ARITHMETIC VOLUMES OF UNITARY SHIMURA VARIETIES

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Abstract. The integral model of a $\mathrm{GU}(n-1,1)$ Shimura variety carries a universal abelian scheme over it, and the dual top exterior power of its Lie algebra carries a natural hermitian metric. We express the arithmetic volume of this metrized line bundle, defined as an iterated self-intersection in the Gillet-Soulé arithmetic Chow ring, in terms of logarithmic derivatives of Dirichlet L-functions.

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1. Introduction

The explicit calculation of arithmetic volumes of Shimura varieties began with the work of Kühn [Kuhn01] and Kramer (independently, in unpublished work). Over the moduli stack $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ of elliptic curves there is a line bundle of weight one modular forms, characterized as the dual Lie algebra of the universal elliptic curve. This line bundle carries a natural hermitian metric, and so determines a class in the codimension one arithmetic Chow of $\mathcal{X}$, in the sense of Gillet-Soulé. Working on a suitable compactification, Kühn and Kramer computed the self-intersection multiplicity of this hermitian line bundle, and gave a simple formula for it in terms of the logarithmic derivative of the Riemann zeta function at $s = -1$.

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More generally, if $\mathcal{X}$ is an integral model of a PEL type (or even Hodge type) Shimura variety, then $\mathcal{X}$ carries a *Hodge bundle*: the dual of the top exterior power of the Lie algebra of the universal abelian scheme over $\mathcal{X}$. The Hodge bundle again carries a natural hermitian metric, and one can ask if its arithmetic volume, defined as the $\dim(\mathcal{X})$-fold self-intersection in the Gillet-Soulé arithmetic Chow group, is again related to logarithmic derivatives of Dirichlet L-functions.

In some instances this is known. If $\mathcal{X}$ is the integral model of a quaternionic Shimura curve, the volume was computed by Kudla-Rapoport-Yang [KRY06]. Hilbert modular surfaces and the Siegel threefold were considered by Bruinier-Burgos-Kühn [BBK07] and Jung-von Pippich [JvP], respectively. The arithmetic volumes of $\mathrm{GSpin}(n,2)$ Shimura varieties, which include all of the above examples as special cases, were computed (up to a $\mathbb{Q}$-linear combination of logarithms of certain bad primes) by Hörmann [Hor14].

The present paper deals with the case of $\mathrm{GU}(n-1,1)$ Shimura varieties. The overall strategy is similar to that used in [BBK07] and [Hor14], in that it uses the theory of Borcherds products to express the metrized Hodge bundle as a linear combination (in the arithmetic Chow group) of smaller unitary Shimura varieties embedded as divisors. This allows one to compute the arithmetic self-intersection of the Hodge bundle using induction on the dimension of the ambient Shimura variety.

The added complications in the $\mathrm{GU}(n-1,1)$ case come mainly from the primes of bad reduction of the Shimura variety; that is to say, from primes dividing the discriminant of the quadratic imaginary field used to define the unitary group. At such primes, the divisors of Borcherds products include quite complicated linear combinations of vertical divisors, which then appear as correction terms when one expresses the Hodge bundle as a linear combination of smaller Shimura varieties. These vertical components in the divisors of Borcherds products, which have no analogue in [BBK07] or [Hor14], were calculated explicitly in [BHK$^+$a], making the present work possible.

1.1. **Statement of the main result.** Let $\boldsymbol{k} \subset \mathbb{C}$ be an imaginary quadratic field of odd discriminant

$$-D = \mathrm{disc}(\boldsymbol{k}).$$

Given an integer $n \geq 2$, there is a regular Deligne-Mumford stack

$$\mathcal{M}_{(n-1,1)} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}),$$

flat of relative dimension $n-1$, parametrizing principally polarized abelian schemes A, endowed with an action $\mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$, and extra data encoding a signature $(n-1,1)$ condition on $\mathrm{Lie}(A)$. See §4.1 for details.

This stack admits a decomposition

$$\mathcal{M}_{(n-1,1)} = \bigsqcup_{W} \mathcal{M}_W$$

into open and closed substacks indexed by strict similarity classes of relevant $\boldsymbol{k}$-hermitian spaces (W,h) of signature $(n-1,1)$, in such a way that the generic fiber of $\mathcal{M}_W$ is a Shimura variety for the unitary similitude group $\mathrm{GU}(W)$. Here *relevant* means that W contains an $\mathcal{O}_{\boldsymbol{k}}$-lattice $\mathfrak{a}\subset W$ that is self-dual, in the sense of (1.3.1). The notion of strict similarity is defined in §4.1, but the important fact is that there is a dichotomy between the cases of n odd and n even. If n is odd the relevant W form a single strict similarity class, and the disjoint union has a single term. If n is even then strict similarity is equivalent to isometry, and the number of terms in the disjoint union is $2^{o(D)-1}$, where

$$o(D)=\#\{\text{prime divisors of } D\}.$$

Fix a relevant W as above, and consider the restriction $\pi : A\to\mathcal{M}_W$ of the universal abelian scheme. Its relative dimension is $n=\dim(A)$, and its *metrized Hodge bundle*

$$\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}\in\widehat{\mathrm{Pic}}(\mathcal{M}_W)$$

is $\omega^{\mathrm{Hdg}}_{A/\mathcal{M}_W}=\pi_*\Omega^{\dim(A)}_{A/\mathcal{M}_W}$ endowed with the hermitian metric

$$\|s_z\|^2=\left|\frac{1}{(2\pi i)^{\dim(A)}}\int_{A_z(\mathbb{C})}s_z\wedge\overline{s}_z\right| \tag{1.1.1}$$

for any $z\in\mathcal{M}_W(\mathbb{C})$ and $s_z\in\omega^{\mathrm{Hdg}}_{A/\mathcal{M}_W,z}\cong H^0(A_z,\Omega^{\dim(A_z)}_{A_z/\mathbb{C}})$.

