-modular interpretation of the individual values $|a_{d,\mathcal{A}}|$. It is convenient to give it as a twisted version of (1.12).

Theorem 1.2. *For any* $\tilde{g}_\psi(z) = \sum_{n \geq n_0} r_\psi^+(n) q^n$ *given by Theorem 1.1 and* $-d < 0$ *any fundamental discriminant different from* $-p$ *we have*

$$\sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log |a_{d,\mathcal{A}}|^{2/w_d} = -\tfrac{1}{4} \sum_{k \in \mathbb{Z}} \delta(k)\, r_\psi^+(\tfrac{pd-k^2}{4}) \tag{1.15}$$

where $a_{d,\mathcal{A}}$ *is defined in (1.11).*

As with Theorem 1.1, this is proved using the methods of [25] and not the analytic technique of [24], which uses the restriction to the diagonal of an Eisenstein series for a Hilbert modular group. In particular, we will define a real-analytic function $\Phi(z)$, which transforms with weight 3/2 and level 4, and use holomorphic projection to obtain an equation between a finite linear combination of $r_\psi^+(n)$'s and an infinite sum, similar to the one in [25]. We also make use of machinery from [32]. One interesting new feature is an elementary counting argument needed to construct a Green's function evaluated at CM points. Actually, Eq. (1.15) is a particular example of a more general identity involving values of certain Borcherds lifts. Although we will not carry this out here, it will become clear that similar methods can be used to prove a level N version in this form.

Let us illustrate our Theorems in the first non-trivial case, which occurs when $p = 23$. The class group $\mathrm{Cl}(F)$ has size 3 and two non-trivial characters ψ and $\overline{\psi}$. The Hilbert class field H is generated by $X^3 - X - 1$ over F. Let $\alpha = 1.32472\ldots$ be the unique real root of $X^3 - X - 1$, which is also the absolute value of the square of a unit in H. The space $S_1(23, \chi_{23})$ is one dimensional and spanned by the cusp form

$$g_\psi(z) = \eta(z)\eta(23z),$$

where $\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the eta function. According to Theorem 1.1, there exists a mock-modular form $\tilde{g}_\psi(z)$ having a simple pole and the following Fourier expansion

$$\tilde{g}_\psi(z) = \sum_{\substack{n\geq -1\\ \chi_{23}(n)\neq 1}} r_\psi^+(n)q^n.$$

The condition on the principal part determines $\tilde{g}_\psi(z)$ uniquely in this case, though this is not true in general. Using Stokes' Theorem and Stark's calculation at the end of [44, II], one can show that

$$r_\psi^+(-1) = \langle g_\psi, g_\psi\rangle = 3\log(\alpha) \quad \text{and} \quad r_\psi^+(0) = -\log(\alpha).$$

From the numerical calculations of $r_\psi^+(n)$, we can predict the values of κ_p and $u(n, \mathcal{A})$ in Theorem 1.1. Let $\tilde{\kappa}_p$ and $\tilde{u}(n, \mathcal{A})$ denote the predicted values for κ_p and $u(n, \mathcal{A})$ respectively. The following table lists $r_\psi^+(n)$, which are calculated numerically, and the predicted values $\tilde{u}(n, \mathcal{A}_0)$ when $\tilde{\kappa}_p = 1$ for the principal class $\mathcal{A}_0 \in \mathrm{Cl}(F)$ for $1 \leq n \leq 23$ and a few other values of n, all with $\chi_{23}(n) \neq 1$. It also contains the norms of $\tilde{u}(n, \mathcal{A}_0)$, which agree with condition (iv) in Theorem 1.1.

n	$r_\psi^+(n)$	$\tilde{u}(n, \mathcal{A}_0)$	$\mathrm{N}_{H/\mathbb{Q}}(\tilde{u}(n, \mathcal{A}_0))^{1/\tilde{\kappa}_p}$
5	1.1001149692823391	π_5	5^2
7	1.7161505040673007	π_7	7^2
10	3.9614773685309742	$5\alpha^{-6}\pi_5^{-1}$	5^4
11	0.052996471463740862	π_{11}	11^2
14	1.6582443878082415	$7\alpha^{-4}\pi_7^{-1}$	7^4
15	-1.9437136922512246	$5\alpha\pi_5^{-1}$	5^4
17	-4.2163115309750479	π_{17}	17^2
19	-2.7119255841404505	π_{19}	19^2
20	5.9051910607821988	$5\alpha^{-7}$	5^6
21	6.7198367256215547	$7\alpha^{-10}\pi_7^{-1}$	7^4
22	-4.2709900863081686	$11\alpha^5\pi_{11}^{-1}$	11^4
23	-3.8460181706191355	π_{23}	23

28	4.21799361484444277	$7\alpha^{-5}$	7^6
34	-2.532478252776036	$17\alpha^8\pi_{17}^{-1}$	17^4
38	-9.942055260392833	$19\alpha^{15}\pi_{19}^{-1}$	19^4
40	-14.92826076712649	$5\alpha^{19}\pi_5$	5^8

TABLE 1. Coefficients of $\tilde{g}_\psi(z)$ for $g_\psi(z) = \eta(z)\eta(23z) \in S_1(23, \chi_{23})$

Here, π_ℓ is a generator of the prime ideal $\mathfrak{l}_{\mathcal{A}_0}$ in $\mathcal{O}_H$ and is given below in terms of α.

ℓ	π_ℓ
5	$2\alpha^2 - \alpha - 1$
7	$\alpha^2 + \alpha - 2$
11	$2\alpha^2 - \alpha$
17	$2\alpha^2 + 3\alpha + 3$
19	$3\alpha^2 + \alpha$
23	$\frac{1}{\sqrt{-23}}(10\alpha^2 + 8\alpha + 1)$

TABLE 2. Generators of $\mathfrak{l}_{\mathcal{A}_0}$.

To illustrate how Theorem 1.2 supports these predictions, consider the classical example $-d = -7$, which also appeared in [48]. First, we can combine Eq. (1.15) and Theorem 1.1 to write

$$4\log|a_{d,\mathcal{A}}|^{2/w_d} = \tfrac{2}{\kappa_p}\sum_{k\in\mathbb{Z}} \delta(k)\log\left|u\left(\tfrac{pd-k^2}{4}, \mathcal{A}\right)\right|.$$

For $p = 23$ and $\mathcal{A} = \mathcal{A}_0$, we can use the predicted values in Table 1 to rewrite the equation above as

$$\begin{aligned} 4\log|a_{7,\mathcal{A}_0}| &= 4\begin{pmatrix} \log|u(40,\mathcal{A}_0)| + \log|u(38,\mathcal{A}_0)| + \log|u(34,\mathcal{A}_0)| + \\ \log|u(28,\mathcal{A}_0)| + \log|u(20,\mathcal{A}_0)| + \log|u(10,\mathcal{A}_0)| \end{pmatrix} \\ &= 4\log|5^3\cdot 7\cdot 17\cdot \pi_{17}^{-1}\cdot 19\cdot\pi_{19}^{-1}\cdot\alpha^{24}|. \end{aligned}$$

This agrees with the exact of value of $a_{7,\mathcal{A}_0}$, which is $5^3\cdot 7\cdot 17\cdot\pi_{17}^{-1}\cdot 19\cdot\pi_{19}^{-1}\cdot\alpha^{24}$. Applying the Galois action to these predicted $u(n,\mathcal{A}_0)$ shows that they also agree with Theorem 1.2.

This and several other numerical examples, together with Eq. (1.14), motivate us to make a conjecture about the factorization of the fractional ideal generated by $u(n,\mathcal{A})$ in Theorem 1.1. It is not hard to see that the following conjecture implies (1.14).

Conjecture. *In (ii) of Theorem 1.1 we have the following.*

(i) The number κ_p is an integer dividing $24H(p)h_H$, where h_H is the class number of H.

(ii) For $\mathcal{B} \in \mathrm{Cl}(F)$, let $\mathfrak{l}_\mathcal{B}$ be a prime ideal above the rational prime ℓ as in (1.13). Then the order of $u(n,\mathcal{A})$ at the place of H corresponding to $\mathfrak{l}_\mathcal{B}$ is

$$\mathrm{ord}_{\mathfrak{l}_\mathcal{B}}(u(n,\mathcal{A})) = \kappa_p\sum_{m\geq 1} r_{(\mathcal{A}^{-1}\mathcal{B})^2}\left(\frac{n}{\ell^m}\right).$$

At all other places of H, $u(n,\mathcal{A})$ is a unit. In particular, $u(n,\mathcal{A}) \in \mathcal{O}_H$ for all $n, \mathcal{A}$.

In general (for prime level $p \equiv 3 \pmod 4$) , the Deligne-Serre Theorem [17] (see also [40]) identifies the L-function of a newform $f \in S_1(p,\chi_p)$ with the Artin L-function of an irreducible, odd, two-dimensional, complex representation ρ_f of the Galois group of a normal extension $K/\mathbb{Q}$. Such a ρ_f gives rise to a projective representation $\tilde{\rho}_f : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{PGL}_2(\mathbb{C})$ whose image is isomorphic to D_{2n}, S_4, or A_5, in which case we say that f is dihedral, octahedral, or icosahedral, respectively. The forms $g_\psi(z)$ are precisely the dihedral forms and for them $K = H$. Since $g_{\psi_1}(z) = g_{\psi_2}(z)$ if and only if $\psi_1 = \psi_2$ or $\psi_1 = \overline{\psi_2}$, there are exactly $(H(p) - 1)/2$ such forms. Theorem 1.1 relates the Fourier coefficients of a mock-modular form with dihedral shadow $g_\psi(z)$ to linear combinations of logarithms of algebraic numbers in the number field $K = H$ determined by the Galois representation. Furthermore, the coefficients in these linear combinations are algebraic numbers in the field generated by the Fourier coefficients of the shadow.

Non-dihedral newforms are often referred to as being "exotic" since their occurrence is rare and unpredictable. We have carried out some numerical calculations for mock-modular forms whose shadows are certain exotic newforms and have observed that they also seem to be related to linear combinations of logarithms of algebraic numbers in the number field K determined by the associated Galois representation. For instance, when $p = 283$ the space $S_1(283, \chi_{283})$ contains a pair of octahedral newforms associated to a Galois representation of $\mathrm{Gal}(K/\mathbb{Q})$, where K is the degree 2 extension of the normal closure of $\mathbb{Q}[X]/(X^4 - X - 1)$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{F}_3)$. Similar to the dihedral case, the Petersson norms of these newforms come from the residue of a certain degree four L-function. As a consequence of proven cases of Stark's conjecture, this residue is the logarithm of a unit inside a subfield F_6 of K. This determines the principal part of a mock modular form whose shadow is one of these octahedral newforms. Using this hint and some numerical calculations, we noticed that the other coefficients also seem to be linear combinations of logarithms of algebraic numbers in the same subfield F_6. Furthermore, these algebraic numbers generate ideals with nice factorizations. Thus, it is natural to expect a statement analogous to Theorem 1.1 and the Conjecture should hold for exotic newforms. In the final section of the paper we will give computational details when $p = 283$.

We end the introduction with a brief outline of the paper. Section 2 gives general properties of a weight one mock-modular form, such as its existence (§2.1) and the relationship between its Fourier coefficients and the regularized inner product via Stokes' theorem (§2.2). We also decompose the space of weight one harmonic Maass forms into a plus space and minus space in §2.3 and include some facts about basis of modular functions of prime level N.

In §3 we study the principal part at infinity of a mock-modular form with shadow $g_\psi(z)$. The proof of Theorem 1.2 is in §4, which can be read independently of the other sections and gives a model of the procedure to produce relationships between Fourier coefficients of weight one mock-modular forms and CM-values of modular functions. Sections 5 and 6 contain the integral weight versions of Theorem 1.2 for level 1 and prime level N respectively. One of their consequences, Proposition 7.1, is the algebraic property satisfied by various linear combinations of Fourier coefficients of weight one mock-modular forms. This is used in §7, along with a modularity lemma, to finish the proof of Theorem 1.1. Finally, an analysis of the specific case $p = 283$ is given in §8.

The main results of this paper were announced by the first author in February 2012 in the Symposium: Modular Forms, Mock Theta Functions, and Applications in Köln. We have since learned that there is substantial overlap with [21] and [46]. In particular, Theorems 1.1 and 1.2 are also obtained in [21] for square-free level but by different methods. Also, many cases of the conjecture above could be resolved by carefully studying the height pairings between Heegner points (see [47]). The relationship between our approach and the theta-lifting technique is similar to the one between [12] and [13]. It would be interesting to generalize these approaches to the exotic cases.

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2. Preliminary Facts of Harmonic Maass Forms

2.1. Existence of Harmonic Maass Forms of Weight One.

We begin with some basic definitions (see [9]). Let $k \in \mathbb{Z}$. For any function $f : \mathcal{H} \to \mathbb{C}$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$, define the weight k slash operator $|_k\gamma$ by

$$(f|_k\gamma)(z) := \tfrac{(\det(\gamma))^{k/2}}{(cz+d)^k} f(\gamma z),$$

where γz is the linear fractional transformation of z under γ. We will write $f|\gamma$ for $f|_k\gamma$ when the weight of f is understood. For $M \in \mathbb{Z}^+$ let $\Gamma_0(M)$ denote the usual congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level M, namely

$$\Gamma_0(M) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) | \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod M\}.$$

Let $\nu : \mathbb{Z}/M\mathbb{Z} \to \mathbb{C}^\times$ be a Dirichlet character such that $\nu(-1) = (-1)^k$ and $\nu(\gamma) := \nu(d)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$. Let $\mathcal{F}_k(M, \nu)$ be the space of smooth functions $f : \mathcal{H} \to \mathbb{C}$ such that for all $\gamma \in \Gamma_0(M)$

$$(f\ |_k\ \gamma)(z) = \nu(\gamma) f(z).$$

Recall from (1.1) and (1.2) the differential operator ξ_k and the weight k hyperbolic Laplacian Δ_k. Let $z = x + iy$. Then Δ_k can be written as

$$-\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then $f(z) \in \mathcal{F}_k(M, \nu)$ is a weight k harmonic weak Maass form of level M and character ν (or more briefly, a weakly harmonic form) if it satisfies the following properties.

(i) $f(z)$ is real-analytic.
(ii) $\Delta_k(f) = 0$.
(iii) The function $f(z)$ has at most linear exponential growth at all cusps of $\Gamma_0(M)$.

Let $H_k(M, \nu)$ be the space of weakly harmonic forms of weight k, level M and character ν, whose image under ξ_k is a holomorphic modular form. Denote by $M^!_k(M, \nu)$, $M_k(M, \nu)$ and $S_k(M, \nu)$ the usual subspaces of weakly holomorphic modular forms, holomorphic modular forms and cusp forms, respectively. A mock-modular form is a formal Laurent series in q,

$$\tilde{g}(z) = \sum_{n \gg -\infty} c^+(n) q^n,$$

such that for some $k \in \mathbb{Z}$, there exists $g(z) = \sum_{n \geq 0} c(n) q^n \in M_{2-k}(M, \overline{\nu})$ satisfying

$$\sum_{n \gg -\infty} c^+(n) q^n - \sum_{n \geq 0} c(n) \beta_k(n, y) q^{-n} \in H_k(M, \nu).$$

The form $g(z)$ is called the *shadow* of $\tilde{g}(z)$. The expression $\sum_{n<0} c^+(n) q^n$ is called the *principal part* of $\tilde{g}(z)$. Let $\mathbb{M}_k(M, \nu)$ be the subspace of mock-modular forms whose shadows are in $M_{2-k}(M, \overline{\nu})$. Since every weakly harmonic form can be written uniquely as the sum of a holomorphic part and a non-holomorphic part, the spaces $H_k(M, \nu)$ and $\mathbb{M}_k(M, \nu)$ are canonically isomorphic to each other.

Property (ii) and Eq. (1.2) gives the following map

$$\xi_k : H_k(M, \nu) \longrightarrow M_{2-k}(M, \overline{\nu}),$$

whose kernel is exactly $M_k^!(M, \nu)$. When $k \neq 1$, the map above is surjective as shown in [6] and [9]. When $k = 1$, one can still prove surjectivity by analytically continuing the weight one Poincaré series, the same family as in [6] for $k = 1$, via spectral expansion or slightly modifying the abstract arguments employing Serre duality in [9]. Since the arguments are standard and not necessary for our main results, we omit the proof of the following proposition. Interested readers can check [34] for more details.

Proposition 2.1. *In the notations above, the following map is a surjection*

$$\xi_1 : H_1(M, \nu) \to S_1(M, \overline{\nu}),$$

i.e. for any cusp form $h(z) \in S_1(M, \overline{\nu})$, there exists $\tilde{h}(z) \in \mathbb{M}_1(M, \nu)$ with shadow $h(z)$.

2.2. **Regularized Petersson Inner Products.** From now on, we fix $M = p$ to be a prime number congruent to 3 modulo 4 and $\nu = \chi_p = \overline{\chi_p}$. The spaces $H_1(p, \chi_p)$, $M_1^!(p, \chi_p)$, $M_1(p, \chi_p)$, $S_1(p, \chi_p)$, $\mathbb{M}_1(p, \chi_p)$ are the same as before, and we will drop χ_p and sometimes p in these notations when they are fixed. In this section, we will relate the regularized inner products between $g(z) \in S_1(p)$ and $f(z) \in M_1^!(p)$ to linear combinations of coefficients of a mock-modular form $\tilde{g}(z)$, whose shadow is $g(z)$, via Stokes' theorem. The regularization technique is standard and has been used in many places before (see for example [4, 7, 9, 11, 20, 27]).

Given $f(z) \in M_1^!(p)$ and $g(z) \in S_1(p)$, the usual Petersson inner product $\langle f, g \rangle$ can be regularized as follows. Since p is prime, $\Gamma_0(p)$ has two inequivalent cusps, 0 and ∞. They are related by the scaling matrix $\sigma_0 = \left(\begin{smallmatrix} & -1/\sqrt{p} \\ \sqrt{p} & \end{smallmatrix} \right)$. Take a fundamental domain of $\Gamma_0(p)\backslash\mathcal{H}$, cut off the portion with $\mathrm{Im}(z) > Y$ for a sufficiently large Y and intersect it with its translate under σ_0. We will call this the truncated fundamental domain $\mathcal{F}(Y)$. Now, define the regularized inner product by

$$\langle f, g \rangle_{\mathrm{reg}} := \lim_{Y \to \infty} \int_{\mathcal{F}(Y)} f(z)\overline{g(z)} y \, \frac{dxdy}{y^2}. \tag{2.1}$$

If $f(z) \in M_1(p)$, then this is the usual Petersson inner product. Now let $\hat{g}(z) \in H_1(p)$ be a preimage of $g(z)$ under ξ_1 with the following Fourier expansions

$$\hat{g}(z) = \sum_{n \in \mathbb{Z}} c_\infty^+(n) q^n - \sum_{n \geq 1} c(g, n) \beta_1(n, y) q^{-n},$$
$$(\hat{g}|_1 W_p)(z) = \sum_{n \in \mathbb{Z}} c_0^+(n) q^n - \sum_{n \geq 1} c(g|_1 W_p, n) \beta_1(n, y) q^{-n},$$

where $W_p = \left(\begin{smallmatrix} & -1 \\ p & \end{smallmatrix} \right)$ is the Fricke involution that acts on $f(z) \in H_1^!(p)$ by

$$(f|_1 W_p)(z) = \tfrac{1}{\sqrt{p}z} f\left(-\tfrac{1}{pz}\right).$$

The expression for $(\hat{g}|_1 W_p)(z)$ follows from the commutativity between ξ_1 and the slash operator. Note that $\sum_{n\in\mathbb{Z}} c_\infty^+(n)q^n$ and $\sum_{n\in\mathbb{Z}} c_0^+(n)q^n$ are mock-modular forms with shadows $g(z)$ and $(g|_1 W_p)(z)$ respectively.

Suppose $f(z) \in M_1^!(p)$ has Fourier expansions $\sum_{n\in\mathbb{Z}} c_\infty(f, n)q^n$ and $\sum_{n\in\mathbb{Z}} c_0(f, n)q^n$ at the cusp infinity and 0 respectively. Then as a special case of Proposition 3.5 in [9], we can express $\langle f, g \rangle_{\mathrm{reg}}$ in terms of these Fourier coefficients.

Lemma 2.2 (See Proposition 3.5 in [9]). *Let $f(z) \in M_1^!(p)$ and $g(z) \in S_1(p)$. In the notations above, we have*

$$\langle f, g\rangle_{\mathrm{reg}} = \sum_{n\in\mathbb{Z}} c_\infty^+(n)c_\infty(f,-n) + c_0^+(n)c_0(f,-n). \tag{2.2}$$

Remark. Notice that the right hand side of Eq. (2.2) depends on the choice of $\hat{g}(z)$, whereas the left hand side only depends on $g(z)$. So if we replace $\hat{g}(z)$ with $h(z) \in M_1^!(p)$, then Lemma 2.2 still holds and we obtain

$$0 = \sum_{n\in\mathbb{Z}} c_\infty(h,n)c_\infty(f,-n) + c_0(h,n)c_0(f,-n),$$

where $h(z)$ has Fourier expansions $\sum_{n\in\mathbb{Z}} c_\infty(h,n)q^n$ and $\sum_{n\in\mathbb{Z}} c_0(h,n)q^n$ at the cusp infinity and 0 respectively.

2.3. **Weight One Plus Space.** In this section, we will canonically decompose the space $H_1(p)$ into the direct sum of two subspaces, which behave nicely with respect to ξ_1 and the regularized inner product. This type of decomposition has appeared in [30, §3.1] for weight two holomorphic modular forms of prime level $p \equiv 1 \pmod 4$, and also in [8] for weakly holomorphic modular forms of any weight and any prime level. This section is a slight generalization of §3 in [8] to the harmonic setting.

For $\epsilon = \pm 1$, we define the following space

$$H_1^\epsilon(p) := \left\{ f(z) = \sum_{n\in\mathbb{Z}} a(n,y)q^n \in H_1(p) : a(n,y) = 0 \text{ whenever } \chi_p(n) = -\epsilon \right\}. \tag{2.3}$$

By imposing the same condition on the Fourier expansions, we can define $M_1^{!,\epsilon}(p)$, $M_1^\epsilon(p)$, $S_1^\epsilon(p)$ and $\mathbb{M}_1^\epsilon(p)$. Notice that $M_1^{!,\epsilon}(p)$ is the same as the space $A_k^\epsilon(p,\chi_p)$ defined on page 51 of [8] for $k = 1$. By the definition of $R_p(n)$ in (1.5), it is clear that $E_1(z) \in M_1^+(p)$. Also, the weight one mock-Eisenstein series $\tilde{E}_1(z)$ belongs to $\mathbb{M}_1^-(p)$. This can be checked from the factorization (1.6) and the fact that $E_1(z) \in M_1^+(p)$.

Recall the standard operator U_p on $H_1(p)$ defined by

$$f(z)|_1 U_p = \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} f|_1 \left(\begin{smallmatrix} 1 & j \\ & p \end{smallmatrix}\right). \tag{2.4}$$

Its action on the Fourier expansion of $f(z) = \sum_m a(m,y)q^m \in H_1(p)$ is

$$(f|_1 U_p)(z) = \sum_m a\left(pm, \tfrac{y}{p}\right) q^m. \tag{2.5}$$

The U_p operator preserves the space $H_1(p)$. This is also true for the Fricke involution W_p, since χ_p is a real character. Applying W_p twice produces a negative sign for odd weight, i.e. $(f|_1 W_p^2)(z) = -f(z)$.

For $\epsilon = \pm 1$, define the operator $pr^\epsilon : H_1(p) \longrightarrow H_1(p)$ by

$$pr^\epsilon(f) := \frac{1}{2}\Big(\epsilon i (f|_1 U_p W_p) + f\Big). \tag{2.6}$$

By considering the actions of U_p and W_p on the associated weakly harmonic form, one can define pr^ϵ on $\mathbb{M}_1(p)$ as well. These operators then decompose $H_1(p)$, resp. $\mathbb{M}_1(p)$, into the direct sum of $H_1^+(p)$ and $H_1^-(p)$, resp. $\mathbb{M}_1^+(p)$ and $\mathbb{M}_1^-(p)$. The lemma below is a generalization of Lemma 3 in [8, §3] to the harmonic setting.

Lemma 2.3. *Let $f \in H_1(p)$. Then for $\epsilon = \pm 1$,*

$$(f|_1 W_p)(z) = -i\epsilon (f|_1 U_p)(z), \tag{2.7}$$

if and only if $f(z) \in H_1^\epsilon(p)$. As a result, the operator pr^ϵ is the identity when restricted to the subspace $H_1^\epsilon(p)$, annihilates the subspace $H_1^{-\epsilon}(p)$ and

$$H_1(p) = pr^+(H_1(p)) \oplus pr^-(H_1(p)) = H_1^+(p) \oplus H_1^-(p). \tag{2.8}$$

Remark. The same decomposition above holds for $\mathbb{M}_1(p)$ and subspaces $S_1(p) \subset M_1(p) \subset M_1^!(p) \subset H_1(p)$ as well.

Proof. Let $f(z) = \sum_{n\in\mathbb{Z}} a(n,y)q^n \in H_1(p)$. It suffices to show that for $\epsilon = \pm 1$, $pr^{-\epsilon}(f) \in H_1^\epsilon(p)$ and

$$pr^{-\epsilon}(f) = 0 \Longleftrightarrow f \in H_1^\epsilon(p).$$

These then imply the first half of the lemma after applying the definition of pr^ϵ and $f \mid_1 W_p^2 = -f$. They also imply the second half since

$$f = pr^+(f) + pr^-(f) \in H_1^+(p) + H_1^-(p)$$

and for any $h \in H_1^+(p) \cap H_1^-(p)$

$$h = pr^+(h) + pr^-(h) = 0 + 0 = 0.$$

With the matrix calculations in Lemma 3 of [8], one has

$$(pr^{-\epsilon} f)(z) = \frac{1}{2}\left(-\epsilon i (f|_1 W_p)(pz) + \sum_{n\in\mathbb{Z}} (1 - \chi_p(n)\epsilon)\, a(n,y) q^n \right) \in H_1^\epsilon(p).$$

So if $pr^{-\epsilon}(f) = 0$, then $a(n,y) = 0$ whenever $\chi_p(n) = -\epsilon$ and $f \in H_1^\epsilon(p)$ by definition. Conversely if $f \in H_1^\epsilon(p)$, then the Fourier coefficients $\xi_1(pr^{-\epsilon}(f)) \in M_1^!(p)$ are supported on multiples of p. A lemma due to Hecke ([37, Lemma, p. 32]) implies that

$$M_1^{!,+}(p) \cap M_1^{!,-}(p) = \{0\}.$$

So $\xi_1(pr^{-\epsilon}(f)) = 0$ and $pr^{-\epsilon}(f)$ is holomorphic. But it is contained in $M_1^{!,+}(p) \cap M_1^{!,-}(p)$, hence vanishes. □

The decomposition above behaves well under the action of ξ_1. Combining with Proposition 2.1, it proves a statement about surjectivity of $\xi_1 : H_1^\epsilon(p) \to M_1^{-\epsilon}(p)$. Furthermore, there is a nice characterization of the space $S_1^\epsilon(p)$ as a consequence of facts about weight one modular forms.

