RESTRICTION OF COHERENT HILBERT EISENSTEIN SERIES

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Abstract. In [30], Tonghai Yang constructed a family of coherent Hilbert Eisenstein series and asked several questions concerning the span of their diagonal restrictions. In this paper, we will answer one of them, where the construction essentially comes from a biquadratic number field K. Surprisingly, the span depends on the non-vanishing of the central critical value of the L-function associated to a weight 2 eigenform.

1. Introduction

As special examples of modular forms, Eisenstein series play an important role in number theory. Their explicit Fourier coefficients contain a great amount of interesting arithmetic information and have been exploited to study arithmetic problems. In [8], Hecke had tried to construct Hilbert Eisenstein series of parallel weight 1 using coherent Eisenstein series [8]. An unfortunate sign resulted in him producing the zero function. Amusingly, this mistake turned out to be essential for Gross-Zagier's work on the factorization of difference of singular moduli [6]). Extending this idea and using a similar Eisenstein series, they construct a kernel function for producing the derivative of L-functions [7]. This allowed them to prove the famous Gross-Zagier formula, which related the derivative of L-series attached to modular elliptic curves to the heights of Heegner points, and is instrumental in studying the BSD conjecture.

Both Eisenstein series mentioned above are examples of incoherent Eisenstein series in the context of Kudla's program [18], which relates the Fourier coefficients of derivatives of Siegel-Eisenstein series to arithmetic degrees of special cycles on Shimura varieties. Combining the restrictions of the derivatives of these incoherent Siegel-Eisenstein series to the diagonal with the techniques of theta-lifting, Bruinier and Yang have generalized considerably the works of Gross and Zagier. They have extended the factorization of the difference of singular moduli to that of CM-values of Hilbert modular functions [1], and the Gross-Zagier formula to higher dimensional Shimura varieties [2].

Another useful application of Eisenstein series is the construction of basis of modular forms from products of Eisenstein series. Kohnen and Zagier showed that the space of modular forms on $\mathrm{SL}_2(\mathbb{Z})$ can be spanned by the products of two Eisenstein series [13]. This was

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considered by Imamoḡlu and Kohnen for level 2 in the context of representing integers as sum of squares [10]. Similar statements have been proved by Kohnen and Martin for prime level p [17] and Westerholt-Raum for arbitrary level [29]. More generally, one could ask the same question by replacing the product of elliptic Eisenstein series with the restrictions of Hilbert Eisenstein series.

In [30], Yang injected ramification into Hecke's method and constructed holomorphic Hilbert Eisenstein series of parallel weight one. Let $N \geq 1$ be an odd, square-free integer and define the following subset of fundamental discriminants

$$\mathcal{D}_N := \left\{ -D < 0 \text{ fundamental discriminant} \,\middle|\, \left(\tfrac{-D}{p}\right) = -1 \text{ for all primes } p \mid N \right\}. \tag{1.1}$$

Given two relatively prime fundamental discriminants $d_1, d_2 \in \mathcal{D}_N$ and a suitable integral ideal $\mathcal{N}$ in the real quadratic field $F := \mathbb{Q}(\sqrt{d_1 d_2})$, the construction in [30] produces a Hilbert Eisenstein series of weight $(1,1)$. Its restriction to the diagonal, which is denoted by $f_{d_1,d_2,\mathcal{N}} \in M_2(N)$, is a modular form on $\Gamma_0(N)$ of weight 2 and trivial nebentypus character. Besides the constant term, all of its Fourier coefficients are integers (see §3 for definition). In addition, the following conjecture was proposed.

Conjecture 1.1. (Conjecture 5.10(1) in [30]). Fix an odd, square-free positive integer N. Then the space $M_2(N)$ is spanned by modular forms $f_{d_1,d_2,\mathcal{N}}$ with

- $(d_1, d_2) \in \mathcal{D}_N^{2,*} := \{(-D_1, -D_2) \in \mathcal{D}_N^2 \mid \gcd(D_1, D_2) = 1\}$,
- $\mathcal{N}$ varying over integral ideals in $\mathbb{Q}(\sqrt{d_1 d_2})$ having an odd number of prime divisors and satisfying $\mathcal{N} \cap \mathbb{Z} = N\mathbb{Z}$.

As remarked in [30], $f_{d_1,d_2,\mathcal{N}}$ is an old form if $\mathrm{Nm}\mathcal{N} \neq N$. Whenever $\mathrm{Nm}_{F/\mathbb{Q}}\mathcal{N} = N$, a simple proposition (Prop. 3.3) shows that $f_{d_1,d_2,\mathcal{N}}$ is contained in the subspace $M_2^+(N) \subset M_2(N)$ of modular forms invariant under the U_N-operator. Under the Petersson inner product, one can write

$$M_2(N) = M_2^{\mathrm{old}}(N) \oplus M_2^{\mathrm{new}}(N) = M_2^{\mathrm{old}}(N) \oplus M_2^+(N) \oplus S_2^-(N),$$

where $M_2^{\mathrm{old}}(N) \subset M_2(N)$ is the subspace of oldforms and $S_2^-(N)$ is spanned by newforms in $S_2(N)$ with eigenvalue -1 under U_N. The space $S_2^-(N)$ is non-empty if there exists an elliptic curve of conductor N with odd rank.

In this paper, we will consider the span of the cuspidal part of

$$F_{D_1,D_2,N}(z) := \frac{1}{2} \sum_{\substack{\mathcal{N} \subset \mathcal{O}_F \\ \mathrm{Nm}\mathcal{N} = N}} f_{-D_1,-D_2,\mathcal{N}}(z) \tag{1.2}$$

with $(-D_1, -D_2) \in \mathcal{D}_N^{2,*}$. When N is prime, there is only one choice of $\mathcal{N}$ up to conjugation in the field F and $F_{D_1,D_2,N} = f_{-D_1,-D_2,\mathcal{N}}$ by Proposition 3.3. In this case, the main result (Theorem 5.1) is as follows.

Theorem 1.2. *When N is an odd prime, let $M_2^{+,0}(N) \subset M_2^+(N)$ be the $\mathbb{C}$-subspace spanned by the unique Eisenstein series $E_{2,N} \in M_2(N)$ (see equation (5.1)) and all eigenforms $G \in S_2(N)$ satisfying $L(G,1) \neq 0$. Then it is also the span of $E_{2,N}$ and $F_{D_1,D_2,N}$ over all $(-D_1,-D_2) \in \mathcal{D}_N^{2,*}$.*

Remark 1.3. Eigenforms in $M_2^+(N)\backslash M_2^{+,0}(N)$ are associated to elliptic curves, whose Mordell-Weil groups have even ranks and are at least two by the BSD conjecture. It it not surprising that they are outside the span of the restrictions of Hilbert Eisenstein series, which have explicit Fourier coefficients. The smallest prime N such that $M_2^{+,0}(N) \subsetneq M_2^+(N)$ is 389.

Remark 1.4. It would be interesting to understand the Eisenstein-part of the $F_{D_1,D_2,N}$'s.

Theorem 5.1 answers the question raised in Conjecture 1.1 when N is prime and enables us to give a refinement of Conjecture 1.1 (see Conjecture 5.2). En route to the proof of Theorem 5.1, we will calculate the Petersson inner product between a newform $G \in S_2(N)$ and $F_{D_1,D_2,N}$. Let $g = \sum_{n\geq 1} a_g(n)q^n$ be the weight 3/2 newform whose Shimura lift is G. We can express $\langle G, F_{D_1,D_2,N}\rangle$ in a nice form. This result is the analogue of inner product formula for derivatives of L-functions (see equation (4) in [5]).

Theorem 1.5. *Let N be an odd, square-free positive integer and $D_1, D_2, g, G, F_{D_1,D_2,N}$ be as above. Then we have*

$$\langle G, F_{D_1,D_2,N}\rangle = \frac{3}{4\pi}\frac{\langle G,G\rangle}{\langle g,g\rangle} a_g(D_1)\overline{a_g(D_2)}L(G,1). \tag{1.3}$$

Remark 1.6. For any fundamental discriminant $-D < 0$, a result of Waldspurger [27] implied that the ratio between $|a_g(D)|^2$ and $\sqrt{D}L(G,D,1)$ is independent of D. Here $L(G,D,1)$ is the D^{th} quadratic twist of the L-series associated to G. Kohnen then generalized this proportionality result to that between cycle integrals and the products of Fourier coefficients [16]. Combining Kohnen's result with Theorem 1.5 above yields

$$\langle G, F_{D_1,D_2,N}\rangle = \frac{1}{4\pi}L(G,1)r_{1,N}(G;-D_1,-D_2) \tag{1.4}$$

where $r_{1,N}(G;-D_1,-D_2)$ is defined by equation (7) in [16] and equals to a finite sum of cycle integrals with a twist by genus character. Note that unlike in [16], we define the Petersson norm in equation (2.5) without dividing by the index of the congruence subgroup. From its definition, it is readily checked that $r_{1,N}(G;-D_1,-D_2) = r_{1,N}(G;-D_2,-D_1)$.

