NOTES ON HILBERT MODULAR FORMS AND APPLICATIONS

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1. Introduction.

These notes are prepared for a mini-course at the Gothenburg center for advanced studies in science and technology in November 2016. Here, we will discuss Hilbert modular forms and some of their arithmetic applications.

1.1. **Hilbert Modular Forms.** Let F be a totally real number field of degree m over $\mathbb{Q}$. Denote its ring of integers, groups of units, different and embeddings by $\mathcal{O}_F, \mathcal{O}_F^*, \mathfrak{d}_F$ and $\{\sigma_1, \dots, \sigma_m\}$ respectively. For any unital ring R, let $M_2(R)$ be all 2 by 2 matrices with entries in R and denote

$$\mathrm{SL}_2(R) := \{\gamma \in M_2(R) : \det \gamma = 1\}.$$

Using the embeddings σ_j, we can embed the group $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(\mathbb{R})^m$, which acts on

$$\mathcal{H}^m := \{\underline{z} = (z_1, \dots, z_m) : z_j = x_j + iy_j, y_j > 0 \text{ for all } 1 \le j \le m\}$$

via the linear fractional transformation.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_1, \cdots, z_m) = \left(\frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)}, \dots \frac{\sigma_m(a)z_m + \sigma_m(b)}{\sigma_m(c)z_m + \sigma_m(d)} \right). \tag{1.1}$$

The volume form invariant under this action is given by

$$d\mu(\underline{z}) = \frac{dx_1 dy_1}{y_1^2} \dots \frac{dx_m dy_m}{y_m^2}.$$

Consider the subgroup $\Gamma_F := \mathrm{SL}_2(\mathcal{O}_F) \subset \mathrm{SL}_2(F)$. It acts on $\mathcal{H}^m$ properly discontinuously. In particular, there are finitely many elliptic fixed points. The quotient $Y_F = \Gamma \backslash \mathcal{H}^m$ is a normal complex variety. The number of cusps is exactly the class number of F by the following lemma.

Lemma 1.1. *The class group $\mathrm{Cl}(F)$ is isomorphic to the equivalent classes of cusps $\Gamma_F \backslash \mathbb{P}^1(F)$ by the bijection $(\alpha\mathcal{O}_F + \beta\mathcal{O}_F) \leftrightarrow (\alpha : \beta)$.*

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Just like the case of $F = \mathbb{Q}$, the variety Y_F can be compactified by adding the cusps $\Gamma_F \backslash \mathbb{P}^1(F)$ and introducing a suitable topology compatible with the subset topology. This is the Bailey-Borel compactification. On the other hand, Hilbert modular varieties are also moduli spaces of principally polarized abelian surfaces with real multiplication. One can use this interpretation to give compactifications over $\mathbb{Z}$.

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and an m-tuple $\underline{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$, denote the usual weight $\underline{k}$ automorphy factor by

$$j_{\underline{k}}(\gamma, \underline{z}) := \prod_{j=1}^{m} (\sigma_j(c) z_j + \sigma_j(d))^{k_j}.$$

For simplicity, we write $j(\gamma, \underline{z}) := j_{(1,\dots,1)}(\gamma, \underline{z})$. Then $f(\gamma \cdot \underline{z}) = \frac{f(\underline{z})}{|j(\gamma,\underline{z})|^2}$, where $f(\underline{z}) = \prod_{1 \le j \le m} \mathrm{Im}(z_j)$ and $j_{\underline{k}}(\gamma, \underline{z}) = j(\gamma, \underline{z})^k$ when $\underline{k} = (k, \dots, k)$. Notice that $j(\gamma, \underline{z})$ is a cocycle on $\mathrm{SL}_2(F)$, i.e.

$$j(\gamma\gamma', \underline{z}) = j(\gamma, \gamma' \cdot \underline{z}) j(\gamma', \underline{z}). \tag{1.2}$$

Definition 1.2. For $\underline{k} \in \mathbb{Z}^m$, a *holomorphic Hilbert modular form* of weight $\underline{k}$ with respect to Γ_F is a holomorphic function $f : \mathcal{H}^m \to \mathbb{C}$ such that

$$(f \mid_{\underline{k}} \gamma)(\underline{z}) := j_{\underline{k}}^{-1}(\gamma, \underline{z}) f(\gamma \cdot \underline{z}) = f(\underline{z}) \tag{1.3}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_F$. Let $M_{\underline{k}}(\Gamma_F)$ denote the space of all such holomorphic functions.

Remark 1.3. A subgroup $\Gamma \subset \mathrm{SL}_2(F)$ is *commensurable* with Γ_F if $\Gamma \cap \Gamma_F$ has finite index in both Γ and Γ_F. The same notion of Hilbert modular forms can be defined with respect to $\Gamma \cap \Gamma_F$ when Γ is a commensurable subgroup.

For $\mu \in \mathcal{O}_F$, we have the matrix $n(\mu) := \left(\begin{smallmatrix} 1 & \mu \\ & 1 \end{smallmatrix}\right) \in \Gamma_F$. Substituting it into Eq. (1.3) implies that $f(\underline{z} + \underline{\mu}) = f(\underline{z})$ for all $f \in M_{\underline{k}}(\Gamma_F)$, where $\underline{\mu} := (\sigma_1(\mu), \dots, \sigma_m(\mu))$. Since f is holomorphic, we can write it uniquely as

$$f(\underline{z}) = \sum_{\nu \in \mathfrak{d}_F^{-1}} c_\nu \mathbf{e}(\underline{\nu} \cdot \underline{z}). \tag{1.4}$$

on $\mathcal{H}^2$. Notice that the inverse different $\mathfrak{d}_F^{-1}$ is the dual of $\mathcal{O}_F$ with respect to the norm. For each $\gamma \in \mathrm{SL}_2(F)$, we can consider the Fourier expansion of f at γ given by

$$(f \mid_{\underline{k}} \gamma)(\underline{z}) = \sum_{\nu} c_\nu(\gamma) \mathbf{e}(\underline{\nu} \cdot \underline{z}).$$

Using this, we can define the notion of cusp form.