Lemma 2.4. *For $\epsilon = \pm 1$, the following map is a surjection*

$$\xi_1 : H_1^\epsilon(p) \to S_1^{-\epsilon}(p).$$

In other words, for any $g \in S_1^{-\epsilon}(p)$, there exists a mock-modular form $\tilde{g} \in \mathbb{M}_1^\epsilon(p)$ whose shadow is g.

In addition, $S_1^+(p)$ contains all dihedral cusp forms. If there are $2d_-$ non-dihedral forms in $S_1(p)$, then the spaces $S_1^+(p)$ and $S_1^-(p)$ have dimensions $\frac{1}{2}(H(p)-1)+d_-$ and d_- respectively, where $H(p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$ as in the introduction.

Proof. By Proposition 2.1, the following map is surjective

$$\xi_1 : H_1(p) \to S_1^{-\epsilon}(p).$$

Since ξ_1 commutes with the action of U_p and W_p and conjugates their coefficients, it commutes with the operator pr^ϵ as follows

$$\xi_1(pr^\epsilon(f)) = pr^{-\epsilon}(\xi_1(f)). \tag{2.9}$$

The operator $pr^{-\epsilon}$ is the identity when restricted to $S_1^{-\epsilon}(p)$. So the preimage of $S_1^{-\epsilon}(p)$ under ξ_1 lies in $H_1^\epsilon(p)$ and the map $\xi_1 : H_1^\epsilon(p) \to S_1^{-\epsilon}(p)$ is surjective for $\epsilon = \pm 1$.

Now if $f \in S_1(p)$ is a dihedral newform, then it is a linear combination of theta series, whose n^{th} coefficient is zero if $\chi_p(n) = -1$. So $f \in S_1^+(p)$ by definition. If $f(z) = \sum_{n\geq 1} c(f,n)q^n$ is an octahedral or icosahedral newform, then $f(z)$ and $\overline{f}(z) := \overline{f(\overline{z})} = \sum_{n\geq 1} \overline{c(f,n)}q^n$ are linearly independent newforms. When $\ell \neq p$ is a prime number, we have the relationship

$$c(f,\ell) = \chi_p(\ell)\overline{c(f,\ell)}. \tag{2.10}$$

This is a consequence of the formula of the adjoint of the ℓ^{th} Hecke operator with respect to the Petersson inner product [38, p.21]. Together with the recursive relation

$$c(f,\ell)c(f,n) = c(f,n\ell) + \chi_p(\ell)c(f,n/\ell),$$

a simple induction shows that whenever $p \nmid n$,

$$c(f,n) = \chi_p(n)\overline{c(f,n)}.$$

So $f + \overline{f} \in S_1^+(p)$ and $f - \overline{f} \in S_1^-(p)$. Since the number of octahedral and icosahedral newforms is always even and set to be $2d_-$, we obtain the formulae for the dimensions. □

Remark: The spaces $S_1^\epsilon(p)$ should be compared to the spaces M_+ and M_- in (9.1.2) in [40]. If all octahedral and icosahedral newforms $f(z) \in S_1(p)$ satisfy

$$(f|_1W_p)(z) = -i\overline{f}(z), \tag{2.11}$$

then M_+ is the span of the weight one Eisenstein series and $S_1^+(p)$ and $M_- = S_1^-(p)$.

Besides compatibility with ξ_1, the decomposition also behaves nicely with respect to the regularized inner product. As a consequence of Lemma 2.2, we have the following proposition.

Proposition 2.5. *Let $f(z) \in M_1^{!,\epsilon}(p)$, $g(z) \in S_1^{\epsilon'}(p)$ and $\tilde{g}(z) \in \mathbb{M}_1^{-\epsilon'}(p)$ with shadow $g(z)$ and Fourier expansion $\sum_{n\in\mathbb{Z}} c^+(n)q^n$. If $\epsilon = \epsilon'$, then*

$$\langle f, g\rangle_{\text{reg}} = \sum_{n\in\mathbb{Z}} \delta(n)c(f,-n)c^+(n). \tag{2.12}$$

Here $\delta(n)$ is 2 if $p|n$ and 1 otherwise. If $\epsilon \neq \epsilon'$, then $\langle f, g\rangle_{\text{reg}} = 0$.

Proof. Let $\hat{g}(z) \in H_1^{-\epsilon'}(p)$ be the harmonic Maass form associated to $\tilde{g}(z)$. Then both assertions are immediate consequences of Lemma 2.2 and

$$(f|_1W_p)(z) = -i\epsilon(f|_1U_p)(z), (\hat{g}|_1W_p)(z) = -i\epsilon'(\hat{g}|_1U_p)(z).$$

Since the adjoint of W_p (resp. U_p) is $W_p^{-1} = -W_p$ (resp. $W_pU_pW_p^{-1}$) with respect to the regularized inner product, it is easy to check that pr^ϵ is self-adjoint, which also proves the second claim. □

When $g(z)$ above is zero, i.e. $\tilde{g}(z)$ is modular, Eq. (2.12) reduces to an orthogonal relationship between Fourier coefficients of weakly holomorphic modular forms. In particular, we can take $\tilde{g}(z)$ to be a cusp form and obtain relations satisfied by $\{c(f,-n) : n \geq 1\}$. These relations turn out to characterize the space $M_1^{!,\epsilon}(p)$, which gives a nice characterization of the space $\mathbb{M}_1^\epsilon(p)$. The following proposition is a slight generalization of Theorem 6 in [8, §3].

Proposition 2.6. *Let $g(z) \in S_1^{\epsilon}(p)$. Then there exists a mock-modular form $\tilde{g}(z) \in \mathbb{M}_1^{-\epsilon}(p)$ with shadow $g(z)$ and prescribed principal part $\sum_{n<0} c^+(n)q^n$ if and only if*

$$\sum_{n<0} \delta(n)c(h,-n)c^+(n) = \langle h, g\rangle_{\mathrm{reg}} \tag{2.13}$$

for every cusp form $h(z) \in S_1^{\epsilon}(p)$.

Proof. When $\tilde{g}$ is modular, this follows from Theorem 6 in [8, §3]. When $\tilde{g}(z)$ is mock-modular, the necessity part follows from Proposition 2.5. Since $h(z)$ is a cusp form, the summation in (2.12) only extends over $n < 0$. Let $\sum_{n<0} c^+(\tilde{g}_1, n)q^n$ be the principal part of $\tilde{g}_1(z) \in \mathbb{M}_1^{-\epsilon}(p)$, whose shadow is $g(z)$. This exists by Lemma 2.4. Then the difference $\sum_{n<0}(c^+(n)-c^+(\tilde{g}_1,n))q^n$ satisfies Eq. (2.13) with the right hand side being 0. So there exists a weakly holomorphic form $d(z) \in M_1^{!,-\epsilon}(p)$ with the prescribed principal part $\sum_{n<0}(c^+(n) - c^+(\tilde{g}_1,n))q^n$. Thus, the sum $\tilde{g}_1(z) + d(z) \in \mathbb{M}_1^{-\epsilon}(p)$ is the desired form with shadow $g(z)$ and principal part $\sum_{n<0} c^+(n)q^n$. □

2.4. **Echelon bases of modular forms.** In this section, we will gather some facts about bases of the spaces of the weight 0 and weight 2 modular forms with level. These will be useful during the proof of Theorem 6.1.

Let N be 1 or an odd prime and $M_2^!(N)$, resp. $M_0^!(N)$, be the space of weakly holomorphic weight 2 modular forms, resp. modular functions, of level N, and trivial nebentypus. Denote its subspaces of cusp forms and holomorphic modular forms by $S_2(N), M_2(N)$ respectively. Let $M_0^{!,\mathrm{new}}(N)$ be the subspace of $M_0^!(N)$ containing weakly holomorphic modular functions $f(z)$ satisfying

$$(f|W_N)(z) = -N(f|U_N)(z), \tag{2.14}$$

where W_N is the Fricke involution and U_N acts on f via

$$(f \mid U_N)(z) = \tfrac{1}{N}\sum_{\mu=0}^{N-1} f\left(\tfrac{z+\mu}{N}\right).$$

Note that $M_0^{!,\mathrm{new}}(1) = \{0\}$. Define the trace down operator Tr_1^N by

$$\mathrm{Tr}_1^N(f)(z) := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]}\sum_{\gamma\in\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} (f \mid \gamma)(z). \tag{2.15}$$

Using the trace down operator, one can decompose $M_0^!(N)$ into the direct sum of $M_0^{!,\mathrm{new}}(N)$ and modular functions of level 1.

Lemma 2.7. *Let $f(z)$ be a modular function of level N, where N is either an odd prime or 1. Then it can be written uniquely as*

$$f(z) = f_1(Nz) + f_2(z),$$

where $f_1 = \mathrm{Tr}_1^N(f \mid W_N) \in M_0^!(1)$ and $f_2(z) \in M_0^{!,\mathrm{new}}(N)$.

Proof. When $N = 1$, the space is $M_0^{!,\mathrm{new}}(N) = \{0\}$ and $f_1(z) = f(z)$. When N is an odd prime, some matrix calculations give us

$$Nf \mid U_N W_N = -f + (N+1)\mathrm{Tr}_1^N(f \mid W_N)(Nz).$$

Using this, one can verify that $f_2(z) \in M_0^{!,\mathrm{new}}(N)$. □

For the rest of the section, N is an odd prime. The space $M_0^{!,\mathrm{new}}(N)$ has a nice q-echelon basis, whose principal parts are closely related to the space $S_2(N)$.

Definition 2.8. A set of numbers $\{\lambda_m \in \mathbb{C} : m \geq 1\}$ is called a *relation* for $S_2(N)$ if

(i) λ_m is zero for all but finitely many m.
(ii) For any cusp form $h(z) = \sum_{m\geq 1} c(h,m)q^m \in S_2(N)$, the numbers $\{\lambda_m\}$ satisfy

$$\sum_{m\geq 1} \delta_N(m)\lambda_m c(h,m) = 0. \tag{2.16}$$

where $\delta_N(m) = N+1$ if $N|m$ and 1 otherwise.

A relation $\{\lambda_m\}_{m\geq 1}$ is called *integral* if $\lambda_m \in \mathbb{Z}$ for all $m \geq 1$. Denote Λ_N the set of all relations for $S_2(N)$.

For any $f(z) \in M_0^{!,\mathrm{new}}(N)$, one knows that the principal part coefficients, $\{c(f,-m) : m \geq 1\}$, is a relation for $S_2(N)$ from works by Petersson. By considering the Fourier expansion of vector-valued Poincaré series, (see [7, 29, 35]), one can also show that the converse is true. So given $\lambda = \{\lambda_m\} \in \Lambda_N$, the expression

$$\sum_{m\geq 1} \lambda_m q^{-m}$$

is necessarily the principal part of the Fourier expansion at infinity of a unique function $f_\lambda(z) \in M_0^{!,\mathrm{new}}(N)$. Alternatively, this statement follows essentially from an application of Serre duality [5, §3].

Suppose $\lambda \in \Lambda_N$ is integral. Then the n^{th} Fourier coefficient of $f_\lambda(z)$ is integral when $n \neq 0$ since f_λ is a rational function of $j(z)$ and $j(Nz)$. Applying Eq. (2.16), with the sum over all $m \geq 0$, to f_λ and the Eisenstein series

$$N \cdot \hat{E}_2(Nz) - \hat{E}_2(z) = (N-1) - 24\sum_{m\geq 1}\left(N\sigma_1\left(\frac{m}{N}\right) - \sigma_1(m)\right)q^m \in M_2(N) \tag{2.17}$$

then shows that (N^2-1) times the constant term in the Fourier expansion of f_λ is an integer. Here, $\hat{E}_2(z)$ is the non-holomorphic Eisenstein series of weight 2, level 1 defined by

$$\hat{E}_2(z) = \tilde{E}_2(z) - \frac{3}{\pi y} = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n - \frac{3}{\pi y}. \tag{2.18}$$

Let g_N be the genus of $\Gamma_0(N)$. From [1], we know that for all $h(z) \in S_2(N)$, $\mathrm{ord}_\infty(h(z)) \leq \dim(S_2(N)) = g_N \leq (N+1)/12$. So the space $S_2(N)$ has a q-echelon basis $\{h_j, 1 \leq j \leq g_N\}$ with

$$h_j(z) = q^j + O(q^{g_N+1}).$$

Using matrix computations similar to that in the proof of Lemma 2.7, one can show that

$$h|_2 U_N = -h|_2 W_N \tag{2.19}$$

for all $h(z) \in M_2(N)$, implying that $U_N \circ U_N$ is the identity operator on $M_2(N)$. Note that since N is prime, $\dim(M_2(N)) = g_N + 1$ with the extra contribution coming from the Eisenstein series in Eq. (2.17). Combining these facts together gives us the following lemma about $M_0^{!,\mathrm{new}}(N)$.

Lemma 2.9. *The space* $M_0^{!,\mathrm{new}}(N)$ *is spanned by two disjoint sets* $\mathcal{S}_{N,1}, \mathcal{S}_{N,2}$ *of modular functions defined by*

$$\begin{aligned}\mathcal{S}_{N,1} &:= \{f_m(z) = q^{-m} + \sum_{n=-g_N}^{\infty} c_m(n)q^n : m \geq 1 + g_N, N^2 \nmid m\},\\ \mathcal{S}_{N,2} &:= \{h_m(z) = q^{-N^2 m} - \tfrac{N+1}{\delta_N(m)}q^{-m} + O(q) : m \geq 1\}.\end{aligned} \tag{2.20}$$

For these $m \geq g_N + 1$*,* $c_m(n) \in \mathbb{Z}$ *when* $n \neq 0$ *and* $(N^2 - 1)c_m(0) \in \mathbb{Z}$*.*

There is a similar duality statement dictating the existence of forms in $M_2^!(N)$.

Lemma 2.10. *For every* $n \geq 0$*, there exists* $P_{n,N}(z) = \sum_{m\in\mathbb{Z}} c(P_{n,N}, m)q^m \in M_2^!(N)$ *such that*

(i) At the cusp infinity, $P_{n,N}(z) = q^{-n} + O(q)$*.*
(ii) $(P_{n,N}|_2W_N)(z) = -(P_{n,N}|_2U_N)(z)$*.*

Furthermore, if $f(z) = \sum_{m\in\mathbb{Z}} c(f, m)q^n \in M_0^{!,\mathrm{new}}(N)$*, then*

$$\sum_{m\in\mathbb{Z}} \delta_N(m)c(f, m)c(P_{n,N}, -m) = 0.$$

Remark. Notice that $(N - 1)P_{0,N}(z) \in M_2(N)$ is the Eisenstein series defined by Eq. (2.17).

3. Coefficients of the Principal Part of $\tilde{g}_\psi(z)$

Let $g_\psi(z) \in S_1^+(p)$ be a dihedral newform associated to a class group character ψ. Lemma 2.4 tells us that there exists a mock-modular form $\tilde{g}_\psi(z) \in \mathbb{M}_1^-(p)$ with $g_\psi(z)$ as shadow. However, $\tilde{g}_\psi(z)$ is only well-defined up to the addition of a form in $M_1^{!,-}(p)$. In this section, we will show that there exists a $\tilde{g}_\psi(z) \in \mathbb{M}_1^-(p)$ such that its *principal part coefficients* satisfy nice properties. These principal part coefficients can be related to L-functions via Kronecker's limit formula. In a sense, this section gives a model of how to study the other coefficients, in which case there will no longer be L-functions to work with.

Let F, H be the same number fields as in §1. For a class $\mathcal{A} \in \mathrm{Cl}(F)$, a CM point $\tau_\mathcal{A} = x_\mathcal{A} + iy_\mathcal{A} \in \mathcal{H} \cap F$ is associated to $\mathcal{A}$ if the $\mathcal{O}_F$-fractional ideal $\mathbb{Z} + \mathbb{Z}\bar{\tau}_\mathcal{A}$ is in the class $\mathcal{A} \in \mathrm{Cl}(F)$. Let $\mathcal{A}_0 \in \mathrm{Cl}(F)$ denote the principal class. For any $\mathcal{A}, \mathcal{B} \in \mathrm{Cl}(F)$, define $u_{\mathcal{A},\mathcal{B}}, u_\mathcal{A} \in H$ by

$$u_{\mathcal{A},\mathcal{B}} := \frac{y_\mathcal{A}^6 \Delta(\tau_\mathcal{A})}{y_\mathcal{B}^6 \Delta(\tau_\mathcal{B})}, \quad u_\mathcal{A} := \prod_{\mathcal{I}\in\mathrm{Cl}(F)} u_{\mathcal{A},\mathcal{I}}, \tag{3.1}$$

where $\Delta(\tau) = \eta^{24}(\tau)$ is the unique normalized cusp form of weight twelve, level one. Although $\tau_\mathcal{A}$ is not unique, its equivalence class under the action of $\mathrm{PSL}_2(\mathbb{Z})$ is well-defined. Since $y^6\eta^{24}(\tau)$ is invariant under the action of $\mathrm{PSL}_2(\mathbb{Z})$, the quantity $u_{\mathcal{A},\mathcal{B}}$ is independent of the choices of $\tau_\mathcal{A}$ and $\tau_\mathcal{B}$. Let $\sigma_\mathcal{C} \in \mathrm{Gal}(H/F)$ be the Galois automorphism associated to the class $\mathcal{C} \in \mathrm{Cl}(F)$ via Artin's isomorphism. Then by the theory of complex multiplication [42, Chap. II §2], $u_{\mathcal{A},\mathcal{B}}$ and $u_\mathcal{A}$ are units in H and satisfy

$$\sigma_\mathcal{C}(u_{\mathcal{A},\mathcal{B}}) = u_{\mathcal{A}\mathcal{C}^{-1},\mathcal{B}\mathcal{C}^{-1}}, \quad \sigma_\mathcal{C}(u_\mathcal{A}) = u_{\mathcal{A}\mathcal{C}^{-1}}. \tag{3.2}$$

For an integer M, let $E_M(z, s)$ be the non-holomorphic Eisenstein series of weight zero, level M defined by

$$E_M(z, s) = \sum_{\gamma\in\tilde{\Gamma}_\infty\backslash\Gamma_0(M)} (\mathrm{Im}(\gamma z))^s, \tag{3.3}$$

where $\tilde{\Gamma}_\infty = \{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}) \in \mathrm{SL}_2(\mathbb{Z})\}$. It has a simple pole at $s = 1$ and the well-known expansion

$$E_M(z, s) = y^s + \varphi_M(s)y^{1-s} + O(e^{-y})$$

as $y \to \infty$, where $z = x + iy$ and

$$\varphi_1(s) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)},$$

$$\varphi_M(s) = \tfrac{M^{-2s}(M-1)}{1-M^{2s}}\varphi_1(s) \text{ when } M \text{ is prime.}$$

For convenience, we write $E(z, s)$ for $E_1(z, s)$. Kronecker's first limit formula states that

$$2\zeta(2s)E(z,s) = \frac{\pi}{s-1} + 2\pi\left(\gamma - \log(2) - \log(\sqrt{y}|\eta(z)|^2)\right) + O(s-1), \tag{3.4}$$

where γ is the Euler constant. The factor of 2 comes from $\pm I \in \tilde{\Gamma}_\infty$. Using (3.4) and Rankin-Selberg unfolding trick, we can relate the inner product between dihedral newforms to logarithm of $u_{\mathcal{A},\mathcal{A}_0}$ as follows.

Proposition 3.1. *Let ψ, ψ' be characters of* $\mathrm{Cl}(F)$*, with ψ non-trivial. If $\psi' = \psi$ or $\overline{\psi}$, then*

$$\langle g_\psi, g_{\psi'}\rangle = -\frac{H(p)}{12}\sum_{\mathcal{A}\in \mathrm{Cl}(F)} \psi^2(\mathcal{A})\log|u_{\mathcal{A},\mathcal{A}_0}| = \frac{1}{12}\sum_{\mathcal{A}\in \mathrm{Cl}(F)}\psi^2(\mathcal{A})\log|u_{\mathcal{A}}| = H(p)I_{\psi^2}, \tag{3.5}$$

where $\mathcal{A}_0 \in \mathrm{Cl}(F)$ is the principal class and

$$I_{\psi^2} := -\sum_{\mathcal{A}\in\mathrm{Cl}(F)} \psi^2(\mathcal{A})\log|\sqrt{y_{\mathcal{A}}}\eta(\tau_{\mathcal{A}})^2|. \tag{3.6}$$

Otherwise, $\langle g_\psi, g_{\psi'}\rangle = 0$.

Proof. Since p is prime, the Eisenstein series $E_p(z, s)$ has a simple pole at $s = 1$ with residue $\frac{3}{\pi(p+1)}$, which is independent of z. So we have the relationship

$$\frac{3}{\pi(p+1)}\cdot\langle g_\psi, g_{\psi'}\rangle = \mathrm{Res}_{s=1}\int_{\Gamma_0(p)\backslash\mathcal{H}} g_\psi(z)\overline{g_{\psi'}(z)}E_p(z,s)y\frac{dxdy}{y^2}.$$

Now, we can use the Rankin-Selberg method to unfold the right hand side and obtain

$$\int_{\Gamma_0(p)\backslash\mathcal{H}} g_\psi(z)\overline{g_{\psi'}(z)}E_p(z,s)y\frac{dxdy}{y^2} = \frac{\Gamma(s)}{(4\pi)^s}\sum_{n\ge1}\frac{r_\psi(n)\overline{r_{\psi'}(n)}}{n^s}$$

Let $\rho_\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{C})$ be the representation induced from ψ. Then it is also the one attached to $g_\psi(z)$ via Deligne-Serre's theorem. Up to Euler factors at p, the right hand side is $L(s, \rho_\psi \otimes \overline{\rho_{\psi'}})$, the L-function of the tensor product of the representations ρ_ψ and $\rho_{\psi'}$. From the character table of the dihedral group $D_{2H(p)}$, we see that

$$\rho_\psi \otimes \overline{\rho_{\psi'}} = \rho_{\psi\psi'} \oplus \rho_{\psi\overline{\psi'}}.$$

So when $\psi' \neq \psi$ or $\overline{\psi}$, the L-function $L(s, \rho_\psi \otimes \overline{\rho_{\psi'}})$ is holomorphic at $s = 1$ and $\langle g_\psi, g_{\psi'}\rangle = 0$. Otherwise, we have

$$\sum_{n\ge1}\frac{r_\psi(n)\overline{r_\psi(n)}}{n^s} = \frac{\zeta(s)L(s,\chi_p)L(s,\rho_{\psi^2})}{\zeta(2s)(1+p^{-s})},$$

Putting these together, we obtain

$$\langle g_\psi, g_\psi\rangle = \frac{p}{2\pi^2}L(1,\chi_p)L(1,\rho_{\psi^2}). \tag{3.7}$$

From the theory of quadratic forms (see Eq. (5) and (6) in [42, Chap. 1 §1]) , we have

$$\sqrt{p}L(s,\rho_{\psi^2}) = \zeta(2s)\sum_{\mathcal{A}\in \mathrm{Cl}(F)} \psi^2(\mathcal{A})E(\tau_{\mathcal{A}},s).$$

Since ψ is non-trivial and $H(p)$ is odd, ψ^2 is non-trivial and (3.4) implies that

$$L(1,\rho_{\psi^2}) = -\frac{2\pi}{\sqrt{p}}\sum_{\mathcal{A}\in \mathrm{Cl}(F)} \psi^2(\mathcal{A})\log(\sqrt{y_{\mathcal{A}}}|\eta(\tau_{\mathcal{A}})|^2).$$

Along with the class number formula for $p>3$

$$L(1,\chi_p) = \frac{2\pi H(p)}{w_p\sqrt{p}} = \frac{\pi H(p)}{\sqrt{p}},$$

we arrive at

$$\langle g_\psi, g_\psi\rangle = -\frac{H(p)}{12}\sum_{\mathcal{A}\in \mathrm{Cl}(F)} \psi^2(\mathcal{A})\log|y_{\mathcal{A}}^6\Delta(\tau_{\mathcal{A}})|. \tag{3.8}$$

Since ψ^2 is non-trivial, this implies Eq. (3.5). □

Remarks.

i) The calculations above also follow from [44, II §6].
ii) By the same procedure, one can analyze the inner product between any pairs of weight one newforms. In particular, if $g(z), h(z)\in S_1(p)$ arise from different types of Galois representations, then $\langle g,h\rangle = 0$.
iii) This proposition is really a proven case of Stark's conjecture in the abelian, rank one case for the abelian extension H/F (see Conjecture 1.1 in [16]).

Next we prove the existence of a preimage whose *principal part coefficients* are special.

Proposition 3.2. *Let ψ be a non-trivial character of* $\mathrm{Cl}(F)$. *Then there exists* $\tilde{g}_\psi(z) = \sum_{n\geq n_0} r_\psi^+(n)q^n \in \mathbb{M}_1^-(p)$ *with shadow $g_\psi(z)$ such that*

(i) When $\chi_p(n)=1$ or $n<-\frac{p+1}{24}$, the coefficient $r_\psi^+(n)$ equals to zero.
(ii) For $n\leq 0$, the coefficients $r_\psi^+(n)$ are of the form

$$r_\psi^+(n) = \frac{1}{12H(p)\kappa_p^-}\sum_{\mathcal{A}\in \mathrm{Cl}(F)} \psi^2(\mathcal{A})\log|u(n,\mathcal{A})|,$$

where $\kappa_p^-\in\mathbb{Z}$ is defined by Eq. (3.9) *and depends only on p. The units $u(n,\mathcal{A})\in\mathcal{O}_H^\times$ depends only on n and $\mathcal{A}$. When $n<0$, $u(n,\mathcal{A})$ is an $H(p)^{th}$ power in $\mathcal{O}_H^\times$.*
(iii) Let $\sigma_{\mathcal{C}}\in \mathrm{Gal}(H/F)$ be the element associated to the class $\mathcal{C}\in \mathrm{Cl}(F)$ via Artin's isomorphism. When $n\leq 0$, it acts on the units $u(n,\mathcal{A})$ by

$$\sigma_{\mathcal{C}}(u(n,\mathcal{A})) = u(n,\mathcal{A}\mathcal{C}^{-1}).$$

Remark. In practice, one could choose $u(n,\mathcal{A})$ and $1/\kappa_p^-\in\mathbb{Z}$ such that $1/\kappa_p^- \mid 12H(p)$ (see the example $p=23$ in §1).