Another important ingredient of the proof of Theorem 5.1 is a non-vanishing result on the Fourier coefficients of an *arbitrary* modular form of half-integral weight. Given a nonzero modular form $f = \sum_{n\geq 1} a_f(n)q^n \in S_k(4N,\chi)$ of half-integral weight $k \geq 3/2$ and nebentypus character $\chi : (\mathbb{Z}/4N\mathbb{Z})^\times \to \mathbb{C}^\times$, we are interested in the non-vanishing of the Fourier coefficients $a_f(D)$ as D ranges over square-free integers. When $k \geq 5/2$ and certain mild conditions on N and χ are imposed, it is shown in [22] that there are infinitely many square-free integers D such that $a_f(D) \neq 0$. Using the same argument loc. cit. and extending the estimates in [3] to weight $3/2$, we will prove the following result necessary for the proof of Theorem 5.1.

Proposition 1.7. *Let N be an odd and square-free integer and $f(z) = \sum_{n \geq 1} a_f(n) q^n \in S_{3/2}(4N)$ an arbitrary, nonzero modular form. Then for any finite set of primes $\mathcal{S}$, there are infinitely many square-free integers D satisfying $a_f(D) \neq 0$ and $\gcd(D, \ell) = 1$ for all $\ell \in \mathcal{S}$.*

Remark 1.8. We will state the proposition above in §2.2 with a slightly more relaxed condition on N and χ as in Theorem 2 in [22].

Remark 1.9. In [23], there is a quantitative version of the non-vanishing result of [22] that we generalized here. With this and a little more work, it is possible to give an effective version of this proposition and make the generating set in Theorem 1.2 finite.

Remark 1.10. With the generalization of the result of Duke and Iwaniec to weight $3/2$, Theorem 1 in [23] now holds with weight $2k = 2$.

The structure of the paper is as follows. In section 2, we will give preliminary information on modular forms of half-integral weight, such as Shimura lift, Fourier coefficients, and prove Proposition 2.4. In section 3, we will recall the results about Hilbert Eisenstein series in [30] and prove Theorem 1.5. Section 4 contains the counting lemma that relates $F_{D_1, D_2, N}$ to the Shimura lift of modular forms of weight $3/2$. Finally, we prove Theorem 5.1, hence Theorem 1.2, and give a refinement of conjecture 1.1 in section 5.

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2. Preliminaries

In this section, we will give some preliminary information on half-integral weight modular forms. We refer the readers to [11] for its definition. For our purpose, we will only consider the spaces with trivial nebentypus character.

2.1. Modular forms of half-integral weight and their Shimura Lifts.

For a positive integer $N, \kappa \geq 1$, let $M_{\kappa+1/2}(4N)$ and $S_{\kappa+1/2}(4N)$ be the spaces of modular forms and cusp forms of weight $\kappa + 1/2$ on $\Gamma_0(N)$ respectively. Denote Kohnen's plus subspace by

$$M^+_{\kappa+1/2}(4N) := \left\{ \sum_{n \geq 0} a(n) q^n \in M_{\kappa+1/2}(4N) : a(n) = 0 \text{ whenever } (-1)^\kappa n \equiv 2, 3 \bmod 4 \right\}.$$

Kohnen introduced this subspace [14,15] to study the Shimura lift between cusp forms of half integral weight and integral weight. This has been generalized to modular forms of various weights and levels in [20, 21, 26, 28].

For a fundamental discriminant $-D < 0$ relatively prime to N, denote the Dirichlet character of $\mathbb{Q}(\sqrt{-D})$ by

$$\chi_D(\cdot) := \left(\frac{(-1)^{(D-1)/2}D}{\cdot}\right) = \left(\frac{\cdot}{D}\right).$$

Then the D^{th} Shimura lift of $g(z) = \sum_{n\geq 0} a_g(n)q^n \in M^+_{\kappa+1/2}(4N)$ is defined by Theorem 2 in [28]

$$\mathscr{S}_D(g) := \frac{a_g(0)}{2} L\left(\chi_D \mathbb{1}_N, 1-\kappa\right) + \sum_{n\geq 1}\left(\sum_{d|n} \mathbb{1}_N(d)\chi_D(d)d^{\kappa-1}a_g\left(D\frac{n^2}{d^2}\right)\right) q^n. \tag{2.1}$$

Here $\mathbb{1}_N$ is the trivial character of conductor N and $L(\chi, s)$ is the L-function associated to the Dirichlet character χ. By varying the choice of D, one can obtain a family of nonzero lifts. When g is an eigenform, all these lifts are multiples of a single integral weight eigenform G. This normalized eigenform G is then called the *Shimura lift* of g and denoted by $\mathscr{S}(g)$.

When N is odd and square-free, Kohnen defined Atkin-Lehner operators for half-integral weight modular forms [15]. Using these, he set up a theory of newforms for half-integral weight modular forms and showed that the subspace of newforms

$$S^{+,\text{new}}_{\kappa+1/2}(4N) \subset S^+_{\kappa+1/2}(4N) := M^+_{\kappa+1/2}(4N) \cap S_{\kappa+1/2}(4N)$$

is canonically isomorphic to the subspace of weight 2κ newforms $S^{\text{new}}_{2\kappa}(N) \subset S_{2\kappa}(N)$ as Hecke modules under the Shimura correspondence [15]. We can summarize the result as the following theorem.

Theorem 2.1 (Theorem 2 in [15])**.** *Suppose N is odd and square-free. Then there is a linear combination of $\mathscr{S}_D$'s that map $S^{+,new}_{\kappa+1/2}(4N)$ isomorphically to $S^{new}_{2\kappa}(N)$. Furthermore, if $g(z) = \sum_{n\geq 1} a_g(n)q^n \in S^{+,new}_{\kappa+1/2}(4N)$ is an eigenform and $G = \mathscr{S}(g) \in S^{new}_{2\kappa}(N)$ the Shimura lift, then $\mathscr{S}_D(g) = a_g(D)\cdot G$ and*

$$L(\chi_D\mathbb{1}_N, s-\kappa+1)\left(\sum_{n\geq 1}\frac{a_g(Dn^2)}{n^s}\right) = a(D)L(G,s). \tag{2.2}$$

When N is odd and square-free, the dimensions of the subspaces of Eisenstein series in $M^+_{3/2}(4N)$ and $M_2(N)$ are both $2^{\omega(N)}-1$, where $\omega(N)$ is the number of distinct prime divisors of N. As shown in [28], the map $\mathscr{S}$ gives a Hecke-equivariant isomorphism between them.

Now, we will record a proposition about the vanishing of the Fourier coefficients of newforms $g \in S^+_{\kappa+1/2}(4N)$.

Proposition 2.2. *Let* $g(z) = \sum_{n\geq 1} a_g(n)q^n \in S^{+,new}_{\kappa+1/2}(4N)$ *be an eigenform and* $\varepsilon_p \in \{1, -1\}$ *be the eigenvalue of* g *under the* $(p^{1-\kappa}U_{p^2})$*-operator for each prime* $p \mid N$*. Then* $a_g(n) = 0$ *if there exists a prime* $p \mid N$ *such that*

$$\left(\frac{(-1)^\kappa n}{p}\right) = \varepsilon_p.$$

Remark 2.3. By Theorem 2 in [15], the $\varepsilon_p p^{\kappa-1}$ is the eigenvalue of $\mathscr{S}(g)$ under the U_p-operator.

