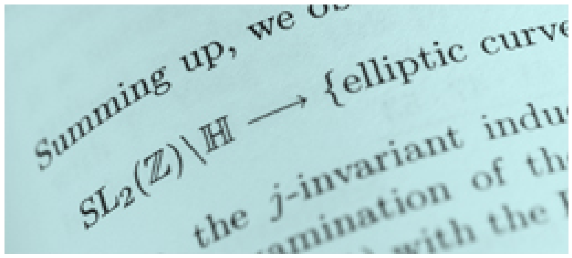
Complex Multiplication

Dozent/in: Prof. Dr. Yingkun Li

Termine: Fr 13:30-15:10 (Starting on Nov. 6)

Location: online. For detail, please check moodle or contact the lecturer.



Number fields are the main actors in algebraic number theory. Class field theory gives a qualitative description of abelian extensions of a number field K. The Kronecker-Weber theorem states that any finite, abelian extension of the number field $\mathbb{Q}$ is contained in a cyclotomic extension, which is generated by the values of the special function $f(z) = e^{2\pi iz}$ at the special points $z \in \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$. Extension of this result to other number field is Hilbert's 12^{th} problem, and not much is known in general.

When K is an imaginary quadratic field though, the theory of complex multiplication gives a beautiful extension of the Kronecker-Weber theorem. Here, $f(z)$ will be a function on the modular curve $\Gamma(N)\backslash\mathbb{H} = Y(N)$, which is (most of the time) a moduli space of elliptic curves, and the special points will be elliptic curves with complex multiplication. For example, take f to be the j-invariant (as in the picture above)

$$j(z) = q^{-1} + 744 + 196884q + O(q^2),\ q = e^{2\pi iz}$$

and $z_0 = \frac{1+\sqrt{163}i}{2}$. The theory of complex multiplication, together with the fact that $\mathbb{Q}(\sqrt{-163})$ has class number one, implies that $j(z_0)$ is an integer, which explains why the number

$$e^{\sqrt{163}\pi} = 262537412640768743.99999999999250.....$$

is very close to an integer.

The goal of this course is to describe and prove the main results in complex multiplication. Basic understanding of complex analysis and algebraic number theory will be assumed. During the first half of the course, we will briefly introduce elliptic curve and modular form, and discuss relevant concepts in algebraic geometry and algebraic number theory. In the second half, we will accomplish the goal above. If time permits at the end, we will also discuss applications, such as to cryptography, and recent developments.

Literatur:

D., Cox, *Primes of the form* $x^2 + ny^2$, John Wiley & Sons, (2013).
N., Koblitz, *Introduction to elliptic curves and modular forms*, Springer, (1993).
R., Schertz, *Complex multiplication*, CUP, (2010).
J., Silverman, *The arithmetic of elliptic curves*, Springer, (2009).
J., Silverman, *Advance topics in elliptic curves*, Springer, (1994).