Proof. From Lemma 2.4, we know that the dimension of $S_1^+(p)$, denoted by d_+, is $\frac{1}{2}(H(p)-1)+d_-$. Let $\{g_1,g_2,\ldots,g_{d_+}\}$ be a basis of $S_1^+(p)$. Suppose we have chosen $n_1,n_2,\ldots,n_{d_+}>0$ such that $\chi_p(n_k)\neq -1$ and the matrix

$$P := [c(g_j,n_k)]_{1\leq j,k\leq d_+}$$

is invertible. Then for any ψ, there exists $\tilde{g}_\psi(z) \in \mathbb{M}_1^-(p)$ with shadow $g_\psi(z)$ and principal part coefficients $r_\psi^+(-n) = 0$ for all positive integers n not in the set $\{n_1, n_2, \dots, n_{d_+}\}$ by Proposition 2.6. Furthermore, these $r_\psi^+(-n)$ can be uniquely determined from solving a $d_+ \times d_+$ system of equations. We will show that such $\tilde{g}_\psi(z)$ can be made to satisfy the proposition.

First, we will show that it is possible to choose such $\{n_k : 1 \le k \le d_+\}$ satisfying $n_k \le (p+1)/24$. Let $\{h_1, h_2, \dots, h_{d_+}\}$ to be a q-echelon basis of $S_1^+(p)$. Then these n_k's can be chosen to be bounded by the supremum of the set

$$\{\mathrm{ord}_\infty h(z) : h(z) \in S_1^+(p)\}.$$

Now, the square of a cusp form in $S_1(p)$ is in $S_2(p)$, the space of weight 2 cusp form of trivial nebentypus on $\Gamma_0(p)$. This gives rise to a holomorphic differential form on the modular curve $X_0(p)$. Atkin showed (see [1, 31, 36]) that ∞ is not a Weierstrass point of the modular curve $X_0(p)$, i.e. $\mathrm{ord}_\infty h(z)$ is no more than the genus of $X_0(p)$. Using the Riemann-Hurwitz formula (see [41]), we know that the genus of $X_0(p)$ is bounded by $(p+1)/12$. So the n_k's can be chosen to be bounded by $(p+1)/24$. Note that with these n_k's, the matrix P above will be non-singular with any basis $\{g_j\}$ of $S_1^+(p)$.

In particular, let $\{g_j(z) : 1 \le j \le (H(p)-1)/2\}$ be the dihedral newforms, and label the class group characters as $\{\psi_j : 1 \le j \le H(p)-1\}$ such that $g_{\psi_j}(z) = g_j(z)$ and $\overline{\psi_j} = \psi_{H(p)-j}$ for $1 \le j \le (H(p)-1)/2$. When $(H(p)+1)/2 \le j \le d_+$, let $g_j(z)$ be linear combinations of non-dihedral newforms such that $g_j(z)$ has integral coefficients and the set $\{g_j(z) : (H(p)+1)/2 \le j \le d_+\}$ is linearly independent. Then for each non-trivial ψ, Proposition 2.5 implies that the values $\{r_\psi^+(-n_k) : 1 \le k \le d_+\}$ satisfy the matrix equation

$$P \cdot [\delta(n_k) r_\psi^+(-n_k)]_{1 \le k \le d_+} = R,$$

where R is the $d_+ \times 1$ matrix $(\langle g_j, g_\psi \rangle)_{1 \le j \le d_+}$. By Proposition 3.1, the matrix R equals to

$$R = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{H(p)}{12} \sum_{\mathcal{A}} \psi^2(\mathcal{A}) \log |u_{\mathcal{A},\mathcal{A}_0}| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Label the non-principal classes as $\mathcal{A}_i$ for $1 \le i \le H(p)-1$ such that $\mathcal{A}_i^{-1} = \mathcal{A}_{H(p)-i}$. Let M' be the matrix $\left(\begin{smallmatrix} M_1 & 0 \\ 0 & Id_{d_-} \end{smallmatrix}\right)$, where M_1 is a non-singular matrix of size $\frac{1}{2}(H(p)-1) \times \frac{1}{2}(H(p)-1)$ defined by

$$[(\psi_j(\mathcal{A}_i) + \psi_j(\mathcal{A}_{H(p)-i}) - 2)/H(p)]_{1 \le i,j \le (H(p)-1)/2}$$

and Id_{d_-} is the $d_- \times d_-$ identity matrix. The product $M'P$ is the matrix

$$M'P = \begin{pmatrix} (r_{\mathcal{A}_1}(n_1) - r_{\mathcal{A}_0}(n_1)) & \cdots & (r_{\mathcal{A}_1}(n_{d_+}) - r_{\mathcal{A}_0}(n_{d_+})) \\ \vdots & \ddots & \vdots \\ (r_{\mathcal{A}_{(H(p)-1)/2}}(n_1) - r_{\mathcal{A}_0}(n_1)) & \cdots & (r_{\mathcal{A}_{(H(p)-1)/2}}(n_{d_+}) - r_{\mathcal{A}_0}(n_{d_+})) \\ c(g_{(H(p)+1)/2}, n_1) & \cdots & c(g_{(H(p)+1)/2}, n_{d_+}) \\ \vdots & \ddots & \vdots \\ c(g_{d_+}, n_1) & \cdots & c(g_{d_+}, n_{d_+}) \end{pmatrix},$$

which has integer coefficients and is also non-singular. The product $M'R$ is the $d_+ \times 1$ matrix

$$M'R = \begin{pmatrix} \frac{1}{12}\sum_{\mathcal{A}} \psi^2(\mathcal{A}) \log |u_{\mathcal{A}}(\mathcal{A}_1)| \\ \vdots \\ \frac{1}{12}\sum_{\mathcal{A}} \psi^2(\mathcal{A}) \log |u_{\mathcal{A}}(\mathcal{A}_{(H(p)-1)/2})| \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where

$$u_{\mathcal{A}}(\mathcal{B}) := u_{\mathcal{A}\sqrt{\mathcal{B}},\mathcal{A}} u_{\mathcal{A}\sqrt{\mathcal{B}}^{-1},\mathcal{A}} \in \mathcal{O}_H^{\times}.$$

Also by (3.2), we have

$$\sigma_{\mathcal{C}}(u_{\mathcal{A}}(\mathcal{B})) = u_{\mathcal{A}\mathcal{C}^{-1}}(\mathcal{B})$$

for any class $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Cl}(F)$.

Since $M'P$ is a non-singular matrix with integer entries, one can write its inverse as $\frac{1}{\kappa_p^-}(\alpha_{k,j})_{1\le k,j\le d_+}$ with $\alpha_{k,j} \in \mathbb{Z}$ and

$$\kappa_p^- := -\det(M'P) \in \mathbb{Z}. \tag{3.9}$$

Since n_k is bounded by $(p+1)/24$ as chosen earlier, $p \nmid n_k$ and $\delta(n_k) = 1$ for all $1 \le k \le d_+$. Thus, $r_\psi^+(-n_k)$ can be written as

$$r_\psi^+(-n_k) = \frac{1}{12H(p)\kappa_p^-} \sum_{\mathcal{A}} \psi^2(\mathcal{A}) \log \left| \prod_{j=1}^{(H(p)-1)/2} u_{\mathcal{A}}(\mathcal{A}_j)^{\alpha_{k,j}} \right|^{H(p)}.$$

From this, we can choose

$$u(-n_k, \mathcal{A}) := \left(\prod_{j=1}^{(H(p)-1)/2} u_{\mathcal{A}}(\mathcal{A}_j)^{\alpha_{k,j}} \right)^{H(p)}.$$

Then $r_\psi^+(-n_k)$ satisfies conditions (ii) and (iii) in the proposition.

Finally, applying Propositions 2.5 and 3.1 to the Eisenstein series $E_1(z) \in M_1^+(p)$ in Eq. (1.4) and the cusp form $g_\psi(z)$ gives us

$$H(p)r_\psi^+(0) + \sum_{k=1}^{d_+} R_p(n_k) r_\psi^+(-n_k) = \langle E_1, g_\psi \rangle = 0.$$

So we can write $r_\psi^+(0) = \frac{1}{12H(p)\kappa_p^-} \log |u(0,\mathcal{A})|$ with

$$u(0,\mathcal{A}) := \prod_{k=1}^{d_+} u(n_k, \mathcal{A})^{-R_p(n_k)} \in \mathcal{O}_H^{\times},$$

which satisfies condition (iii) as well. □

As a consequence of the analysis in Proposition 3.2, we have the following corollary.

Corollary 3.3. *Let $g(z) \in S_1^+(p)$ be a cusp form with integral Fourier coefficients at infinity. Then for any character $\psi : \mathrm{Cl}(F) \longrightarrow \mathbb{C}^\times$, one can write*

$$\langle g, g_\psi \rangle = \frac{1}{12\kappa_p^-} \sum_{\mathcal{A}\in \mathrm{Cl}(F)} \log |u_g(\mathcal{A})|,$$

where the unit $u_g(\mathcal{A}) \in \mathcal{O}_H^\times$ *is independent of* ψ *and satisfies* $\sigma_{\mathcal{C}}(u_g(\mathcal{A})) = u_g(\mathcal{A}\mathcal{C}^{-1})$ *for all* $\mathcal{C} \in \mathrm{Cl}(F)$*.*

Proof. If ψ is trivial, then g_ψ is the Eisenstein series and its inner product with $g(z)$ is 0. Otherwise, write $g(z) = \sum_{n\geq 1} c(n)q^n$ with $c(n) \in \mathbb{Z}$. In the notation of the proof of Proposition 3.2, we can write

$$\langle g, g_\psi\rangle = \begin{pmatrix} c(n_1) & c(n_2) & \dots & c(n_{d_+})\end{pmatrix} \cdot (M'P)^{-1}(M'R).$$

From the shape of $M'R$ in Proposition 3.2 and $\det(M'P) = -\kappa_p^-$, we can deduce the corollary. □

4. Values of Borcherds lift and Proof of Theorem 1.2

In this section we will prove Theorem 1.2. It will be deduced as a corollary of a more general identity for special values of certain Borcherds lifts of weight 1/2 weakly holomorphic forms from an application of Rankin's method to forms with poles. We remark that the method also yields another proof of Eq. (1.12), whose analytic proof in [24] uses Hecke-Eisenstein series of weight one. The method can also be generalized to higher levels in order to obtain some refinements of certain results of [26] on height pairings of Heegner divisors.

4.1. **Binary Quadratic Forms, Borcherds Lift and Theorem 4.1.** Let $-d < 0$ be a discriminant, not necessarily fundamental, and $\mathcal{Q}_{-d}$ the set of positive definite integral binary quadratic forms

$$q(x, y) = ax^2 + bxy + cy^2$$

with $-d = b^2 - 4ac$, where $a, b, c \in \mathbb{Z}$. For any $q \in \mathcal{Q}_{-d}$, the associated CM point is defined to be

$$\tau_q = \frac{-b + \sqrt{-d}}{2a} \in \mathcal{H},$$

where $\mathcal{H}$ is the upper half plane. Clearly $q(\tau_q, 1) = 0$. The modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts on $q \in \mathcal{Q}_{-d}$ by a linear change of variables, which induces linear fractional transformation on τ_q. Let w_q be the number of stabilizers of q and $H(d)$ the number of equivalence classes of quadratic forms, which is the Hurwitz class number. Those classes represented by primitive forms (those with $\gcd(a, b, c) = 1$) comprise a finite abelian group under composition, which is called the class group. When $-d$ is fundamental, this class group is canonically isomorphic to the ideal class group of $\mathbb{Q}(\sqrt{-d})$ by sending $[q]$ to the class $\mathcal{A} \in \mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))$ containing the fractional ideal $\mathbb{Z} + \mathbb{Z}\overline{\tau_q}$.

Let $M^!_{1/2}$ be the space of weakly holomorphic modular forms of weight 1/2 and level 4 satisfying Kohnen's plus space condition. It has a canonical basis $\{f_d\}_{d\geq 0}$ with $d \equiv 0, 3 \pmod 4$ and Fourier expansions

$$f_d(z) = q^{-d} + \sum_{n\geq 1} c(f_d, n)q^n.$$

Let $f(z) \in M^!_{1/2}$ be a weakly holomorphic form with integral Fourier coefficients $c(f, n)$. In [3], Borcherds constructed an infinite product $\Psi_f(z)$ using $c(f, n)$ as exponents, and showed that it is a modular form of weight $c(f, 0)$ and some character. The divisors of $\Psi_f(z)$ are supported on cusps and imaginary quadratic irrationals. In particular, if τ is a quadratic irrational of discriminant $-D < 0$, then its multiplicity in $\Psi_f(z)$ is

$$\mathrm{ord}_\tau(\Psi_f) = \sum_{k>0} c(f, -Dk^2).$$

For example, when $f(z) = f_d(z)$ with $d > 0$, the Borcherds product $\Psi_{f_d}(z)$ equals to

$$\prod_{q \in \mathcal{Q}_{-d}/\Gamma} (j(z) - j(\tau_q))^{1/w_q}, \tag{4.1}$$

Note that when $-d$ is fundamental, w_d, the number of roots of unity in $\mathbb{Q}(\sqrt{-d})$, is equal to $2w_q$ for all $q \in \mathcal{Q}_{-d}$.

Given $f(z) \in M^!_{1/2}$, define a modular form $f^{\mathrm{lift},\theta}(z) \in M^!_1(p)$ by

$$f^{\mathrm{lift},\theta}(z) := U_4(f(pz)\theta(z)), \tag{4.2}$$

where U_4 is the standard U-operator. It is easy to verify that $f^{\mathrm{lift},\theta}(z) \in M^{!,+}_1(p)$ from its Fourier expansion.

Let ψ be a non-trivial character of $\mathrm{Cl}(F)$ and $g_\psi(z) \in S_1(p)$ the associated weight one newform. The main theorem of this section relates the regularized inner product $\langle f^{\mathrm{lift},\theta}, g_\psi\rangle_{\mathrm{reg}}$ to the value of the Borcherds lift $\Psi_f(z)$.

Theorem 4.1. *Let $f(z) \in M^!_{1/2}$ be a weakly holomorphic modular form with integral Fourier coefficients $c(f, n)$, and $\Psi_f(z)$ its Borcherds lift. Suppose ψ is a non-trivial character of $\mathrm{Cl}(F)$. Then we have*

$$\langle f^{\mathrm{lift},\theta}, g_\psi\rangle_{\mathrm{reg}} = -2 \lim_{\epsilon \to 0} \sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \left(\log |\Psi_f(\tau_{\mathcal{A}} + \epsilon)|^2 + C_f \log |y_{\mathcal{A}}| \right), \tag{4.3}$$

where $C_f = \sum_{k \in \mathbb{Z}} c(f, -pk^2)$ is the constant term of $f^{\mathrm{lift},\theta}(z)$, and $\tau_{\mathcal{A}}$ is a CM point associated to the class $\mathcal{A}$.

Remark. It will clear from the proof that the limit in Eq. (4.3) exists and is independent of the choice of $\tau_{\mathcal{A}}$.

Let

$$\tilde{g}_\psi(z) = \sum_{n \in \mathbb{Z}} r^+_\psi(n) q^n \in \mathbb{M}^-_1(p)$$

be a mock-modular form with shadow $g_\psi(z)$. By Proposition 2.5, the regularized inner product $\langle f^{\mathrm{lift},\theta}, g_\psi\rangle_{\mathrm{reg}}$ can be expressed in terms of $r^+_\psi(n)$ as

$$\langle f^{\mathrm{lift},\theta}, g_\psi\rangle_{\mathrm{reg}} = \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} r^+_\psi\left(\frac{-pm - k^2}{4}\right) c(f, m)\delta(k). \tag{4.4}$$

By setting $f = f_d$ for $-d$ a fundamental discriminant and choosing $\tilde{g}_\psi(z)$ as in Proposition 3.2, one can see that Theorem 1.2 is a direct consequence of Theorem 4.1 and Eq. (4.4).

The plan of the proof goes as follows. First, we will recall from [24] the construction of the automorphic Green's function as the limit of an infinite sum and express the Borcherds lift in terms of such sum. Then we will prove an identity between $\langle f^{\mathrm{lift},\theta}, g_\psi\rangle_{\mathrm{reg}}$ and the limit of another similar infinite sum through Eq. (4.4). Finally, an elementary counting argument will connect these two infinite sums and finish the proof of Theorem 4.1, from which Theorem 1.2 is deduced. In some sense, the proof is in the same spirit as Zagier's proof of Borcherds' theorem in [49].

4.2. **Automorphic Green's Function.** Here, we will follow the construction in [24, §5]. For two distinct points $z_j = x_j + iy_j \in \mathcal{H}$, the invariant hyperbolic distance $d(z_1, z_2)$ between them is defined by

$$\cosh d(z_1, z_2) = \frac{(x_1 - x_2)^2 + y_1^2 + y_2^2}{2y_1y_2}. \tag{4.5}$$

Note $d(z_1, z_2) = d(\gamma z_1, \gamma z_2)$ for all $\gamma \in \mathrm{PSL}_2(\mathbb{R})$. The Legendre function of the second kind $Q_{s-1}(t)$ is defined by

$$\begin{aligned} Q_{s-1}(t) &= \int_0^\infty (t + \sqrt{t^2-1}\cosh u)^{-s} du, \ \mathrm{Re}(s) > 1, t > 1, \\ Q_0(t) &= \tfrac{1}{2}\log\left(1 + \tfrac{2}{t-1}\right). \end{aligned} \tag{4.6}$$

For two distinct points $z_1, z_2 \in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H}$, the following convergent series defines the automorphic Green's function

$$G_s(z_1, z_2) := \sum_{\gamma \in \mathrm{PSL}_2(\mathbb{Z})} g_s(z_1, \gamma z_2), \ \mathrm{Re}(s) > 1, \tag{4.7}$$

where

$$g_s(z_1, z_2) := -2Q_{s-1}(\cosh d(z_1, z_2)). \tag{4.8}$$

Recall that $E(\tau, s)$ is defined in (3.3) and $\varphi_1(s)$ is the coefficient of y^{1-s} in the Fourier expansion of $E(\tau, s)$. Proposition 5.1 in [24] tells us that for distinct $z_1, z_2 \in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H}$, the values of the j-function are related to the values of the automorphic Green's function by

$$\log|j(z_1) - j(z_2)|^2 = \lim_{s\to 1}(G_s(z_1, z_2) + 4\pi E(z_1, s) + 4\pi E(z_2, s) - 4\pi\varphi_1(s)) - 24. \tag{4.9}$$

Along with Eq. (4.7), this gives us the following proposition.

Proposition 4.2. *Let $d, D > 0$ be congruent to 0 or 3 modulo 4 and $Q \in \mathcal{Q}_{-D}$. If $\tau_Q \neq \tau_q$ for any $q \in \mathcal{Q}_{-d}$, then*

$$\log|\Psi_{f_d}(\tau_Q)|^2 = \lim_{s\to 1}\left(\sum_{k > \sqrt{dD}} \rho_Q(k, d)(-2)Q_{s-1}\left(\frac{k}{\sqrt{dD}}\right) + H(d)4\pi E(\tau_Q, s) + R(d, s)\right), \tag{4.10}$$

where $R(d, s) = \sum_{q \in \mathcal{Q}_{-d}/\Gamma}(4\pi E(\tau_q, s) - 4\pi\varphi_1(s) - 24)$ and $\rho_Q(k, d)$ is the counting function defined by

$$\rho_Q(k, d) := \#\{q \in \mathcal{Q}_{-d} | \ \cosh d(\tau_q, \tau_Q) = \tfrac{k}{\sqrt{dD}}\} \tag{4.11}$$

and is independent of the choice of the representative Q.

Remark. A similar equation holds when $\tau_Q = \tau_q$ for some $q \in \mathcal{Q}_{-d}$ (see Eq. (4.25)).

Proof. For τ_Q, τ_q as above, it is easy to verify that

$$\sqrt{dD}\cosh d(\tau_Q, \tau_q) \in \mathbb{Z}.$$

Furthermore, $\cosh d(\tau_Q, \tau_q) = 1$ precisely when $\tau_Q = \tau_q$. Otherwise, $\cosh d(\tau_Q, \tau_q) > 1$.

Now let $z_1 = \tau_Q, z_2 = \tau_q$ in Eq. (4.9) and sum over $q \in \mathcal{Q}_{-d}/\Gamma$. With the following observation

$$\begin{aligned}\sum_{q\in\mathcal{Q}_{-d}/\Gamma} \frac{1}{w_q} G_s(\tau_Q,\tau_q) &= \sum_{q\in\mathcal{Q}_{-d}/\Gamma}\sum_{\gamma\in\Gamma}\frac{1}{w_q}(-2)Q_{s-1}(\cosh d(\tau_Q,\gamma\tau_q))\\ &= \sum_{q\in\mathcal{Q}_{-d}}(-2)Q_{s-1}(\cosh d(\tau_Q,\tau_q))\\ &= \sum_{k>\sqrt{dD}}\rho_Q(k,d)(-2)Q_{s-1}\left(\frac{k}{\sqrt{dD}}\right),\end{aligned}$$

we have Eq. (4.10). □

4.3. **Infinite Sum Expression of the Regularized Inner Product.** For convenience, we denote

$$a(d,\psi) := \langle f_d^{\mathrm{lift},\theta}, g_\psi\rangle_{\mathrm{reg}}. \tag{4.12}$$

The following lemma relates $a(d,\psi)$ with the limit of an infinite sum involving the $r_\psi(n)$'s.

Lemma 4.3. *Let $d \geq 1$ be an integer congruent to 0 or 3 modulo 4. Then we have*

$$a(d,\psi) - \frac{a(0,\psi)H(d)}{H(0)} = 2\lim_{s\to1}\left(\sum_{k>\sqrt{pd}}\delta(k)r_\psi\left(\frac{-pd+k^2}{4}\right)2Q_{s-1}\left(\frac{k}{\sqrt{pd}}\right)\right). \tag{4.13}$$

The idea of the proof is to define a non-holomorphic function $\Phi^*(z)$ of weight 3/2 and level 4, which satisfies Kohnen's plus space condition and a mildly decaying condition at the cusps. Then its holomorphic projection vanishes since there is no holomorphic cusp form of weight 3/2, level 4 satisfying the plus space condition. This, along with Eq. (4.4), gives us the desired identity. The function $\Phi^*(z)$ is analogous to $\mathcal{G}_D(z)$ in [32], where the inner product between $\mathcal{G}_D(z)$ and a weight $k+1/2$, level 4 eigenform $g(z)$ gives the special value of the L-function associated to the Shimura lift of $g(z)$.

Proof. Let $\tilde{g}_\psi(z) \in \mathbb{M}_1^-(p)$ be a mock-modular form with shadow $g_\psi(z)$ and

$$\mathrm{ord}_\infty \tilde{g}_\psi(z) \geq -\tfrac{p}{4}.$$

Forms satisfying these conditions exist by Proposition 3.2. The restriction on the order of $\tilde{g}_\psi(z)$ at infinity simplifies the proof and is not important. But the requirement that $\tilde{g}_\psi(z)$ be in the minus space is crucial for the validity of the statement.

Denote $\hat{g}_\psi(z) \in H_1^-(p)$ the associated harmonic Maass form and define the function $\Phi(z)$ by

$$\begin{aligned}\Phi(z) &:= \mathrm{Tr}_4^{4p}((\hat{g}_\psi|U_p)(4z)\theta(pz))\\ &= \frac{1}{p+1}\left((\hat{g}_\psi \mid U_p)(4z)\theta(pz) + (\hat{g}_\psi(4z)\theta(z)) \mid U_p\right).\end{aligned}$$

Here Tr_4^{4p} is the trace down operator from level $4p$ to level 4 of weight 3/2, similarly defined as in Eq. (2.15). The function $\Phi(z)$ transforms like a weight 3/2 modular form of level 4 and should be compared to $\mathcal{G}_D(z)$ in [32], which was defined by applying the trace down operator Tr_4^{4D} to the product of the holomorphic weight k Eisenstein series and $\theta(|D|z)$. If one replaces the weight k holomorphic Eisenstein series by the weight one real-analytic Eisenstein series with an s variable, and consider the derivative with respect to s, then it is something quite similar to this $\Phi(z)$.