Proof. By Proposition 4 on page 39 and Theorem 1 on page 64 of [15], the operator $p^{1-\kappa}U_{p^2}$ is a hermitian involution on $S^{+,\text{new}}_{\kappa+1/2}(4N)$ and that g has eigenvalue ± 1 under $p^{1-\kappa}U_{p^2}$ if and only if $a_g(n) = 0$ for all n satisfying $\left(\frac{(-1)^\kappa n}{p}\right) = \pm 1$. □

2.2. **Non-vanishing of Fourier Coefficients.** In this section, we will extend the estimate in [3] and apply the argument in [22] to prove the following useful proposition. The main difference between this and Theorem 2 in [22] is the case of weight 3/2. In that case, there are one-variable theta series

$$f(z) = \sum_{m\geq 1} m\psi(m)q^{m^2 t} \in S_{3/2}(4r^2 t, \psi_t), \tag{2.3}$$

with ψ is a character modulo r satisfying $\psi(-1) = -1$, t is a positive integer and $\psi_t(d) := \psi(d)\left(\frac{t}{d}\right)\left(\frac{-1}{d}\right)$. They do not satisfy the non-vanishing result that we will state. This is reflected in the proof of Lemma 3.8 in [22], as these theta series are the only cusp forms whose Shimura lifts are Eisenstein series [25].

Proposition 2.4. *Let* r *and* t *be odd, square-free and relatively prime integers and* χ_r, χ_{4t} *characters modulo* r *and* $4t$ *respectively. Suppose* χ_r *is primitive. Then for any integer* $\kappa \geq 1$*, any finite set of primes* $\mathcal{S}$*, and any nonzero cusp form* $f(z) = \sum_{n\geq 1} a_f(n)q^n \in S_{\kappa+1/2}(4r^2 t, \chi_r\chi_{4t})$*, which is not a theta series of the form* (2.3)*, there exist infinitely many square-free integers* D *such that* $a_f(D) \neq 0$ *and* $\gcd(D, \ell) = 1$ *for all* $\ell \in \mathcal{S}$*.*

Remark 2.5. When $\kappa \geq 2$, the proposition above is inherent in the proof of Theorem 2 in [22] as there is no theta series. This also happens when $\kappa = 1$ and N is square-free, in which case Proposition 2.4 becomes Proposition 1.7.

An important ingredient in the proof of Proposition 2.4 is an estimate due to Duke and Iwaniec [3] (see Theorem 3 in [22]), which was only proved for $\kappa \geq 2$. With this result for $\kappa = 1$, the arguments in [22] can be applied mutatis mutandis. However, it is crucial that the Shimura lift f is a cusp form, since Lemma 3.8 in [22] uses the Deligne bound for the Fourier coefficients of integral weight eigenforms. This does not happen precisely when f is a theta series [25]. Thus, they are excluded from the statement of the proposition. We will now state the estimate in [3] as phrased in [22] and prove it for $\kappa = 1$. This will then complete the proof of Proposition 2.4.

Theorem 2.6 (Duke-Iwaniec [3], Saha [22])**.** *Let $N, \kappa \geq 1$ be positive integers, χ a Dirichlet character modulo $4N$ and $f \in S_{\kappa+1/2}(4N, \chi)$ an arbitrary cusp form with Fourier expansion $f(z) = \sum_{n\geq 1} f_n n^{\frac{\kappa}{2}-\frac{1}{4}} q^n$. Then there exists a positive square-free integer $B_{N,\kappa}$ divisible by all primes dividing $4N$ such that*

$$\sum_{n\geq 1} f_{nr}\overline{f_{ns}} e^{-n/X} = \delta_{r,s} \cdot C_\kappa \cdot \frac{\langle f, f\rangle}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4N)]} \cdot X + O_{f,\epsilon}\left((rsX)^{\frac{1}{2}+\epsilon}\right) \tag{2.4}$$

for any positive square-free integers r, s satisfying $r \equiv s \pmod 4$ and $\gcd(rs, B_{N,\kappa}) = 1$. Here $\delta_{r,s}$ is the Kronecker delta function, $C_\kappa := \frac{12\cdot(4\pi)^{\kappa-1/2}}{\Gamma(\kappa+1/2)}$ is an explicit constant and $\langle f, f\rangle$ is the Petersson norm of f defined by

$$\langle f, f\rangle := \int_{\Gamma_0(4N)\backslash\mathcal{H}} |f(z)|^2 y^{\kappa+\frac{1}{2}} \frac{dxdy}{y^2}. \tag{2.5}$$

Proof. When $\kappa \geq 2$, this is just Theorem 3 in [22], which follows directly from combining Theorem 5, Lemma 4 and Lemma 5 in [3]. When $\kappa = 1$, these estimates from [3] still hold. But one needs to be careful with convergence at various places of the argument. Here, we will give the appropriate modifications, with which the same arguments in [3] apply. Unless noted otherwise, the page and equation numbers below are from [3].

The technical issue for $\kappa = 1$ is the convergence of the Poincaré series $P_m(z; \Gamma, \nu)$ on page 794. This could be resolved with the Hecke trick (see Proposition 4 in [16] and remark on page 257 of [16]) and the expression of the Fourier coefficient given by equation (31) converges for $k = 3/2$, i.e. $\kappa = 1$. However, it needs not to be absolutely convergent, which is needed to justify the proof of Theorem 5 as there is an interchange of summations in equation (41).

To avoid this, one can modify the coefficient $p_n(m)$ in equation (31) into

$$p_n(m, \upsilon) := \delta_{mn} + 2\pi i^k \sum_{c>0} c^{-1} S(m, n; c) J_{3/2-1+\upsilon}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

with $\upsilon \in \mathbb{C}$, $\mathrm{Re}(\upsilon) \gg 0$ and define the series

$$\tilde{P}_m(z, \upsilon) := \sum_{n\geq 1} \left(\frac{n}{m}\right)^{(3/2-1)/2} p_n(m, \upsilon) q^n. \tag{2.6}$$

From the trivial estimate $|S(m, n, c)| \leq c$ and the asymptotic $J_\alpha(x) \sim x^\alpha$ for $\mathrm{Re}(\alpha) > 0$ and $|x|$ small, we know that $p_n(m, \upsilon)$ converges absolutely for $\mathrm{Re}(\upsilon) > 1/2$. One could analytically continue it to $\mathrm{Re}(\upsilon) > -1/2$ by comparing it to Selberg's Kloosterman zeta function

$$\sum_{c\geq 1} \frac{S(m, n; c)}{c^{2s}},$$

which converges absolutely for $\mathrm{Re}(s) > 1$ and can be meromorphically continued to $\mathrm{Re}(s) > 1/2$. This zeta function has at most finitely many simple poles, all of which lie on the real interval $(1/2, 1)$ and correspond exactly to the exceptional eigenvalues of the weight-k hyperbolic Laplacian [4]. As a consequence of Selberg's well-known 3/16 Theorem [24], the analytically continued $p_n(m, \upsilon)$ has no pole for $\upsilon \in (0, \frac{1}{2})$. At $\upsilon = 0$, $\tilde{P}_m(z, \upsilon)$ is the m^{th} Poincaré series of weight 3/2.

Now Theorem 5 could be stated with υ-variable added. The left hand side of equation (48) then becomes

$$\mathscr{L}_{rs}(f \otimes g_{m,\upsilon}, \omega) := \sum_{n=1}^{\infty} f_{rn} \overline{p_{sn}(m, \upsilon)} \omega(n), \tag{2.7}$$

where $g_{m,\upsilon}(z) := m^{\frac{1}{4}} \tilde{P}_m(z, \upsilon)$ and $\omega(x) := e^{-x/X}$. This expression converges for $\mathrm{Re}(\upsilon) \geq 0$ and defines a holomorphic function in $\overline{\upsilon}$. On the right hand side of equation (48), the only change is to replace $j_\omega\left(\frac{2\pi}{cq}\sqrt{\frac{mu}{t}}, \frac{2\pi}{cq}\sqrt{\frac{nw}{v}}\right)$ with $j_\omega\left(\frac{2\pi}{cq}\sqrt{\frac{mu}{t}}, \frac{2\pi}{cq}\sqrt{\frac{nw}{v}}, \overline{\upsilon}\right)$, where

$$j_\omega(A, B, \upsilon) := \sqrt{AB} \int \omega(x) J_{3/2-1+\upsilon}(2A\sqrt{x}) J_{3/2-1+\upsilon}(2B\sqrt{x}) dx$$

is the appropriate modification of equation (47).