The stack $\mathcal{M}_W$ has a canonical toroidal compactification $\bar{\mathcal{M}}_W$, with boundary a Cartier divisor smooth over $\mathcal{O}_{\boldsymbol{k}}$. Although the Hodge bundle has a distinguished extension to the compactification, the metric on it does not extend smoothly. Instead, the metric is pre-log singular along the boundary, in the sense of Definition 1.20 of [BBK07]. This implies the integrability of its Chern form: the smooth $(1,1)$ form on $\mathcal{M}_W(\mathbb{C})$ defined locally by

$$\mathrm{ch}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W})=\frac{1}{2\pi i}\partial\overline{\partial}\log\|s\|^2=-dd^c\log\|s\|^2 \tag{1.1.2}$$

for any nonzero holomorphic section s. Define the *complex volume* of the metrized Hodge bundle

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W})=\int_{\mathcal{M}_W(\mathbb{C})}\mathrm{ch}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W})^{n-1}$$

as the (orbifold) integral of the top exterior power of its Chern form.

The pre-log singular conditions also allow us to view

$$\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}\in\widehat{\mathrm{CH}}^1(\bar{\mathcal{M}}_W,\mathscr{D}_{\mathrm{pre}})$$

as a class in the codimension one arithmetic Chow group of Burgos-Kramer-Kühn [BKK07]. These arithmetic Chow groups come with intersection pairings, and in codimension $n=\dim(\bar{\mathcal{M}}_W)$ there is an arithmetic degree

$$\widehat{\deg}:\widehat{\mathrm{CH}}^n(\bar{\mathcal{M}}_W,\mathscr{D}_{\mathrm{pre}})\to\mathbb{R}.$$

This allows us to define the *arithmetic volume*

$$\widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}) = \widehat{\deg}\big(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W} \cdots \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}\big)$$

as the arithmetic degree of the n-fold iterated intersection.

Our main result is the calculation of these complex and arithmetic volumes in terms of Dirichlet L-functions. To state it, for any place $\ell \le \infty$ denote by

$$\mathrm{inv}_\ell(W) = (\det W, -D)_\ell$$

the local invariant of W. Our assumption that W contains a self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice implies that $\mathrm{inv}_\ell(W) = 1$ for all finite primes $\ell \nmid D$. As $\mathrm{inv}_\infty(W) = -1$, and the product of all local invariants is 1, we obtain

$$\prod_{\ell \mid D} \mathrm{inv}_\ell(W) = -1. \tag{1.1.3}$$

The quadratic Dirichlet character determined by $\boldsymbol{k}/\mathbb{Q}$ is denoted

$$\varepsilon : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}.$$

For an integer $k \ge 1$ set

$$\mathbf{a}_k(s) = \frac{D^{k/2}\Gamma(s+k)L(2s+k,\varepsilon^k)}{2^k \pi^{s+k}}, \tag{1.1.4}$$

where, if k is even, we understand $L(s,\varepsilon^k) = \zeta(s)$. Define

$$\begin{aligned} \mathbf{A}_W(s) &= \mathbf{a}_1(s) \cdots \mathbf{a}_n(s) \\ &\quad \times \begin{cases} \prod_{\ell \mid D} \left(1 + \left(\frac{-1}{\ell}\right)^{\frac{n}{2}} \mathrm{inv}_\ell(W) \ell^{-s-\frac{n}{2}}\right) & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{1.1.5}$$

If n is odd there is no dependence on W beyond its dimension.

All $\mathbf{a}_k(0)$ are positive rational numbers, and hence so is $\mathbf{A}_W(0)$. The following is our main result. It appears in the text as Theorem 8.5.1.

Theorem A. *The metrized Hodge bundle has complex volume*

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}) = \mathbf{A}_W(0) \cdot \begin{cases} 2^{n-1} & \text{if } n \text{ is odd} \\ 2^{n-o(D)} & \text{if } n \text{ is even} \end{cases}$$

and arithmetic volume

$$\widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}) = \left(2\frac{\mathbf{A}'_W(0)}{\mathbf{A}_W(0)} - nC_0(n) + \log(D)\right) \cdot \mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}),$$

where we have set

$$C_0(n) = 2\log\left(\frac{4\pi e^\gamma}{\sqrt{D}}\right) + (n-4)\left(\frac{L'(0,\varepsilon)}{L(0,\varepsilon)} + \frac{\log(D)}{2}\right).$$

Theorem A is closely related to a conjecture of Kudla-Rapoport [KR14] on the arithmetic degrees of 0-cycles on integral models of unitary Shimura varieties. Briefly, to a $\boldsymbol{k}$-hermitian space V of signature $(n-1,1)$ one can attach a Shimura variety $\mathcal{S}_V$ (see the discussion in §1.2 below), which carries a family of arithmetic 0-cycles $\widehat{\mathcal{Z}}_V(T)$ indexed by $n \times n$ hermitian matrices T. The conjecture of Kudla-Rapoport predicts that the arithmetic degrees of these 0-cycles should agree with the Fourier coefficients of the derivative of an Eisenstein series on the quasi-split unitary group $U(n,n)$.