Now, we can calculate the Fourier expansion of $\Phi(z)$ as

$$\Phi(z) = \frac{1}{p+1}\sum_{n\in\mathbb{Z}}(b(n,y)+a(n))q^n,$$
$$b(n,y) = -\sum_{k\in\mathbb{Z}}\delta(k)r_\psi(m)\beta_1\left(\tfrac{k^2-pn}{4}, \tfrac{4y}{p}\right),$$
$$a(n) = \sum_{k\in\mathbb{Z}}\delta(k)r_\psi^+\left(\frac{pn-k^2}{4}\right).$$

Note that we have used the fact

$$\overline{r_\psi(m)} = r_\psi(m)$$

for all $m \geq 1$ when expressing $b(n,y)$. Since $p \equiv 3 \pmod 4$, $\Phi(z)$ satisfies Kohnen's plus space condition and $n \equiv 0$ or $3 \pmod 4$. Also, the right hand side of Eq. (4.4) equals to $a(d)$ for $f(z) = f_d(z)$ as $\mathrm{ord}_\infty \tilde{g}_\psi(z) \geq -p/4$, So we have

$$a(d) = a(d,\psi). \tag{4.14}$$

Let $\mathcal{F}(z)$ be the weight 3/2 Eisenstein series studied in [30], which has the following Fourier expansion

$$\mathcal{F}(z) = \sum_{n=0}^{\infty}H(n)q^n + y^{-1/2}\sum_{m=-\infty}^{\infty}\frac{1}{16\pi}\beta_{3/2}(m^2,y)q^{-m^2},$$

and satisfies Kohnen's plus space condition. Recall that β_k is defined in §1. For all $n \in \mathbb{Z}$, notice that $b(n,y)q^n$ is exponentially decaying as $y \to \infty$. Also, $a(n)$ vanishes for all $n < 0$. Thus, the function

$$\Phi^*(z) := \Phi(z) - \frac{a(0)\mathcal{F}(z)}{H(0)}$$

is $O(y^{-1/2})$ as $y \to \infty$. The same decaying property holds at the other two cusps of $\Gamma_0(4)$ as well, since $\Phi^*(z)$ satisfies Kohnen's plus space condition. So we can consider the holomorphic projection of $\Phi^*(z)$ to the Kohnen plus space $S^+_{3/2}(\Gamma_0(4))$. Define the weight 3/2 Poincaré series by

$$\mathcal{P}_d(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} j(\gamma,z)^{-3}e^{2d\pi i\gamma z}\mathrm{Im}(\gamma z)^{s/2},$$

where for $\gamma \in \Gamma_0(4)$

$$j(\gamma,z) := \frac{\theta(\gamma z)}{\theta(z)}.$$

This series converges absolutely for $\mathrm{Re}(s) > 1$ and can be analytically continued to $\mathrm{Re}(s) \geq 0$. As $s \to 0$, the inner product $\sqrt{d}\langle\Phi^*(z), \mathcal{P}_d(z,s)\rangle$ is the d^{th} Fourier coefficient of a cusp form in $S^+_{3/2}(\Gamma_0(4))$, since $\Phi^*(z)$ is already in the plus space. Given $S^+_{3/2}(\Gamma_0(4)) = \{0\}$, we know the limit is zero and obtain the following equation after applying Rankin-Selberg unfolding,

$$\lim_{s\to0}\left(\frac{\Gamma(\frac{1+s}{2})}{(4\pi d)^{1/2+s/2}}\left(a(d) - \frac{a(0)H(d)}{H(0)}\right) + \int_0^\infty b(d,y)e^{-4\pi dy}y^{1/2+s/2}\frac{dy}{y}\right) = 0. \tag{4.15}$$

After some manipulations, we have

$$\int_0^\infty \beta_1(d,\mu y)e^{-4\pi dy}y^{1/2+s/2}\frac{dy}{y} = \frac{\Gamma(\frac{1+s}{2})}{(4\pi d)^{1/2+s/2}}\varrho_s(\mu),$$

where the function $\varrho_s(\mu)$ is defined by

$$\varrho_s(\mu) := \int_1^\infty \frac{du}{(\mu u+1)^{\frac{1+s}{2}} u}, \quad \mu > 0. \tag{4.16}$$

After substituting $\mu = \frac{k^2}{pd} - 1$, we arrive at the following expression

$$-\int_0^\infty b(d,y)e^{-4\pi dy} y^{1/2+s/2}\frac{dy}{y} = \frac{\Gamma(\frac{1+s}{2})}{(4\pi d)^{\frac{1+s}{2}}} 2 \sum_{k>\sqrt{pd}} \delta(k) r_\psi\left(\frac{-pd+k^2}{4}\right)\varrho_s\left(-1+\frac{k^2}{pd}\right). \tag{4.17}$$

Here, we used the fact

$$\beta_1(d,\alpha y) = \beta_1(d\alpha, y)$$

for all $\alpha, y, d \in \mathbb{R}_{>0}$. Since $r_\psi(m) = 0$ whenever $m \leq 0$, the sum changed from $k \in \mathbb{Z}$ to $k > \sqrt{pd}$ and produced a factor of 2. Now substituting (4.14) and (4.17) into (4.15) gives us

$$a(d,\psi) - \frac{a(0,\psi)H(d)}{H(0)} = 2\lim_{s\to 0}\left(\sum_{k>\sqrt{pd}} \delta(k) r_\psi\left(\frac{-pd+k^2}{4}\right)\varrho_s\left(-1+\frac{k^2}{pd}\right)\right).$$

With the following comparisons (see [24, §7] for similar arguments).

$$\varrho_0(\mu) = 2Q_0(\sqrt{\mu+1}),$$
$$Q_{s-1}(\sqrt{\mu+1}) - \frac{s\Gamma(s)^2}{2^{2-s}\Gamma(2s)}\varrho_{s-1}(\mu) = O(\mu^{-1/2-s/2}),$$

we could substitute $\varrho_s(-1+\frac{k^2}{pd})$ with $2Q_{s-1}(\frac{k}{\sqrt{pd}})$ in the limit and obtain Eq. (4.13). □

4.4. **Counting CM Points with Distinct Discriminants.** In preparation for the proof of Theorem 4.1, we will count the number of CM points on a hyperbolic circle in terms of the representation numbers of positive definite binary quadratic forms. Such a counting argument is needed to construct a Green's function at special points. This construction follows the ideas of [24], but in the counting argument given there one also sums over all classes of a given discriminant. Even when one discriminant is prime, which we are assuming, their proof involves quite an intricate application of algebraic number theory. Surprisingly, the refined version we need for a fixed class has a completely elementary proof using the classical theory of composition of binary quadratic forms. It has the further advantage that it applies without extra effort when the other discriminant is not fundamental. Although we will not give details here, the argument adapts to give a corresponding refinement of the level N case in [25].

First, we will recall some facts about positive definite binary quadratic forms. Good references for this theory include the books by Buell [2] and Cox [15]. Let $-D < 0$ be a discriminant, $Q \in \mathcal{Q}_{-D}$ a primitive binary quadratic form and Q^2 denote a representative of the square class of Q. Associated to Q is the counting function

$$r_Q(k) = \tfrac{1}{2}\#\{(x,y) \in \mathbb{Z}^2 | \ Q(x,y) = k\},$$

which is finite and can be non-zero only for non-negative integers k. Clearly $r_Q(k)$ is a class invariant. When $-D < -4$ is fundamental and $\mathcal{A} \in \mathrm{Cl}(\mathbb{Q}(\sqrt{-D}))$ is the class associated to $[Q]$, the counting function $r_Q(k)$ is the same as $r_{\mathcal{A}}(k)$, the number of integral ideals in the class $\mathcal{A}$ having norm equals to k. The goal now is to relate r_{Q^2} to the counting function ρ_Q defined in (4.11).

Proposition 4.4. *Let Q be a positive definite binary quadratic form of discriminant $-D$ where $D = p \equiv 3 \pmod 4$ is a prime. Suppose that $-d < 0$ is a discriminant and that $k \geq 0$. Then*

$$\rho_Q(k,d) = \delta(k)\, r_{Q^2}(\tfrac{k^2-pd}{4}). \tag{4.18}$$

Our proof of this result is an entirely elementary exercise in the classical theory of binary quadratic forms. It relies on a remarkable algebraic identity given in (4.23) below for a particular quadratic form in the square class, whose existence can be coaxed out of the work of Dirichlet [18].

Lemma 4.5. *Every class of primitive positive definite binary quadratic forms of discriminant $-D$ contains a representative of the form*

$$Q(x,y) = Ax^2 + Bxy + ACy^2 \tag{4.19}$$

with $-D = B^2 - 4A^2C$, $A > 0$ and $\gcd(A,B) = 1$. Furthermore, the class of the square of Q is represented by the form

$$Q^2(x,y) = A^2x^2 + Bxy + Cy^2. \tag{4.20}$$

Proof. It is a well–known result of Gauss ([22, §228], see also [2, Prop 4.2 p.50]) that a primitive form of discriminant $-D$ properly represents a positive integer A prime to $-D$ and is thus, upon an appropriate change of variables, equivalent to a form of the shape (A,b,c) with $b^2 - 4Ac = -D$. In particular we have that $\gcd(A,b) = 1$.

Next, by means of a simple translation transformation $x \mapsto x + ty, y \mapsto y$ we will arrange that $(A,b,c) \sim (A,B,AC)$ as desired. This transformation leaves A alone and changes b to $B = b + 2At$. We now choose t to force

$$B^2 = (b + 2At)^2 \equiv -D \pmod{4A^2}.$$

Since $c = \frac{b^2+D}{4A}$ this is equivalent to solving for t the congruence

$$tb \equiv c \pmod A,$$

which is possible since $\gcd(b,A) = 1$.

The fact that (A^2,B,C) represents the square class of (A,B,AC) is a consequence of a classical result of Dirichlet on the convolution of "united" forms (see [2, p.57]). ☐

Turning now to the proof of Proposition 4.4, it is easily checked using (4.5) that for

$$q(x,y) = ax^2 + bxy + cy^2 \in \mathcal{Q}_{-d}$$

and $Q \in \mathcal{Q}_{-D}$ as in (4.19) above we have

$$\sqrt{dD}\cosh d(\tau_q, \tau_Q) = 2Ac + 2ACa - Bb. \tag{4.21}$$

It follows that the statement of Theorem 4.4 is equivalent to the equality

$$\#\{(a,b,c) \in \mathbb{Z}^3 |\; b^2 - 4ac = -d \text{ and } 2Ac + 2ACa - Bb = k\} = \delta(k)\, r_{Q^2}(\tfrac{k^2-dD}{4}), \tag{4.22}$$

when $D = p$. Note that $a > 0$ for any (a,b,c) in the set since $k \geq 0$.

In order to prove this we will establish a direct bijection between the solutions to the relevant equations. A calculation verifies the truth of the following key identity:

$$4\,Q^2(x,y) = (2Ac + 2ACa - Bb)^2 - (B^2 - 4A^2C)(b^2 - 4ac) \tag{4.23}$$

where Q^2 is as in (4.20) above and

$$x = c - Ca, \quad y = Ba - Ab.$$

Thus every solution (a,b,c) from (4.22) gives rise to a solution (x,y) of

$$Q^2(x,y) = \frac{k^2 - dD}{4}. \tag{4.24}$$

In fact, there is no need to assume that $D = p$ for this part of the argument.

This assumption simplifies our treatment of the converse, to which we now turn. Suppose we are given a solution (x,y) of (4.24). The result is trivial when $d = pm^2$ and $k = pm$ for some integer m, so assume otherwise. Then $(-x,-y)$ is another distinct solution. Observe that since $p = 4A^2C - B^2$ we have

$$4A^2Q^2(x,y) \equiv A^2(4A^2x^2 + 4Bxy + 4Cy^2) \equiv (2A^2x + By)^2 \pmod{p}.$$

Thus

$$\begin{aligned}\big(Ak - (2A^2x + By)\big)\big(Ak + (2A^2x + By)\big) \equiv& A^2k^2 - (2A^2x + By)^2 \\ \equiv& A^2k^2 - 4A^2Q^2(x,y) \equiv 0 \pmod{p}.\end{aligned}$$

Since Q is assumed to be primitive we know that $p \nmid A$.

If $p \nmid k$ we can find an integer a such that

$$An = \pm(2A^2x + By) + pa$$

in precisely one of the two cases of $\pm$. Suppose as we may that $Ak = 2A^2x + By + pa$. Thus we have

$$a = \frac{Ak - 2A^2x - By}{p}.$$

Then since $p = 4A^2C - B^2$ we have

$$k = 2A(x + Ca) + 2ACa - B\frac{Ba - y}{A}.$$

Since $\gcd(A,B) = 1$ we must have that $A \mid (Ba - y)$ and we may define

$$b = (Ba - y)/A, \quad c = x + Ca.$$

Then we have $2Ac + 2ACa - Bb = k$. A computation also shows that $b^2 - 4ac = -d$. If $p \mid k$ then we get two distinct triples (a,b,c) for $\pm(x,y)$ which both satisfy these. Obviously the definition of (a,b,c) inverts the map we started with to get (x,y), at least for those pairs that correspond to an (a,b,c). This finishes the proof of Proposition 4.4.

Remark. To give a level N version of Proposition 4.4, we can start with the more general identity

$$4N\,Q^*(x,y) = (2ANc + 2ACNa - Bb)^2 - (B^2 - 4A^2NC)(b^2 - 4Nac)$$

where $Q^*(x,y) = A^2Nx^2 + Bxy + Cy^2$ and

$$x = c - Ca, \quad y = Ba - Ab.$$

4.5. **Proof of Theorem 4.1.** First, notice that both sides of Eq. (4.3) are additive with respect to $f(z)$. So it suffices to prove the theorem when $f(z) = f_d(z)$ for all $d \geq 0$ and $d \equiv 0$ or 3 modulo 4.

When $d = 0$, $f_d(z) = \theta(z)$ is the Jacobi theta function, $f^{\mathrm{lift},\theta}(z) = 2\vartheta_{\mathcal{A}_0}(z)$ is twice the weight one theta series associated to the principal class $\mathcal{A}_0 \in \mathrm{Cl}(F)$. The Borcherds lift of $\theta(z)$ is $\eta^2(z)$ and Eq. (4.3) becomes

$$\langle 2\vartheta_{\mathcal{A}_0}, g_\psi\rangle = -4 \sum_{\mathcal{A}\in\mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log\left|\sqrt{y_{\mathcal{A}}}\eta^2(\tau_{\mathcal{A}})\right| = 4I_{\psi^2},$$

which is justified by Eq. (3.6), (3.8) and the fact that

$$\langle \vartheta_{\mathcal{A}_0}, g_\psi \rangle = \frac{1}{H(p)} \sum_{\psi' : \mathrm{Cl}(F) \to \mathbb{C}^\times} \langle g_{\psi'}, g_\psi \rangle.$$

When $-d < 0$ is a discriminant *not* of the form $-pm^2$ for any integer m, we know that $\tau_Q \neq \tau_q$ for all $Q \in \mathcal{Q}_{-p}, q \in \mathcal{Q}_{-d}$. In this case, $C_{f_d} = 0$ and the limit in ϵ on the right hand side of Eq. (4.3) is not necessary. Now we can substitute $D = p, \tau_Q = \tau_{\mathcal{A}}$ for $\mathcal{A} \in \mathrm{Cl}(F)$ in Eq. (4.10), apply Eq. (4.18) and sum over all $\mathcal{A}$ with a non-trivial character ψ^2. This yields

$$\sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log |\Psi_{f_d}(\tau_{\mathcal{A}})|^2 = \lim_{s \to 1} \left(\sum_{k > \sqrt{pd}} \delta(k) r_\psi \left(\frac{k^2 - pd}{4} \right) (-2) Q_{s-1} \left(\frac{k}{\sqrt{pd}} \right) + \right.$$
$$\left. + H(d) 4\pi \lim_{s \to 1} \sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) (E(\tau_{\mathcal{A}}, s) + R(d, s)). \right)$$

Adding twice the equation above to Eq. (4.13) cancels the infinite sum on the right hand side and gives us

$$\sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log |\Psi_{f_d}(\tau_{\mathcal{A}})|^2 + a(d, \psi) - \frac{a(0, \psi) H(d)}{H(0)} =$$
$$2H(d) 4\pi \lim_{s \to 1} \sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) (E(\tau_{\mathcal{A}}, s) + R(d, s)).$$

The function $R(d, s)$ is independent of $\mathcal{A}$ and will disappear from the right hand side. Then the rest of the right hand side exactly cancels the term $-\frac{a(0,\psi)H(d)}{H(0)}$ on the left hand side by Kronecker's limit formula (Eq. (3.4)) and we obtain Eq. (4.3).

When $d = pm^2$ for some positive integer m, neither side of Eq. (4.10) makes sense, since for any $Q \in \mathcal{Q}_{-p}$, $\Psi_{f_d}(z)$ has a simple zero at $z = \tau_Q$ and $\rho_Q(pm, d) = 1 = \frac{1}{2} C_{f_d}$ by Proposition 4.4. To accommodate this, we can slightly modify Eq. (4.10) to obtain

$$\begin{aligned} &\lim_{\epsilon \to 0} (\log |\Psi_{f_d}(\tau_Q + \epsilon)|^2 - \rho_Q(pm, d) g_1(\tau_Q + \epsilon, \tau_Q)) = \\ &\lim_{s \to 1} \left(\sum_{k \geq pm+1} \rho_Q(k, d) (-2) Q_{s-1} \left(\frac{k}{\sqrt{pd}} \right) + H(d) 4\pi E(\tau_Q) + R(d, s) \right), \end{aligned} \tag{4.25}$$

for any $Q \in \mathcal{Q}_{-p}$. Since $g_1(z + \epsilon, z) = -\log \left(1 + \frac{4y^2}{\epsilon^2} \right)$ and ψ^2 is non-trivial, we have

$$\sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \lim_{\epsilon \to 0} g_1(\tau_{\mathcal{A}} + \epsilon, \tau_{\mathcal{A}}) = -2 \sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log(y_{\mathcal{A}}). \tag{4.26}$$

Applying the same procedure to Eq. (4.25) as before gives us Eq. (4.3) in this case.

5. Rankin's method and values of Green's function I

In this section, we will prove Theorem 5.1, which is the integral weight analogue of Theorem 4.1. In this case, the weakly holomorphic form in Eq. (4.3) will be replaced with lifts of level one modular functions, and the regularized inner product is related to the values of the Borcherds lifts of these modular functions.

This result is more or less a twisted version of the Gross-Zagier formula at level one, where there is no weight two cusp form or L-function. It prepares the way for §6, where we treat the level N case. Furthermore, the information provided by this theorem will be a stepping stone to the proof of Theorem 1.1 in §7.

For $m \geq 0$, let $j_m(z) = q^{-m} + O(q)$ be the unique modular function of level one with a pole of order m at infinity, e.g. $j_0(z) = 1$ and $j_1(z) = j(z) - 744$. For each $\mathcal{B} \in \mathrm{Cl}(F)$, choose an associated CM point $\tau_{\mathcal{B}} = x_{\mathcal{B}} + iy_{\mathcal{B}} \in \mathcal{H}$ and define $j_m^{\mathrm{lift},\mathcal{B}}(z) \in M_1^{!,+}(p)$ by

$$j_m^{\mathrm{lift},\mathcal{B}}(z) := j_m(pz)\vartheta_{\mathcal{B}}(z). \tag{5.1}$$

Let $\mathrm{M}_2(\mathbb{Z})$ be the space of 2×2 matrices with integer coefficients. For $m \geq 1, (z_1, z_2) \in (\mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H})^2$, define the function $\Psi_m(z_1, z_2)$ by

$$\Psi_m(z_1, z_2) := \prod_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{M}_2(\mathbb{Z}) \\ \det(\gamma) = m}} \left(j(z_1) - j(\gamma z_2)\right). \tag{5.2}$$

It is the value of the modular polynomial $\varphi_m(X, Y)$ at $X = j(z_1), Y = j(z_2)$, and also the Borcherds lift of $j_m(z)$ to a function on the degenerated Hilbert modular surface. The main result of this section is as follows.

Theorem 5.1. *Let ψ be a non-trivial character of $\mathrm{Cl}(F)$ and $m \geq 1$. Then*

$$\langle j_m^{\mathrm{lift},\mathcal{B}}, g_\psi \rangle_{\mathrm{reg}} = -2 \lim_{\epsilon \to 0} \sum_{\mathcal{A}' \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}') \begin{pmatrix} \log|\Psi_m(\tau_{\mathcal{A}'\mathcal{B}'^{-1}} + \epsilon, \tau_{\mathcal{A}'\mathcal{B}'})| + \\ c(j_m^{\mathrm{lift},\mathcal{B}}, 0) \log|y_{\mathcal{A}'\mathcal{B}'^{-1}}| \end{pmatrix}, \tag{5.3}$$

where $\mathcal{B}' \in \mathrm{Cl}(F)$ is the unique class such that $\mathcal{B}'^2 = \mathcal{B}$ and $c(j_m^{\mathrm{lift},\mathcal{B}}, 0) = r_{\mathcal{B}}(pm)$ is the constant term of $j_m^{\mathrm{lift},\mathcal{B}}(z)$.

Remark. When $\mathcal{B} = \mathcal{A}_0$ is the principal class in $\mathrm{Cl}(F)$, we can write

$$\begin{aligned} 2j_m^{\mathrm{lift},\mathcal{A}_0}(z) &= j_m(pz)U_4(\theta(pz)\theta(z)) \\ &= \left(j_m(4z)\theta(z)\right)^{\mathrm{lift},\theta} \\ &= \sum_{\substack{t \in \mathbb{Z} \\ t^2 \leq 4m}} f_{4m-t^2}^{\mathrm{lift},\theta}(z). \end{aligned}$$

Suppose $r_{\mathcal{A}_0}(m) = 0$, then m is not a perfect square and the right hand side of Eq. (5.3) becomes

$$-2 \sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log|\Psi_m(\tau_{\mathcal{A}}, \tau_{\mathcal{A}})|.$$

By Kronecker's identity, this is the same as

$$-2 \sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \left(\sum_{\substack{t \in \mathbb{Z} \\ t^2 \leq 4m}} \log|\Psi_{f_{4m-t^2}}(\tau_{\mathcal{A}})| \right).$$

Thus, this case of Theorem 5.1 is a consequence of Theorem 4.1.

The procedure of the proof will be the same as that of Theorem 4.1 with all half-integral weight objects replaced by their integral counterparts. The analogue of the counting argument, Proposition 4.4, is also given in [25, §II.3].

Proof of Theorem 5.1. Let $\tilde{g}_\psi(z) \in \mathbb{M}_1^-(p)$ be a mock-modular form as in Proposition 3.2 with Fourier expansion

$$\tilde{g}_\psi(z) = \sum_{n \in \mathbb{Z}} r_\psi^+(n) q^n$$

and $\hat{g}_\psi(z)$ the associated harmonic Maass form. For a class $\mathcal{B} \in \mathrm{Cl}(F)$, define

$$\begin{aligned}\Phi_{\mathcal{B}}(\hat{g}_\psi, z) &:= (p+1)\mathrm{Tr}_1^p(-i(\hat{g}_\psi|W_p)(z)\vartheta_{\mathcal{B}}(z)) \\ &= (\hat{g}_\psi|U_p)(z)\vartheta_{\mathcal{B}}(z) + (\hat{g}_\psi(z)\vartheta_{\mathcal{B}}(z))|U_p.\end{aligned}$$

This should be compared to $\tilde{\Phi}_s(z)$ in Proposition (1.2) of [25, §IV]. In that case, the derivative of $\tilde{\Phi}_s(z)$ at $s = 1$ when $N = 1$ is more or less our $\Phi_{\mathcal{B}}(\hat{g}_\psi, z)$ for ψ trivial.

Here, $\Phi_{\mathcal{B}}(\hat{g}_\psi, z)$ transforms like a level one, weight two modular form and has the following Fourier expansion at infinity

$$\begin{aligned}\Phi_{\mathcal{B}}(\hat{g}_\psi, z) &= \sum_{n \in \mathbb{Z}} (b_{\mathcal{B}}(\hat{g}_\psi, n, y) + a_{\mathcal{B}}(\hat{g}_\psi, n)) q^n, \\ b_{\mathcal{B}}(\hat{g}_\psi, n, y) &= -\sum_{k \in \mathbb{Z}} \delta(k) r_\psi(k) \beta_1\left(k, \tfrac{y}{p}\right) r_{\mathcal{B}}(pn + k), \\ a_{\mathcal{B}}(\hat{g}_\psi, n) &= \sum_{k \in \mathbb{Z}} \delta(k) r_\psi^+(k)\, r_{\mathcal{B}}(pn - k).\end{aligned}$$

By the choice of $\tilde{g}_\psi(z)$, we have $\mathrm{ord}_\infty(\tilde{g}_\psi(z)) \geq -\frac{p+1}{24} > -p$. So the sum $a_{\mathcal{B}}(\hat{g}_\psi, n)$ is zero for all $n < 0$. It is easy to see from the definition above that $j_m^{\mathrm{lift},\mathcal{B}} \in M_1^{!,+}(p)$. So we can apply Proposition 2.5 to obtain

$$\langle j_m^{\mathrm{lift},\mathcal{B}}, g_\psi \rangle_{\mathrm{reg}} = a_{\mathcal{B}}(\hat{g}_\psi, m). \tag{5.4}$$

In particular when $m = 0$, we have $j_0^{\mathrm{lift},\mathcal{B}} = 1^{\mathrm{lift},\mathcal{B}} = \vartheta_{\mathcal{B}}(z)$ and

$$\langle \vartheta_{\mathcal{B}}, g_\psi \rangle = a_{\mathcal{B}}(\hat{g}, 0).$$

Now, let $\hat{E}_2(z)$ be the non-holomorphic Eisenstein series of level one, weight two defined in Eq. (2.18). Then we can use Poincaré series to apply holomorphic projection to the function

$$\Phi_{\mathcal{B}}^*(\hat{g}_\psi, z) := \Phi_{\mathcal{B}}(\hat{g}_\psi, z) - a_{\mathcal{B}}(\hat{g}_\psi, 0)\hat{E}_2(z),$$

since it satisfies the growth condition $\Phi_{\mathcal{B}}^*(\hat{g}_\psi, z) = O(1/y)$ at the cusp infinity. For $m > 0$, let $\mathcal{P}_{m,1}(z, s)$ be the Poincaré series of level one, weight two defined by

$$\mathcal{P}_{m,1}(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{PSL}_2(\mathbb{Z})} (y^s e^{2\pi i m z})|_2 \gamma.$$

As s goes to zero, the quantity $m\langle \Phi_{\mathcal{B}}^*(\hat{g}_\psi, z), \mathcal{P}_{m,1}(z, s)\rangle$ is the m^{th} Fourier coefficient of a cusp form of weight two, level one. Since the space of such forms is trivial, we have

$$\lim_{s \to 0} \langle \Phi_{\mathcal{B}}^*(\hat{g}_\psi, z), \mathcal{P}_{m,1}(z, s)\rangle = 0. \tag{5.5}$$

Using the Fourier expansion of $\Phi_{\mathcal{B}}(\hat{g}_\psi, z)$ and Rankin-Selberg unfolding, we can rewrite Eq. (5.5) as

$$a_{\mathcal{B}}(\hat{g}_\psi, m) + 24\sigma_1(m) a_{\mathcal{B}}(\hat{g}_\psi, 0) = \lim_{s \to 0} \sum_{k \geq 1} \delta(k) r_{\mathcal{B}}(pm + k) r_\psi(k) \varrho_{2s+1}\left(\tfrac{k}{pm}\right), \tag{5.6}$$

where $\varrho_s(\mu)$ was defined by Eq. (4.16). Comparing $Q_{s-1}(t)$ and $\varrho_{2s-1}(\mu)$ near $s=1$, we see

$$\varrho_1(\mu) = 2Q_0(1+2\mu),$$
$$\tfrac{2\Gamma(2s)}{s\Gamma(s)^2}Q_{s-1}(1+2\mu) - \varrho_{2s-1}(\mu) = O(\mu^{-1-s}).$$

That means we can substitute $2Q_{s-1}\left(1+\frac{2k}{pm}\right)$ for $\varrho_{2s-1}\left(\frac{k}{pm}\right)$ in the limit as s approaches 1. Together with (1.8) and (5.4), Eq. (5.6) becomes,

$$\begin{aligned}(5.7)\qquad &\langle j_m^{\mathrm{lift},\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}} + 24\sigma_1(m)\langle\vartheta_{\mathcal{B}}, g_\psi\rangle = \\ &-\lim_{s\to 1}\sum_{\mathcal{A}}\psi(\mathcal{A})\sum_{k\geq 1}\delta(k)r_{\mathcal{B}}(pm+k)r_{\mathcal{A}}(k)\left(-2Q_{s-1}\left(1+\tfrac{2k}{pm}\right)\right),\end{aligned}$$

which is the integral weight analogue of Eq. (4.13).