Definition 1.4. A holomorphic Hilbert modular form f is called a *cusp form* if the constant term of its Fourier expansion $c_0(\gamma)$ vanishes for all $\gamma \in \mathrm{SL}_2(F)$.

Suppose for now $m \geq 2$. Then $\mathcal{O}_F^*$ is infinite and we can find a unit $\varepsilon \in \mathcal{O}_F^*$ which is not a root of unity. Then substituting $m(\varepsilon) := \left(\begin{smallmatrix} \varepsilon & \\ & \varepsilon^{-1} \end{smallmatrix}\right) \in \Gamma_F$ into Eq. (1.3) and (1.4) implies that

$$c_{\varepsilon^2\nu} = \left(\prod_{1\leq j\leq m} \sigma_j(\varepsilon)^{-k_j}\right) c_\nu$$

for all $\nu \in \mathfrak{d}_F^{-1}$. Suppose there is any $\nu \in \mathfrak{d}_F^{-1}$ such that $\nu \neq 0$ is not totally positive and $c_\nu \neq 0$, then we can choose a unit ε such that $\mathrm{tr}(\nu\varepsilon^2) < 0$. In that case, the following series

$$\sum_{n\geq 0} c_{\nu\varepsilon^2} \mathbf{e}(i\mathrm{tr}(\nu\varepsilon^2))$$

is divergent. Therefore, we must have $c_\nu = 0$. This is known as the Götzky-Koecher principle [2]. As a consequence, every holomorphic Hilbert modular form of weight $\underline{k}$ is a cusp form unless it has parallel weight $\underline{k} = (k, \ldots, k)$.

1.2. **Holomorphic Eisenstein Series.**

We will now follow closely section 1.5 of [3] to construct holomorphic Hilbert Eisenstein series of parallel, even integral weight $k > 2$.

Definition 1.5. To each non-zero fractional ideal $\mathfrak{m}$ of F and even integer $k > 2$, we define the function $E(\underline{z}, \mathfrak{m}, k)$ by
(1.5)

$$E(\underline{z}, \mathfrak{m}, k) := \mathrm{Nm}(\mathfrak{m})^k {\sum_{c,d\in\mathfrak{m}}}' j\left(\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right), \underline{z}\right)^{-k} = \mathrm{Nm}(\mathfrak{m})^k {\sum_{c,d\in\mathfrak{m}}}' \left(\prod_{1\leq j\leq m} (\sigma_j(c)z_j + \sigma_j(d))\right)^{-k},$$

where the $\sum'$ means sum over $((\mathfrak{m} \times \mathfrak{m}) - \{0\})/\mathcal{O}_F^*$.

It is not a priori clear that the sum above converges for $k > 2$ (it is clear for $k > 2m$). This is the consequence of the following lemmas.

Lemma 1.6 (Landau's Lemma)**.** *Let $\{\lambda_n : n \in \mathbb{N}\}$ be a monotone increasing sequence of positive numbers and c_n arbitrary positive real numbers. Suppose $f(s) := \sum_{n\geq 1} c_n\lambda_n^{-s}$ converges absolutely for $\mathrm{Re}(s) > s_0$ and analytically continues to a holomorphic function in a neighborhood around $s = s_0$. Then there exists $\delta > 0$ such that the series defining $f(s)$ converges absolutely for $\mathrm{Re}(s) > s_0 - \delta$.*

Sketch of Proof. The derivative of $f(s)$ at $s = s_0+1$ can be evaluated in terms of the Dirichlet series. Substituting this into the Taylor series expansion of $f(s)$ at $s = s_0 + 1$ produces a convergent double sum, with positive summands. This allows one to switch the order of summation. Evaluating at $s = s_0 - \epsilon$ for some small ϵ then gives the series expression for $f(-\epsilon)$. □

Lemma 1.7 (Lemma in section 1.5 of [3])**.** *The function*

$$\Phi(s, \mathfrak{m}) := (\pi^{-s}\Gamma(s))^m \sum_{c,d\in\mathfrak{m}}{}' |j(\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right), \underline{z})|^{-k}$$

has analytic continuation in s with a pole at $s = 1$.

Proposition 1.8. *The function $E(\underline{z}, \mathfrak{m}, k)$ is a Hilbert modular form of parallel weight k for Γ_F. It has the following Fourier expansion*

$$E(\underline{z}, \mathfrak{m}, k) = \zeta(\mathfrak{m}, k) + D_F^{-1/2}\left(\frac{(-2\pi i)^k}{\Gamma(k)}\right)^m \sum_{0\ll\nu\in\mathfrak{d}_F^{-1}} \sigma_{k-1}(\nu; \mathfrak{m})\mathbf{e}(\underline{\nu}\cdot\underline{z}), \tag{1.6}$$

where D_F is the discriminant of $\mathcal{O}_F$ over $\mathbb{Z}$, $\mathfrak{m}^\vee := \{a \in F : \mathrm{Tr}_{F/\mathbb{Q}} a\mathfrak{m} \subset \mathbb{Z}\}$ is the dual of $\mathfrak{m}$ and

$$\zeta(\mathfrak{m}, s) := \sum_{\mathfrak{m}|(\alpha)} \mathrm{Nm}((\alpha)/\mathfrak{m})^{-s}, \ \sigma_s(\nu; \mathfrak{m}) = \sum_{\delta\in\mathfrak{m}^\vee/\mathcal{O}_F^*,\, (\delta)\mathfrak{m}|(\nu)} |\mathrm{Nm}((\delta)\mathfrak{m})|^s.$$

Remark 1.9. When $k = 2$ and $m \geq 2$, the sum defining $E(\underline{z}, \mathfrak{m}, k)$ can be made to converge by adding the factor $|j(\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right), \underline{z})|^{-s}$ to the summand and analytically continue this to $s = 0$. This trick of Hecke can be used to show that the Fourier expansion in Eq. (1.6) still defines a Hilbert modular form of parallel weight 2 for Γ_F.