For those 0-cycles with $\det(T) \neq 0$, and supported at primes $p \nmid D$, the Kudla-Rapoport conjecture is now a theorem of Li-Zhang [LZ19]. The relevance of Theorem A is to the degenerate cases in which $\det(T) = 0$, and especially to the most degenerate case $T = 0$. In this case the associated arithmetic 0-cycle is, up to some correction factors at the primes $p \mid D$, the n-fold iterated intersection of a certain hermitian line bundle $\widehat{\mathcal{L}}_V$ on $\mathcal{S}_V$. In other words, the Kudla-Rapoport conjecture predicts that the arithmetic volume of $\widehat{\mathcal{L}}_V$ is essentially the constant term of the derivative of an Eisenstein series.

The calculation of the arithmetic volume of $\widehat{\mathcal{L}}_V$, which is intertwined with the proof of Theorem A, can be found in Corollary 8.4.5. We have not attempted to compare this formula with the constant term of an Eisenstein series, in part because it is not clear precisely which Eisenstein series one should work with. The Eisenstein series should be constructed from local data, and it remains an open problem to determine precisely which data one should choose at $p \mid D$ in order to make the Kudla-Rapoport conjecture true for *all* choices of T. Similarly, because of subtleties at the primes dividing D, the precise relation between the arithmetic degree of $\widehat{\mathcal{Z}}_V(0)$ and the arithmetic volume of $\widehat{\mathcal{L}}_V$ is not completely clear.

We expect that the proof of Theorem A makes accessible more degenerate cases of the Kudla-Rapoport conjecture, beyond the most degenerate case $T = 0$. For example, it seems likely that our methods could be used to approach cases in which T has small rank, or in which

$$T = \begin{pmatrix} T' & \\ & 0_{n-d} \end{pmatrix}$$

with T' a nonsingular $d \times d$ hermitian matrix that is not too complicated (e.g. squarefree determinant). We hope to return to these questions in future work.

1.2. **Outline of the paper.** Fix a $\boldsymbol{k}$-hermitian space V of signature $(n-1,1)$, and a self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice $L \subset V$.

We construct in §2 an Eisenstein series $E_L(\tau, s, n)$ of weight n, valued in a finite dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$. Its Fourier coefficients can be expressed in terms of local representation densities of the lattice L, which we then compute in order to make the coefficients completely explicit.

In §3 we introduce a complex Shimura variety $\mathrm{Sh}_K(H, \mathcal{D})$ associated to the unitary group $H = \mathrm{U}(V)$. It carries a family of special divisors $Z(m)$ indexed by positive $m \in \mathbb{Z}$, and a metrized line bundle $\widehat{\mathcal{L}}$. To a harmonic Maass form f of weight $2-n$ we attach a divisor $Z(f)$, defined as a linear combination of $Z(m)$'s. We then construct a Green function $\Phi(f)$ for $Z(f)$

using the machinery of regularized theta lifts, and show in Theorem 3.4.2 that the normalized integral

$$\mathcal{I}(f) \stackrel{\text{def}}{=} \mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}})^{-1} \int_{\mathrm{Sh}_K(H,\mathcal{D})} \Phi(f)\, \mathrm{ch}(\widehat{\mathcal{L}})^{n-1} \tag{1.2.1}$$

can be expressed in terms of the Fourier coefficients of the derivative of $E_L(\tau, s, n)$ at $s_0 = (n-1)/2$. The proof follows ideas of Kudla [Ku03], who computed similar integrals on orthogonal Shimura varieties.

The complex Shimura variety $\mathrm{Sh}_K(H, \mathcal{D})$ is not a moduli space of abelian varieties, but it can be covered by another Shimura variety that does have a moduli interpretation. In §4 we work with $\boldsymbol{k}$-hermitian spaces W_0 and W of signatures $(1,0)$ and $(n-1,1)$. The Shimura varieties associated to the unitary similitude groups $\mathrm{GU}(W_0)$ and $\mathrm{GU}(W)$ have moduli interpretations, which can be used to construct regular integral models $\mathcal{M}_{W_0}$ and $\mathcal{M}_W$ over $\mathcal{O}_{\boldsymbol{k}}$. We recall the definition of these integral models, their toroidal compactifications, and the structure of their reductions at $p \mid D$. Then we recall those aspects of the Gillet-Soulé arithmetic intersection theory (as extended by Burgos-Kramer-Kuhn [BKK07]) that are needed in the sequel.

In §5 we choose W_0 and W so that

$$V \cong \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$$

as hermitian spaces. This allows us to define an open and closed substack

$$\mathcal{S}_V \subset \mathcal{M}_{W_0} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_W,$$

which is an integral model of the Shimura variety associated to the subgroup $G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W)$ of pairs for which the similitude characters agree. There is a natural surjection $G \to H$, which induces a finite étale cover

$$\mathcal{S}_V(\mathbb{C}) \to \mathrm{Sh}_K(H, \mathcal{D}). \tag{1.2.2}$$

We next define the family of *Kudla-Rapoport divisors* $\mathcal{Z}_V(m)$ on $\mathcal{S}_V$. The complex fibers of these divisors are the pullbacks of the divisors already defined on $\mathrm{Sh}_K(H, \mathcal{D})$. Similarly, we define a metrized line bundle $\widehat{\mathcal{L}}_V$ on $\mathcal{S}_V$, compatible with the one already defined on $\mathrm{Sh}_K(H, \mathcal{D})$. There is another metrized line bundle on $\mathcal{S}_V$: the universal abelian scheme over $\mathcal{M}_W$ pulls back to an abelian scheme $A \to \mathcal{S}_V$, which determines a metrized Hodge bundle $\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$.