Now the counting argument in Proposition (3.11) in [24] tells us that

$$\delta(k)r_{\mathcal{B}}(pm+k)r_{\mathcal{A}}(k) = \rho^m(\mathcal{A},\mathcal{B},k),$$

where $\rho^m(\mathcal{A},\mathcal{B},k)$ counts the number of $\gamma \in \mathrm{M}_2(\mathbb{Z})/\{\pm 1\}$ such that $\det(\gamma) = m$ and

$$\cosh d(\tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}, \gamma\tau_{\sqrt{\mathcal{A}\mathcal{B}}}) = 1 + \tfrac{2k}{pm}.$$

In particular when $k = 0$, the number of $\gamma \in \mathrm{M}_2(\mathbb{Z})/\{\pm 1\}$ such that $\det(\gamma) = m$ and $\gamma\tau_{\sqrt{\mathcal{A}\mathcal{B}}} = \tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}$ is exactly $r_{\mathcal{B}}(pm) = r_{\mathcal{B}}(m)$. Since $k \geq 1$ in the summation, Eq. (5.7) becomes

$$\begin{aligned}(5.8)\qquad \langle j_m^{\mathrm{lift},\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}} + 24\sigma_1(m)\langle\vartheta_{\mathcal{B}}, g_\psi\rangle &= -\lim_{s\to 1}\sum_{\mathcal{A}}\psi(\mathcal{A})\sum_{\substack{\gamma\in\mathrm{M}_2(\mathbb{Z})/\{\pm 1\}\\ \det(\gamma)=m,\\ \gamma\tau_{\sqrt{\mathcal{A}\mathcal{B}}}\neq\tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}}} g_s\left(\tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}, \gamma\tau_{\sqrt{\mathcal{A}\mathcal{B}}}\right) \\ &= -\lim_{\epsilon\to 0}\lim_{s\to 1}\sum_{\mathcal{A}}\psi(\mathcal{A})\left(G_s^m(\tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}+\epsilon, \tau_{\sqrt{\mathcal{A}\mathcal{B}}}) - r_{\mathcal{B}}(m)g_s(\tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}+\epsilon, \tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}})\right)\end{aligned}$$

where g_s is defined in (4.8) and

$$G_s^m(z_1,z_2) = \sum_{\substack{\gamma\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{M}_2(\mathbb{Z})\\ \det(\gamma)=m}} g_s(z_1,\gamma z_2).$$

From this expression, it is clear that the choice of these CM points do not matter.

Applying the m^{th} Hecke operator to z_2 on both sides of Eq. (4.9) gives us [24, Eq. (5.2)]

$$(5.9)\qquad \log|\Psi_m(z_1,z_2)|^2 = \lim_{s\to 1}\begin{pmatrix} G_s^m(z_1,z_2) + 4\pi\sigma_1(m)E(z_1,s) + \\ 4\pi m^s\sigma_{1-2s}(m)E(z_2,s) - 4\pi\sigma_1(m)\varphi_1(s)\end{pmatrix} - 24\sigma_1(m),$$

where $\sigma_\nu(m) = \sum_{d|m} d^\nu$. This implies

$$\begin{aligned}\lim_{s\to 1}\sum_{\mathcal{A}}\psi(\mathcal{A})G_s^m(\tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}+\epsilon, \tau_{\sqrt{\mathcal{A}\mathcal{B}}}) &= \sum_{\mathcal{A}}\psi(\mathcal{A})\log|\Psi_m(\tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}+\epsilon, \tau_{\sqrt{\mathcal{A}\mathcal{B}}})|^2 - \\ &4\pi\sigma_1(m)\lim_{s\to 1}\sum_{\mathcal{A}}\psi(\mathcal{A})\left(E(\tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}+\epsilon, s) + E(\tau_{\sqrt{\mathcal{A}\mathcal{B}}}, s)\right).\end{aligned}$$

From Eq. (3.5) and $H(p)\vartheta_{\mathcal{B}}(z) = \sum_{\psi'}\overline{\psi'(\mathcal{B})}g_{\psi'}(z)$, it is easy to see that

$$\langle\vartheta_{\mathcal{B}}, g_\psi\rangle = (\psi(\mathcal{B}) + \overline{\psi(\mathcal{B})})I_{\psi^2}.$$

By Eq. (3.4), we have

$$\lim_{s\to 1} 4\pi \sum_{\mathcal{A}} \psi(\mathcal{A})(E(\tau_{\sqrt{\mathcal{A}\mathcal{B}^{-1}}}, s) + E(\tau_{\sqrt{\mathcal{A}\mathcal{B}}}, s)) = 24(\psi(\mathcal{B}) + \overline{\psi(\mathcal{B})})I_{\psi^2} = 24\langle \vartheta_{\mathcal{B}}, g_\psi\rangle.$$

After substituting these into Eq. (5.8), canceling $24\sigma_1(m)\langle\vartheta_{\mathcal{B}}, g_\psi\rangle$ and changing $\mathcal{A}, \mathcal{B}$ to $(\mathcal{A}')^2, (\mathcal{B}')^2$ respectively, we obtain Eq. (5.3). □

6. Rankin's method and values of Green's Function II

In this section, we will give an analogue of Theorem 5.1 for modular functions of prime level N. This is necessary when we prove Theorem 1.1 in §7. The main result here, Theorem 6.1, is also interesting on its own as the analogue of the Gross-Zagier formula with trivial Heegner divisor on $J_0(N)$, the Jacobian of the modular curve $X_0(N)$.

Let N be an odd prime number such that $\chi_p(N) = 1$. For a modular function $f(z)$ of level N and $\mathcal{B} \in \mathrm{Cl}(F)$, one can define $f^{\mathrm{lift},N,\mathcal{B}}(z) \in M_1^!(p)$ by

$$\begin{aligned} f^{\mathrm{lift},N,\mathcal{B}}(z) &:= [\Gamma_0(p) : \Gamma_0(Np)] \cdot \mathrm{Tr}_p^{Np}((f|W_N)(pz)\vartheta_{\mathcal{B}}(Nz)) \\ &= (f \mid W_N)(pz)\vartheta_{\mathcal{B}}(Nz) + (f(pz)\vartheta_{\mathcal{B}}(z)) \mid U_N, \end{aligned} \tag{6.1}$$

where $W_N = \left(\begin{smallmatrix} & -1 \\ N & \end{smallmatrix}\right)$ is the Fricke involution. Notice that the form $f^{\mathrm{lift},N,\mathcal{B}}(z)$ is the same as $f^{\mathrm{lift},\mathcal{B}}(z)$ from §5 when $N = 1$. If $f = \sum_{m\in\mathbb{Z}} c(f,m)q^m \in M_0^{!,\mathrm{new}}(N)$ as in §2.4, then $f \mid W_N = -Nf \mid U_N$. Define modular functions $f_0, f_N \in M_0^!(N)$ by

$$f_N(z) := \sum_{m'\geq 0} c(f, -Nm')j_{m'}(Nz), \quad f_0(z) := f(z) - f_N(z). \tag{6.2}$$

Then the $(-Nm')^{\mathrm{th}}$ Fourier coefficient of $f_0(z)$ at infinity is zero for all $m' \geq 0$.

Since N satisfies $\chi_p(N) = 1$, one can write $N = \mathfrak{n}\bar{\mathfrak{n}}$ in $\mathcal{O}_F$ and $\mathcal{N} := [\mathfrak{n}] \in \mathrm{Cl}(F)$. Let $\mathcal{B}', \mathcal{N}' \in \mathrm{Cl}(F)$ be the unique square roots of $\mathcal{B}$ and $\mathcal{N}$ respectively. Denote the Heegner points $(\mathcal{O}_F, \mathfrak{n}, \mathcal{A}'\mathcal{B}'\mathcal{N}')$ and $(\mathcal{O}_F, \mathfrak{n}, \mathcal{A}'(\mathcal{B}')^{-1}\mathcal{N}')$ by $\tau_j = \tau_j(\mathcal{A}', \mathcal{B}', \mathcal{N}')$ for $j = 1, 2$ respectively.

The main result of this section is the following generalization of Theorem 5.1.

Theorem 6.1. *Let $f(z)$ be a modular function in the $\mathbb{C}$-span of $\mathcal{S}_{N,1}$ given by Eq.* (2.20) *and $\Psi_{f,\mathcal{N}}(z_1, z_2)$ be the modular function defined in* (6.6)*. Then*

$$\langle f_0^{\mathrm{lift},N,\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}} = -\sum_{\mathcal{A}'\in\mathrm{Cl}(F)} \psi^2(\mathcal{A}')\log|\Psi_{f,\mathcal{N}}(\tau_1, \tau_2)| + C_{f,\mathcal{N},\mathcal{B},\psi}, \tag{6.3}$$

where $C_{f,\mathcal{N},\mathcal{B},\psi}$ is the constant

$$\begin{aligned} C_{f,\mathcal{N},\mathcal{B},\psi} :=& -c(f,0)\left(N\psi(\mathcal{B}\mathcal{N}) + N\psi(\mathcal{B}^{-1}\mathcal{N}^{-1}) - \psi(\mathcal{B}\mathcal{N}^{-1}) - \psi(\mathcal{B}^{-1}\mathcal{N})\right) I_{\psi^2} \\ &+ \left((2N-2)c(f,0) + 4c(f_0^{\mathrm{lift},N,\mathcal{B}}, 0)\right)\psi(\mathcal{B}\mathcal{N}^{-1})I_{\psi^2} \end{aligned} \tag{6.4}$$

and I_{ψ^2} is as in Eq. (3.6)*.*

Remark. If $\mathcal{B}$ and $\mathcal{N}$ are both the identity class in $\mathrm{Cl}(F)$, the constant $C_{f,N,\mathcal{B},\psi}$ becomes $4c(f_0^{\mathrm{lift},N,\mathcal{B}}, 0)I_{\psi^2}$.

The structure of the proof is the same as that of Theorem 5.1. First, we will recall some facts about Heegner points and height pairings on $J_0(N)$. The modular function $\Psi_{f,\mathcal{N}}$ will come from the height pairing between Heegner divisors. Then, we will introduce the automorphic Green's function for level N, which will be slightly different from that for level one in §5. The counting argument again follows from Proposition (3.11) in [25, Chap. II]. After that, we will prove Lemma 6.3, which is the analogue of Lemma 4.3 and Eq. (5.7). The calculations will be more involved here. Finally, we will combine Lemma 6.2 and Lemma 6.3 to deduce Theorem 6.1.

6.1. **Heegner Points and Height Pairings on Jacobian.** Now, we will recall some facts about Heegner points on the modular curve from [23] and their height pairings on the Jacobian. Let $X_0(N)$ be the natural compactification of $Y_0(N)$, the open modular curve over $\mathbb{Q}$ classifying pairs of elliptic curves (E, E') with an order N cyclic isogeny $\phi : E \to E'$. The complex points of $X_0(N)$ have a structure of a compact Riemann surface and can be identified with $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ modulo the action of $\Gamma_0(N)$. Heegner points on $X_0(N)$ correspond to pair of elliptic curves (E, E') having complex multiplication by the same ring $\mathcal{O}$ in some imaginary quadratic field F. This occurs only when there exists an $\mathcal{O}$-ideal $\mathfrak{n}$ dividing N such that $\mathcal{O}/\mathfrak{n}$ is cyclic of order N. In this case, $E(\mathbb{C})$ is isomorphic to $\mathbb{C}/\mathfrak{a}$ with $\mathfrak{a}$ an invertible $\mathcal{O}$-submodule in F. Since this $\mathfrak{a}$ can be chosen independent of homothety by elements in $F^\times$, we only need to consider $[\mathfrak{a}]$, the class of $\mathfrak{a}$ in $\mathrm{Pic}(\mathcal{O})$. So a Heegner point on $X_0(N)$ can be expressed as the triplet $(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])$. The index $c := [\mathcal{O}_F : \mathcal{O}]$ is called the conductor of this Heegner point. To find the image of such a Heegner point in $\mathcal{H}$, choose an oriented basis $\langle \omega_1, \omega_2 \rangle$ of $\mathfrak{a}$ such that $\mathfrak{a}\mathfrak{n}^{-1}$ has basis $\langle \omega_1, \omega_2/N \rangle$. Then $\tau_{[\mathfrak{a}],\mathfrak{n}} := \omega_1/\omega_2 \in \Gamma_0(N)\backslash\mathcal{H}$ is the image of this Heegner point.

Heegner points of conductor c are rational over the field $H_c := F(j(\mathcal{O})) \subset \mathbb{C}$. The theory of complex multiplication spells out the Galois action as follows. Let $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})$ and $\sigma_\mathfrak{b} \in \mathrm{Gal}(H_c/F)$ the corresponding automorphism under the Artin isomorphism. Then

$$(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])^{\sigma_\mathfrak{b}} = (\mathcal{O}, \mathfrak{n}, [\mathfrak{a}\mathfrak{b}^{-1}]).$$

For prime $\ell \nmid N$, the Hecke correspondences on $X_0(N)$ stabilize divisors supported on Heegner points of F with conductors prime to N. The action is given by

$$T_\ell(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}]) = \sum_{\mathfrak{a}/\mathfrak{b} = \mathbb{Z}/\ell} (\mathcal{O}_\mathfrak{b}, \mathfrak{n}_\mathfrak{b}, [\mathfrak{b}]),$$

where the sum is over the $(\ell + 1)$ sublattices $\mathfrak{b} \subset \mathfrak{a}$ of index ℓ, $\mathcal{O}_\mathfrak{b} = \mathrm{End}(\mathfrak{b})$ and $\mathfrak{n}_\mathfrak{b} = (\mathfrak{n}\mathcal{O}_\mathfrak{b}) \cap \mathcal{O}_\mathfrak{b}$. When $d \mid N$, the Atkin-Lehner involution of $X_0(N)$, W_d also acts on Heegner points. In particular, the action of W_N is

$$W_N(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}]) = (\mathcal{O}, \overline{\mathfrak{n}}, [\mathfrak{a}]\mathcal{N}),$$

where $\mathcal{N} \in \mathrm{Pic}(\mathcal{O})$ denotes the class of $\mathfrak{n}$.

For our purpose, $F = \mathbb{Q}(\sqrt{-p})$ and $\mathcal{O} = \mathcal{O}_F$. Then we require $\chi_p(N) = 1$ for Heegner points to exist. In that case, N splits into $\mathfrak{n}\overline{\mathfrak{n}}$ in F with

$$\mathfrak{n} = \mathbb{Z}N + \mathbb{Z}\tfrac{b_\mathfrak{n} + \sqrt{-p}}{2}.$$

Let $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_F) = \mathrm{Cl}(F)$. A point $\tau \in \mathcal{H}$ corresponding to the Heegner point $(\mathcal{O}_F, \mathfrak{n}, \mathcal{A})$ must satisfy an equation

$$A\tau^2 + B\tau + C = 0,$$

with $B^2 - 4AC = -p$, $N|A$, $B \equiv b_\mathfrak{n} \pmod{2N}$ and $\gcd(A/N, B, NC) = 1$. We will denote this image by $\tau(\mathcal{A}, \mathfrak{n})$. Notice $\tau(\mathcal{A}, \mathfrak{n})$ is well-defined up to the action of $\Gamma_0(N)$.

Let $J_0(N)$ be the Jacobian of $X_0(N)$. Its complex points is a compact Riemann surface and can be viewed as the set of divisors of degree zero modulo the set of principal divisors on $X_0(N)$. Let $\langle , \rangle_\mathbb{C}$ be the height symbol on $X_0(N)(\mathbb{C})$. It is the unique bi-additive, symmetric, continuous real-valued function defined on the set of divisors of degree zero and satisfies

$$\left\langle \sum_i n_i(x_i), b \right\rangle_{\mathbb{C}} = \sum_i n_i \log |\Psi(x_i)|^2$$

if $b = (\Psi)$ is a principal divisor.

Let $f = \sum_{n \in \mathbb{Z}} c(f,n) q^n$ be a modular function of prime level N spanned by functions in $\mathcal{S}_{N,1}$ as defined in (2.20). Then $c(f, -N^2 n') = 0$ for all positive integers n'. Define T_f to be

$$T_f := \left(\sum_{\substack{m \geq 1 \\ N \nmid m}} c(f, -m) T_m - (N+1) \sum_{\substack{m' \geq 1 \\ N \nmid m'}} c(f, -Nm') T_{m'} W_N \right). \tag{6.5}$$

Given any newform $h(z) \in S_2(N)$, we know that $(h|W_N)(z) = -(h|U_N)(z)$. It is then easy to check that $T_f(h(z)) = 0$. Since N is prime, $S_2(N)$ is spanned by newforms. So T_f annihilates any $h(z) \in S_2(N)$.

For $j = 1, 2$, let $z_j \in \mathcal{H}$ and $x_j \in X_0(N)(\mathbb{C})$ the corresponding points on the modular curve. Then $T_f((x_2) - (0))$ is a principal divisor on $J_0(N)$, since the actions of the Hecke operators and Fricke involution on the Jacobian are the same as those on $S_2(N)$. Thus, there exists a modular function $\Psi_{f,\mathcal{N}}(z_1, z_2)$ on $(\Gamma_0(N) \backslash \mathcal{H})^2$ such that the divisor of $\Psi_{f,\mathcal{N}}(\cdot, z_2) : \Gamma_0(N) \backslash \mathcal{H} \longrightarrow \mathbb{C}$ is $T_f((x_2) - (0))$ and

$$\log |\Psi_{f,\mathcal{N}}(z_1, z_2)| = \langle (x_1) - (\infty), T_f((x_2) - (0)) \rangle_{\mathbb{C}}. \tag{6.6}$$

When z_1, z_2 are Heegner points of discriminant $-p$ on $X_0(N)(\mathbb{C})$, the divisor $T_f((x_2) - (0))$ and the value $\Psi_{f,\mathcal{N}}(z_1, z_2)$ are both defined over H, the Hilbert class field of F. As in §5, we will relate these special values of $\Psi_{f,\mathcal{N}}$ to an infinite sum via the automorphic Green's function.

6.2. **Level N Green's function and Counting CM Points of Equal Discriminant.** In Theorem 4.1 and 5.1, one needs to take a limit in ϵ when the CM points coincide. Nevertheless, the limit is still the value of a modular function. To obtain this modular function directly without the limit in ϵ, one could deform the definition of the Green's function as done in [25, §II.5]. We will follow this approach here and define the automorphic Green's function of level N to be

$$G_{N,s}(z_1, z_2) := \sum_{\substack{\gamma \in \Gamma_0(N)/\{\pm 1\} \\ z_1 \neq \gamma z_2}} g_s(z_1, \gamma z_2) + \sum_{\substack{\gamma \in \Gamma_0(N)/\{\pm 1\} \\ z_1 = \gamma z_2}} g_s(z_1),$$

where $g_s(z_1, z_2)$ is defined in Eq. (4.8) and

$$\begin{aligned} g_s(z) &:= \lim_{w \to z} (g_s(z, w) - \log |2\pi i \eta(z)^4 (z - w)|^2 \\ &= -\log |2\pi (z - \overline{z}) \eta(z)^4|^2 + 2\tfrac{\Gamma'}{\Gamma}(s) - 2\tfrac{\Gamma'}{\Gamma}(1). \end{aligned}$$

When $\gcd(N, m) = 1$, the m^{th} Hecke operator T_m acts on z_2 in $G_{N,s}(z_1, z_2)$ and defines $G^m_{N,s}(z_1, z_2)$ as follows. Let R_N be the subset of $\mathrm{M}_2(\mathbb{Z})$ containing all matrices whose lower left entry is divisible by N. For $z_1, z_2 \in \mathcal{H}$ and $k \geq 0$, let $\rho^m_N(z_1, z_2, k)$ be the counting function defined as

$$\rho^m_N(z_1, z_2, k) := \#\{\gamma \in R_N/\{\pm 1\} : \det(\gamma) = m, \cosh d(z_1, \gamma z_2) = 1 + \tfrac{2Nk}{pm}\}. \tag{6.7}$$

Then $G^m_{N,s}(z_1,z_2)$ can be written as

$$\begin{aligned} G^m_{N,s}(z_1,z_2) &:= \sum_{\substack{\gamma\in R_N/\{\pm1\}\\ \det\gamma=m\\ z_1\neq\gamma z_2}} g_s(z_1,\gamma z_2) + \sum_{\substack{\gamma\in R_N/\{\pm1\}\\ \det\gamma=m\\ z_1=\gamma z_2}} g_s(z_1) \\ &= -2\sum_{k>0}\rho^m_N(z_1,z_2,k)Q_{s-1}\left(1+\tfrac{2Nk}{pm}\right) + \rho^m_N(z_1,z_2,0)g_s(z_1). \end{aligned}$$

Notice for arbitrary $z_1, z_2 \in \mathcal{H}$, there could be *non-integer* k such that $\rho^m_N(z_1,z_2,k)$ is non-zero. When $z_1, z_2 \in \mathcal{H}$ are Heegner points of discriminant $-p$ though, $\rho^m_N(z_1,z_2,k)$ is necessarily supported on integral k's.

Let $x_j \in X_0(N)(\mathbb{C})$ be the points corresponding to $z_j \in \mathcal{H}$. By Proposition (2.23) in [25, p.242], one can relate the Green's function to the height pairing between $(x_1)-(\infty)$, $(x_2)-(0)$ and $W_N((x_2)-(0))$ on the Jacobian as follows

$$\langle (x_1)-(\infty), T_m((x_2)-(0))\rangle_{\mathbb{C}} = \lim_{s\to1}\begin{pmatrix} G^m_{N,s}(z_1,z_2) + 4\pi\sigma_1(m)E_N(W_Nz_1,s) \\ +4\pi m^s\sigma_{1-2s}(m)E_N(z_2,s) + R(N,m,s)\end{pmatrix}, \tag{6.8}$$

$$\langle (x_1)-(\infty), T_{m'}W_N((x_2)-(0))\rangle_{\mathbb{C}} = \lim_{s\to1}\begin{pmatrix} G^{m'}_{N,s}(z_1,W_Nz_2) + 4\pi\sigma_1(m')E_N(z_1,s) \\ +4\pi (m')^s\sigma_{1-2s}(m')E_N(W_Nz_2,s) \\ +R(N,m',s)\end{pmatrix}. \tag{6.9}$$

Here $E_N(z,s)$ is the Eisenstein series defined in (3.3) and $R(N,m,s)$ is an explicit function depending on N, m and s only.

For $j = 1, 2$, let $\mathcal{A}_j \in \mathrm{Cl}(F)$ and $\tau_j \in \Gamma_0(N)\backslash\mathcal{H}$ be the image of the Heegner points $x_j = (\mathcal{O}_F, \mathfrak{n}, \mathcal{A}_j)$. Let $\mathfrak{a}_j$ be integral ideals in the class $\mathcal{A}_j$ such that $\mathfrak{n}|\mathfrak{a}_j$, $\mathrm{N}(\mathfrak{a}_j) = A_j$. Then by Proposition (3.11) in [25], the counting function $\rho^m_N(\tau_1,\tau_2,k)$ satisfies

$$\begin{aligned} \rho^m_N(\tau_1,\tau_2,k) &= \#\{(\alpha,\beta)\in((\overline{\mathfrak{a}}_1)^{-1}\mathfrak{a}_2^{-1}\times\mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1}\mathfrak{n})/\{\pm1\} \mid \mathrm{N}_{F/\mathbb{Q}}(\alpha) = \tfrac{Nk+pm}{A_1A_2}, \\ &\qquad \mathrm{N}_{F/\mathbb{Q}}(\beta) = \tfrac{Nk}{A_1A_2}, A_1A_2(\alpha-\beta)\equiv 0 \pmod{\mathfrak{d}}\}, \\ &= r_{\mathcal{A}_1\mathcal{A}_2^{-1}}(Nk+pm)r_{\mathcal{A}_1\mathcal{A}_2\mathcal{N}^{-1}}(k)\delta(k) \end{aligned}$$

for $k \ge 0$, where $\mathfrak{d} = \sqrt{-p}\mathcal{O}_F$ is the different of F. The first equality is established by the bijection

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in R_N/\{\pm1\} \mapsto \begin{matrix}\alpha = c\overline{\tau}_1\tau_2 + d\overline{\tau}_1 - a\tau_2 - b, \\ \beta = c\tau_1\tau_2 + d\tau_1 - a\tau_2 - b.\end{matrix}$$

The Fricke involution W_N sends x_j to the Heegner point $x'_j := (\mathcal{O}_F, \overline{\mathfrak{n}}, \mathcal{A}_j\mathcal{N}^{-1})$. Set $\mathcal{A}'_2 = \mathcal{A}_2\mathcal{N}^{-1}$ and $\mathfrak{a}'_2$ an ideal in the class $\mathcal{A}'_2$ such that $\overline{\mathfrak{n}} \mid \mathfrak{a}'_2$. then the same bijection establishes

$$\begin{aligned} \rho^{m'}_N(\tau_1,\tau'_2,k') &= \#\{(\alpha,\beta)\in((\overline{\mathfrak{a}}_1)^{-1}(\mathfrak{a}'_2)^{-1}\overline{\mathfrak{n}}\times\mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1})/\{\pm1\} \mid \mathrm{N}_{F/\mathbb{Q}}(\alpha) = \tfrac{Nk'}{A_1A_2}, \\ &\qquad \mathrm{N}_{F/\mathbb{Q}}(\beta) = \tfrac{Nk'-pm'}{A_1A_2}, A_1A_2(\alpha-\beta)\equiv 0 \pmod{\mathfrak{d}}\}, \\ &= r_{\mathcal{A}_1(\mathcal{A}'_2)^{-1}\mathcal{N}^{-1}}(k')r_{\mathcal{A}_1\mathcal{A}'_2}(Nk'-pm')\delta(k') \\ &= r_{\mathcal{A}_1\mathcal{A}_2^{-1}}(k')r_{\mathcal{A}_1\mathcal{A}_2\mathcal{N}^{-1}}(Nk'-pm')\delta(k') \end{aligned}$$

for $k' \geq pm'/N$. After applying these counting arguments to $G^m_{N,s}(\tau_1, \tau_2)$ and $G^{m'}_{N,s}(\tau_1, W_N\tau_2)$, we obtain

$$\begin{aligned} G^m_{N,s}(\tau_1,\tau_2) =& -2\sum_{k\geq 1} r_{\mathcal{A}_1\mathcal{A}_2^{-1}}(Nk+pm) r_{\mathcal{A}_1\mathcal{A}_2\mathcal{N}^{-1}}(k)\delta(k)Q_{s-1}\left(1+\tfrac{2Nk}{pm}\right) + \\ & r_{\mathcal{A}_1\mathcal{A}_2^{-1}}(pm) g_s(\tau_2), \end{aligned} \tag{6.10}$$

$$\begin{aligned} G^{m'}_{N,s}(\tau_1,W_N\tau_2) =& -2\sum_{k'>pm'/N} r_{\mathcal{A}_1\mathcal{A}_2^{-1}}(k') r_{\mathcal{A}_1\mathcal{A}_2\mathcal{N}^{-1}}(Nk'-pm')\delta(k')Q_{s-1}\left(-1+\tfrac{2Nk'}{pm'}\right) + \\ & r_{\mathcal{A}_1\mathcal{A}_2^{-1}}(\tfrac{pm'}{N}) g_s(\tau_1). \end{aligned} \tag{6.11}$$

Since $N \nmid m'$ in the definition of $G^{m'}_{N,s}(z_1, z_2)$, the term $r_{\mathcal{A}_1\mathcal{A}_2^{-1}}(\frac{pm'}{N})g_s(\tau_1)$ above vanishes. Putting these together gives us the following lemma, which is the level N analogue of Eq. (5.9).