To show that this modified right hand side also converges for $\mathrm{Re}(\upsilon) \geq 0$, one could write it into the sum of $\mathscr{L}^0_{rs}(f \otimes g_{m,\upsilon}, \omega) + \mathscr{L}^*_{rs}(f \otimes g_{m,\upsilon}, \omega)$ depending on the vanishing of the parameter h in equation (50). The same proof of Lemma 4 yields

$$\mathscr{L}^0_{rs}(f \otimes g_{m,\upsilon}, \omega) = \delta_{r,s} \frac{12 \cdot f_m \cdot (3/2 - 1 + \overline{\upsilon})^{-1}}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} X + O_{f,\epsilon}((rX)^{1/2+\epsilon}),$$

Similarly for $\mathscr{L}^*_{rs}(f \otimes g_{m,\upsilon}, \omega)$, replacing the estimate of $j_\omega(A, B)$ in the proof of Lemma 5 by

$$j_\omega(A, B, \upsilon) \ll X^{1/2} \min\left\{ \left(\frac{A}{B}\right)^2, \left(\frac{B}{A}\right)^2, ABX \right\}$$

yields the bound $\mathscr{L}^*_{rs}(f \otimes g_{m,\upsilon}, \omega) \ll_{f,\epsilon} (mrsX)^{1/2+\epsilon}$. Thus in the range $\mathrm{Re}(\upsilon) \geq 0$, the right hand side of the modified equation (48) converges and defines a holomorphic function in $\overline{\upsilon}$, which agrees with the value of $\mathscr{L}_{rs}(f \otimes g_{m,\upsilon}, \omega)$ at $\overline{\upsilon} = 0$ after analytic continuation in $\overline{\upsilon}$.

Writing f as a finite, linear combination of $\tilde{P}_m(z, 0)$'s produces equation (2.4). One could choose $B_{N,\kappa}$ to be the product of $4N$ and all primes less than or equal to the Sturm bound of $S_{\kappa+1}(4N, \chi)$. $\square$

2.3. **A Family of Weight $3/2$ Modular Forms.** Let $-D<0$ be an odd, fundamental discriminant. For a positive integer n and divisor $D' \mid D$, define the arithmetic function

$$\rho_{D,D'}(n) = \begin{cases} -L(\chi_{D/D'},0)L(\chi_{D'},0) & n=0, \\ \sum_{d\mid n}\chi_{D/D'}(d)\chi_{D'}(n/d) & n\geq 1. \end{cases} \tag{2.8}$$

As a consequence of the factorization of the Dedekind zeta function of $\mathbb{Q}(\sqrt{-D})$, we know that $\rho_{D,1}(n)$ is number of the integral ideals in $\mathbb{Q}(\sqrt{-D})$ with norm n for $n\geq 1$. The generating series of $\rho_{D,D'}(n)$ is an Eisenstein series

$$E_{D,D'}(z) := \sum_{n\geq 0}\rho_{D,D'}(n)q^n \in M_1(D,\chi_D), \tag{2.9}$$

which satisfies $U_\ell E_{D,D'} = \chi_{D/D'}(\ell)E_{D,D'}$ for any $\ell \mid D'$. Furthermore for any prime $\ell \mid D'$ and $n=\ell^r n'$ with $\gcd(n',\ell)=1$, we have the relationship

$$\rho_{D,D'}(n) = \left(\frac{D/\ell}{\ell}\right)^r\left(\frac{n'}{\ell}\right)\rho_{D,D'/\ell}(n). \tag{2.10}$$

Finally for any prime p satisfying $\chi_D(p)=-1$, we have

$$\rho_{D,D'}(pn) = \rho_{D,D'}(n/p) \tag{2.11}$$

for all $D' \mid D$ and $n\geq 0$.

Let $N\geq 1$ be as in Theorem 2.1. Define the function $\Phi_{N,D}(z)$ by

$$\Phi_{N,D}(z) := \mathrm{Tr}^{4ND}_{4N}\left(E_{D,1}(4Nz)\theta(Dz)\right), \tag{2.12}$$

where $\theta(z) := \sum_{k\in\mathbb{Z}} q^{k^2}$ is the theta series of weight $1/2$ and Tr is the trace operator. The following result tells us the Fourier coefficients of $\Phi_{N,D}(z)$.

Proposition 2.7. *If $D\in\mathcal{D}_N$, then the n^{th} Fourier coefficient of $\Phi_{N,D}(z)\in M^+_{3/2}(4N)$, denoted by $b(n,N,D)$, can be written as*

$$b(n,N,D) = \sum_{D'\mid D}\sum_{k\in\mathbb{Z}}\chi_{D'}(-N)\rho_{D,D'}\left(\frac{Dn-(Dk/D')^2}{4N}\right). \tag{2.13}$$

Proof. The calculation is very similar to the level $N=1$ case in the appendix of [12]. Using similar coset decomposition as on page 196 loc. cit., we can show that

$$\Phi_{N,D}(z) = \sum_{D'\mid D}\chi_{D/D'}(D')\chi_{D'}(-N)U_{D'}\left(E_{D,D'}(4Nz)\theta((D/D')z)\right).$$

Since $\rho_{D,D'}(D'm) = \chi_{D/D'}(D')\rho_{D,D'}(m)$ for all $m\geq 0$, we obtain equation (2.13). □

Corollary 2.8. *Suppose $p\mid N$ is a prime satisfying $\chi_D(p)=-1$. Then for all $n\geq 0$,*

$$b(p^2n,N,D) = b(n,N,D). \tag{2.14}$$

Proof. Notice that the sum in $b(p^2 n, N, D)$ must be over those $k \in \mathbb{Z}$ divisible by p. Using equation (2.11), we can eliminate the p^2 factor in $\rho_{D,D'}$ and arrive at $b(n, N, D)$. □

Later, we will calculate the Petersson inner product between $\Phi_{N,D}$ and a newform in $S^+_{3/2}(4N)$. Let $\mathrm{pr}^+ : M_{k+1/2}(4N) \to M^+_{k+1/2}(4N)$ be Kohnen's projection operator defined by

$$\mathrm{pr}^+ g := \tfrac{1-(-1)^k i}{6} \mathrm{Tr}^{16N}_{4N} V(g) + \frac{g}{3}, \qquad V(g)(z) := g\left(z + \tfrac{1}{4}\right).$$

For convenience, we will record a lemma, which will be used later.

Lemma 2.9. *For $e \mid 4N$, define*

$$\Phi_{N,D,e}(z) := \mathrm{Tr}^{4ND}_{4N}\left(E_{D,1}\left(\frac{4Nz}{e}\right)\theta(Dz)\right) \in M_{3/2}(4N).$$

Then $\Phi_{N,D,e} \in M_{3/2}(4N)$ is an old form when $\gcd(e, N) > 1$ and

$$\mathrm{pr}^+ \Phi_{N,D,2}(z) = \tfrac{2}{3}\left(\chi_D(2) + 1\right)\Phi_{N,D}(z). \tag{2.15}$$

Proof. The proof is the level-N version of the computation on page 195 of [12] for D odd and square-free. Using the fact $V \mathrm{Tr}^{4ND}_{4N} g = \mathrm{Tr}^{16ND}_{16N} g$, we have

$$\mathrm{pr}^+ \Phi^*_{N,D}(z) = \mathrm{Tr}^{4ND}_{4N}\left(\tfrac{1-(-1)^k i}{6}\mathrm{Tr}^{16ND}_{4ND} V\left(E_D(2Nz)\theta(Dz)\right) + \frac{E_D(2Nz)\theta(Dz)}{3}\right),$$

which could be calculated as on page 195 of [12]. □

2.4. **Inner Product Calculations.** In this section, we will express the Petersson inner product between $\Phi_{N,D}(z)$ and a newform $g \in S^+_{3/2}(4N)$ in terms of the special values of L-series associated to $\mathscr{S}(g) \in S_2(N)$. Specifically, we have the following result. The calculations are routine and follow from those on pages 270-272 in [7].

Proposition 2.10. *If N is an odd, square-free integer and $D \in \mathcal{D}_N$, then*

$$\langle g, \Phi_{N,D}\rangle = \frac{3}{4\pi} a_g(D) L(G, 1), \tag{2.16}$$

where $g(z) = \sum_{n \ge 1} a_g(n) q^n \in S^+_{3/2}(4N)$ is a newform and $G := \mathscr{S}(g) \in S_2(N)$ its Shimura lift.