The main result of §5 is Theorem 5.5.1, which explains the precise connection between $\widehat{\mathcal{L}}_V$ and $\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$. What we prove is that, up to numerical equivalence (Definition 4.5.1), they differ by shifting the metrics by an explicit constant, and adding vertical divisors supported in characteristics $p \mid D$. Going back and forth between these line bundles, in order to exploit the most desirable properties of each, is key to the proof of Theorem A. More precisely, we introduce in Theorem 5.5.1 a third metrized line bundle $\widehat{\mathcal{K}}_V$, constructed so that it enjoys the best properties of $\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$ and $\widehat{\mathcal{L}}_V$. In the generic fiber, it agrees with the square of $\widehat{\mathcal{L}}_V$, so one can construct rational sections of it using the theory of Borcherds products. On the other hand, like the metrized Hodge bundle, it has trivial arithmetic intersection with all components of the exceptional divisor $\mathrm{Exc}_V \subset \mathcal{S}_V$ of Definition 5.1.5. This implies that certain error terms that appear in later calculations disappear after taking arithmetic intersection with $\widehat{\mathcal{K}}_V$.

In §6 we prove Theorems 6.1.2 and 6.1.3, which are essential to the induction arguments used to prove Theorem A. The idea is that if p is a prime split in $\boldsymbol{k}$, the Kudla-Rapoport divisor $\mathcal{Z}_V(p)$ should be closely related to a Shimura variety $\mathcal{S}_{V'}$ defined in the same way as $\mathcal{S}_V$, but with $\dim(V') = \dim(V) - 1$. This is true, up to some error terms coming from divisors supported in characteristics dividing pD. Using such a relation, we are able to express the height of the divisor $\mathcal{Z}_V(p)$ with respect to $\widehat{\mathcal{K}}_V$ in terms of the arithmetic volume of the metrized line bundle $\widehat{\mathcal{K}}_{V'}$ on the lower-dimensional Shimura variety $\mathcal{S}_{V'}$. A similar relation holds with $\widehat{\mathcal{K}}_V$ replaced by $\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$. Because of the many subtleties in the moduli problem defining the integral models, this is perhaps the most technical part of the proof of Theorem A.

In §7 we recall the Borcherds products on $\mathcal{S}_V$, which were studied in detail in [BHK$^+$a]. Given a meromorphic modular form $f(\tau) = \sum_{m \gg 0} c(m) q^m$ of weight $2 - n$ and level $\Gamma_0(D)$, holomorphic outside the cusp ∞, one can construct a *Borcherds product* $\psi(f)$. This is a rational section of a suitable power $\mathcal{L}_V^{\otimes k(f)}$ on the toroidal compactification $\bar{\mathcal{S}}_V$, whose divisor is

$$\mathrm{div}(\psi(f)) = \sum_{m>0} c(-m) \mathcal{Z}_V(m), \tag{1.2.3}$$

up to an error term consisting of a linear combination of boundary components and vertical divisors in characteristics dividing D.

In Theorem 7.1.2 we show that f may be chosen so that the Borcherds product has many desirable properties: the weight $k(f)$ is nonzero and can be expressed in terms of the functions (1.1.4), the only divisors $\mathcal{Z}_V(m)$ appearing in (1.2.3) are those with m a prime congruent to 1 modulo D, no boundary components appear in $\mathrm{div}(\psi(f))$, and the vertical divisors appearing in the error term have a particularly simple form. See especially Proposition 7.3.2.

In §8, we put everything together to prove Theorem A via induction on $n \geq 2$ (the case $n = 1$ can be deduced from the Chowla-Selberg formula; see the proof of Theorem 8.5.1). When $n = 2$ the Shimura variety $\mathcal{S}_V$ is closely related to a modular curve or quaternionic Shimura curve, and Theorem A can be deduced from the results of Kühn, Kramer, and Kudla-Rapoport-Yang alluded to above; this is carried out in [How20].

Because of the relations between $\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$, $\widehat{\mathcal{L}}_V$, and $\widehat{\mathcal{K}}_V$ worked out in §5, in order to prove Theorem A in general, it suffices to compute the complex and arithmetic volumes of $\widehat{\mathcal{K}}_V$, which can be found in Theorems 8.3.2 and 8.4.1. To do this, we use the fact that $\widehat{\mathcal{K}}_V$ agrees with $\widehat{\mathcal{L}}_V^{\otimes 2}$ in the generic fiber to view the Borcherds product $\psi(f)^{\otimes 2}$ as a rational section of $\widehat{\mathcal{K}}_V^{\otimes k(f)}$. In the codimension one arithmetic Chow group of $\mathcal{S}_V$, this allows us to represent a multiple of $\widehat{\mathcal{K}}_V$ by the divisor (1.2.3) endowed with the Green function $\Phi_V(f)$ obtained by pulling back via (1.2.2) the Green function appearing in (1.2.1), up to some contributions from vertical divisors in characteristics dividing D (known by the results of §7).