Example 1.10. When $F = \mathbb{Q}$ and $\mathfrak{m} = \mathbb{Z}$, we recover the classical Eisenstein series E_k, which is normalized to have the Fourier expansion

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^\infty \sigma_{k-1}(n)q^n,$$

where the B_k's are the Bernoulli numbers with the generating series $\frac{t}{e^t-1}\sum_{k\geq 0} B_k \frac{t^k}{k!}$ (e.g. $B_4 = -1/30, B_6 = 1/42, B_8 = -1/30$, etc). They appear as a consequence of the formula

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2\cdot k!}.$$

of the special value of the Riemann zeta function. For $4 \leq k \leq 10$, the space $M_k(\Gamma_\mathbb{Q})$ is one dimensional. When $k = 12$, there is a cusp form given by

$$\Delta(z) := \frac{E_4^3 - E_6^2}{1728} = q\prod_{n=1}^\infty (1-q^n)^2 4, \ q = \mathbf{e}(z).$$

When $k = 0$, the space of modular functions on $\Gamma_\mathbb{Q}\backslash\mathcal{H}$ is generated by the absolute invariant

$$j(z) = E_4^3\Delta^{-1} = q^{-1} + 744 + O(q).$$

1.3. **Theta Series as Hilbert Modular Forms.** It is well-known that the following theta function

$$\theta(z) := \sum_{\nu \in \mathbb{Z}} \mathbf{e}\left(\frac{\nu^2}{2} z\right)$$

is an elliptic modular form of weight $\frac{1}{2}$. In general, we can replace $(\mathbb{Z}, \frac{\nu^2}{2})$ with any positive definite lattice (L, Q) in the construction to produce holomorphic elliptic modular form of weight $\mathrm{rank}(L)/2$. The same procedures works over a totally real field F.

For a positive integer n, let $\Lambda \subset F^n$ be a lattice and $Q : F^n \to F$ a quadratic form with associated bilinear form

$$B(\alpha, \beta) := Q(\alpha + \beta) - Q(\alpha) - Q(\beta).$$

We say that (Λ, Q) is *even-integral* if $Q(\Lambda) \subset \mathfrak{d}_F^{-1}$, i.e. $B(\Lambda \times \Lambda) \subset \mathfrak{d}_F^{-1}$. Under the standard basis of F^n, the quadratic form Q is represented by a non-singular, symmetric matrix M_Q of size $n \times n$. We use Q^{-1} to represent the quadratic form such that $M_{Q^{-1}} = M_Q^{-1}$. Then we have the following result.

Theorem 1.11 (Theorem in section 5.4 of [3]). *Let $(\mathcal{O}_F^n, Q)$ be a positive, definite even-integral lattice of rank $n \in 2\mathbb{Z}$. Suppose* $\det Q$ *is a square in $F^\times$ and Q^{-1} is even-integral on $\mathfrak{d}_F^{-1}$. Then the theta function*

$$\Theta(\underline{z}; Q) := \sum_{\nu \in \mathcal{O}_F^n} \mathbf{e}\left(\underline{Q(\nu)} \cdot \underline{z}\right)$$

is a Hilbert modular form of weight $n/2$ for Γ_F.

Example 1.12. Consider the quadratic form Q on $\mathbb{Z}^8$ represented by the matrix M_Q

$$\begin{pmatrix} 8 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Then Q is positive definite with determinant 1 and $\Theta(z; Q)$ is a non-zero holomorphic modular form of weight 4 on $\mathrm{SL}_2(\mathbb{Z})$. Since $M_4(\mathrm{SL}_2(\mathbb{Z}))$ is one dimensional, we have the identity

$$E_4(z) = \Theta(z; Q).$$

This is a very special case of the Siegel-Weil formula, which identifies Eisenstein series with integrals of theta functions.

2. Rationality of $\zeta_F(1-k)$.

For a number field F, recall that its Dedekind zeta function is defined by

$$\zeta_F(s) := \sum_{\mathfrak{a}\subset\mathcal{O}_F} (\mathrm{Nm}\,\mathfrak{a})^{-s}.$$

It is absolutely convergent for $\mathrm{Re}(s) > 1$ and has analytic continuation to a meromorphic function of $s \in \mathbb{C}$. When F has r_1 real places and r_2 complex places, then the completed L-function

$$\Lambda_F(s) := |D_F|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_F(s)$$

satisfies the functional equation $\Lambda_F(s) = \Lambda_F(1-s)$, where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2), \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$ and D_F is the discriminant of the field F.

In this section, we will follow [4] and use the properties of Hilbert Eisenstein series to prove the following theorem.