Proof. For $M \in \mathbb{Z}$, let $\mathcal{E}_{M,D}(z, s)$ be the real-analytic Eisenstein series of weight one defined for $\mathrm{Re}(s) > \frac{1}{2}$ by

$$\mathcal{E}_{M,D}(z, s) := \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ MD \mid c \\ \gcd(MD, d) = 1}} \frac{\chi_D(d)}{cz + d} \cdot \frac{y^s}{|cz + d|^{2s}}. \tag{2.17}$$

It has analytic continuation to $s \in \mathbb{C}$. The value at $s = 0$, which was computed by Hecke (see equations (26) and (26) on page 475 of [9]), is a holomorphic Eisenstein series of weight one. When $\gcd(M, D) = 1$, it is given by

$$\mathcal{E}_{M,D}(z, 0) = \frac{2\pi}{\sqrt{D}} \sum_{e \mid M} \frac{\mu(e)\chi_D(e)}{e} E_D\left(\frac{Mz}{e}\right). \tag{2.18}$$

We begin by manipulating the expression $\iint_{\Gamma_\infty \backslash \mathcal{H}} g(z)\overline{\theta(Dz)} y^{s+2} \frac{dxdy}{y^2}$. On the one hand, direct calculation and equation (2.2) show that

$$\iint_{\Gamma_\infty \backslash \mathcal{H}} g(z)\overline{\theta(Dz)} y^{s+3/2} \frac{dxdy}{y^2} = \frac{2\Gamma(s+\frac{1}{2})a(D)}{(4\pi D)^{s+1/2}} \cdot \frac{L(G, 2s+1)}{L(\chi_D \mathbb{1}_N, 2s+1)}.$$

On the other hand, this expression can be written as

$$\iint_{\Gamma_\infty \backslash \mathcal{H}} g(z)\overline{\theta(Dz)} y^{s+3/2} \frac{dxdy}{y^2} = \iint_{\mathcal{F}_{4ND}} g(z)\overline{\theta(Dz)} y^{3/2} \sum_{\gamma = \pm\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(4ND)} \overline{\chi_D(d) y^s \mid_1 \gamma} \frac{dxdy}{y^2},$$

where $\mathcal{F}_{4ND}$ be the fundamental domain of $\Gamma_0(4ND)\backslash\mathcal{H}$. Simple manipulations yield that

$$\sum_{\gamma = \pm\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(4ND)} \chi_D(d) y^s \mid_1 \gamma = \frac{\mathcal{E}_{4N,D}(z,s)}{L(\chi_D \mathbb{1}_{4N}, 2s+1)}.$$

Putting these together gives us

$$\iint_{\mathcal{F}_{4ND}} g(z)\overline{\theta(Dz)\mathcal{E}_{4N,D}(z,s)} y^{3/2} \frac{dxdy}{y^2} = \frac{2\Gamma(s+\frac{1}{2})a(D)}{(4\pi D)^{s+1/2}} \cdot \left(1 - \frac{\chi_D(2)}{2^{2s+1}}\right) L(G, 2s+1).$$

At $s = 0$, we obtain

$$\sum_{e \mid 4N} \frac{\mu(e)\chi_D(e)}{e} \iint_{\mathcal{F}_{4N}} g(z)\overline{\Phi_{N,D,e}(z)} y^{3/2} \frac{dxdy}{y^2} = \frac{a(D)}{2\pi} \cdot \left(1 - \frac{\chi_D(2)}{2}\right) L(G, 1).$$

By Lemma 2.9, $\Phi_{N,D,e}$ is an old form when $\gcd(e, N) > 1$. In that case, the integral vanishes since $g(z)$ is a newform. When $e = 4$, the summand also vanishes since $\mu(4) = 0$. Thus, the equation above becomes

$$\langle g, \Phi_{N,D}\rangle - \frac{\chi_D(2)}{2}\langle g, \Phi_{N,D,2}\rangle = \frac{a(D)}{2\pi} \cdot \left(1 - \frac{\chi_D(2)}{2}\right) L(G, 1).$$

Since $g \in S^+_{3/2}(4N)$ is in Kohnen's plus space and pr^+ is self-adjoint with respect to the Petersson inner product, we have $\langle g, \Phi_{N,D,2}\rangle = \langle \mathrm{pr}^+ g, \mathrm{pr}^+ \Phi_{N,D,2}\rangle = \langle g, \mathrm{pr}^+ \Phi_{N,D,2}\rangle$. By Lemma 2.9, we obtain equation (2.16). □

3. Hilbert Eisenstein Series

In this section, we will establish a connection between the restriction of Hilbert Eisenstein series studied in [30] and the half-integral weight modular forms $\Phi_{N,D}$ studied in §2.3. The following result is Theorem 5.1 in [30].

Theorem 3.1 (Theorem 5.1 in [30])**.** *Let N be a square-free positive integer. Let $-D_1, -D_2 < 0$ be two fundamental discriminants of imaginary quadratic fields, $F = \mathbb{Q}(\sqrt{D_1 D_2})$ a real quadratic field and $K = \mathbb{Q}(\sqrt{-D_1}, \sqrt{-D_2})$ a CM extension. Assume that*

$$\gcd(D_1, D_2) = 1, \text{ and } \left(\frac{-D_1}{p}\right) = \left(\frac{-D_2}{p}\right) = -1 \text{ forevery } p \mid N.$$

Let $\mathcal{N}$ be an integral ideal of F with an odd number of prime factors in F such that $\mathcal{N} \cap \mathbb{Z} = N\mathbb{Z}$. Then

$$f_{-D_1,-D_2,\mathcal{N}}(z) := L(0, \chi_{K/F}) + \sum_{m=1}^{\infty} A(m, D_1, D_2, \mathcal{N}) q^m \tag{3.1}$$

is a holomorphic (elliptic) modular form of weight 2 for $\Gamma_0(N)$ with the trivial Nebentypus character, where

$$A(m, D_1, D_2, \mathcal{N}) := 2 \sum_{t = \frac{k + m\sqrt{D_1 D_2}}{2} \in \mathcal{N}, |k| < m\sqrt{D_1 D_2}} \rho_{K/F}(t\mathcal{N}^{-1}). \tag{3.2}$$

Remark 3.2. Since $\rho_{K/F}(\mathfrak{a}) = \rho_{K/F}(\mathfrak{a}')$ for any ideal $\mathfrak{a}$ and its conjugate $\mathfrak{a}'$ in F, we have $A(m, D_1, D_2, \mathcal{N}) = A(m, D_1, D_2, \mathcal{N}')$ directly from the definition.

As remarked in [30], if there is an integer d and integral ideal $\mathcal{N}_1$ such that $\mathcal{N} = \mathcal{N}_1 d$, then $f_{-D_1,-D_2,\mathcal{N}}(z) = f_{-D_1,-D_2,\mathcal{N}_1}(dz)$ is an old form in $M_2(N)$. Thus, it is natural to just consider those $\mathcal{N}$ such that $\mathrm{Nm}\mathcal{N} = N$. In this case, N satisfies the following condition

$$N \text{ is odd, square-free and } \omega(N) \text{ is odd.} \tag{3.3}$$

From its definition, it is clear that $A(m, D_1, D_2, \mathcal{N}') = A(m, D_1, D_2, \mathcal{N})$. As a consequence of the following proposition, we know that such $f_{-D_1,-D_2,\mathcal{N}}$ are in $M_2^+(N)$.

Proposition 3.3. *Let $p \mid N$ be a prime such that $p = \mathfrak{p}\mathfrak{p}'$ in F and $\mathfrak{p} \mid \mathcal{N}$, $\mathfrak{p}' \nmid \mathcal{N}$. Then*

$$U_p(f_{-D_1,-D_2,\mathcal{N}}) = f_{-D_1,-D_2,\mathcal{N}\mathfrak{p}'/\mathfrak{p}}.$$

In particular, $f_{-D_1,-D_2,\mathcal{N}} \in M_2^+(N)$.

Proof. It suffices to show that the coefficients $A(m, D_1, D_2, \mathcal{N})$ satisfy

$$A(pm, D_1, D_2, \mathcal{N}) = A(m, D_1, D_2, \mathcal{N}\mathfrak{p}'/\mathfrak{p}).$$

In the summation defining $A(pm, D_1, D_2, \mathcal{N})$ in equation (3.2), the conditions $\frac{k + pm\sqrt{D_1 D_2}}{2} \in \mathcal{N}$ and $|k| < pm\sqrt{D_1 D_2}$ implies that $k = pk'$ for some $k' \in \mathbb{Z}$ satisfying $|k'| < m\sqrt{D_1 D_2}$.