Thus the arithmetic volume of $\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V)$ can be computed by intersecting the codimension $n-1$ arithmetic cycle $\widehat{\mathcal{K}}_V^{n-1}$ against the divisor (1.2.3) endowed with $\Phi_V(f)$. This intersection can then be expressed in terms of the arithmetic heights of the $\mathcal{Z}_V(m)$'s with respect to $\widehat{\mathcal{K}}_V^{n-1}$, and the integral $\mathcal{I}(f)$ of (1.2.1). Using the results of §6, the arithmetic heights are essentially the arithmetic volumes of Shimura varieties in one dimension lower, which can be expressed in terms of the functions $\mathbf{a}_k(s)$ of (1.1.4) by the induction hypothesis. The integral $\mathcal{I}(f)$ was expressed in terms of the coefficients of $E_L(\tau, s, n)$ in §3, and the formulas from §2 for these coefficients allow us to write $\mathcal{I}(f)$ in terms of the functions $\mathbf{a}_k(s)$. Thus the arithmetic volume of $\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V)$ can be expressed in terms of $\mathbf{a}_k(s)$, and so the same is true of both the metrized Hodge bundle and $\widehat{\mathcal{L}}_V$.

1.3. **Notation and conventions.** Throughout the paper $\boldsymbol{k} \subset \mathbb{C}$ is a quadratic imaginary field of discriminant $-D$, and $\varepsilon : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ is the associated quadratic character.

The symbol V always denotes a $\boldsymbol{k}$-hermitian space of signature $(n-1,1)$ with $n \geq 1$, and the hermitian form is denoted $\langle -, - \rangle$. Hermitian forms are always linear in the first variable and conjugate-linear in the second, and are assumed to be nondegenerate.

Beginning in §4, and continuing for the rest of the paper, we assume that D is odd and the hermitian space admits an $\mathcal{O}_{\boldsymbol{k}}$-lattice $L \subset V$ that is *self-dual*, in the sense that

$$L = \{x \in V : \langle x, L \rangle \subset \mathcal{O}_{\boldsymbol{k}}\}. \tag{1.3.1}$$

These restrictions are not imposed in §2 and §3 unless stated explicitly.

The term *stack* means Deligne-Mumford stack.

2. Eisenstein series and theta functions

Fix a $\boldsymbol{k}$-hermitian space V of signature $(n-1,1)$, with $n \geq 1$. After reviewing some basics of Eisenstein series, theta functions, and the Siegel-Weil formula, we attach a particular Eisenstein series to a lattice $L \subset V$, and express its Fourier coefficients in terms of representation densities. We then compute these representation densities in the case of a self-dual lattice, and so obtain explicit formulas for the Fourier coefficients; these formulas have a different shape depending on whether n is even or odd.

2.1. **A seesaw dual reductive pair.** Let V_0 be the unique symplectic space over $\mathbb{Q}$ of dimension 2, and denote the symplectic form by $\langle x,y\rangle_0$. Set $V_{0\boldsymbol{k}} = V_0 \otimes_{\mathbb{Q}} \boldsymbol{k}$, and extend the symplectic form $\boldsymbol{k}$-linearly in the first argument and $\boldsymbol{k}$-conjugate-linearly in the second argument. This defines a skew-hermitian form on $V_{0\boldsymbol{k}}$.

The $\mathbb{Q}$-vector space underlying V carries a $\mathbb{Q}$-bilinear form

$$[x,y] = \mathrm{tr}_{\boldsymbol{k}/\mathbb{Q}}\langle x,y\rangle \tag{2.1.1}$$

of signature $(2n-2,2)$, with associated quadratic form

$$Q(x) = \langle x,x\rangle. \tag{2.1.2}$$

This data determine a seesaw dual reductive pair

$$\begin{array}{ccc} G = \mathrm{U}(V_{0\boldsymbol{k}}) & & H' = \mathrm{O}(V) \\ | & \times & | \\ G' = \mathrm{Sp}(V_0) & & H = \mathrm{U}(V). \end{array}$$

In particular, there are compatible Weil representations of $G(\mathbb{A})\times H(\mathbb{A})$ and $G'(\mathbb{A})\times H'(\mathbb{A})$ on the space $S(V(\mathbb{A}))$ of Schwartz-Bruhat functions on $V(\mathbb{A})$. This will be made explicit in §2.3.

Let e,f be a basis of V_0 with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and use this to identify

$$G \cong \mathrm{U}(1,1) \stackrel{\mathrm{def}}{=} \left\{ g \in \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\,\mathrm{GL}_2 : g\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

and $G' \cong \mathrm{SU}(1,1) \cong \mathrm{SL}_2$. Let $P = NM$ be the parabolic subgroup of G with Levi factor

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} : a \in \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\,\mathbb{G}_m \right\}$$

and unipotent radical

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{G}_a \right\}.$$

For each prime $p \le \infty$ of $\mathbb{Q}$, let $K_p \subset G(\mathbb{Q}_p)$ be the maximal compact subgroup defined by

$$K_p = \begin{cases} G(\mathbb{Q}_p) \cap \mathrm{GL}_2(\mathcal{O}_{\boldsymbol{k},p}), & \text{if } p<\infty, \\ G(\mathbb{R}) \cap \mathrm{U}(2,\mathbb{R}), & \text{if } p=\infty, \end{cases}$$

and put $K = \prod_{p\le\infty} K_p$. If we let $\boldsymbol{k}^1$ be the torus of norm one elements in $\boldsymbol{k}^\times = \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m$, there is a natural homomorphism

$$\mathrm{SU}(1,1) \times \boldsymbol{k}^1 \xrightarrow{(g,a)\mapsto g\cdot m(a)} \mathrm{U}(1,1),$$

and the image of $\boldsymbol{k}^1$ is the center of $\mathrm{U}(1,1)$. This homomorphism induces an isomorphism

$$\mathrm{SO}(2,\mathbb{R}) \times \mathrm{U}(1,\mathbb{R})/\{\pm 1\} \cong K_\infty.$$

Let $\mathcal{H}$ be the complex upper half-plane. The first isomorphism in

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2,\mathbb{R}) \cong G(\mathbb{R})/K_\infty.$$

sends $\tau = u + iv \in \mathcal{H}$ to the element

$$g_\tau = n(u)m(v^{1/2}) \in \mathrm{SL}_2(\mathbb{R}) \tag{2.1.3}$$

satisfying $g_\tau \cdot i = \tau$. The second is induced $\mathrm{SL}_2(\mathbb{R}) = G'(\mathbb{R}) \subset G(\mathbb{R})$.