Theorem 2.1. *When F is totally real and $k \ge 2$ is an even integer, the value $\zeta(1-k)$ is rational.*

By the observation $\zeta_F(s) = \sum_{[\mathfrak{m}]\in \mathrm{Cl}(F)} \zeta(\mathfrak{m}, s)$ with $\mathrm{Cl}(F)$ being the class group of F, and the functional equation

$$\zeta_F(1-k) = D_F^{k-1/2}\left(\frac{2\Gamma(k)}{(2\pi i)^k}\right)^m \zeta_F(k),$$

we just need to consider the rationality of $\zeta(\mathfrak{m}, k)D_F^{1/2}(2\pi)^{-km}$. From Prop. 1.8, we know that this is exactly the constant term of $D_F^{1/2}(2\pi)^{-km}E(\underline{z}, \mathfrak{m}, k)$.

The key observation of Siegel is that when $\underline{z} = (z, \dots, z)$ is on the diagonal, the function $E(\underline{z}, \mathfrak{m}, k)$ is an elliptic modular form in the variable z of weight km for $\Gamma_{\mathbb{Q}}$. Furthermore, the Fourier coefficients $c(n)$ in

$$D_F^{1/2}(2\pi)^{-km}E((z, \dots, z), \mathfrak{m}, k) = \sum_{n\ge 0} c(n)q^n \in M_{mk}(\Gamma_{\mathbb{Q}})$$

are all rational whenever $n \ge 1$. For example, when $m = 2$ and $k = 2, 4$, we can explicitly write

$$\zeta_F(1-k) = \frac{1}{30k}\sum_{\substack{\nu\in\mathfrak{d}_F^{-1}\\ \nu\gg 0\\ \mathrm{Tr}(\nu)=1}} \sum_{\mathfrak{a}|(\nu)\mathfrak{d}_F} \mathrm{Nm}(\mathfrak{a})^{k-1}.$$

Now for any $h \in 2\mathbb{N}$, let

$$r := \dim M_h = \lfloor\frac{h}{4}\rfloor + \lfloor\frac{h}{3}\rfloor - \frac{h}{2} + 1,$$

which is also the number of integer solutions (a,b) to $4a+6b=h$. Define a modular form in $M^!_{-h+2}$ by

$$T_h := E_{12r-h+2}\Delta^{-r} = C_h(r)q^{-r} + \cdots + C_h(1)q^{-1} + C_h(0) + O(q).$$

Since all Fourier coefficients of E_k and Δ are in $\mathbb{Q}$, so are those of T_h. We first have the following result.

Proposition 2.2. *For any $M = \sum_{n\geq 0} a(n)q^n \in M_h$, we have*

$$\sum_{1\leq n\leq r} a(n)C_h(r) = 0. \tag{2.1}$$

Notice that the sum on the left hand side above is the constant term of T_hM, which is a modular form of weight 2 and holomorphic on $\mathcal{H}$. Therefore $\omega = T_h(z)M(z)dz$ is an invariant differential 2-form on the modular curve, which satisfies $\omega = df$ for some modular function f, since the modular curve has genus 0. It is then clear that the constant term of the Fourier expansion of $\omega = df$ is zero.

The following result is due to Siegel [4].

Theorem 2.3. *We have $C_h(0)\neq 0$.*

Proof. If $h\equiv 2 \bmod 4$, then $h\equiv 2,6,10 \bmod 12$, $12r-h+2 = 0,8,4$ and $E_{12r-h+2} = E_0, E_4^2, E_4$ respectively. Since all the Fourier coefficients of E_4 and Δ^{-1} are positive, the same holds true for T_h, which means $C_h(0)\neq 0$ in particular.

If $h\equiv 0 \bmod 4$, then $h\equiv 4t \bmod 12$ with $t=0,1,2$ and $12r = h-4t+12$. Using the identities

$$E_{14} = E_4^tE_{14-4t} = -q\frac{dj}{dq}\Delta,\ E_4 = (\Delta j)^{1/3},$$

we can rewrite

$$\begin{aligned}
T_h &= E_{12r-h+2}\Delta^{-r} = -E_{14-4t}E_{14}^{-1}\Delta^{1-r}\left(q\frac{dj}{dq}\right) = -E_4^{-t}\Delta^{1-r}\left(q\frac{dj}{dq}\right)\\
&= -\Delta^{1-r-t/3}j^{t/3}\left(q\frac{dj}{dq}\right) = \frac{3}{t-3}\Delta^{1-r-t/3}\left(q\frac{dj^{1-t/3}}{dq}\right)\\
&= \frac{3}{t-3}\left(q\frac{d(\Delta^{1-r-t/3}j^{1-t/3})}{dq} - (1-r-t/3)j^{1-t/3}\Delta^{-r-t/3}q\frac{d\Delta}{dq}\right)\\
&= \frac{3}{t-3}\left(q\frac{d(\Delta^{-r}E_4^{3-t})}{dq} - (1-r-t/3)E_4^{3-t}q\frac{d\Delta^{-r}}{dq}\right).
\end{aligned}$$

The constant term of the first summand is zero. Since the Fourier coefficients of E_4 and Δ^{-1} are all positive, the constant term of $E_4^{3-t}q\frac{d\Delta^{-r}}{dq}$ is non-zero. Since $1-r-t/3\neq 0$, the second summand has non-zero constant term Fourier coefficient. This finishes the proof. ☐

Now, if we apply the result above to $M = D_F^{1/2}(2\pi)^{-km}E((z,\dots,z),\mathfrak{m},k)$, then we can write

$$\zeta(\mathfrak{m},k)D_F^{1/2}(2\pi)^{-km} = -C_{km}(0)^{-1}\left(\sum_{n=1}^{r} c(n)C_{km}(n)\right) \in \mathbb{Q}.$$

Therefore, it follows that $\zeta_F(1-k) \in \mathbb{Q}$.