Setting $t' := t/p$, we have $t' \in \mathcal{N}/\mathfrak{p}$. If $t' \notin \mathfrak{p}'$, then $\mathrm{ord}_{\mathfrak{p}'}(t\mathcal{N}^{-1}) = 1$. This implies $\rho_{K/F}(t\mathcal{N}^{-1}) = 0$ since $\mathfrak{p}'$ is inert in K. Thus, we can write

$$\begin{aligned} A(pm, D_1, D_2, \mathcal{N}) &= 2 \sum_{t' = \frac{k' + m\sqrt{D_1 D_2}}{2} \in \mathcal{N}/\mathfrak{p}, |k'| < m\sqrt{D_1 D_2}} \rho_{K/F}(pt'\mathcal{N}^{-1}) \\ &= 2 \sum_{t' = \frac{k' + m\sqrt{D_1 D_2}}{2} \in \mathcal{N}\mathfrak{p}'/\mathfrak{p}, |k'| < m\sqrt{D_1 D_2}} \rho_{K/F}((\mathfrak{p}')^2 t' (\mathcal{N}\mathfrak{p}'/\mathfrak{p})^{-1}) \\ &= A(m, D_1, D_2, \mathcal{N}\mathfrak{p}'/\mathfrak{p}), \end{aligned}$$

where the last step follows from $\rho_{K/F}((\mathfrak{p}')^2\mathfrak{a}) = \rho_{K/F}(\mathfrak{a})$ for all ideal $\mathfrak{a}$ in F. □

Define the quantity $A_N(m, D_1, D_2)$ by

$$A_N(m, D_1, D_2) := \frac{1}{2} \sum_{\substack{\mathcal{N} \subset \mathcal{O}_F \\ \mathrm{Nm}\mathcal{N} = N}} A(m, D_1, D_2, \mathcal{N}). \tag{3.4}$$

It is the m^{th} Fourier coefficient of the weight 2 modular form $F_{D_1, D_2, N}$. From definition, it is easy to see that $F_{D_1, D_2, N} = F_{D_2, D_1, N}$. So we suppose that D_1 is odd. Using Lemma 4.1, we can relate $F_{D_1, D_2, N}$ to the Shimura lift of Φ_{N, D_1} by giving $A_N(m, D_1, D_2)$ a different expression.

Proposition 3.4. *Suppose D_1 is odd and N satisfies* (3.3)*. Then under the same conditions as in Theorem 3.1, we have*

$$F_{D_1, D_2, N}(z) = \mathscr{S}_{D_2}(\Phi_{N, D_1}(z)). \tag{3.5}$$

Proof. By definition of $\mathscr{S}_{D_2}$, it suffices to show that for all $m \geq 1$

$$A_N(m, D_1, D_2) = \sum_{d \mid m} \mathbb{1}_N(d) \left(\tfrac{-D_2}{d}\right) b((m/d)^2 D_2, N, D_1), \tag{3.6}$$

where $b(n, N, D_1)$ is defined in equation (2.13). For any prime $p \mid N$, the set $\{\mathcal{N} \subset \mathcal{O}_F : \mathrm{Nm}\mathcal{N} = N\}$ is permuted under the map $\mathcal{N} \mapsto \frac{\mathcal{N}p}{(\mathcal{N}, p)}$. By remark 3.2 and Proposition 3.3, we have

$$A_N(pm, D_1, D_2) = \tfrac{1}{2} \sum_{\substack{\mathcal{N} \subset \mathcal{O}_F \\ \mathrm{Nm}\mathcal{N} = N}} A(pm, D_1, D_2, \mathcal{N}) = A_N(m, D_1, D_2).$$

By Corollary 2.8, the right hand side of equation (3.6) is unchanged when m is replaced by pm. Thus, it suffices to prove equation (3.6) when $\gcd(N, m) = 1$.

Substituting equation (4.1) into equation (3.2) yields

(3.7)
$$A_N(m, D_1, D_2) = \sum_{\substack{\mathcal{N} \subset \mathcal{O}_F \\ \mathrm{Nm}\mathcal{N} = N \\ t = \frac{k + m\sqrt{D_1 D_2}}{2} \in \mathcal{N}}} \sum_{d \mid t\mathcal{N}^{-1}} \left(\tfrac{-D_2}{d}\right) \rho_1\left(\frac{m^2 D_1 D_2 - k^2}{4Nd^2}\right) \delta_{D_1}\left(\frac{m^2 D_1 D_2 - k^2}{4Nd^2}, D_2\right).$$

The condition $|k| < m\sqrt{D_1 D_2}$ is dropped from the first summation since $\rho_1(n) = 0$ whenever $n < 0$. Since $(N, m) = 1$, we have for any $d \in \mathbb{N}$

$$d \mid t \cdot \mathcal{N}^{-1} \Rightarrow (d \mid t \& d \mid t') \Rightarrow d \mid \gcd(k, m) \Rightarrow d \nmid N \Rightarrow d \mid t \cdot \mathcal{N}^{-1}.$$

Furthermore, exactly one of t and t' will appear in the summation if and only if $\frac{m^2 D_1 D_2 - k^2}{4N} \in \mathbb{Z}$. So we can rewrite equation (3.7) into the following form.

$$A_N(m, D_1, D_2) = \sum_{d \mid m} \left(\tfrac{-D_2}{d}\right) \sum_{k' \in \mathbb{Z}} \rho_1\left(\frac{(m/d)^2 D_1 D_2 - (k')^2}{4N}\right) \delta_{D_1}\left(\frac{(m/d)^2 D_1 D_2 - (k')^2}{4N}, D_2\right).$$

For a fixed $d \mid m$ and $k' \in \mathbb{Z}$, suppose there exists a prime $\ell \mid D_1$ such that

$$\left(\tfrac{-D_2}{\ell}\right) = -1 \text{ and } \mathrm{ord}_\ell\left(\tfrac{(m/d)^2 D_1 D_2 - (k')^2}{4N}\right) \text{ is odd.} \tag{3.8}$$

Write $m/d = \ell^a \cdot u, k' = \ell^b \cdot v$ such that $\ell \nmid uv$. Then $\mathrm{ord}_\ell((m/d)^2 D_1 D_2) = 2a + 1$ and condition (3.8) implies that $2a + 1 < 2b$. The quantity $\rho_1\left(\frac{(m/d)^2 D_1 D_2 - (k')^2}{4N}\right)$ becomes

$$\rho_1\left(\tfrac{(m/d)^2 D_1 D_2 - (k')^2}{4N}\right) = \rho_1\left(\tfrac{\ell^{2a+1} u^2 (D_1/\ell) D_2 - \ell^{2b} v^2}{4N}\right) = \rho_1\left(\tfrac{u^2 (D_1/\ell) D_2 - \ell^{2b-(2a+1)} v^2}{4N}\right) = 0.$$

The last equality follows from

$$\left(\frac{(u^2 (D_1/\ell) D_2 - \ell^{2b-(2a+1)} v^2)/(4N)}{D_1}\right) = -1.$$

Thus in the sum over $d \mid m$ and $k' \in \mathbb{Z}$, it suffices to consider those $(m/d, k')$ violating condition (3.8), in which case we have

$$\delta_{D_1}\left(\tfrac{(m/d)^2 D - (k')^2}{4N}, D_2\right) = \delta_{D_1}((k')^2, D_2) = 2^{\omega(\gcd(k', D_1))}.$$

Thus when $\gcd(N, m) = 1$, the quantity $A_N(m, D_1, D_2)$ can be rewritten as

$$A_N(m, D_1, D_2) = \sum_{d \mid m} \left(\tfrac{-D_2}{d}\right) \sum_{k' \in \mathbb{Z}} \rho_1\left(\frac{D_1\left((m/d)^2 D_2\right) - (k')^2}{4N}\right) 2^{\omega(\gcd(k', D_1))}. \tag{3.9}$$

On the other hand, let $\ell \mid \gcd(k', D_1)$ be a prime. If $\mathrm{ord}_\ell\left(\frac{(m/d)^2 D_1 D_2 - (k')^2}{4N}\right)$ is even, then equation (2.10) implies that

$$\rho_1\left(\frac{D_1 D_2 (m/d)^2 - (k')^2}{4N}\right) = \left(\frac{-N}{\ell}\right)\rho_{D_1,\ell}\left(\frac{D_1 D_2 (m/d)^2 - (k')^2}{4N}\right).$$

The same holds if $\mathrm{ord}_\ell\left(\frac{(m/d)^2 D_1 D_2 - (k')^2}{4N}\right)$ is odd and $\left(\frac{-D_2}{\ell}\right) = 1$. So for pairs of $(m/d, k')$ violating condition (3.8), we have

$$\rho_1\left(\frac{D_1 D_2 (m/d)^2 - (k')^2}{4N}\right) 2^{\omega(\gcd(k',D_1))} = \sum_{D' \mid \gcd(k',D_1)} \left(\frac{-N}{D'}\right)\rho_{D_1,D'}\left(\frac{D_1 D_2 (m/d)^2 - (k')^2}{4N}\right).$$

The same argument above shows that the right hand side vanishes if $(m/d, k')$ satisfies condition (3.8). Thus, the equality above holds for all $(m/d, k')$. After summing over $k' \in \mathbb{Z}$, making a change of variable and switching the summations, we arrive at

$$\sum_{k' \in \mathbb{Z}} \rho_1\left(\frac{D_1\left((m/d)^2 D_2\right) - (k')^2}{4N}\right) 2^{\omega(\gcd(k',D_1))} = b(D_2 (m/d)^2, N, D_1).$$

Substituting this into equation (3.9) gives us equation (3.6) for $(m, N) = 1$, which finishes the proof. □

Combining the proposition above with the inner product calculation in Proposition 2.10, we can prove Theorem 1.5.