2.2. **Eisenstein series for** $\mathrm{U}(1,1)$**.** Given an $s \in \mathbb{C}$ and a character $\chi : \mathbb{A}_{\boldsymbol{k}}^\times/\boldsymbol{k}^\times \to \mathbb{C}^\times$, let

$$I(s,\chi) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi|\cdot|_{\mathbb{A}_{\boldsymbol{k}}}^s)$$

be the induced representation, realized on the space of smooth K-finite functions Φ on $G(\mathbb{A})$ satisfying

$$\Phi(n(b)m(a)g,s) = \chi(a)|a|_{\mathbb{A}_{\boldsymbol{k}}}^{s+\frac{1}{2}}\Phi(g,s)$$

for $b \in \mathbb{A}$ and $a \in \mathbb{A}_{\boldsymbol{k}}^\times$. Here $|a|_{\mathbb{A}_{\boldsymbol{k}}}$ denotes the norm on $\mathbb{A}_{\boldsymbol{k}}^\times$. In particular, at the archimedean place we have the normalized absolute value $|a|_\infty = a\bar{a}$.

Recall that a section of $I(s,\chi)$ is called *standard* if its restriction to K is independent on s. For a standard section Φ of $I(s,\chi)$, the associated Siegel Eisenstein series

$$E(g,s,\Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma g, s)$$

converges absolutely for $\mathrm{Re}(s) > 1/2$, and defines an automorphic form on $G(\mathbb{A})$. The Eisenstein series has meromorphic continuation to all $s \in \mathbb{C}$. We now describe this Eisenstein series in more classical terms, by restricting it to the subgroup $\mathrm{SL}_2 \cong G' \subset G$.

Assume that $\Phi(s) = \Phi_\infty(s) \otimes \Phi_f(s)$ is a factorizable standard section with $\Phi_f = \otimes_{p<\infty}\Phi_p(s)$, and that Φ_∞ is a normalized standard section of weight $\ell \in \mathbb{Z}$. In other words, $\Phi_\infty(1,s) = 1$, and for all

$$k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{SO}(2,\mathbb{R}) \subset K_\infty$$

we have $\Phi_\infty(gk,s) = \Phi_\infty(g,s)\cdot \underline{k}^\ell$, where

$$\underline{k} = a + ib \in \mathbb{C}^\times. \tag{2.2.1}$$

Abbreviate $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \subset G(\mathbb{A})$, and let $\Gamma_\infty = P(\mathbb{Q}) \cap \Gamma$ be is subgroup of upper triangular matrices.

Lemma 2.2.1. *If Φ_∞ is a normalized standard section of weight $\ell \in \mathbb{Z}$ as above, then*

$$E(g_\tau,s,\Phi) = j(g_\tau,i)^{-\ell} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma\tau)^{s+\frac{1}{2}-\frac{\ell}{2}} \cdot j(\gamma,\tau)^{-\ell}\cdot \Phi_f(\gamma).$$

Here $g_\tau \in \mathrm{SL}_2(\mathbb{R})$ is as in (2.1.3)*, and $j(\gamma,\tau) = c\tau + d$ the usual automorphy factor associated to $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.*

Proof. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we write $\gamma g_\tau = n(\beta)m(\alpha)k$ with $\beta \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$, and $k \in \mathrm{SO}(2,\mathbb{R})$. Then a computation shows that

$$\alpha = \frac{v^{1/2}}{|c\tau + d|}, \qquad \underline{k} = \frac{c\bar{\tau} + d}{|c\tau + d|},$$

and hence

$$\Phi_\infty(\gamma g_\tau, s) = v^{\ell/2} \cdot \frac{v^{s+\frac{1}{2}-\frac{\ell}{2}}}{|c\tau + d|^{2s+1-\ell}} \cdot (c\tau + d)^{-\ell}.$$

The inclusions $\Gamma \subset \mathrm{SL}_2(\mathbb{Q}) \subset G(\mathbb{Q})$ induce bijections

$$\Gamma_\infty \backslash \Gamma \cong (P(\mathbb{Q}) \cap \mathrm{SL}_2(\mathbb{Q})) \backslash \mathrm{SL}_2(\mathbb{Q}) \cong P(\mathbb{Q}) \backslash G(\mathbb{Q}),$$

and hence

$$\begin{aligned} E(g_\tau, s, \Phi) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Phi_\infty(\gamma g_\tau, s) \Phi_f(\gamma) \\ &= v^{\ell/2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma\tau)^{s+\frac{1-\ell}{2}} \cdot (c\tau + d)^{-\ell} \cdot \Phi_f(\gamma) \\ &= j(g_\tau, i)^{-\ell} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma\tau)^{s+\frac{1-\ell}{2}} \cdot j(\gamma, \tau)^{-\ell} \cdot \Phi_f(\gamma), \end{aligned}$$

as desired. □

2.3. **The Siegel-Weil formula.** Write ψ for the standard additive character of $\mathbb{A}/\mathbb{Q}$, satisfying $\psi_\infty(x) = e^{2\pi i x}$. Let

$$\varepsilon_{\mathbb{A}} : \mathbb{A}^\times / \mathbb{Q}^\times \to \{\pm 1\}$$

be the idele class character determined by the quadratic extension $\boldsymbol{k}/\mathbb{Q}$, and fix a character

$$\chi : \mathbb{A}_{\boldsymbol{k}}^\times / \boldsymbol{k}^\times \to \mathbb{C}^\times$$

such that $\chi|_{\mathbb{A}^\times} = \varepsilon_{\mathbb{A}}^n$.