3. Hilbert Modular Forms of Small Weights

3.1. **Orthogonal Shimura Varieties and Special Divisors.** When $m = 2$, the surface Y_F can be viewed as a Shimura variety associated to the rational orthogonal group of signature $(2,2)$. For example, consider the $\mathbb{Q}$ vector space

$$V = \left\{ A = \begin{pmatrix} a & \lambda \\ \lambda' & b \end{pmatrix} \in M_2(F) : A' = {}^tA \right\},$$

where $F \subset \mathbb{R}$ is a real quadratic field, $'$ is the conjugation in $\mathrm{Gal}(F/\mathbb{Q})$ and ${}^tA$ is the transpose of A. Equipped with the determinant as the quadratic form, V becomes a rational quadratic space of signature $(2,2)$. Since $F \subset \mathbb{R}$, we have $V_\mathbb{R} = M_2(\mathbb{R})$ and bilinear form associated to the determinant is given by

$$B(A_1, A_2) = a_1d_2 + d_1a_2 - b_1c_2 - c_1b_2,$$

where $A_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right) \in V_\mathbb{R}$ for $j = 1,2$. The group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts on $A \in V_\mathbb{R}$ via

$$(\gamma_1,\gamma_2)\cdot A = \gamma_2 A^t\gamma_1.$$

If we embed the group $\mathrm{SL}_2(F) \subset \mathrm{SL}_2(\mathbb{R})$ via $\gamma \mapsto (\gamma,\gamma')$ into $\mathrm{SL}_2(\mathbb{R})^2$, then it acts on $A \in V$ via

$$\gamma \cdot A = \gamma' A^t\gamma. \tag{3.1}$$

Notice that this action is an isometry, i.e. $Q(A) = Q(\gamma\cdot A)$. This embeds $\mathrm{SL}_2(F)$ into the $SO_V(\mathbb{Q})$. The Grassmannian $\mathcal{D}$ of negative two-planes in $V_\mathbb{R} = V \otimes \mathbb{R}$ can be identified with $\mathcal{H}^2$ via the isomorphism

$$w : \mathcal{H}^2 \to \mathcal{D}$$
$$\underline{z} = (z,z') \mapsto \mathbb{R}\mathrm{Re}Z + \mathbb{R}\mathrm{Im}Z,\ Z(\underline{z}) := \begin{pmatrix} zz' & z \\ z' & 1 \end{pmatrix}.$$

This puts a complex structure and hyperbolic metric on $\mathcal{D}$. It is easy to check that for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(F)$

$$\gamma \cdot Z(\underline{z}) = Z(\gamma\cdot\underline{z})j(\gamma,\underline{z}).$$

Therefore, $w(\gamma \cdot \underline{z}) = w(\underline{z})$. Similarly, we can identify the complement of $w(\underline{z})$ by

$$\begin{aligned} w^\perp &: \mathcal{H}^2 \to \mathcal{D} \\ \underline{z} = (z, z') &\mapsto \mathbb{R}\mathrm{Re}Z^\perp + \mathbb{R}\mathrm{Im}Z^\perp,\ Z^\perp(\underline{z}) := \begin{pmatrix} z\overline{z'} & z \\ \overline{z'} & 1 \end{pmatrix}. \end{aligned}$$

For simplicity of notation, we will write w and $w^\perp$.

Given a vector $\lambda \in V_\mathbb{R}$ and a subspace $U \subset V_\mathbb{R}$, let λ_U be the projection of λ onto U. Then any vector $\lambda \in V_\mathbb{R}$ can be written as

$$\lambda = \lambda_w + \lambda_{w^\perp}$$

and $Q(\lambda) = Q(\lambda_w) + Q(\lambda_{w^\perp})$. Since w is negative definite, $Q(\lambda_w) \le 0$ and $Q(\lambda_{w^\perp}) \ge 0$ for any $\lambda \in V_\mathbb{R}$ and $w \in \mathcal{D}$. Therefore, the following quadratic form on $V_\mathbb{R}$ is positive definite

$$Q_w(\lambda) := -Q(\lambda_w) + Q(\lambda_{w^\perp}).$$

This is called the majorant of Q and in general not defined over $\overline{\mathbb{Q}}$.

Take a full-rank lattice $L \subset V$ such that $Q(L) \subset \mathbb{Z}$. Let L^* be its dual lattice, which contains L. The abelian group L^*/L is called the discriminant group of L. Then the elements of $SO_V(\mathbb{Q})$ fixing L^*/L form a discrete subgroup Γ_L, called the discriminant kernel, which acts discontinuously on $\mathcal{D}$. The quotient $Y_L := \Gamma_L \backslash \mathcal{D}$ is a locally symmetric space. For a rational number m and $h \in L^*/L$, consider the following subset of L^*

$$L_{m,h} := \{\lambda \in L + h : Q(\lambda) = m\}.$$

This is clearly Γ_L-invariant, and so is the following analytic set

$$T_{m,h} := \{(z, z') \in \mathcal{H}^2 : w(z, z') \perp \lambda \text{ for all } \lambda \in L_{m,h}\}.$$

This descends to an algebraic divisor on Y_L and the sum $\sum_{h \in L^*/L} T_{m,h}$ is called a Hirzebruch-Zagier divisors.

Example 3.1. If we take

$$L = \left\{ A = \begin{pmatrix} a & \lambda \\ \lambda' & b \end{pmatrix} \in M_2(F) : a, b \in \mathbb{Z}, \lambda \in \mathcal{O}_F \right\},$$

then Γ_F is contained in Γ_L. In this case, $L^*/L \cong \mathcal{O}_F/\mathfrak{d}_F$.