Proof of Theorem 1.5. By Theorem 2.1, $\langle G, \mathscr{S}_D(g)\rangle = \overline{a_g(D)}\langle G, G\rangle$ for any fundamental discriminant $-D < 0$, Thus, for any $f \in M^+_{3/2}(4N)$, we have

$$\langle G, \mathscr{S}_D(f)\rangle = \frac{\langle G, G\rangle}{\langle g, g\rangle}\overline{a_g(D)}\langle g, f\rangle.$$

Substituting in $D = D_2$, $f = \Phi_{N,D_1}$ to the equation above and applying Propositions 2.10 and 3.4 yields equation (1.3). □

4. Counting Lemma

Lemma 4.1. *Let $d_1, d_2 < 0$ be relatively prime fundamental discriminants such that d_1 is odd. Define the number fields*

$$K := \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}),\ F := \mathbb{Q}(\sqrt{d_1 d_2}),\ K_1 := \mathbb{Q}(\sqrt{d_1}),\ K_2 := \mathbb{Q}(\sqrt{d_2})$$

and the ideal counting functions

$$\rho_{K/F}(\mathfrak{a}) := \{\mathfrak{A} \subset \mathcal{O}_K : \mathrm{Nm}_{K/F}(\mathfrak{A}) = \mathfrak{a} \subset \mathcal{O}_F\},$$
$$\rho_j(m) := \{\mathfrak{a} \subset \mathcal{O}_{K_j} : \mathrm{Nm}(\mathfrak{a}) = m\},\ j = 1, 2.$$

Then we have the following relationship

$$\rho_{K/F}(\mathfrak{a}) = \sum_{d \mid \mathfrak{a},\, d \in \mathbb{N}} \left(\tfrac{d_2}{d}\right) \rho_1\left(\mathrm{Nm}(\mathfrak{a}/d)\right) \delta_{d_1}(\mathrm{Nm}(\mathfrak{a}/d), d_2), \tag{4.1}$$

where

$$\delta_{d_1}(m, d_2) := \prod_{\ell \mid d_1 \text{ prime}} \delta_\ell(m, d_2),$$

$$\delta_\ell(m, d_2) := \begin{cases} 1 & \ell \nmid m, \\ 1 + \left(\frac{d_2}{\ell}\right) & \mathrm{ord}_\ell(m) \text{ odd}, \\ 2 & \mathrm{ord}_\ell(m) \geq 2 \text{ even}. \end{cases}$$

Remark 4.2. If d_2 is odd, then equation (4.1) holds after switching the indices 1 and 2.

Proof. Let ℓ be a rational prime and $\mathfrak{l} \subset \mathcal{O}_F, \mathfrak{L} \subset \mathcal{O}_K$ primes lying above it. Denote the conjugate of $\mathfrak{l}$ in $\mathcal{O}_F$ by $\mathfrak{l}'$, which could be the same as $\mathfrak{l}$. Since both sides of equation (4.1) are multiplicative, it suffices to prove it when $\mathfrak{a} = \mathfrak{l}^a(\mathfrak{l}')^b$ for integers $a, b \geq 0$. Fix an isomorphism between $\mathrm{Gal}(K/\mathbb{Q})$ and $(\mathbb{Z}/2\mathbb{Z})^2$ such that K_1 is fixed by $(1,0)$, K_2 is fixed by $(0,1)$ and F is fixed by $(1,1)$.

The proof is based on analyzing individual cases depending on the splitting behavior of the prime ℓ in K. If ℓ is unramified in K, i.e. $\ell \nmid d_1 d_2$, then there are four possibilities for the decomposition group $G_\mathfrak{L}$ of $\mathfrak{L}$

- **(U1):** $G_\mathfrak{L}$ is trivial.
- **(U2):** $G_\mathfrak{L}$ is generated by $(1,0)$.
- **(U3):** $G_\mathfrak{L}$ is generated by $(0,1)$.
- **(U3):** $G_\mathfrak{L}$ is generated by $(1,1)$.

In case (U1), the prime ideal ℓ splits completely in K. In particular, $\mathfrak{l}$ differs from $\mathfrak{l}'$ and both of them split in K. So the LHS of equation (4.1) equals to $(a+1)(b+1)$ when $\mathfrak{a} = \mathfrak{l}^a(\mathfrak{l}')^b$. On the other hand, set $s := \max(a,b), t := \min(a,b)$ and we have

$$\begin{aligned} \text{RHS of (4.1)} &= \sum_{j=0}^{\min(a,b)} \rho_1(\ell^{a+b-2j}) \cdot \left(\tfrac{d_2}{\ell^j}\right) = \sum_{j=0}^{\min(a,b)} a + b - 2j + 1 = \sum_{j=0}^{t} s + t - 2j + 1 \\ &= (s+1)(t+1) = (a+1)(b+1). \end{aligned}$$

In case (U2), the prime ℓ splits in K_1 and is inert in F. Thus, $\left(\frac{d_2}{\ell}\right) = -1$. Also, $\mathfrak{l} = \mathfrak{l}' = \ell\mathcal{O}_F$ splits in K. So it suffices to consider $\mathfrak{a} = \mathfrak{l}^a$, in which case the left hand side is $(a+1)$ and

$$\text{RHS of (4.1)} = \sum_{j=0}^{a} \rho_1(\ell^{2(a-j)})(-1)^j = \sum_{j=0}^{a}(2(a-j)+1)(-1)^j = a+1.$$

In case (U3), the prime ℓ splits in K_2 and is inert in F. Then $\left(\frac{d_2}{\ell}\right) = 1$ and $\rho_1(\ell^r) = 1$ for any integer $r \geq 0$ and both sides of equation (4.1) are $a+1$ when $\mathfrak{a} = \mathfrak{l}^a = (\mathfrak{l}')^a = \ell^a \mathcal{O}_F$.

Case (U4) is similar to case (U1), in which $\mathfrak{l}$ and $\mathfrak{l}'$ are distinct. But ℓ is inert in both K_1 and K_2, hence $\mathfrak{l}$ and $\mathfrak{l}'$ are both inert in K. When $\mathfrak{a} = \mathfrak{l}^a(\mathfrak{l}')^b$, the LHS of equation (4.1) is 1 if $a \equiv b \equiv 0 \pmod 2$ and 0 otherwise. For the RHS, we have

$$\text{RHS of (4.1)} = \sum_{j=0}^{\min(a,b)} \rho_1(\ell^{a+b-2j}) \left(\tfrac{d_2}{\ell^j}\right).$$

If $a+b \equiv 1 \pmod 2$, then $\rho_1(\ell^{a+b-2j}) = 0$ for all $0 \leq j \leq \min(a,b)$ and the RHS vanishes. Otherwise, we have

$$\text{RHS of (4.1)} = \sum_{j=0}^{\min(a,b)} (-1)^j = \begin{cases} 1 & a \equiv b \equiv 0 \pmod 2, \\ 0 & a \equiv b \equiv 1 \pmod 2). \end{cases}$$

When ℓ is ramified in K, we have $\ell \mid d_1 d_2$ and $\ell\mathcal{O}_F = \mathfrak{l}^2, \mathfrak{l} = \mathfrak{l}'$. So it enough to verify equation (4.1) for $\mathfrak{a} = \mathfrak{l}^a$. There are now two cases

(R1): $\ell \nmid d_1$.
(R2): $\ell \nmid d_2$.

