Oberwolfach Report: Real-Dihedral Harmonic Maass Forms and CM-Values of Hilbert Modular Functions

Yingkun Li

In the theory of modular forms, those of weight $k = 1$ are important because of their connection to Galois representations. By the Theorem of Deligne-Serre [6], one can functorially attach to each weight one newform f a continuous, odd, irreducible representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C}).$$

Let $\tilde{\rho}_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be the associated projective representation. If the image of $\tilde{\rho}_f$ is isomorphic to a dihedral group, then ρ_f is induced from a character of $\mathrm{Gal}(\overline{F}/F)$ for some quadratic field F in M. We say that f or ρ_f is real-dihedral if F is a real quadratic field.

A *harmonic Maass form* of weight $k \in \mathbb{Z}$ is a real-analytic function $\mathcal{F} : \mathbb{H} \longrightarrow \mathbb{C}$ such that it is modular and annihilated by the hyperbolic Laplacian Δ_k of weight k

$$\begin{aligned} \Delta_k &:= y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \xi_{2-k} \circ \xi_k, \\ \xi_k &:= 2iy^k\overline{\partial_{\overline{z}}}, \end{aligned} \tag{0.1}$$

where we write $z = x + iy$. Furthermore, it is only allowed to have polar-type singularities in the cusps. They were introduced in [2] to study theta-liftings. Every harmonic Maass form $\mathcal{F}$ can be written canonically as the sum of a holomorphic part $\tilde{f}$ and a non-holomorphic part f^*. The holomorphic part $\tilde{f}$ is also known as a *mock-modular form*, which has been extensively studied by many people [1, 3, 7] after Zwegers' groundbreaking thesis [12] (see [11] for a good exposition) and has connections to many different areas of mathematics (see [8] for a comprehensive overview). When $k = 1$, we call $\mathcal{F}$ real-dihedral if $\xi_1(\mathcal{F})$ is a real-dihedral newform.

We are interested in studying a family of real-dihedral harmonic Maass forms and relate their Fourier coefficients to CM-values of Hilbert modular functions. Suppose $D \equiv 1 \pmod 4, p \equiv 5 \pmod 8$ are primes satisfying conditions

$$\begin{aligned} F = \mathbb{Q}(\sqrt{D}) &\text{ has class number one,} \\ p\mathcal{O}_F &= \mathfrak{p}\mathfrak{p}', \\ \mathrm{ord}_{\mathfrak{p}}(u_F^{(p-1)/4} - 1) &> 0, \end{aligned}$$

where $u_F > 1$ is the fundamental unit of F. Let $\chi_D(\cdot) = \left(\frac{\cdot}{D}\right)$ be the quadratic character of conductor D and ϕ_p the character of conductor p and order 4. The space of cusp forms $S_1(Dp, \chi_D\phi_p)$ is one-dimensional and spanned by a newform

$$f_\varphi(z) := \sum_{\mathfrak{a} \subset \mathcal{O}_F} \varphi(\mathfrak{a}) q^{\mathrm{Nm}(\mathfrak{a})} = \sum_{n \geq 1} c_\varphi(n) q^n, \tag{0.2}$$

where $q = e^{2\pi iz}$ and φ is a ray class group character of F. When $D = 5, p = 29$, the form f_φ was studied by Stark in the context of producing explicit generators of class fields of real-quadratic fields from special values of L-functions [9, 10].

Since $S_1(Dp, \chi_D \phi_p)$ is one-dimensional, there exists a harmonic Maass form $\mathcal{F}_\varphi(z)$ such that $\xi_1(\mathcal{F}_\varphi) = f_\varphi$ and its holomorphic part $\tilde{f}_\varphi$ has the following Fourier expansion at infinity

$$\tilde{f}_\varphi(z) = c^+_\varphi(-1)q^{-1} + c^+_\varphi(0) + \sum_{\substack{n \geq 2 \\ \chi_D(n) \neq -1}} c^+_\varphi(n) q^n.$$

Furthermore, with a mild condition on the growths of $\mathcal{F}_\varphi$ at other cusps of $\Gamma_0(Dp)$, the form $\mathcal{F}_\varphi$ is *unique* and the coefficients $c^+_\varphi(-1), c^+_\varphi(0)$ can be written explicitly as algebraic multiples of $\log u_F$.

Let $F_2 = \mathbb{Q}(\sqrt{p})$, $\mathcal{O}_{F_2}$ its ring of integers and X_{F_2} the open Hilbert modular surface whose complex points are $\mathrm{SL}_2(\mathcal{O}_{F_2})\backslash\mathbb{H}^2$. It is a connected component of the moduli space parametrizing isomorphisms of abelian surfaces with real multiplication. Let M_8 denote the field fixed by $\ker \tilde{\rho}_\varphi$. It contains two pairs of CM extensions K_4/F_2 and $\tilde{K}_4/\tilde{F}_2$, which are reflex fields of each other under the appropriate CM types $\Sigma = \{1, \sigma\}$ and $\tilde{\Sigma} = \sigma^3\Sigma = \{1, \sigma^{-1}\}$. Here, σ is an element of order 4 in the dihedral group $\mathrm{Gal}(M_8/\mathbb{Q}) \cong D_8$ of order 8.