3.2. **Weil Representation and Theta Function.** Let S_L be the $\mathbb{C}$ vector space with the basis $\{\mathfrak{e}_h : h \in L^*/L\}$ indexed by the discriminant group. There is an action ρ_L of $\mathrm{SL}_2(\mathbb{Z})$ on S_L known as the Weil representation. On the generators $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$, it is given by

$$\begin{aligned} \rho_L(T)(\mathfrak{e}_\mu) &= \mathbf{e}(Q(\mu))\mathfrak{e}_\mu, \\ \rho_L(S)(\mathfrak{e}_\mu) &= \frac{1}{\sqrt{D}} \sum_{\nu \in L^*/L} \mathbf{e}(-(\mu, \nu))\mathfrak{e}_\nu. \end{aligned} \tag{3.2}$$

We can now define the theta function and use it to construct Hilbert modular forms. In the notations above, we can consider the following vector-valued function of $\tau = u + iv \in \mathcal{H}$ and $\underline{z} = (z, z') \in \mathcal{H}^2$

$$\begin{aligned}
&\Theta(\tau, \underline{z}; L) := \sum_{h \in L^*/L} \Theta_h(\tau, \underline{z}; L)\mathfrak{e}_h, \\
(3.3) \qquad &\Theta_h(\tau, \underline{z}; L) := v \sum_{\lambda \in L+h} \mathbf{e}(Q(\lambda_{w^\perp(\underline{z})})\tau + Q(\lambda_{w(\underline{z})})\overline{\tau}) = \sum_{\lambda \in L+h} \mathbf{e}(Q(\lambda)\tau) e^{-4\pi v Q_{w(\underline{z})}(\lambda)}.
\end{aligned}$$

Since Q_w is positive definite, the sum above converges absolutely. As a consequence of Poisson summation, we have the following result.

Theorem 3.2 (See [1])**.** *The theta function $\Theta(\tau, \underline{z}; L)$ is a modular function in $\tau \in \mathcal{H}$ with respect to the Weil representation ρ_L. Each of its component $\Theta_h(\tau, \underline{z}; L)$ is a function of $w(\underline{z}) \in Y_L$.*

If we take the example above, then there is a finite map $Y_F \to Y_L$. We can view $\Theta(\tau, \underline{z}; L)$ as a Hilbert modular function on Y_F.

Remark 3.3. In general, one can replace the Gaussian with a polynomial times a Gaussian and obtain Hilbert modular forms of higher weights.

3.3. **Regularized Theta Lift.** Since $\Theta(\tau, \underline{z}; L)$ is a modular function in τ with moderate growth at the cusps, we can integrate it against a modular function $f(\tau)$ on the truncated fundamental domain $\mathcal{F}_T := \{\tau \in \mathcal{H} : |\mathrm{Re}(\tau)| \le 1/2, \mathrm{Im}(\tau) < T, |\tau| > 1\}$ with respect to the invaraint measure. If we take the limit as $T \to \infty$, the result does not always exist. Instead, we suppose that the limit

$$\lim_{T \to \infty} \int_{\mathcal{F}_T} f(\tau) v^{-s} d\mu(\tau)$$

exists for $\mathrm{Re}(s) \gg 0$ and can be analytically continued to a meromorphic vector-valued function in $s \in \mathbb{C}$. This assumption is easily satisfied for any weakly holomorphic modular form. Then we can suitably regularize this integral by taking the constant term of its Laurent expansion at $s = 0$. We denote this integral by

$$\Phi_L(\underline{z}, f) := \int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f(\tau)\overline{\Theta(\tau, \underline{z}; L)} \frac{dudv}{v^2} = \mathrm{Const}_{s=0} \lim_{T \to \infty} \int_{\mathcal{F}_T} f(\tau)\overline{\Theta(\tau, \underline{z}; L)} v^{-s} \frac{dudv}{v^2}.$$

This idea was due to Harvey and Moore. It was picked up by Borcherds, where he used the regularized integral to construct automorphic forms on locally symmetric spaces associated to arbitrary orthogonal groups over $\mathbb{Q}$. One important feature is that these automorphic forms have singularities along special submanifold.

In the case of the lattice L in example 3.1, one of the main results of Borcherds read as follows.

Theorem 3.4 (Borcherds 1998)**.** *Suppose that* $f(\tau) = \sum_{h \in L^*/L} \sum_{n \gg -\infty} c_f(n,h) q^n \mathfrak{e}_h \in M^!_{0,\rho_L}$ *satisfies* $c_f(n,h) \in \mathbb{Z}$ *for* $n \le 0$. *Then there exists a meromorphic Hilbert modular form* $\Psi(\underline{z}, L)$ *on* Y_L *of weight* $\frac{c_f(0,0)}{2}$ *with the divisor* $\sum_{m>0} c_f(-m,h) T_{m,h}$ *such that*

$$-4 \log |\Psi(\underline{z}, f)| = \Phi(\underline{z}, f) + c_f(0,0)(\log y_1 y_2 + \Gamma'(1) + \log(2\pi)). \tag{3.4}$$

Furthermore, $\Psi(\underline{z}, f)$ *has an infinite product expansion in a neighborhood of each cusp of* Y_L.