Lemma 6.2. *Let $f = \sum_{m\in\mathbb{Z}} c(f, m)q^m \in M^{!,\mathrm{new}}(N)$ be a modular function in the $\mathbb{C}$ span of $\mathcal{S}_{N,1}$ defined in (2.20) and $\Psi_{f,\mathcal{N}}(z_1, z_2)$ as in Eq. (6.6). Let $\mathcal{B}, \mathcal{N} \in \mathrm{Cl}(F)$ be as above and denote their unique square roots in $\mathrm{Cl}(F)$ by $\mathcal{B}'$ and $\mathcal{N}'$ respectively. Then*

$$\sum_{\mathcal{A}'\in\mathrm{Cl}(F)} \psi^2(\mathcal{A}')\log|\Psi_{f,\mathcal{N}}(\tau_1,\tau_2)| = \Sigma_{f,N,\mathcal{B},\psi} + U_{f,N,\mathcal{B},\psi}, \tag{6.12}$$

where

$$\begin{aligned} \Sigma_{f,N,\mathcal{B},\psi} :=& -\lim_{s\to 1}\sum_{\substack{m\geq 1\\ N\nmid m}} c(f,-m)\sum_{k\geq 1}\delta(k)r_{\mathcal{B}}(pm+Nk)r_\psi(k)2Q_{s-1}\left(1+\tfrac{2kN}{pm}\right) + \\ & (N+1)\lim_{s\to 1}\sum_{m'\geq 1} c(f,-Nm')\sum_{k'\geq 1}\delta(k')r_{\mathcal{B}}(k')r_\psi(Nk'-pm')2Q_{s-1}\left(-1+\tfrac{2k'N}{pm'}\right), \\ U_{f,N,\mathcal{B},\psi} :=& -c(f,0)(N\psi(\mathcal{B}\mathcal{N})+N\psi(\mathcal{B}^{-1}\mathcal{N}^{-1})-\psi(\mathcal{B}^{-1}\mathcal{N})-\psi(\mathcal{B}\mathcal{N}^{-1}))I_{\psi^2} \\ & -24\left(\sum_{\substack{m'\geq 1\\ N\nmid m'}} c(f,-Nm')\sigma_1(m')\right)(\psi(\mathcal{B})+\psi(\mathcal{B}^{-1}))(\psi(\mathcal{N})+\psi(\mathcal{N}^{-1}))I_{\psi^2} \\ & +\left(\sum_{\substack{m\geq 1\\ N\nmid m}} c(f,-m)r_{\mathcal{B}}(m)\right)\psi(\mathcal{B}\mathcal{N}^{-1})4I_{\psi^2}. \end{aligned}$$

Proof. Using the definition of $\mathcal{S}_{N,1}, \Psi_{f,\mathcal{N}}$, Eq. (6.8) and (6.9), one can rewrite the left hand side of Eq. (6.12) as

$$\begin{aligned} \text{LHS of (6.12)} =& \lim_{s\to 1}\sum_{\substack{m\geq 1\\ N\nmid m}} c(f,-m)\sum_{\mathcal{A}'}\psi^2(\mathcal{A}')\begin{pmatrix} G^m_{N,s}(\tau_1,\tau_2)+4\pi\sigma_1(m)E_N(W_N\tau_1,s) \\ +4\pi m^s\sigma_{1-2s}(m)E_N(\tau_2,s)\end{pmatrix} \\ & -(N+1)\lim_{s\to 1}\sum_{\substack{m'\geq 1\\ N\nmid m'}} c(f,-Nm')\sum_{\mathcal{A}'}\psi^2(\mathcal{A}')\begin{pmatrix} G^{m'}_{N,s}(\tau_1,W_N\tau_2)+4\pi\sigma_1(m')E_N(\tau_1,s) \\ +4\pi(m')^s\sigma_{1-2s}(m')E_N(W_N\tau_2,s)\end{pmatrix} \\ =& \Sigma_{f,N,\mathcal{B},\psi} + U'_{f,N,\mathcal{B},\psi} \end{aligned}$$

where the second equality follows from Eq. (6.10), (6.11) and

$$\begin{aligned}U'_{f,N,\mathcal{B},\psi} :=& 4\pi \sum_{\substack{m\geq 1\\ N\nmid m}} c(f,-m)\sigma_1(m) \lim_{s\to 1}\sum_{\mathcal{A}'}\psi^2(\mathcal{A}')(E_N(W_N\tau_1,s)+E_N(\tau_2,s))\\ &- 4\pi(N+1)\sum_{\substack{m'\geq 1\\ N\nmid m'}} c(f,-Nm')\sigma_1(m') \lim_{s\to 1}\sum_{\mathcal{A}'}\psi^2(\mathcal{A}')(E_N(\tau_1,s)+E_N(W_N\tau_2,s))\\ &+ \sum_{\substack{m\geq 1\\ N\nmid m}} c(f,-m) r_{\mathcal{B}}(pm) \lim_{s\to 1}\sum_{\mathcal{A}'}\psi^2(\mathcal{A}') g_s(\tau_2).\end{aligned}$$

Note that $W_N\tau_j$ corresponds to the Heegner point $(\mathcal{O}_F, \overline{\mathfrak{n}}, \mathcal{A}_j\mathcal{N}^{-1})$. Using the elementary identity (see Eq. (2.16) in [25, Chap. II])

$$E_N(z,s) = \frac{1}{N^{2s}-1}(N^s E(W_N z,s) - E(z,s)),$$

and Kronecker's limit formula (Eq. (3.4)), we have

$$\begin{aligned}U'_{f,N,\mathcal{B},\psi} :=& \left(\sum_{\substack{m\geq 1\\ N\nmid m}} c(f,-m)\sigma_1(m)\right)\frac{24}{N^2-1}\begin{pmatrix} N\psi(\mathcal{B}\mathcal{N}) + N\psi(\mathcal{B}^{-1}\mathcal{N}^{-1})\\ -\psi(\mathcal{B}^{-1}\mathcal{N}) - \psi(\mathcal{B}\mathcal{N}^{-1})\end{pmatrix} I_{\psi^2}\\ &- \left(\sum_{\substack{m'\geq 1\\ N\nmid m'}} c(f,-Nm')\sigma_1(m')\right)\frac{24}{N-1}\begin{pmatrix} N\psi(\mathcal{B}^{-1}\mathcal{N}) + N\psi(\mathcal{B}\mathcal{N}^{-1})\\ -\psi(\mathcal{B}^{-1}\mathcal{N}^{-1}) - \psi(\mathcal{B}\mathcal{N})\end{pmatrix} I_{\psi^2}\\ &+ \left(\sum_{\substack{m\geq 1\\ N\nmid m}} c(f,-m) r_{\mathcal{B}}(pm)\right)\psi(\mathcal{B}\mathcal{N}^{-1}) 4 I_{\psi^2},\end{aligned}$$

where I_{ψ^2} is defined by Eq. (3.6). Using the relationship

$$24\left(\sum_{\substack{m\geq 1\\ N\nmid m}} c(f,-m)\sigma_1(m) + \sum_{\substack{m'\geq 1\\ N\nmid m'}} c(f,-Nm')(\sigma_1(Nm') - N\sigma_1(m'))(N+1)\right) = (1-N^2)c(f,0),$$

we can rewrite $U'_{f,N,\mathcal{B},\psi}$ as $U_{f,N,\mathcal{B},\psi}$. □

6.3. **Infinite Sum Expression of Regularized Inner Product II.** For $m \geq 0$, some calculations tell us that

$$\begin{aligned}(j_m(Nz))^{\mathrm{lift},N,\mathcal{B}} &= j_m(pz)T_N(\vartheta_{\mathcal{B}})(z)\\ &= j_m(pz)(\vartheta_{\mathcal{B}\mathcal{N}}(z) + (\vartheta_{\mathcal{B}\mathcal{N}^{-1}})(z))\\ &= j_m^{\mathrm{lift},\mathcal{B}\mathcal{N}}(z) + j_m^{\mathrm{lift},\mathcal{B}\mathcal{N}^{-1}}(z).\end{aligned} \tag{6.13}$$

where T_N is the N^{th} Hecke operator. So denote the inner product

$$\rho_{N,\mathcal{B},\psi}(m) := \langle j_m(pz)T_N(\vartheta_{\mathcal{B}}), g_\psi\rangle_{\mathrm{reg}}. \tag{6.14}$$

In particular for $j_0(z) = 1$, the inner product $\rho_{N,\mathcal{B},\psi}(0)$ becomes

$$\begin{aligned}\rho_{N,\mathcal{B},\psi}(0) &= \langle \vartheta_{\mathcal{B}\mathcal{N}}, g_\psi\rangle + \langle \vartheta_{\mathcal{B}(\mathcal{N})^{-1}}, g_\psi\rangle \\ &= (\psi(\mathcal{B}) + \psi(\mathcal{B}^{-1}))(\psi(\mathcal{N}) + \psi(\mathcal{N}^{-1}))I_{\psi^2}.\end{aligned}$$

Let $f \in M_0^{!,\mathrm{new}}(N)$. The following lemma relates $\langle f^{\mathrm{lift},N,\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}}$ to $\rho_{N,\mathcal{B},\psi}(m)$ and the infinite sum defined in Lemma 6.2. This is the level N version of Eq. (5.7).

Lemma 6.3. *Suppose $f(z) = \sum_{m\in\mathbb{Z}} c(f,m)q^m \in M_0^{!,\mathrm{new}}(N)$. In the notations above, we have*

$$\langle f^{\mathrm{lift},N,\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}} = -\Sigma_{f,N,\mathcal{B},\psi} + \sum_{m'\geq 1} c(f,-Nm')\left(\rho_{N,\mathcal{B},\psi}(m') + 24\sigma_1(m')\rho_{N,\mathcal{B},\psi}(0)\right) \tag{6.15}$$

where $\Sigma_{f,N,\mathcal{B},\psi}$ is defined in Lemma 6.2.

Proof. Let $\tilde{g}_\psi(z) \in \mathbb{M}_1^-(p)$ be a mock-modular form as in Proposition 3.2 with Fourier expansion $\tilde{g}_\psi(z) = \sum_{n\in\mathbb{Z}} r_\psi^+(n)q^n$ and $\hat{g}_\psi(z)$ the associated harmonic Maass form. By Eq. (5.4) and (5.7), we have the following identities

$$\begin{aligned}\rho_{N,\mathcal{B},\psi}(n) =& \sum_{k\in\mathbb{Z}} r_\psi^+(k)\left(r_\mathcal{B}\left(\tfrac{pn-k}{N}\right) + r_\mathcal{B}(pNn - Nk)\right)\delta(k),\\ \rho_{N,\mathcal{B},\psi}(n) =& -24\sigma_1(n)\rho_{N,\mathcal{B},\psi}(0) + \lim_{s\to 1}\sum_{k\geq 1}\delta(k)c(T_N(\vartheta_\mathcal{B}), pn+k)r_\psi(k)2Q_{s-1}\left(1 + \tfrac{2k}{pn}\right).\end{aligned}$$

The first equality holds for $n \geq 0$ while the second one holds for $n \geq 1$. Similar to the proof of Theorem 5.1, define

$$\begin{aligned}\Phi_{N,\mathcal{B}}(\hat{g}, z) :=&(p+1)\mathrm{Tr}_N^{Np}(-i(\hat{g}|W_p)(Nz)\vartheta_\mathcal{B}(z))\\ =&(\hat{g}|U_p)(Nz)\vartheta_\mathcal{B}(z) + (\hat{g}(z)\vartheta_\mathcal{B}(Nz))|U_p.\end{aligned} \tag{6.16}$$

When $N = 1$, this is the same as $\Phi_\mathcal{B}(\hat{g}_\psi, z)$ defined in the proof of Theorem 5.1. It transforms like a modular form of level N, weight two and has the following Fourier expansion at infinity

$$\begin{aligned}\Phi_{N,\mathcal{B}}(\hat{g}_\psi, z) =& \sum_{m\in\mathbb{Z}}(b_{N,\mathcal{B}}(\hat{g}_\psi, m, y) + a_{N,\mathcal{B}}(\hat{g}_\psi, m))q^m,\\ b_{N,\mathcal{B}}(\hat{g}_\psi, m, y) =& -\sum_{k\in\mathbb{Z}}\delta(k)r_\psi(k)\beta_1\left(k, \tfrac{Ny}{p}\right)r_\mathcal{B}(pm + Nk),\\ a_{N,\mathcal{B}}(\hat{g}_\psi, m) =& \sum_{k\in\mathbb{Z}} r_\psi^+(k)\, r_\mathcal{B}(pm - Nk)\delta(k).\end{aligned}$$

Since $f \in M_0^{!,\mathrm{new}}(N)$, one could use Eq. (2.14) and (6.1) to calculate the $-k^{\mathrm{th}}$ Fourier coefficient of $f^{\mathrm{lift},N,\mathcal{B}}$ as

$$c(f^{\mathrm{lift},N,\mathcal{B}}, -k) = \sum_{m\in\mathbb{Z}} c(f,-m)\left(r_\mathcal{B}(pm - Nk) - Nr_\mathcal{B}\left(\tfrac{pm-Nk}{N^2}\right)\right). \tag{6.17}$$

From this and the condition $\chi_p(N) = 1$, it is easy to see that $f^{\mathrm{lift},N,\mathcal{B}}(z) \in M_1^{!,+}(p)$. So we can apply Proposition 2.5 to obtain

$$\begin{aligned}
\langle f^{\mathrm{lift},N,\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}} =& \sum_{m\in\mathbb{Z}} c(f,-m) \left(\sum_{k\in\mathbb{Z}} r_\psi^+(k) r_{\mathcal{B}}(pm - Nk)\delta(k) - N \sum_{k\in\mathbb{Z}} r_\psi^+(k) r_{\mathcal{B}}\left(\tfrac{pm-Nk}{N^2}\right)\delta(k)\right)\\
=& - N \sum_{m'\in\mathbb{Z}} c(f, -Nm') \sum_{k\in\mathbb{Z}} r_\psi^+(k) \left(r_{\mathcal{B}}\left(\tfrac{pm'-k}{N}\right) + r_{\mathcal{B}}(pNm' - Nk)\right)\delta(k)\\
&+ \sum_{m\in\mathbb{Z}} \delta_N(m) c(f,-m) a_{N,\mathcal{B}}(\hat{g}_\psi, m)\\
=& \Sigma'_{f,N,\mathcal{B},\psi} - N \sum_{m'\geq 0} c(f,-Nm')\rho_{N,\mathcal{B},\psi}(m'), \qquad (6.18)
\end{aligned}$$

where $\Sigma'_{f,N,\mathcal{B},\psi}$ is defined by

$$\Sigma'_{f,N,\mathcal{B},\psi} := \sum_{m\in\mathbb{Z}} \delta_N(m) c(f,-m) a_{N,\mathcal{B}}(\hat{g}_\psi, m). \qquad (6.19)$$

The sum over $m' \in \mathbb{Z}$ changes to be over only $m' \geq 0$ in Eq. (6.18) since $r_\psi^+(k) = 0$ for all $k \leq -p$ by the choice of $\tilde{g}_\psi(z)$.

For $n \geq 0$, let $P_{n,N}(z) = q^{-n} + O(q) \in M_2^!(N)$ be the weakly holomorphic modular form as in Lemma 2.10. Consider $\Phi^*_{N,\mathcal{B}}(\hat{g}_\psi, z)$, defined by

(6.20)
$$\Phi^*_{N,\mathcal{B}}(\hat{g}_\psi, z) := \Phi_{N,\mathcal{B}}(\hat{g}_\psi, z) - \sum_{n\geq 0} a_{N,\mathcal{B}}(\hat{g}_\psi, -n) P_{n,N}(z) - \frac{N\rho_{N,\mathcal{B},\psi}(0)}{N^2-1}(\hat{E}_2(z) - \hat{E}_2(Nz)).$$

Now it is easy to check that $\Phi^*_{N,\mathcal{B}}(\hat{g}_\psi, z)$ is $O(1/y)$ at the cusp infinity. At the cusp 0, some calculations show that

$$\begin{aligned}
\Phi_{N,\mathcal{B}}(\hat{g}_\psi, z)|W_N &= (\hat{g}(z)\vartheta_{\mathcal{B}}(Nz)) \mid U_p + (\hat{g} \mid U_p)(z)\vartheta_{\mathcal{B}}(Nz),\\
\Phi_{N,\mathcal{B}}(\hat{g}_\psi, z)|U_N &= (\hat{g}(z)(\vartheta_{\mathcal{B}} \mid U_N)(z)) \mid U_p + (\hat{g} \mid U_p)(z)(\vartheta_{\mathcal{B}} \mid U_N)(z).
\end{aligned}$$

Thus using the relationship $\vartheta_{\mathcal{B}}(Nz) + (\vartheta_{\mathcal{B}} \mid U_N)(z) = \vartheta_{\mathcal{BN}}(z) + \vartheta_{\mathcal{BN}^{-1}}(z)$, we have

$$\Phi_{N,\mathcal{B}}(\hat{g}_\psi, z)|W_N + \Phi_{N,\mathcal{B}}(\hat{g}_\psi, z)|U_N = \Phi_{1,\mathcal{BN}}(\hat{g}_\psi, z) + \Phi_{1,\mathcal{BN}^{-1}}(\hat{g}_\psi, z).$$

From the proof of Theorem 5.1, we know that $\Phi_{1,\mathcal{BN}}(\hat{g}_\psi, z) + \Phi_{1,\mathcal{BN}^{-1}}(\hat{g}_\psi, z)$ has no pole and a constant term $\rho_{N,\mathcal{B},\psi}(0)$. So $\Phi^*_{N,\mathcal{B}}(\hat{g}_\psi, z)$ is also $O(1/y)$ at the cusp 0.

For $m \geq 1$, let $\mathcal{P}_{m,N}(z,s)$ be the Poincaré series of level N, weight two defined by

$$\mathcal{P}_{m,N}(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} (y^s e^{2\pi imz})|_2\gamma.$$

It is characterized by the property that

$$\lim_{s\to 0}\langle h(z), \mathcal{P}_{m,N}(z,s)\rangle = \frac{c(h,m)}{4\pi m}$$

for any $h(z) = \sum_{m\geq 1} c(h,m)q^m \in S_2(N)$. Since $\{c(f,-m) : m \geq 1\} \in \Lambda_N$ is a relation for $S_2(N)$ (see Def. 2.8), we know that

$$\lim_{s\to 0}\left\langle h(z), \sum_{m\geq 1} m\delta_N(m) c(f,-m)\mathcal{P}_{m,N}(z,s)\right\rangle = 0, \qquad (6.21)$$

for any $h(z) \in S_2(N)$. So $\lim_{s\to 0}\sum_{m\geq 1} m\delta_N(m)c(f,-m)\mathcal{P}_{m,N}(z,s) \in S_2(N)$ is 0.

Now we can consider the inner product between $\Phi^*_{N,\mathcal{B}}(\hat{g}_\psi, z)$ and this linear combination of Poincaré series and obtain

$$4\pi \lim_{s\to 0}\sum_{m\geq 1} m\delta_N(m)c(f,-m)\langle \Phi^*_{N,\mathcal{B}}(\hat{g}_\psi, z), \mathcal{P}_{m,N}(z,s)\rangle = 0. \tag{6.22}$$

Since $\Phi^*_{N,\mathcal{B}}(\hat{g}_\psi, z)$ is $O(1/y)$ at both cusps, we can apply Rankin-Selberg unfolding as in the proof of Theorem 5.1. The limit on the left hand side of Eq. (6.22) then breaks up into two parts. The first part is

$$\sum_{m\geq 1}\delta_N(m)c(f,-m)\begin{pmatrix} a_{N,\mathcal{B}}(\hat{g}_\psi, m) - \sum_{n\geq 0} a_{N,\mathcal{B}}(\hat{g}_\psi, -n)c(P_{n,N}, m) \\ -\dfrac{24N\rho_{N,\mathcal{B},\psi}(0)}{N^2-1}\left(\sigma_1\left(\frac{m}{N}\right) - \sigma_1(m)\right)\end{pmatrix} \tag{6.23}$$

Since $(f|W_N)(z) = -N(f|U_N)(z)$ and $P_{n,N}(z) = q^{-n} + O(q)$, Lemma 2.10 tells us that for $n \geq 0$

$$\sum_{m\geq 1}\delta_N(m)c(f,-m)c(P_{n,N}, m) = -\delta_N(n)c(f,n).$$

So expression (6.23) becomes

$$\Sigma'_{f,N,\mathcal{B},\psi} - \frac{24N\rho_{N,\mathcal{B},\psi}(0)}{N^2-1}\sum_{m\geq 1}\delta_N(m)c(f,-m)\left(\sigma_1\left(\tfrac{m}{N}\right) - \sigma_1(m)\right). \tag{6.24}$$

The second part of the left hand side of Eq. (6.22) involves the limit of an infinite sum, which can be evaluated as

$$\begin{aligned}
&-\lim_{s\to 0}\sum_{m\geq 1}\delta_N(m)c(f,-m)\sum_{k\geq 1}\delta(k)r_{\mathcal{B}}(pm+Nk)r_\psi(k)\varrho_{2s+1}\left(\frac{kN}{pm}\right)\\
&\quad= -\lim_{s\to 1}\sum_{m\geq 1}\delta_N(m)c(f,-m)\sum_{k\geq 1}\delta(k)r_{\mathcal{B}}(pm+Nk)r_\psi(k)2Q_{s-1}\left(1+\frac{2kN}{pm}\right)\\
&\quad= -(N+1)\lim_{s\to 1}\sum_{m'\geq 1}c(f,-Nm')\sum_{k'\geq 1}\delta(k)\begin{pmatrix} r_{\mathcal{B}}(N(pm'+k)) \\ +r_{\mathcal{B}}\left(\dfrac{pm'+k}{N}\right)\end{pmatrix} r_\psi(k)2Q_{s-1}\left(1+\tfrac{2kN}{pm'}\right)\\
&\qquad+\Sigma_{f,\mathcal{N},\mathcal{B},\psi}\\
&\quad=\Sigma_{f,N,\mathcal{B},\psi} - (N+1)\sum_{m'\geq 1}c(f,-Nm')(\rho_{N,\mathcal{B},\psi}(m') + 24\sigma_1(m')\rho_{N,\mathcal{B},\psi}(0)).
\end{aligned} \tag{6.25}$$

Adding expressions (6.24) and (6.25) together yields 0 and can be used to solve for $\Sigma'_{f,N,\mathcal{B},\psi}$

$$\begin{aligned}\Sigma'_{f,N,\mathcal{B},\psi} =&\frac{24N\rho_{N,\mathcal{B},\psi}(0)}{N^2-1}\sum_{m\geq 1}\delta_N(m)c(f,-m)\left(\sigma_1\left(\tfrac{m}{N}\right)-\sigma_1(m)\right)\\ &-\Sigma_{f,N,\mathcal{B},\psi}+(N+1)\sum_{m'\geq 1}c(f,-Nm')(\rho_{N,\mathcal{B},\psi}(m')+24\sigma_1(m')\rho_{N,\mathcal{B},\psi}(0))\\ =&-\Sigma_{f,N,\mathcal{B},\psi}+\sum_{m'\geq 1}c(f,-Nm')\left(\rho_{N,\mathcal{B},\psi}(m')+24\sigma_1(m')\rho_{N,\mathcal{B},\psi}(0)\right)\\ &+N\sum_{m'\geq 0}c(f,-Nm')\rho_{N,\mathcal{B},\psi}(m'),\end{aligned}$$

where we have used the following equation obtained from applying Lemma 2.10 to $P_{0,N}(z) \in M_2^!(N)$ and $f(z) \in M_0^{!,\mathrm{new}}(N)$

$$c(f,0) = \tfrac{24}{N^2-1}\sum_{m\geq 1}\left(N\sigma_1\left(\tfrac{m}{N}\right)-\sigma_1(m)\right)\delta_N(m)c(f,-m). \tag{6.26}$$

Substituting this expression of $\Sigma'_{f,N,\mathcal{B},\psi}$ into (6.18) gives us Eq. (6.15). □

6.4. **Proof of Theorem 6.1.** In terms of $\rho_{N,\mathcal{B},\psi}(0)$, the quantity $U_{f,N,\mathcal{B},\psi}$ in Eq. (6.12) can be written as

$$\begin{aligned}U_{f,N,\mathcal{B},\psi} =&-c(f,0)(N\psi(\mathcal{BN})+N\psi(\mathcal{B}^{-1}\mathcal{N}^{-1})-\psi(\mathcal{B}^{-1}\mathcal{N})-\psi(\mathcal{BN}^{-1}))I_{\psi^2}\\ &-24\left(\sum_{\substack{m'\geq 1\\ N\nmid m'}}c(f,-Nm')\sigma_1(m')\right)\rho_{N,\mathcal{B},\psi}(0)+\left(\sum_{\substack{m\geq 1\\ N\nmid m}}c(f,-m)r_{\mathcal{B}}(m)\right)\psi(\mathcal{BN}^{-1})4I_{\psi^2}.\end{aligned}$$

Since f is in the span of $\mathcal{S}_{N,1}$, $c(f,-Nm')=0$ for all $m'\geq 1$ divisible by N. Adding Eq. (6.12) and (6.15) will cancel the term $24\left(\sum_{m'\geq 1}c(f,-Nm')\sigma_1(m')\right)\rho_{N,\mathcal{B},\psi}(0)$ and give us

$$\begin{aligned}\langle f^{\mathrm{lift},N,\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}} =&-\sum_{\mathcal{A}'\in\mathrm{Cl}(F)}\psi^2(\mathcal{A}')\log|\Psi_{f,\mathcal{N}}(\tau_1,\tau_2)|+\sum_{m'\geq 1}c(f,-Nm')\rho_{N,\mathcal{B},\psi}(m')\\ &+C'_{f,\mathcal{N},\mathcal{B},\psi},\end{aligned} \tag{6.27}$$

where

$$\begin{aligned}C'_{f,\mathcal{N},\mathcal{B},\psi} =&\ -c(f,0)(N\psi(\mathcal{BN})+N\psi(\mathcal{B}^{-1}\mathcal{N}^{-1})-\psi(\mathcal{B}^{-1}\mathcal{N})-\psi(\mathcal{BN}^{-1}))I_{\psi^2}\\ &+\left(\sum_{\substack{m\geq 1\\ N\nmid m}}c(f,-m)r_{\mathcal{B}}(m)\right)\psi(\mathcal{BN}^{-1})4I_{\psi^2}.\end{aligned}$$

From the definition of f_0 in Eq. (6.2) and the fact that $c(f,-N^2m'')=0$ for all $m''>0$, it is easy to see that

$$\begin{aligned}\langle f_0^{\mathrm{lift},N,\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}} &= \langle f^{\mathrm{lift},N,\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}}-\sum_{m'\geq 1}c(f,-Nm')\rho_{N,\mathcal{B},\psi}(m'),\\ c(f_0^{\mathrm{lift},N,\mathcal{B}},0) &= \frac{1-N}{2}c(f,0)+\sum_{m\geq 1, N\nmid m}c(f,-m)r_{\mathcal{B}}(m).\end{aligned}$$

Substituting these into Eq. (6.27) gives us $C'_{f,\mathcal{N},\mathcal{B},\psi}=C_{f,\mathcal{N},\mathcal{B},\psi}$ and Eq. (6.3).