Let $\mathrm{Cl}_0(K_4)$ be the kernel of the norm map $\mathrm{Nm} : \mathrm{Cl}(K_4) \longrightarrow \mathrm{Cl}(F_2)$ on class groups. Each class in $\mathrm{Cl}_0(K_4)$ gives rise to an isomorphism class of abelian surfaces on X_{F_2} with complex multiplication by (K_4, Σ), which is a "big" CM point in the sense of [4]. For $\mathcal{A} \in \mathrm{Cl}_0(K_4)$, let $Z_{\mathcal{A},\Sigma} \in X_{F_2}(\mathbb{C})$ denote the corresponding CM point. Since the 2-rank of $\mathrm{Cl}(K_4)$ is 1, it has a unique quadratic character ψ_2. Then we could define the twisted CM 0-cycle $\mathcal{CM}(K_4, \psi_2)$ by

$$\mathcal{CM}(K_4, \Sigma, \psi_2) := \sum_{\mathcal{A} \in \mathrm{Cl}_0(K_4)} \psi_2(\mathcal{A}) Z_{\mathcal{A},\Sigma}, \tag{0.3}$$

$$\mathcal{CM}(K_4, \psi_2) := \sum_{j=0}^{3} \mathcal{CM}(K_4, \sigma^j\Sigma, \psi_2). \tag{0.4}$$

It is algebraic and defined over the real quadratic field F. For $m \in \mathbb{N}$, let T_m be the m^{th} Hirzebruch-Zagier divisor on X_{F_2}. Given any normalized integral Hilbert modular function $\Psi(z_1, z_2)$ on X_{F_2} in the sense of Theorem 1.1 in [5] with divisor

$$\sum_{\substack{m \geq 1 \\ \gcd(pD, m) = 1}} c(-m) T_m,$$

where $c(-m) \in \mathbb{Z}$, we will show that the value of Ψ at $\mathcal{CM}(K_4, \psi_2)$ are related to the coefficients $c_\varphi^+(n)$ by

$$\log|\Psi(\mathcal{CM}(K_4,\psi_2))| = -\frac{c_\varphi(p)h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} \sum_{m\geq 1} c(-m) b_\varphi(m), \tag{0.5}$$

where $h_{\tilde{F}_2}$ and $h^+_{\tilde{F}_2}$ are the class number and narrow class number of $\tilde{F}_2 = \mathbb{Q}(\sqrt{Dp})$ respectively, and

$$b_\varphi(m) := \sum_{d|m} a_\varphi\left(\frac{m^2}{d^2}\right)\phi_p(d), \tag{0.6}$$

$$a_\varphi(n) := \sum_{k\in\mathbb{Z}} c_\varphi^+\left(\frac{Dn-pk^2}{4}\right)\delta_D(k), \tag{0.7}$$

$$\delta_D(k) := \begin{cases} 1 & D \not| k, \\ 2 & D \mid k. \end{cases} \tag{0.8}$$

References

[1] Bringmann, K.; Ono, K., *Lifting cusp forms to Maass forms with an application to partitions.* Proc. Natl. Acad. Sci. USA 104 (2007), no. 10, 3725-3731

[2] Bruinier, J.; Funke, J., *On two geometric theta lifts.* Duke Math. J. 125 (2004), no. 1, 45-90.

[3] Bruinier, J.; Ono, K., *Heegner Divisors, L-Functions and Harmonic Weak Maass Forms*, Annals of Math. 172 (2010), 2135-2181.

[4] Bruinier, J.; Kudla, S.; Yang, T. H., *Special values of Green functions at big CM points*, IMRN, 2012(9), 1917-1967.

[5] Bruinier, J.; Yang, T. H., *CM-Values of Hilbert Modular Functions*, Invent. Math. 163 (2006), 229-288.

[6] Deligne, P.; Serre, J. P., *Formes modulaires de poids 1.* (French) Ann. Sci. École Norm. Sup. (4) 7 (1974), 507-530 (1975).

[7] Duke, W.; Imamoḡlu, Ö.; Tóth, Á., *Cycle integrals of the j-function and mock modular forms.* Ann. of Math. (2) 173 (2011), no. 2, 947-981.

[8] Ono, K., *Unearthing the visions of a master: harmonic Maass forms and number theory.* Current developments in mathematics, 2008, 347–454, Int. Press, Somerville, MA, 2009.

[9] Stark, H. M., *Class fields and modular forms of weight one.* Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), pp. 277–287. Lecture Notes in Math., Vol. 601, Springer, Berlin, 1977.

[10] Stark, H. M., *Class fields for real quadratic fields and L-series at 1*, Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, 1975), 355-375, Academic Press, 1977.

[11] Zagier, Don, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann).* Seminaire Bourbaki. Vol. 2007/2008. Astérisque No. 326 (2009), Exp. No. 986, vii-viii, 143–164 (2010).

[12] Zwegers, S.P., *Mock Theta Functions,* Utrecht PhD Thesis, (2002) ISBN 90-393-3155-3.