Example 3.5. Take $F = \mathbb{Q} \oplus \mathbb{Q}$, then the lattice L in Example 3.1 is the unimodular lattice $M_2(\mathbb{Z})$ and $\Theta(\tau, \underline{z}; L)$ is a scalar-valued modular function in $\tau \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$. The m^{th} Hirzebruch-Zagier divisor is just the Hecke correspondence. When $m = 1$, we have

$$T_1 = \{(z, z') \in \mathcal{H}^2 : a + dzz' - bz - cz' = 0 \text{ for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\}.$$

This condition is equivalent to $z' = \frac{bz-a}{dz-c}$, which means T_1 is simply the diagonal of $Y_L = (\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H})^2$. Let $J(\tau) := j(\tau) - 744 = q^{-1} + O(q)$ be a modular function. Then $\Phi(\underline{z}, J)$ is a Hilbert modular form on Y_L with logarithmic singularity along the diagonal. The function $\Psi(\underline{z}, L)$ in this case is then given by $j(z) - j(z')$ and Equation (3.4) implies that

$$-4 \log |j(z) - j(z')| = \Phi(z, z', J) = \int^{\text{reg}}_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}} J(\tau) \overline{\Theta(\tau, z, z'; L)} d\mu(\tau). \tag{3.5}$$

4. Hilbert Eisenstein Series of Weight One

As we have seen, the sum used to define the Eisenstein series $E(\underline{z}, \mathfrak{m}, k)$ only converges for $k > 2$. When $k = 2$ and $[F : \mathbb{Q}] \ge 2$, the Fourier expansion in Eq. (1.6) still defines a Hilbert modular form of parallel weight 2. For $k = 1$, Hecke had tried to use the same technique to produce holomorphic Hilbert modular forms of weight $(1,1)$ for Γ_F with F a real quadratic field. However, the same construction yielded the zero function, and a sign mistake prevented him from seeing it. In this section, we will look at the non-holomorphic Hilbert modular form of weight $(1,1)$ produced through Hecke's trick.

Suppose $F = \mathbb{Q}(\sqrt{D})$ where $D = D_1 D_2$ and $-D_j < 0$ are co-prime, fundamental discriminants. Let $\chi_j(\cdot) := \left(\frac{D_j}{\cdot}\right)$ be the Dirichlet characters and $K = \mathbb{Q}(\sqrt{-D_1}, \sqrt{-D_2})$ be the unramified extension of F. Then the extension K/F corresponds to a character χ of $\mathrm{Cl}(F)$ by class field theory. More concretely for a prime ideal $\mathfrak{p}$ of F, $\chi(\mathfrak{p})$ equals to 1 (resp. -1) if $\mathfrak{p}$ is split (resp. inert) in K. Hecke considered the following function

$$E_s(z, z') = \sum_{[\mathfrak{a}] \in \mathrm{Cl}(F)} \chi(\mathfrak{a}) \mathrm{Nm}(\mathfrak{a})^{1+2s} \sideset{}{'}\sum_{m,n \in \mathfrak{a}^2} \frac{y^2 (y')^2}{(mz+n)(m'z'+n')|mz+n|^{2s} |m'z'+n'|^{2s}}.$$

This converges for $\mathrm{Re}(s) > 1$ and can be continued to a holomorphic function in s from its Fourier expansion

$$E_s(z,z') = L(1+2s,\chi) y^s (y')^s + D^{-1/2} y^{-s} (y')^{-s} \left(\begin{array}{l} L(s,\chi)\phi_s(0)^2 + \\ \displaystyle\sum_{\substack{\nu \in \mathfrak{d}_F^{-1}, \\ \nu \neq 0}} \sigma_{-2s,\chi}((\nu)\mathfrak{d}_F)\phi_s(\nu y)\phi_s(\nu' y')\mathbf{e}(\nu x + \nu' x') \end{array} \right)$$

where $L(s,\chi) = \frac{\zeta_K(s)}{\zeta_F(s)} = L(s,\chi_1)L(s,\chi_2)$ and

$$\phi_s(t) = \int_{-\infty}^{\infty} \frac{\mathbf{e}(-wt)}{(w+i)(w^2+1)^s} dw, \ \sigma_{s,\chi}(\mathfrak{a}) = \sum_{\mathfrak{b} \mid \mathfrak{a}} \chi(\mathfrak{b}) \mathrm{Nm}(\mathfrak{b})^s.$$

Since $\phi_0(t) = -\pi i e^{-2\pi t}(1 + \mathrm{sgn}(t))$, the terms with ν not totally positive all vanish. Furthermore, $\sigma_{0,\chi}(\mathfrak{b}) = r_{K/F}(\mathfrak{b})$ counts the number of ideals in K with relative norm $\mathfrak{b}$. Since K/F is a CM extension, there is no ideal in K whose relative norm to F is $(\nu)\mathfrak{d}_F$ when $\nu \gg 0$. Therefore, the Fourier coefficients with $\nu \gg 0$ also vanish at $s = 0$, which means $E_0(z,z') = 0$ identically. This then prompts us to consider its derivative given by

$$\begin{aligned} \frac{\partial}{\partial s} E_s(z,z')|_{s=0} =& 2L(1,\chi)\log(yy') + 4C_\chi + \frac{8\pi^2}{\sqrt{D}} \sum_{\nu \in \mathfrak{d}_F^{-1},\ \nu \gg 0} \sigma'_\chi((\nu)\mathfrak{d}_F)\mathbf{e}(\nu z + \nu' z') \\ & - \frac{4\pi^2}{\sqrt{D}} \sum_{\nu \in \mathfrak{d}_F^{-1},\ \nu > 0 > \nu'} r_{K/F}((\nu)\mathfrak{d}_F)\Gamma(0, 4\pi|\nu'|y')\mathbf{e}(\nu z + \nu' z') \\ & - \frac{4\pi^2}{\sqrt{D}} \sum_{\nu \in \mathfrak{d}_F^{-1},\ \nu' > 0 > \nu} r_{K/F}((\nu)\mathfrak{d}_F)\Gamma(0, 4\pi|\nu|y)\mathbf{e}(\nu z + \nu' z'), \end{aligned} \tag{4.1}$$

where $C_\chi = L'(1,\chi) + L(1,\chi)\left(\frac{\log D}{2} - \log\pi - \gamma\right)$ with γ the Euler's constant, $\Gamma(s,x) := \int_x^\infty e^{-t} t^s \frac{dt}{t}$ is the incomplete Gamma function and

$$\sigma'_\chi(\mathfrak{a}) = \sum_{\mathfrak{b} \mid \mathfrak{a}} \chi(\mathfrak{b}) \log \mathrm{Nm}(\mathfrak{b}).$$