7. Algebraicity, Modularity, and Proof of Theorem 1.1

We are now in position to prove Theorem 1.1. First, we will use the theory of complex multiplication and Theorems 5.1 and 6.1 to show that regularized inner products, such as the ones on the left hand sides of Eq. (5.3) and (6.3), can be put into the form of the right hand of Eq. (1.10) when the modular function has rational Fourier coefficients. Then, we will prove Lemma 7.2, which tells us when a generating series is a modular form of weight one. Finally, we will combine these two results to deduce Theorem 1.1.

7.1. Algebraicity of Regularized Inner Product.

The main result of this section is as follows.

Proposition 7.1. *Let $\mathcal{B} \in \mathrm{Cl}(F)$, N be 1 or an odd prime satisfying $\left(\frac{N}{p}\right) = 1$ and $f \in M_0^!(N)$. Write $f(z) = f_1(Nz) + f_2(z)$ as in Lemma 2.7. If f_j has integral Fourier coefficients in the principal part for $j = 1, 2$, then for each $\mathcal{A}' \in \mathrm{Cl}(F)$, there exists $u_{f,\mathcal{B}}(\mathcal{A}') \in H$ independent of the character ψ such that*

$$\langle f^{\mathrm{lift},N,\mathcal{B}}, g_\psi \rangle_{\mathrm{reg}} = \frac{1}{12H(p)\kappa_p^- \varepsilon_N} \sum_{\mathcal{A}' \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}') \log |u_{f,\mathcal{B}}(\mathcal{A}')|, \tag{7.1}$$

where $\kappa_p^- \in \mathbb{Z}$ is defined in Eq. (3.9) and

$$\varepsilon_N = \begin{cases} N^2 - 1, & \text{when} \quad N \neq 1, \\ 1, & \text{when} \quad N = 1. \end{cases}$$

Furthermore for any $\mathcal{C} \in \mathrm{Cl}(F)$, the algebraic integer $u_{f,\mathcal{B}}(\mathcal{A}')$ satisfies

$$\sigma_{\mathcal{C}}(u_{f,\mathcal{B}}(\mathcal{A}')) = u_{f,\mathcal{B}}(\mathcal{A}'\mathcal{C}^{-1}) \tag{7.2}$$

where $\sigma_{\mathcal{C}} \in \mathrm{Gal}(H/F)$ is associated to $\mathcal{C}$ via Artin's isomorphism.

Proof. Since we could write $f(z)$ as the integral linear combination of $j_m(Nz)$ and some functions in $\mathcal{S}_{N,1}, \mathcal{S}_{N,2}$ with integral principal part Fourier coefficients, there are really three cases to consider.

Case 1: $f(z) = j_m(Nz)$.
When $N = 1$, we have $f^{\mathrm{lift},N,\mathcal{B}}(z) = j_m^{\mathrm{lift},\mathcal{B}}(z)$ as defined in Eq. (5.1). If $m = 0$, then $f^{\mathrm{lift},N,\mathcal{B}}(z) = \vartheta_{\mathcal{B}}(z)$ and

$$\begin{aligned} \langle f^{\mathrm{lift},N,\mathcal{B}}, g_\psi \rangle =& (\psi(\mathcal{B}) + \psi(\mathcal{B}^{-1})) I_{\psi^2} \\ =& \frac{1}{12H(p)} \sum_{\mathcal{A}' \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}') \log |u_{0,\mathcal{B}}(\mathcal{A}')|, \end{aligned}$$

where $u_{0,\mathcal{B}}(\mathcal{A}) := (u_{\mathcal{A}'\mathcal{B}'} u_{\mathcal{A}'(\mathcal{B}')^{-1}})^{-1}$ with $u_{\mathcal{A}_1} \in H^\times$ defined by Eq. (3.1) and $\mathcal{B}' \in \mathrm{Cl}(F)$ the unique square root of $\mathcal{B}$. By the theory of complex multiplication, $u_{f,\mathcal{B}}(\mathcal{A}')$ satisfies Eq. (7.2).

If $m \geq 1$, then for any $r \geq 0$ and $\tau, \tau' \in \mathcal{H}$, define the level one modular function $\Psi_m^*(z; \tau, \tau', r)$ by

$$\Psi_m^*(z; \tau, \tau', r) := \frac{\Psi_m(\tau, z)}{(j(\tau') - j(z))^r},$$

where Ψ_m is defined in Eq. (5.2). In terms of $\Psi_m^*(z;\tau,\tau',r)$, we can use Eq. (5.3) to re-write $\langle j_m^{\mathrm{lift},\mathcal{B}}, g_\psi\rangle$ as

$$-2\sum_{\mathcal{A}'\in\mathrm{Cl}(F)}\psi^2(\mathcal{A}')\left(\log|\Psi_m^*(\tau_{\mathcal{A}'\mathcal{B}'};\tau_{\mathcal{A}'\mathcal{B}'^{-1}},\tau_{\mathcal{A}'\mathcal{B}'},r_{\mathcal{B}}(m))| + r_{\mathcal{B}}(m)\log|y_{\mathcal{A}'\mathcal{B}'}j'(\tau_{\mathcal{A}'\mathcal{B}'})|\right)$$

The quantity $\Psi_m^*(\tau_{\mathcal{A}'\mathcal{B}'};\tau_{\mathcal{A}'\mathcal{B}'}^{-1},\tau_{\mathcal{A}'\mathcal{B}'},r_{\mathcal{B}}(m))$ is well-defined and non-zero, since the order of $\Psi_m(\tau_{\mathcal{A}'\mathcal{B}'^{-1}},z)$ at $z=\tau_{\mathcal{A}'\mathcal{B}'}$ is exactly $r_{\mathcal{B}}(m)$ from the proof of Theorem 5.1. By Eq. (3.2) and the fact that

$$(j'(z))^6 = j(z)^4(j(z)-1728)^3\Delta(z),$$

we have

$$\sigma_{\mathcal{C}}\left(\prod_{\mathcal{C}'\in\mathrm{Cl}(F)}\frac{y_{\mathcal{A}'\mathcal{B}'}^6 j'(\tau_{\mathcal{A}'\mathcal{B}'})^6}{y_{\mathcal{C}'}^6\Delta(\tau_{\mathcal{C}'})}\right) = \left(\prod_{\mathcal{C}'\in\mathrm{Cl}(F)}\frac{y_{\mathcal{C}^{-1}\mathcal{A}'\mathcal{B}'}^6 j'(\tau_{\mathcal{C}^{-1}\mathcal{A}'\mathcal{B}'})^6}{y_{\mathcal{C}'}^6\Delta(\tau_{\mathcal{C}'})}\right)\in H.$$

Similarly, the modular function $\Psi_m^*(z;\tau_{\mathcal{A}'\mathcal{B}'^{-1}},\tau_{\mathcal{A}'\mathcal{B}'},r_{\mathcal{B}}(m))$, which is defined over H, is sent to $\Psi_m^*(z;\tau_{\mathcal{C}^{-1}\mathcal{A}'\mathcal{B}'^{-1}},\tau_{\mathcal{C}^{-1}\mathcal{A}'\mathcal{B}'},r_{\mathcal{B}}(m))$ under $\sigma_{\mathcal{C}}$. So if we let

$$u_{m,\mathcal{B}}(\mathcal{A}') := \Psi_m^*(\tau_{\mathcal{A}'\mathcal{B}'};\tau_{\mathcal{A}'\mathcal{B}'^{-1}},\tau_{\mathcal{A}'\mathcal{B}'},r_{\mathcal{B}}(m))^{-24H(p)}\cdot\left(\prod_{\mathcal{C}'\in\mathrm{Cl}(F)}\frac{y_{\mathcal{A}'\mathcal{B}'}^6 j'(\tau_{\mathcal{A}'\mathcal{B}'})^6}{y_{\mathcal{C}'}^6\Delta(\tau_{\mathcal{C}'})}\right)^{-4r_{\mathcal{B}}(m)},$$

then we can write

$$\langle j_m^{\mathrm{lift},\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}} = \frac{1}{12H(p)}\sum_{\mathcal{A}'\in\mathrm{Cl}(F)}\psi^2(\mathcal{A}')\log|u_{m,\mathcal{B}}(\mathcal{A}')|$$

with $u_{m,\mathcal{B}}(\mathcal{A}')\in H$ satisfying Eq. (7.2).

When $N>1$, Eq. (6.13) tells us that

$$f^{\mathrm{lift},N,\mathcal{B}}(z) = (j_m(Nz))^{\mathrm{lift},N,\mathcal{B}} = j_m^{\mathrm{lift},\mathcal{B}\mathcal{N}}(z) + j_m^{\mathrm{lift},\mathcal{B}\mathcal{N}^{-1}}(z).$$

In terms of the results for $N=1$, we could write

$$\langle f^{\mathrm{lift},N,\mathcal{B}}, g_\psi\rangle_{\mathrm{reg}} = \frac{1}{12H(p)}\sum_{\mathcal{A}'\in\mathrm{Cl}(F)}\psi^2(\mathcal{A}')\log|u_{f,\mathcal{B}}(\mathcal{A}')|,$$

where $u_{f,\mathcal{B}}(\mathcal{A}') = u_{m,\mathcal{B}\mathcal{N}}(\mathcal{A}')u_{m,\mathcal{B}\mathcal{N}^{-1}}(\mathcal{A}')$ also satisfies Eq. (7.2).

Case 2: $f(z) = f_2(z)\in\mathcal{S}_{N,2}$.

By Lemma 2.9, there exist integers $r\ge 0, m'\ge 1$ such that $\gcd(N,m')=1$ and

$$f(z) = q^{-N^{r+2}m'} - \tfrac{N+1}{\delta_N(N^r m')}q^{-N^r m'} + O(q)$$

at the cusp infinity. From Eq. (6.17), it follows that the $-n^{\mathrm{th}}$ Fourier coefficient of $f^{\mathrm{lift},N,\mathcal{B}}(z)$ can be written as

$$\begin{aligned} c(f^{\mathrm{lift},N,\mathcal{B}},-n) = &- N\left(r_{\mathcal{B}}\left(\tfrac{pN^{r+1}m'-n}{N}\right) - \tfrac{N+1}{\delta_N(N^r m')}r_{\mathcal{B}}\left(\frac{\frac{pN^r m'}{N}-n}{N}\right)\right)\\ &+\left(r_{\mathcal{B}}(-Nn+pN^{r+2}m') - \tfrac{N+1}{\delta_N(N^r m')}r_{\mathcal{B}}(-Nn+pN^r m')\right).\end{aligned}$$

When $r = 0$ and $n \geq 0$, we can rewrite the equation above as

$$\begin{aligned} c(f^{\mathrm{lift},N,\mathcal{B}}, -n) = & -(N+1)\left(r_{\mathcal{B}}\left(pm' - \tfrac{n}{N}\right) + r_{\mathcal{B}}(pm' - Nn)\right) \\ & + \left(r_{\mathcal{B}}\left(\tfrac{pNm'-n}{N}\right) + r_{\mathcal{B}}(N(pNm'-n))\right) \\ = & -(N+1)c\left(T_N(j_{m'}^{\mathrm{lift},\mathcal{B}}), -n\right) + c\left(j_{Nm'}(pz)T_N(\vartheta_{\mathcal{B}})(z), -n\right). \end{aligned}$$

Since the principal part and the constant term uniquely determine a form in $M_1^{!,+}(p)$ up to a form in $S_1^+(p)$, we have

$$f^{\mathrm{lift},N,\mathcal{B}}(z) = -(N+1)T_N\left(j_{m'}^{\mathrm{lift},\mathcal{B}}(z)\right) + j_{Nm'}^{\mathrm{lift},\mathcal{B}\mathcal{N}}(z) + j_{Nm'}^{\mathrm{lift},\mathcal{B}\mathcal{N}^{-1}}(z) + g_f(z)$$

for some $g_f(z) \in S_1^+(p)$ with integral linear combinations of $c(f, n)$'s as Fourier coefficients. When $r \geq 1$, the situation is similar. In this case, we have

$$f^{\mathrm{lift},N,\mathcal{B}}(z) = -(N+1)T_N\left(j_{N^r m'}^{\mathrm{lift},\mathcal{B}}(z)\right) + (Nj_{N^{r-1}m'}(pz) + j_{N^{r+1}m'}(pz))T_N(\vartheta_{\mathcal{B}}) + g_f(z)$$

for some $g_f(z) \in S_1^+(p)$ with integral linear combinations of $c(f, n)$'s as Fourier coefficients.

The modular function $f(z)$ is a rational function in $j(z)$ and $j(Nz)$. Since $c(f, n) = c(f_2, n) \in \mathbb{Z}$ for all $n \leq 0$, we know that $c(f, n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. Applying Corollary 3.3 to g_f, we see that the Proposition holds in this case. The presence of g_f is responsible for the κ_p^- in the denominator on the right hand side of Eq. (7.1).

Case 3: $f(z) = f_2(z) \in \mathcal{S}_{N,1}$.

This case follows from Theorem 6.1 and the same analysis in case 1. In the first summand of the right hand side of Eq. (6.27), the value $\Psi_{f,\mathcal{N}}(\tau_1, \tau_2)$ is defined over H. So it can be written in the same form as the right hand side of Eq. (7.1). The Galois action is also satisfied as τ_j corresponds to $(\mathcal{O}_F, \mathfrak{n}, \mathcal{A}'\mathcal{B}'\mathcal{N}')$ and $(\mathcal{O}_F, \mathfrak{n}, \mathcal{A}'(\mathcal{B}')^{-1}\mathcal{N}')$ for $j = 1, 2$ respectively. Applying the analysis in case 1, we see that the second summand $\sum_{m' \geq 1} c(f, -Nm')\rho_{N,\mathcal{B},\psi}(m')$ and the third summand $C_{f,N,\mathcal{B},\psi}$ can also be written in the same form as the right hand side of Eq. (7.1) with appropriate Galois action. Thus the proposition holds in this case. The presence of $c(f, 0)$ is responsible for the factor $N^2 - 1$ in the denominator of the right hand side of Eq. (7.1), since $(N^2 - 1)c(f, 0) \in \mathbb{Z}$ by Lemma 2.9. $\square$

7.2. **Modularity Lemma.** Let $\tilde{g}_\psi(z) = \sum_{n \in \mathbb{Z}} r_\psi^+(n)q^n$ be a mock-modular form with fixed principal part as in Proposition 3.2. Expression (6.18) gives rise to the following equation involving a linear combination of $r_\psi^+(n)$'s and regularized inner products

$$(7.3) \quad \begin{aligned} \sum_{m \in \mathbb{Z}} \delta_N(m)c(f, -m) \sum_{n \in \mathbb{Z}} \delta(n) r_{\mathcal{B}}(pm - Nn) r_\psi^+(n) = & \langle f^{\mathrm{lift},N,\mathcal{B}}, g_\psi \rangle_{\mathrm{reg}} \\ & + N \sum_{m' \geq 0} c(f, -Nm')\rho_{N,\mathcal{B},\psi}(m'). \end{aligned}$$

In this section, we will show that as solutions to the above equations for various $\mathcal{B}, N$ and f, the set $\{r_\psi^+(n) : n \geq 1\}$ is unique up to adjustments by Fourier coefficients of elements in $S_1^-(p)$. Then one can choose an appropriate $\tilde{g}_\psi(z)$ satisfying Theorem 1.1.

For a class $\mathcal{B} \in \mathrm{Cl}(F)$, an odd prime N satisfying $\chi_p(N) = 1$ and a formal power series

$$\mathcal{D}(z) = \sum_{n \geq 1} d(n)q^n \in \mathbb{C}[\![q]\!],$$

define the formal power series $\Phi_{N,\mathcal{B}}(\mathcal{D}, z) \in \mathbb{C}[\![q]\!]$ by

$$\Phi_{N,\mathcal{B}}(\mathcal{D}, z) := (\mathcal{D}|U_p)(Nz)\vartheta_{\mathcal{B}}(z) + (\mathcal{D}(z)\vartheta_{\mathcal{B}}(Nz))|U_p,$$

where U_p acts formally on q-expansions by

$$(\mathcal{D} \mid U_p)(z) = \sum_{n\geq 1} d(pn)q^n.$$

If $\mathcal{D}(z) \in S_1^-(p)$, then $\Phi_{N,\mathcal{B}}(\mathcal{D}, z)$ agrees with the definition in Eq. (6.16), hence is in $S_2(N)$. In this case, the set $\{d(n) : n \geq 1\}$ is a solution to Eq. (7.3) with the right hand side replaced by 0.

Conversely, if $d(n) = 0$ whenever $\chi_p(n) = 1$ and $\Phi_{N,\mathcal{B}}(\mathcal{D}, z)$ is in $S_2(N)$ for sufficiently many N, then it turns out $\mathcal{D}(z) \in S_1^-(p)$. This will be proved in the following modularity lemma, which is the key to the proof of Theorem 1.1.

Lemma 7.2. *For a set of complex numbers*

$$\{d(n) : n \geq 1, d(n) = 0 \text{ whenever } \chi_p(n) = 1\},$$

let $\mathcal{D}(z) = \sum_{n\geq 1} d(n)q^n$ *be the associated formal power series. Suppose that for some class* $\mathcal{B} \in \mathrm{Cl}(F)$,

$$\Phi_{N,\mathcal{B}}(\mathcal{D}, z) \in S_2(N)$$

for all primes $N \equiv 1 \pmod{p}$ *and* $N = 1$. *Then* $\mathcal{D}(z)$ *is a weight one cusp form in* $S_1^-(p)$.

Proof. Let $N_0 := 1$ and pick an arbitrary set of odd primes $\{N_j : 1 \leq j \leq (p-1)/2, N_j \equiv 1 \pmod{p}\}$. For $0 \leq j \leq (p-1)/2$, denote $\Phi_{N_j,\mathcal{B}}(\mathcal{D}, z)$ by $\Phi_{j,\mathcal{B}}(z)$. From its definition and the fact that $N_j \equiv 1 \pmod{p}$, we have

$$\Phi_{j,\mathcal{B}}(z) = \tfrac{1}{p}\sum_{k=0}^{p-1} \left(\vartheta_{\mathcal{B}}\left(\tfrac{z+k}{p}\right) + \vartheta_{\mathcal{B}}(z)\right) \mathcal{D}\left(\tfrac{N_j z+k}{p}\right). \tag{7.4}$$

Set

$$M := \prod_{j'=1}^{(p-1)/2} N_{j'},\ M_j := \tfrac{M}{N_j}.$$

Then $\Phi_{j,\mathcal{B}}(pM_j z)$ is a modular form of weight two, level pM for all $0 \leq j \leq (p-1)/2$. Define the following matrices with entries in $\mathbb{C}[\![q]\!]$ by

$$L := \left(\vartheta_{\mathcal{B}}\left(M_j z + \tfrac{k}{p}\right) + \vartheta_{\mathcal{B}}(pM_j z)\right)_{\substack{0\leq j\leq (p-1)/2 \\ 0\leq k\leq p-1.}}$$

$$X := \left(\mathcal{D}\left(Mz + \tfrac{k}{p}\right)\right)_{0\leq k\leq p-1}$$

$$R := (p\Phi_{j,\mathcal{B}}(pM_j z))_{0\leq j\leq (p-1)/2}\,.$$

The dimensions of L, X and R are $(p+1)/2 \times p, p \times 1$ and $(p+1)/2 \times 1$ respectively. Then we can rewrite Eq. (7.4) in the following matrix expression

$$L \cdot X = R. \tag{7.5}$$

Let $S = \left(\frac{1}{p}e^{-2\pi i kk'/p}\right)_{0\leq k,k'\leq p-1}$ be a $p \times p$ matrix and $\mathcal{D}_k(z)$ the formal power series

$$\mathcal{D}_k(z) := \sum_{\substack{n\geq 1, \\ n\equiv k \pmod{p}}} d(n)q^n$$

for $0 \le k \le p-1$. Then $S^{-1} = \left(e^{2\pi i k' k/p}\right)_{0\le k',k\le p-1}$ and

$$L \cdot S = \left(\delta(k') \sum_{n \equiv k' \pmod p} r_{\mathcal{B}}(n) q^{M_j n}\right)_{\substack{0\le j\le (p-1)/2 \\ 0\le k'\le p-1.}},$$

$$S^{-1}\cdot X = \left(p\mathcal{D}_{k'}(Mz)\right)_{0\le k'\le p-1}.$$

After permuting columns, the matrix $L \cdot S$ is composed of a $(p+1)/2 \times (p-1)/2$ zero submatrix and a $(p+1)/2 \times (p+1)/2$ submatrix L' of the form

$$L' := \left(\delta(n_\nu) \sum_{n \equiv n_\nu \pmod p} r_{\mathcal{B}}(n) q^{M_j n}\right)_{0\le j,\nu\le (p-1)/2}.$$

Here $n_0 = 0$ and $0 < n_1 < n_2 < \cdots < n_{(p-1)/2} < p$ are all the quadratic residues modulo p. Since the series $D_k(z)$ is identically zero when $\chi_p(k) = 1$, the $p \times 1$ matrix $S^{-1} \cdot X$ has $(p-1)/2 \times 1$ zero submatrix and a $(p+1)/2 \times 1$ submatrix of the form

$$\left(p\mathcal{D}_{n_\nu}(Mz)\right)_{0\le \nu\le (p-1)/2}.$$

For each $0 \le \nu \le (p-1)/2$, let $n_{\mathcal{B},\nu}$ be the smallest positive integer such that

$$n_{\mathcal{B},\nu} \equiv n_\nu \pmod p, \; r_{\mathcal{B}}(n_{\mathcal{B},\nu}) \ne 0$$

and $n_{\mathcal{B}} := \max_\nu(n_{\mathcal{B},\nu})$. By changing the index ν if necessary, we could suppose $n_{\mathcal{B},\nu} < n_{\mathcal{B},\nu'}$ whenever $\nu < \nu'$. Now, choose the primes N_j, $1 \le j \le (p-1)/2$ such that

$$\sum_{j=t+1}^{(p-1)/2} \frac{1}{N_j} < \frac{1}{n_{\mathcal{B}} N_t}. \tag{7.6}$$

for all $0 \le t \le (p-3)/2$.

Let ι be any permutation of $\{0, 1, 2, \ldots, (p-1)/2\}$, which is not the identity permutation. Then there exists a unique positive integer $t_\iota \le (p-3)/2$ such that $\iota(t_\iota) > t_\iota$ and $\iota(t) = t$ for all $t < t_\iota$. The monotonicity of $\{n_{\mathcal{B},\nu}\}$ then implies that $n_{\mathcal{B},t_\iota} < n_{\mathcal{B},\iota(t_\iota)}$. From inequality (7.6), we can deduce

$$\sum_{j=0}^{(p-1)/2} M_j n_{\mathcal{B},j} < \left(\sum_{j=0}^{t_\iota} M_j n_{\mathcal{B},j}\right) + M_{t_\iota} \le \sum_{j=0}^{t_\iota} M_j n_{\mathcal{B},\iota(j)} < \sum_{j=0}^{(p-1)/2} M_j n_{\mathcal{B},\iota(j)}. \tag{7.7}$$

That means $\det(L') = O(q^{\sum_{j=0}^{(p-1)/2} M_j n_{\mathcal{B},j}})$ is a non-zero power series, and L' is invertible in the ring of Laurent series over $\mathbb{C}$. Also, the entries in L' and R can be considered as holomorphic functions on $\mathcal{H}$. Thus, the formal power series $p\mathcal{D}_{n_\nu}(Mz)$, which are the entries in $(L')^{-1} \cdot R$, are meromorphic functions in z on $\mathcal{H}$, with possible poles at the zeros of the power series $\det(L')$.

This argument could then be used to show that the formal power series $\mathcal{D}_{n_\nu}$ are all holomorphic on $\mathcal{H}$, which implies that $\mathcal{D}$ is holomorphic on $\mathcal{H}$ since

$$\mathcal{D} = \sum_{\nu=0}^{(p-1)/2} D_{n_\nu}.$$

For a particular $z_0 = x_0 + iy_0 \in \mathcal{H}$, it suffices to show that we could pick the N_j's such that $\det(L')$ does not vanish at $q = e^{2\pi i z_0}$. First, choose N_j large enough such that

$$L' = \left(\delta(n_\nu) r_{\mathcal{B}}(n_{\mathcal{B},\nu}) e^{(M_j n_{\mathcal{B},\nu})2\pi i z_0} + O(e^{-(M_j(n_{\mathcal{B},\nu}+p))2\pi y_0})\right)_{0\le j,\nu\le (p-1)/2},$$

with the constant in the O-term independent of z_0 and N_j. If we further require the N_j's to satisfy inequality (7.6), then inequality (7.7) implies that

$$\sum_{j=0}^{(p-1)/2} M_j n_{\mathcal{B},\iota(j)} - \sum_{j=0}^{(p-1)/2} M_j n_{\mathcal{B},j} > M_{(p-1)/2}$$

for any non-trivial permutation ι. Thus, after making $M_{(p-1)/2}$ large enough by increasing the N_j's, we could make sure that the power series $\det(L')$ has the main term

$$\left(\prod_{\nu=0}^{(p-1)/2} \delta(n_\nu) r_{\mathcal{B}}(n_{\mathcal{B},\nu})\right) e^{\left(\sum_{\nu=0}^{(p-1)/2} M_\nu n_{\mathcal{B},\nu}\right)2\pi i z_0}$$

at $q = e^{2\pi i z_0}$ and does not vanish there.