As explained in [HKS96] and [Ich07], the choices of ψ and χ determine a Weil representation $\omega = \omega_{\psi,\chi}$ of the group $G(\mathbb{A}) \times H(\mathbb{A})$ on the space of Bruhat-Schwartz functions $S(V(\mathbb{A}))$. Recalling the $\mathbb{Q}$-bilinear form (2.1.1) and associated quadratic form (2.1.2), the action is given by the formulas

$$\begin{aligned} \omega(m(a))\varphi(x) &= \chi(a)|a|_{\mathbb{A}_{\boldsymbol{k}}}^{n/2} \varphi(xa), \\ \omega(n(b))\varphi(x) &= \psi(bQ(x))\varphi(x), \\ \omega(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\varphi(x) &= \int_{V(\mathbb{A})} \varphi(y)\psi([x,y])\, dy, \\ \omega(h)\varphi(x) &= \varphi(h^{-1}x) \end{aligned}$$

for $\varphi \in S(V(\mathbb{A}))$ and $x \in V(\mathbb{A})$. Here $a \in \mathbb{A}_{\boldsymbol{k}}^\times$, $b \in \mathbb{A}$, $h \in H(\mathbb{A})$, and the Fourier transform is taken with respect to the self-dual measure dy on $V(\mathbb{A})$.

To a Schwartz function φ there is an associated theta function

$$\theta(g,h,\varphi) = \sum_{x\in V(\mathbb{Q})} \omega(g,h)\varphi(x). \tag{2.3.1}$$

Abbreviating $s_0 = (n-1)/2$, there is a $G(\mathbb{A})$-intertwining operator

$$\lambda : S(V(\mathbb{A})) \to I(s_0,\chi)$$

defined by $\lambda(\varphi)(g) = (\omega(g,1)\varphi)(0)$. We extend $\lambda(\varphi)$ to a standard section of $I(s,\chi)$ by setting

$$\lambda(\varphi)(g,s) = |a(g)|_{\mathbb{A}_{\boldsymbol{k}}}^{s-s_0}(\omega(g,1)\varphi)(0).$$

The following theorem is a case of the Siegel-Weil formula; see [Weil65] or Theorem 1.1 of [Ich07].

Theorem 2.3.1. *Assume that $n > 2$, or that V is anisotropic. For $\varphi \in S(V(\mathbb{A}))$ and $g \in G(\mathbb{A})$ we have*

$$E(g,\lambda(\varphi),s_0) = \kappa \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g,h,\varphi)\,dh,$$

where Haar measure is normalized by $\mathrm{vol}(H(\mathbb{Q})\backslash H(\mathbb{A})) = 1$*, and*

$$\kappa = \begin{cases} 1 & \text{if } n > 1 \\ 2 & \text{if } n = 1. \end{cases} \tag{2.3.2}$$

2.4. **A special Eisenstein series.** We consider Eisenstein series constructed from particular Schwartz functions on $V(\mathbb{A})$. At the archimedian place we take the Gaussian associated to a majorant and at the non-archimedian places we take the characteristic function of a coset of an $\mathcal{O}_{\boldsymbol{k}}$-lattice.

Let $L \subset V$ be an $\mathcal{O}_{\boldsymbol{k}}$-lattice on which the hermitian form is $\mathcal{O}_{\boldsymbol{k}}$-valued. Its dual lattice under the $\mathbb{Q}$-bilinear form (2.1.1) is denoted

$$L' \stackrel{\text{def}}{=} \{x \in V : [x,L] \subset \mathbb{Z}\}. \tag{2.4.1}$$

For $\mu \in L'/L$ let

$$\varphi_\mu = \mathrm{char}(\mu + \hat{L}) \in S(V(\mathbb{A}_f))$$

be the characteristic function of $\mu + \hat{L} \subset V(\mathbb{A}_f)$, and denote by S_L the subspace of $S(V(\mathbb{A}_f))$ generated by all φ_μ with $\mu \in L'/L$. Hence we may identify

$$S_L \cong \mathbb{C}[L'/L].$$

The restriction to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ of the Weil representation of $\mathrm{SL}_2(\mathbb{A}_f) \subset G(\mathbb{A}_f)$ takes the subspace S_L to itself, giving rise to a representation

$$\omega_L : \Gamma \to \mathrm{Aut}(S_L). \tag{2.4.2}$$

The corresponding dual representation

$$\omega_L^\vee : \Gamma \to \mathrm{Aut}(S_L^\vee)$$

is given by $\omega_L^\vee(\gamma)(f) = f \circ \omega_L^{-1}(\gamma)$ for $f \in S_L^\vee$. On the space S_L we also have the conjugate representation $\bar{\omega}_L$ defined by

$$\bar{\omega}_L(\gamma)(\varphi) = \overline{\omega_L(\gamma)(\bar{\varphi})}$$

for $\varphi \in S_L$. If we use the standard $\mathbb{C}$-bilinear pairing

$$\Big\langle \sum_\mu a_\mu \varphi_\mu, \sum_\mu b_\mu \varphi_\mu \Big\rangle = \sum_\mu a_\mu b_\mu \tag{2.4.3}$$

to identify $S_L \cong S_L^\vee$ as $\mathbb{C}$-vector spaces, then $\bar{\omega}_L = \omega_L^\vee$, and this agrees with the representation denoted ρ_L in [Bo98], [Br02], and [BY09].