This function is then a non-holomorphic Hilbert modular form of parallel weight one. Restricting it to the diagonal and normalize suitably then gives us a non-holomorphic modular form of weight 2

$$\begin{aligned} F(\tau) &:= \frac{\sqrt{D}}{8\pi^2}\frac{\partial}{\partial s}E_s(\tau,\tau)|_{s=0} \\ &= \frac{\sqrt{D}}{2\pi^2}(L(1,\chi)\log v + C_\chi) + \sum_{\nu\in\mathfrak{d}_F^{-1},\,\nu\gg 0}\sigma'_\chi((\nu)\mathfrak{d}_F)\mathbf{e}(\mathrm{Tr}(\nu)\tau) \\ &\quad - \sum_{\nu\in\mathfrak{d}_F^{-1},\,\nu>0>\nu'} r_{K/F}((\nu)\mathfrak{d}_F)\Gamma(0,4\pi|\nu'|v)\mathbf{e}(\mathrm{Tr}(\nu)\tau). \end{aligned} \tag{4.2}$$

4.1. **Factorization of Difference of Singular Moduli.** For $j = 1,2$, let $-D_j < 0$ be co-prime, fundamental discriminants. Denote K/F the CM extension as before and let $K_j := \mathbb{Q}(\sqrt{-D_j})$ be the imaginary quadratic subfields of K, with $\mathrm{Cl}(K_j)$ the respective class groups. Let w_j be the number of roots of unities in K_j. Each class $[\mathfrak{a}] \in \mathrm{Cl}(K_j)$ corresponds to a point in $[z_\mathfrak{a}]\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$. Since the class numbers of K_j are finite, there are finitely many such points, which are called CM points. As a consequence of the theory of complex multiplication, the value of the $j(z_\mathfrak{a})$ is an algebraic number. In particular, it generates the Hilbert class field H_j of the imaginary quadratic field K_j. For each class $[\mathfrak{b}] \in \mathrm{Cl}(K_j)$, let $\sigma_{[\mathfrak{b}]} \in \mathrm{Gal}(H_j/K_j)$ be the corresponding element under class field theory. Then, these values satisfy

$$\sigma_{[\mathfrak{b}]}j(z_\mathfrak{a}) = j(z_{\mathfrak{a}\mathfrak{b}^{-1}}).$$

Therefore, the following quantity

$$J(-D_1,-D_2) := \prod_{[\mathfrak{a}_j]\in\mathrm{Cl}(K_j)}(j(z_{\mathfrak{a}_1}) - j(z_{\mathfrak{a}_2}))^{4/(w_1w_2)}$$

is a rational number and we are interested in its factorization. Note that $j(i) = 1728$ and $J(z) = j(z) - j(i)$. The main result, which is due to Gross and Zagier, is as follows.

Theorem 4.1.

$$\begin{aligned} -\log|J(-D_1,-D_2)|^2 &= \sum_{\nu\in\mathfrak{d}_F^{-1},\nu\gg0,\mathrm{tr}(\nu)=1}\sigma'_\chi((\nu)\mathfrak{d}_F) \\ &= \sum_{x^2<D,x^2\equiv D \bmod 4}\ \sum_{n|\frac{x^2-D}{4}}\varepsilon(n)\log n. \end{aligned}$$

where $\varepsilon(n) = \chi(\mathfrak{n})$ if $n = \mathrm{Nm}(\mathfrak{n})$ and zero otherwise.

Proof. Let $L_k := -2iv^2 \frac{\partial}{\partial \overline{\tau}}$ be the weight lowering operator. As a special case of the Siegel-Weil formula, we have the following equation

$$L_k F(\tau) = \frac{2}{w_{-D_1} w_{-D_2}} \sum_{\substack{[z_1],[z_2],\\ \mathrm{disc}(z_j)=-D_j}} \Theta(\tau, (z_1, z_2); L).$$

Using this and Equation (3.5), the theorem is then evident as

$$\begin{aligned} -\log|J(-D_1,-D_2)|^2 &= -\frac{8}{w_1 w_2} \sum_{\substack{\underline{z}=(z_1,z_2)\\ \mathrm{disc}(z_j)=-D_j}} \log|j(z_1)-j(z_2)| = \frac{-2}{w_1 w_2} \sum_{\underline{z}} \Phi_L(\underline{z}, J) \\ &= \frac{-2}{w_{-D_1} w_{-D_2}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}}^{\mathrm{reg}} J(\tau) \sum_{\underline{z}} \Theta(\tau, \underline{z}; L) \frac{du dv}{v^2} \\ &= -\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}}^{\mathrm{reg}} J(\tau) \frac{\partial F(\tau)}{\partial \overline{\tau}} d\tau d\overline{\tau} \\ &= \mathrm{Const}(J(\tau)F(\tau)) = \sum_{\nu \in \mathfrak{d}_F^{-1}, \nu \gg 0, \mathrm{tr}(\nu)=1} \sigma'_\chi((\nu)\mathfrak{d}_F). \end{aligned}$$

□

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