So for a fixed set of N_j's satisfying inequality (7.6), the submatrix L' has full rank in $M_{(p+1)/2,(p+1)/2}(\mathbb{C}[\![q]\!])$. The null space of L is spanned by the column vectors

$$\left(\tfrac{1}{p} e^{2\pi i k n_\nu/p}\right)_{0\le k\le p-1}$$

with $1 \le \nu \le (p-1)/2$. This also shows that L has rank $(p+1)/2$ in $M_{(p-1)/2,(p-1)}(\mathbb{C})$ after substituting $q = e^{2\pi i z}$ for all z in a dense subset $U \in \mathcal{H}$.

Let $\gamma = \left(\begin{smallmatrix} a & b \\ pMc & d\end{smallmatrix}\right) \in \Gamma_0(pM)$, then applying $|_2\gamma$ to both sides of Eq. (7.4) gives us a new equation with the same right hand side. Let $\bar{c}$ denote the multiplicative inverse of c modulo p and we have the following standard transformation of $\vartheta_{\mathcal{B}}(z)$

$$\vartheta_{\mathcal{B}}\left(M_j z + \tfrac{k}{p}\right) |_1 \left(\begin{smallmatrix} a & b \\ pMc & d\end{smallmatrix}\right) = \begin{cases} \chi_p(a+ck)\vartheta_{\mathcal{B}}\left(M_j z + \frac{\overline{(a+ck)}dk}{p}\right), & \text{if } a+ck \not\equiv 0 \bmod p \\ i\sqrt{p}\chi_p(-c)\vartheta_{\mathcal{B}}(pM_j z), & \text{if } a+ck \equiv 0 \bmod p \end{cases}$$

$$\vartheta_{\mathcal{B}}(pM_j z) |_1 \left(\begin{smallmatrix} a & b \\ pMc & d\end{smallmatrix}\right) = \tfrac{\chi_p(-c)i\sqrt{p}}{p}\vartheta_{\mathcal{B}}\left(M_j z + \tfrac{\bar{c}d}{p}\right).$$

Using these properties, Eq. (7.5) transforms into

$$L \cdot T \cdot X' = R,$$

where

$$T := (T_{kk'})_{0\le k,k'\le p-1},$$

$$T_{kk'} := \tfrac{\chi_p(-c)i}{\sqrt{p}}(\delta'_{\bar{c}d}(k) - \tfrac{g_p(k)}{p}) + \begin{cases} \chi_p(a+ck')(\delta'_{\overline{(a+ck')dk'}}(k') - \frac{g_p(k)}{p}), & \text{if } k \not\equiv -a\bar{c} \bmod p \\ \frac{i\chi_p(-c)}{2\sqrt{p}} g_p(k), & \text{if } k \equiv -a\bar{c} \bmod p \end{cases}$$

$$\delta'_\alpha(\beta) := \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{otherwise} \end{cases}$$

$$g_p(k) := \tfrac{1}{2}\sum_{n=0}^{p-1}(1+\chi_p(n))e^{2\pi i n k/p} = \begin{cases} \frac{p}{2}, & \text{if } p \mid k \\ \frac{i\sqrt{p}\chi_p(k)}{2}, & \text{if } p \nmid k \end{cases}$$

$$X' := \left(\mathcal{D}\left(Mz + \tfrac{k'}{p}\right) |_1 \gamma\right)_{0\le k'\le p-1}.$$

So when $q = e^{2\pi iz}$ for any $z \in U \subset \mathcal{H}$, the column vector $X - T \cdot X'$ with entries in $\mathbb{C}$ is in the null space of L, which is spanned by the column vectors $\left(\frac{1}{p}e^{2\pi ikn_\nu/p}\right)_{0\le k\le p-1}$ with $1 \le \nu \le (p-1)/2$.

Now define a $1 \times p$ row vector V by

$$V := \left(\tfrac{1}{p}(g_p(k) + \tfrac{1}{2})\right)_{0\le k\le p-1}.$$

It is orthogonal to the null space of L when $z \in U$. So for all $z \in U$,

$$V \cdot X = V \cdot T \cdot X'.$$

Simple calculation shows that

$$\begin{aligned} V \cdot T &= \chi_p(d)V, \\ V \cdot X &= \chi_p(d)\mathcal{D}(Mz), \\ V \cdot X' &= \chi_p(d)(\mathcal{D}(Mz)|_1\gamma). \end{aligned}$$

So we know that $(\mathcal{D}(Mz)|_1\gamma) = \chi_p(d)\mathcal{D}(Mz)$ for all $z \in U$. Since $U \subset \mathcal{H}$ is dense and both sides are holomorphic, we have $(\mathcal{D}(Mz)|_1\gamma) = \chi_p(d)\mathcal{D}(Mz)$ for all $z \in \mathcal{H}$. This is true for any $\gamma \in \Gamma_0(pM)$, or equivalently

$$(\mathcal{D}|_1 \left(\begin{smallmatrix} a & Mb \\ pc & d \end{smallmatrix}\right))(z) = \chi_p(d)\mathcal{D}(z)$$

for all $\left(\begin{smallmatrix} a & Mb \\ pc & d \end{smallmatrix}\right) \in \Gamma_0(p), b \in \mathbb{Z}$.

Let $\Gamma_0(p, M) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p) : M|b\}$. It is not too hard to see that $\Gamma_0(p)$ is generated by $\Gamma_0(p, M)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right) \in \Gamma_0(p)$. Since $\mathcal{D}(z)$ is also invariant under the action of T, it has level p. By the shape of the Fourier expansion of $\mathcal{D}(z)$, it is in $S_1^-(p)$. □

Corollary 7.3. *Let $\mathcal{B} \in \mathrm{Cl}(F)$ be any class. Then the rational vector space $M_1^{!,+}(p)$ is spanned by the set*

$$\{f^{\mathrm{lift},N,\mathcal{B}}(z) : \mathcal{B} \in \mathrm{Cl}(F), f(z) \in S_{N,1} \cup S_{N,2}, N \equiv 1 \pmod p \text{ prime}\} \cup \{j_m^{\mathrm{lift},\mathcal{B}} : m \ge 0\}$$

Proof. Suppose the subspace of $M_1^{!,+}(p)$ spanned by the set above, denoted by $M_1^{!,+,\mathrm{lift}}(p)$, is strictly smaller. Then Proposition 2.6 implies that there exists $\mathcal{D}(z) = \sum_{n\ge1} d(n)q^n \in \mathbb{Q}[\![q]\!]$ satisfying

(i) $d(n) = 0$ for all n with $\chi_p(n) = 1$,
(ii) $\sum_{n\ge1} c(G, -n)d(n)\delta(n) = 0$ for all $G(z) = \sum_{n\in\mathbb{Z}} c(G, n)q^n \in M_1^{!,+,\mathrm{lift}}(p)$,
(iii) $\mathcal{D}(z) \notin S_1^-(p)$.

For N being 1 or any odd prime satisfying $\chi_p(N) = 1$, the statement $\Phi_{N,\mathcal{B}}(\mathcal{D}, z) \in S_2(N)$ is equivalent to

$$0 = \sum_{m\in\mathbb{Z}} \delta_N(m)c(f, -m) \sum_{n\ge1} \delta(n)d(n)r_{\mathcal{B}}(pm - Nn) = \sum_{n\ge1} c(f^{\mathrm{lift},N,\mathcal{B}}, -n)d(n)\delta(n).$$

for all $f(z) \in S_{N,1} \cup S_{N,2}$ by Lemma 2.9. Specialize G to $f^{\mathrm{lift},N,\mathcal{B}}$ for all such f, condition (ii) then implies that $\Phi_{N,\mathcal{B}}(\mathcal{D}, z) \in S_2(N)$ for all primes $N \equiv 1 \pmod p$ and $N = 1$. Together with condition (i) and Lemma 7.2, we know that $\mathcal{D}(z) \in S_1^-(p)$, which contradicts (iii). Thus, $M_1^{!,+}(p) = M_1^{!,+,\mathrm{lift}}(p)$. □

Remark. A version of this corollary with finitely many N's could provide a bound on κ_p in Theorem 1.1.

7.3. **Proof of Theorem 1.1.** Denote the dimension of $S_1^-(p)$ by d_- and the q-echelon basis by

$$\{h_t(z) \in S_1^-(p) : h_t(z) = q^{n_t} + O(q^{n_{d_-}+1}), 1 \le t \le d_-\}.$$

Choose a mock-modular form $\tilde{g}_\psi(z) = \sum_{n \ge n_0} r_\psi^+(n) q^n \in \mathbb{M}_1^-(p)$ such that it has a fixed principal part as in Proposition 3.2 and $r_\psi^+(n_t)$ satisfies conditions (ii) and (iii) in Theorem 1.1 for $1 \le t \le d_-$. We claim that this $\tilde{g}_\psi(z)$ is a desired choice. Note that if $S_1^-(p) = \emptyset$, then there is only one $\tilde{g}_\psi(z) \in \mathbb{M}_1^-(p)$ with a fixed principal part.

Let $n > 0$ be any integer such that $\chi_p(n) \neq 1$ and $n \neq n_t$ for any $1 \le t \le d_-$. Then by Proposition 2.6, there exists a weakly holomorphic form $G(z) \in M_1^{!,+}(p)$ with the Fourier expansion

$$G(z) = q^{-n} + \sum_{k \ge -n+1} c(G, k) q^k$$

at infinity and $c(G, k) \in \mathbb{Q}$. By Corollary 7.3, we could find $\mathcal{B} \in \mathrm{Cl}(F)$, $\alpha_j \in \mathbb{Q}$, $f_j \in M_0^{!,\mathrm{new}}(N_j)$ and $f_0 \in M_0^!(1)$, all with rational Fourier coefficients, such that

$$G(z) = \sum_{j=0}^{J} \alpha_j f_j^{\mathrm{lift}, N_j, \mathcal{B}}(z).$$

By Proposition 2.5 and , we have

$$\delta(n) r_\psi^+(n) + \sum_{k \le n-1} \delta(k) c(G, -k) r_\psi^+(k) = \langle G, g_\psi \rangle_{\mathrm{reg}} = \sum_{j=0}^{(p-1)/2} \alpha_j \langle f^{\mathrm{lift}, N_j, \mathcal{B}_j}, g_\psi \rangle_{\mathrm{reg}}.$$

After applying Proposition 7.1 and a simple inductive argument on n, we can write $r_\psi^+(n)$ in the form

$$r_\psi^+(n) = -\frac{1}{\kappa_{p,n}} \sum_{\mathcal{A} \in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log |u(n, \mathcal{A})|^2$$

for all $n \in \mathbb{Z}$ with $\kappa_{p,n} \in \mathbb{Z}, u(n, \mathcal{A}) \in H^\times$ independent of ψ and $\sigma_{\mathcal{C}}(u(n, \mathcal{A})) = u(n, \mathcal{A}\mathcal{C}^{-1})$ for all $\mathcal{C} \in \mathrm{Cl}(F)$.

In a similar fashion, we will show that one can choose $\kappa_{p,n}$ independently of n. Fix a finite set of primes

$$\mathcal{M}_p := \{N_v : 1 \le v \le p-1, \chi_p(v) = 1, N_v \equiv v \pmod{p}\}.$$

and define

$$\kappa_p := \mathrm{lcm}\{\kappa_{p,k} : k \le (p+1)/12\} \cdot \mathrm{lcm}\{N^2 - 1 : N \in \mathcal{M}_p\}.$$

Then for $n \le (p+1)/12$, we could replace $\kappa_{p,n}$ with κ_p and $u(n, \mathcal{A})$ with $u(n, \mathcal{A})^{\kappa_p/\kappa_{p,n}}$. Notice that for $n \le 0$, the algebraic number $u(n, \mathcal{A})$ is a perfect $(N^2-1)^{\mathrm{th}}$ power in $H^\times$ for any $N \in \mathcal{M}_p$.

When $n > (p+1)/12$ and $p \nmid n$, pick $N \in \mathcal{M}_p$ and a positive integer m such that $pm - Nn = 1$. Notice $N \nmid m$ and $m \ge g_N + 1$ where g_N is the genus of the modular curve $X_0(N)$. With such a choice of N and m, let $f_m \in \mathcal{S}_{N,1}$ be a modular function as in (2.20).

Then $\delta(n)\delta_N(m)r_{\mathcal{A}_0}(pm - Nn) = 1$ and Eq. (7.3) becomes

$$\begin{aligned} r^+_\psi(n) = &- \sum_{0<k<n} \delta(k) \left(\sum_{m''=1}^{g_N} \delta_N(m'')c(f_m, -m'')r_{\mathcal{A}_0}(pm'' - Nk) \right) r^+_\psi(k) \\ &- \sum_{k\le 0} \delta(k) \left(\sum_{m''\le g_N} \delta_N(m'')c(f_m, -m'')r_{\mathcal{A}_0}(pm'' - Nk) \right) r^+_\psi(k) \\ &+ \langle f_m^{\mathrm{lift},N,\mathcal{A}_0}, g_\psi \rangle_{\mathrm{reg}} + Nc(f_m, 0)\rho_{N,\mathcal{A}_0,\psi}(0). \end{aligned}$$

In the summation on the right hand side, the coefficient of $r^+_\psi(k)$ is an integer when $k > 0$ by Lemma 2.9 and is rational with denominator dividing $(N^2 - 1)$ when $k \le 0$. When $n = pn'$ with $n' \ge 1$ an integer, we could substitute $f = j_{n'}(z)$ into Eq. (5.4) and have

$$r^+_\psi(n) = \langle j_{n'}^{\mathrm{lift},\mathcal{B}}, g_\psi \rangle_{\mathrm{reg}} - \sum_{k<n} \delta(k) r_{\mathcal{B}}(pn' - k) r^+_\psi(k).$$

By Propositions 3.2, 7.1 and a simple induction on n, we could choose $\kappa_{p,n} = \kappa_p$ for $n > (p + 1)/12$ while keeping $u(n, \mathcal{A}) \in H^\times$.

With property (ii) and (iii) known, one could slightly change the choices of κ_p and $u(n, \mathcal{A})$ such that property (iv) is also satisfied. By changing κ_p to $2H(p)\kappa_p$, we could suppose that $(2H(p)) \mid \kappa_p$ and $u(n, \mathcal{A})$ is a $(H(p))^{\mathrm{th}}$ power in $H^\times$. For each $n \ne 0$, we know that $\exp(R^+_p(n)) \in \mathbb{Q}$ by Eq. (1.6). Then there exists $c_n \in \mathbb{Q}$ depending only on n and p such that

$$-\frac{\log \mathrm{N}_{H/\mathbb{Q}}(c_n u(n, \mathcal{A}))}{\kappa_p} = R^+_p(n).$$

Since ψ^2 is non-trivial, we could replace $u(n, \mathcal{A})$ with $c_n u(n, \mathcal{A})$ if necessary to make it satisfy property (iv), as well as properties (ii) and (iii).

For all $n \in \mathbb{Z}$, define $r^+_{\mathcal{A}}(n)$ by

$$\begin{aligned} r^+_{\mathcal{A}}(n) :&= \frac{1}{H(p)} \left(R^+_p(n) + \sum_{\psi \text{ non-trivial}} \overline{\psi(\mathcal{A})} r^+_\psi(n) \right) \\ &= \frac{1}{H(p)} \left(R^+_p(n) + \frac{1}{\kappa_p} \log |\mathrm{N}_{H/\mathbb{Q}}(u(n, \mathcal{A}))| \right) - \frac{1}{\kappa_p} \log |u(n, \sqrt{\mathcal{A}})|^2, \end{aligned}$$

where the second equality is due to the transitive action of $\mathrm{Gal}(H/F)$ on $\{u(n, \mathcal{A}) : \mathcal{A} \in \mathrm{Cl}(F)\}$. The generating series $\sum_{n\in\mathbb{Z}} r^+_{\mathcal{A}}(n)q^n$ is a mock-modular form with shadow $\vartheta_{\mathcal{A}}(z)$. When $n > 0$, property (iv) implies that

$$r^+_{\mathcal{A}}(n) = -\frac{1}{\kappa_p} \log |u(n, \sqrt{\mathcal{A}})|^2$$

for all $\mathcal{A} \in \mathrm{Cl}(F)$, which yields

$$R^+_p(n) = -\frac{1}{\kappa_p} \sum_{\mathcal{A}\in\mathrm{Cl}(F)} \log |u(n, \sqrt{\mathcal{A}})|^2 = \sum_{\mathcal{A}\in\mathrm{Cl}(F)} r^+_{\mathcal{A}}(n).$$

This is the analogue of Eq. (1.9) for mock-modular forms as mentioned in the introduction.

8. Case $p = 283$: Some numerical calculations

Finally, we will present another numerical example to demonstrate that statements similar to the Conjecture in §1 should be true for octahedral newforms. These calculations were conducted in SAGE [45]. When $p = 283$, $\mathrm{Cl}(F)$ has order 3 and the space $S_1(p)$ is 3-dimensional spanned by a dihedral newform $h(z)$ and octahedral newforms $f_\pm(z) = f_1(z) \pm \sqrt{-2} f_2(z)$, where

$$\begin{aligned} h(z) &= q + q^4 - q^7 + q^9 + O(q^{10}),\\ f_1(z) &= q - q^4 + 2q^6 - q^7 - q^9 + O(q^{10}),\\ f_2(z) &= q^2 - q^3 - q^5 + O(q^{10}). \end{aligned}$$

By Lemma 2.4, $h(z), f_1(z) \in S_1^+(p)$ and $f_2(z) \in S_1^-(p)$. By the Deligne-Serre Theorem, $f_\pm(z)$ arises from $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$, where K is the degree 2 extension of the normal closure of $\mathbb{Q}[X]/(X^4 - X - 1)$ such that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{F}_3)$.

Let F_6 be the subfield of K fixed by a subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ isomorphic to $\mathbb{Z}/8\mathbb{Z}$, and H_3 the cubic subfield of F_6. Explicitly, we can write

$$\begin{aligned} H_3 &= \mathbb{Q}[Y]/(Y^3 + 4Y + 1),\\ F_6 &= \mathbb{Q}[X]/(X^6 - 3X^5 + 6X^4 - 7X^3 + 10X^2 - 7X + 6), \end{aligned}$$

with the embedding

$$\begin{aligned} H_3 &\hookrightarrow F_6\\ Y &\mapsto X^2 - X + 1. \end{aligned}$$

Fix a complex embedding of $F_6 \hookrightarrow \mathbb{C}$ such that $X = t, Y = \theta$ with $t, \theta \in \mathbb{C}$ having positive imaginary part. Since F_6 is totally complex, the unit group of F_6 has rank 2 and is generated by

$$u_1 = t^2 - t + 1, u_2 = t^3 + t - 1.$$

As in the case of dihedral newforms, one can express the Petersson norms of $f_+(z)$ and $f_-(z)$ as

$$\langle f_+, f_+ \rangle = \langle f_-, f_- \rangle = \frac{p+1}{12} \mathrm{Res}_{s=1} \frac{\zeta(s) L(s, F_6)}{L(s, H_3)\zeta(2s)(1 + p^{-s})}.$$

Notice the ratio $L(s, F_6)/L(s, H_3)$ is the L-function of the quadratic Hecke character of H_3 associated to the Hilbert quadratic norm residue symbol of F_6/H_3. So it is holomorphic at $s = 1$ and the right hand side can be evaluated to be $8 \log |u_1 u_2^2|$.

By Proposition 2.6, there exist mock modular forms $\tilde{f}_1(z) \in \mathbb{M}_1^-(p)$ and $\tilde{f}_2(z) \in \mathbb{M}_1^+(p)$ with shadows f_1, f_2 respectively such that

$$\begin{aligned} \tilde{f}_1(z) &= c_1^+(-4)q^{-4} + c_1^+(-1)q^{-1} + c_1^+(0) + \sum_{n>0, \chi_p(n) \neq 1} c_1^+(n) q^n,\\ \tilde{f}_2(z) &= c_2^+(-2)q^{-2} + c_2^+(0) + \sum_{n>0, \chi_p(n) \neq -1} c_2^+(n) q^n. \end{aligned}$$

Using Proposition 2.5, we find that

$$-c_1^+(-4) = c_1^+(-1) = c_2^+(-2) = 2 \log |u_1 u_2^2|.$$

Since $M_1^+(p) \subset \mathbb{M}_1^+(p)$ and $M_1^-(p) \subset \mathbb{M}_1^-(p)$ are both non-empty, the principal part of $\tilde{f}_j(z)$ does not determine it uniquely. However, once $c_1^+(2)$ is chosen, then $\tilde{f}_1$ is fixed. Similarly, once $c_2^+(0), c_2^+(1)$ and $c_2^+(4)$ are chosen, $\tilde{f}_2$ is fixed. Since we expect $c_j^+(n)$ to be logarithms of absolute values of algebraic numbers in F_6, it is natural to choose $c_1^+(2), c_2^+(0), c_2^+(1)$ and $c_2^+(4)$ this way and study the other coefficients. So for $j = 1, 2$, we will write

$$c_j^+(n) = \frac{1}{\kappa_j} \log |u_j(n)|,$$

where $\kappa_j \in \mathbb{Z}$ and $u_j(n) \in \mathbb{C}^\times$ is some complex number. Then we fix $\tilde{f}_j(z)$ by letting $\kappa_1 = 2, \kappa_2 = u_2(0) = u_2(1) = 1$ and

$$u_2(4) = u_1(2) = -\frac{7}{16}t^5 + \frac{5}{4}t^4 - \frac{7}{8}t^3 + \frac{47}{16}t^2 - \frac{13}{16}t + \frac{19}{8}.$$

From the numerical calculations, we can make predictions of the algebraicity of $u_j(n)$. In the table below, we list $c_j^+(\ell)$ for $j = 1, 2$ and various primes ℓ. Also, we list the predicted fractional ideals generated by $\tilde{u}_j(n)$ in F_6.

When $\chi_p(\ell) \neq 1$, the ideal (ℓ) splits into $\mathfrak{L}_0\mathfrak{L}_1$ in H_3, and $\mathfrak{L}_1$ splits into $\mathfrak{l}_{\ell,1}\mathfrak{l}_{\ell,2}$ in F_6. When $\chi_p(\ell) = 1$ and $c(f_1, \ell) = \pm 1$, the ideal (ℓ) splits into $\mathfrak{l}_1\mathfrak{l}_2$ in F_6. When $\chi_p(\ell) = 1$ and $c(f_1, \ell) = 0$, the ideal (ℓ) splits into $\mathfrak{L}_{\ell,0}\mathfrak{L}_{\ell,1}$ in H_3 and $\mathfrak{L}_{\ell,1}$ splits into $\mathfrak{l}_{\ell,1}\mathfrak{l}_{\ell,2}$ in F_6, where $\mathfrak{l}_{\ell,j}$ has order 4 in $\mathrm{Cl}(F_6)$, the class group of F_6. Since F_6 and H_3 have class numbers 8 and 2 respectively, the fractional ideals in Table 3 and Table 4 are all principal. For example, the value we chose for $u_1(2)$ generates the fractional ideal $(\mathfrak{l}_{2,1}/\mathfrak{l}_{2,2})^4$, where $\mathfrak{l}_{2,1}$ and $\mathfrak{l}_{2,2}$ have order 8 in $\mathrm{Cl}(F_6)$. So it is necessary to take $\kappa_1 = 2$. The numerical pattern also justifies this choice of $\tilde{f}_j(z)$.

TABLE 3. Coefficients of $\tilde{f}_1(z)$

ℓ	$c(f_2, \ell)$	$c_1^+(\ell)$	$(\tilde{u}_1(\ell))$
2	1	1.2075349695016218	$(\mathfrak{l}_{2,1}/\mathfrak{l}_{2,2})^4$
3	-1	-0.44226603950742649	$(\mathfrak{l}_{3,1}/\mathfrak{l}_{3,2})^4$
5	-1	-3.9855512247433431	$(\mathfrak{l}_{5,1}/\mathfrak{l}_{5,2})^4$
17	0	-3.2181607607379323	$(\mathfrak{l}_{17,1}/\mathfrak{l}_{17,2})^4$
19	1	5.3481233955176073	$(\mathfrak{l}_{19,1}/\mathfrak{l}_{19,2})^4$
31	1	-1.7192005338244623	$(\mathfrak{l}_{31,1}/\mathfrak{l}_{31,2})^4$
37	0	0.32541651822318252	$(\mathfrak{l}_{37,1}/\mathfrak{l}_{37,2})^4$
43	1	-4.6200896216743352	$(\mathfrak{l}_{43,1}/\mathfrak{l}_{43,2})^4$
47	-1	-1.0203031328088645	$(\mathfrak{l}_{47,1}/\mathfrak{l}_{47,2})^4$
53	0	5.8419201851710110	$(\mathfrak{l}_{53,1}/\mathfrak{l}_{53,2})^4$
67	0	-3.9318486618330462	$(\mathfrak{l}_{67,1}/\mathfrak{l}_{67,2})^4$
79	0	7.5720154112893967	$(\mathfrak{l}_{79,1}/\mathfrak{l}_{79,2})^4$
107	0	-0.81774052769784944	$(\mathfrak{l}_{107,1}/\mathfrak{l}_{107,2})^4$
109	-1	4.8808523053451562	$(\mathfrak{l}_{109,1}/\mathfrak{l}_{109,2})^4$
283	0	6.86483405137284	$(\mathfrak{l}_{283,1}/\mathfrak{l}_{283,2})^4$

TABLE 4. Coefficients of $\tilde{f}_2(z)$

ℓ	$c(f_1, \ell)$	$c_2^+(\ell)$	$(\tilde{u}_2(\ell))$
7	-1	-3.27983974462451	$\mathfrak{l}_{7,1}/\mathfrak{l}_{7,2}$
11	1	-2.56257986300244	$\mathfrak{l}_{11,1}/\mathfrak{l}_{11,2}$

13	1	-5.57196302179201	$\mathfrak{l}_{13,1}/\mathfrak{l}_{13,2}$
23	-1	1.01652189648251	$\mathfrak{l}_{23,1}/\mathfrak{l}_{23,2}$
29	-1	1.54494007675715	$\mathfrak{l}_{29,1}/\mathfrak{l}_{29,2}$
41	1	0.771808245755645	$\mathfrak{l}_{41,1}/\mathfrak{l}_{41,2}$
71	0	-4.99942007705695	$(\mathfrak{l}_{71,1}/\mathfrak{l}_{71,2})^2$
73	0	-1.64986308549260	$(\mathfrak{l}_{73,1}/\mathfrak{l}_{73,2})^2$
83	-2	8.97062724307569	1
89	-1	-0.399183274865547	$\mathfrak{l}_{89,1}/\mathfrak{l}_{89,2}$
101	0	5.99108448704487	$(\mathfrak{l}_{101,1}/\mathfrak{l}_{101,2})^2$
283	1	4.48531362153791	1
643	2	2.32782185606303e-10	1
773	2	5.08403073372794e-9	1
859	-2	-8.97062719191082	1

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