In case (R1), the right hand side of equation (4.1) simply becomes

$$\text{RHS of (4.1)} = \rho_1(\ell^a).$$

This is $a+1$ if ℓ splits in K_1 and agrees with $\rho_{K/F}(\mathfrak{l}^a)$ since $\mathfrak{l}$ splits in K. On the other hand, if ℓ is inert in K_1, it becomes

$$\rho_1(\ell^a) = \begin{cases} 1 & a \equiv 0 \pmod 2, \\ 0 & a \equiv 1 \pmod 2, \end{cases}$$

which agrees with $\rho_{K/F}(\mathfrak{l}^a)$ since $\mathfrak{l}$ is inert in K.

Finally in case (R2), $\rho_1(\ell^k) = 1$ for all $k \in \mathbb{N}$. The right hand side of equation (4.1) becomes

$$\text{RHS of (4.1)} = \sum_{j=0}^{\lfloor a/2 \rfloor} \left(\tfrac{d_2}{\ell^j}\right) \delta_\ell(\ell^{a-2j}, d_2) = \begin{cases} a+1 & \left(\tfrac{d_2}{\ell}\right) = 1, \\ 1 & \left(\tfrac{d_2}{\ell}\right) = -1 \text{ and } a \equiv 0 \pmod 2, \\ 0 & \left(\tfrac{d_2}{\ell}\right) = -1 \text{ and } a \equiv 1 \pmod 2, \end{cases}$$

which agrees with $\rho_{K/F}(\mathfrak{l}^a)$. $\square$

5. Main Theorem and Its Proof

Let N be an odd, square-free integer and $S_2^+(N) := M_2^+(N) \cap S_2(N)$. Write $S_2(N) = S_2^{\mathrm{new}}(N) \oplus S_2^{\mathrm{old}}(N)$, where $S_2^{\mathrm{old}}(N)$ and $S_2^{\mathrm{new}}(N)$ are the subspaces spanned by oldforms and newforms respectively. As a consequence of Deligne's bound of Fourier coefficients of integral weight eigenforms, the absolute values of the U_N-eigenvalues of eigenforms in $S_2^{\mathrm{old}}(N)$ are greater than 1. So $S_2^{\mathrm{new}}(N)$ is spanned by the eigenforms whose U_N-eigenvalues are ± 1 and we can write

$$S_2^{\mathrm{new}}(N) = S_2^+(N) \oplus S_2^-(N),$$

where $S_2^-(N)$ consists of cusp forms whose eigenvalue under U_N is -1.

Suppose we write $N = \prod_j p_j$ as the product of distinct primes. Given $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{\omega(N)})$ with $\varepsilon_j \in \{1, -1\}$, we define the subspace $S_2^{\varepsilon}(N) \subset S_2^{\mathrm{new}}(N)$ to be

$$S_2^{\varepsilon}(N) := \left\{ f \in S_2^{\mathrm{new}}(N) : U_{p_j} f = \varepsilon_j f \text{ for all } 1 \le j \le \omega(N) \right\}.$$

Thus, we can write

$$S_2^{\pm}(N) = \bigoplus_{\substack{\varepsilon \in \{1,-1\}^{\omega(N)} \\ \prod_j \varepsilon_j = \pm 1.}} S_2^{\varepsilon}(N).$$

Given an eigenform G in $S_2^+(N)$ (resp. $S_2^-(N)$), the sign of the functional equation of the complete L-series associated to G is $+$ (resp. $-$), and $L(G, s)$ vanishes to even (resp. odd) order at $s = 1$. Let $S_2^{+,0}(N) \subset S_2^+(N)$ be the subspace spanned by newforms $G \in S_2(N)$ such that $L(G, 1) \neq 0$.

The subspace of Eisenstein series in $M_2^+(N)$ is one dimensional and spanned by

$$E_{2,N}(z) := \frac{\varphi(N)}{24} + \sum_{n \ge 1} \left(\sum_{d \mid n, \gcd(d,N)=1} d \right) q^n, \tag{5.1}$$

where $\varphi(N)$ is the Euler totient function. Denote

$$M_2^{(1,\dots,1),0}(N) := S_2^{(1,\dots,1),0}(N) \oplus \mathbb{C} E_{2,N}, \tag{5.2}$$

$$M_2^{+,0}(N) := S_2^+(N) \oplus \mathbb{C} E_{2,N}. \tag{5.3}$$

When N is prime, the space $M_2^{+,0}(N)$ equals to $M_2^{(1,\dots,1),0}(N)$ and agrees with the one in Theorem 1.2. The following theorem would then imply Theorem 1.2.

Theorem 5.1. *Suppose $N \in \mathbb{N}$ satisfies* (3.3). *Then $M_2^{(1,\dots,1),0}(N)$ is the span of $E_{2,N}$ and*

$$\left\{ F_{D_1,D_2,N} \,\middle|\, (-D_1, -D_2) \in \mathcal{D}_N^{2,*} \right\}. \tag{5.4}$$

Proof. By Proposition 3.3, we have $F_{D_1,D_2,N} \in M_2^{(1,\dots,1)}(N)$. Let $\{G_j\} \subset S_2(N)$ be a basis of newforms. Then

$$F_{D_1,D_2,N} = 2^{\omega(N)-1} L(0,\chi_{K/F}) \frac{24}{\varphi(N)} E_{2,N} + \sum_j \frac{\langle G_j, F_{D_1,D_2,N}\rangle}{\langle G_j, G_j\rangle} G_j.$$

If $G_j \notin S_2^{(1,\dots,1)}(N)$, then $\langle G_j, F_{D_1,D_2,N}\rangle = 0$ since the U-operator is self-adjoint with respect to the Petersson inner product. If $G_j \notin S_2^{+,0}(N)$, then Theorem 1.5 implies that $\langle G_j, F_{D_1,D_2,N}\rangle = 0$. Thus, $F_{D_1,D_2,N} \in M_2^{(1,\dots,1),0}(N)$.

Assume there exists nonzero $G \in M_2^{(1,\dots,1),0}(N)$ not in the span of $E_{2,N}$ and the $F_{D_1,D_2,N}$'s. By the decomposition (5.3), we can suppose that $G \in S_2^{(1,\dots,1),0}(N)$ is orthogonal to $F_{D_1,D_2,N}$ for all $(-D_1,-D_2) \in \mathcal{D}_N^{2,*}$. Let $\{F_j(z)\} \subset S_2^{(1,\dots,1),0}(N)$ be the basis of newforms and $\{f_j(z) = \sum_{n\geq 1} a_j(n)q^n\} \subset S_{3/2}(4N)$ the half-integral weight newforms under the Shimura correspondence. Write

$$G = \sum_j \alpha_j F_j$$

with $\alpha_j \in \mathbb{C}$. Since $F \neq 0$, we can index the F_j's such that $\alpha_1 \neq 0$. By Propositions 2.2 and 2.4, we can choose $D_2 \in \mathcal{D}_N$ such that $a_1(D_2) \neq 0$.

Define a cusp form $f \in S_{3/2}(4N)$ by

$$f := \sum_j \beta_j f_j = \sum_{n\geq 1} a_f(n) q^n, \qquad \beta_j := \alpha_j \frac{\langle F_j, F_j\rangle}{\langle f_j, f_j\rangle} \overline{a_j(D_2)} L(F_j,1). \tag{5.5}$$

By the choice of D_2, $\beta_1 \neq 0$ and f is nonzero. Thus, Propositions 2.2 and 2.4 imply that there exists $-D_1 \in \mathcal{D}_N$ relatively prime to D_2 such that $a_f(D_1) \neq 0$. On the other hand, Theorem 1.5 tells us that

$$a_f(D_1) = \sum_j \alpha_j \frac{\langle F_j, F_j\rangle}{\langle f_j, f_j\rangle} \overline{a_j(D_2)} L(F_j,1) a_j(D_1) = \sum_j \alpha_j \langle F_j, F_{D_1,D_2,N}\rangle = \langle G, F_{D_1,D_2,N}\rangle = 0.$$

Thus, there is no such G and $M_2^{(1,\dots,1),0}(N)$ is spanned by $E_{2,N}$ and the modular forms in (5.4). □

Finally, we give the following refinement of Conjecture 1.1

Conjecture 5.2. Let N be a positive integer satisfying (3.3). Then $M_2^{+,0}(N)$ is spanned by

$$\left\{ f_{d_1,d_2,\mathcal{N}} \,\middle|\, (d_1,d_2) \in \mathcal{D}_N^{2,*}, \mathcal{N} \subset \mathcal{O}_F, \mathrm{Nm}\mathcal{N} = N \right\}.$$

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