For $\ell \in \mathbb{Z}$ and $\tau \in \mathcal{H}$, we define an S_L-valued Eisenstein series

$$E_L(\tau, s, \ell) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma\tau)^{s + \frac{1-\ell}{2}} (c\tau + d)^{-\ell} \cdot \bar{\omega}_L(\gamma)^{-1} \varphi_0 \tag{2.4.4}$$

of weight ℓ and representation $\bar{\omega}_L$. Note that this Eisenstein series has a slightly different normalization than the Eisenstein series in equation (2.17) of [BY09], since it is adapted to the unitary setting.

Recall the Maass lowering and raising operators in weight ℓ defined by

$$L_\ell = -2iv^2 \frac{\partial}{\partial \bar{\tau}}, \qquad R_\ell = 2i \frac{\partial}{\partial \tau} + \ell v^{-1}.$$

They lower (respectively raise) the weight of an automorphic form by 2. It is easily seen that

$$\begin{aligned} L_\ell E_L(\tau, s, \ell) &= \left(s + \frac{1-\ell}{2} \right) E_L(\tau, s, \ell - 2) \\ R_\ell E_L(\tau, s, \ell) &= \left(s + \frac{1+\ell}{2} \right) E_L(\tau, s, \ell + 2). \end{aligned} \tag{2.4.5}$$

The Eisenstein series $E_L(\tau, s, \ell)$ is an eigenform of the hyperbolic Laplacian Δ_ℓ of weight ℓ (normalized as in (3.1) of [BF04]) with eigenvalue $(\frac{\ell-1}{2})^2 - s^2$. The following is a consequence of (2.4.5).

Lemma 2.4.1. *Assume that $n > 2$, or that V is anisotropic. The Eisenstein series $E_L(\tau, s_0, n)$ of weight $n = \dim(V)$ at $s_0 = (n-1)/2$ is holomorphic in τ. Its derivative*

$$E_L'(\tau, s_0, n) \stackrel{\text{def}}{=} \frac{\partial}{\partial s} E_L'(\tau, s, n)\big|_{s=s_0}$$

satisfies

$$L_n(E_L'(\tau, s_0, n)) = E_L(\tau, s_0, n-2).$$

Remark 2.4.2. The Eisenstein series $E_L(\tau, s, n-2)$ is coherent as it arises via the Siegel-Weil formula from the global hermitian space V. On the other hand, the Eisenstein series $E_L(\tau, s, n)$ is incoherent as it is associated to the incoherent collection of local hermitian spaces obtained by replacing V_∞ by the positive definite hermitian space of dimension n and keeping the non-archimedian part. In particular, at $s = 0$, the center of symmetry of the functional equation, the Eisenstein series $E_L(\tau, s, n)$ vanishes identically.

2.5. **Coefficients of Eisenstein series.** When the weight of the Eisenstein series (2.4.4) is $n = \dim(V)$, we denote its Fourier expansion by

$$E_L(\tau, s, n) = \sum_{\mu \in L'/L} \sum_{m \in \mathbb{Z} + \langle \mu, \mu \rangle} B(m, \mu; s, v) q^m \varphi_\mu. \tag{2.5.1}$$

We summarize some well-known facts about the coefficients; details can be found in [BrKu03] and [KY10].

If $m \neq 0$, there is a factorization

$$B(m, \mu; s, v) = B(m, \mu, s) \cdot \mathcal{W}_m(s, v) \tag{2.5.2}$$

in which the non-archimedian contribution $B(m, \mu, s)$ is independent of v. The second factor is the archimedian Whittaker function

$$\mathcal{W}_m(s, v) = (4\pi|m|v)^{-n/2} e^{2\pi m v} \cdot W_{\mathrm{sgn}(m)\frac{n}{2}, s}(4\pi|m|v), \tag{2.5.3}$$

where

$$W_{\kappa,\mu}(z) = e^{-z/2} z^{1/2+\mu} U(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z)$$

denotes the classical confluent hypergeometric function as in [AbSt84, Chapter 13], and

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1}\, dt \tag{2.5.4}$$

for $\Re(a) > 0$, $|\arg(z)| < \pi/2$.

If $m = 0$, the Fourier coefficient has the form

$$B(0, \mu; s, v) = \delta_{\mu,0} \cdot v^{s+\frac{1}{2}-\frac{n}{2}} + B(0, \mu, s) \cdot v^{-s+\frac{1}{2}-\frac{n}{2}}, \tag{2.5.5}$$

where $B(0, \mu, s)$ is independent of v, and $\delta_{\mu,0}$ is the Kronecker delta.

Remark 2.5.1. Note that our normalization of the archimedian Whittaker function in (2.5.3) differs from the normalization in [KY10, Proposition 2.3]. The normalizing factor $\Gamma(s + \frac{n}{2} + \frac{1}{2})^{-1}$ for $m > 0$, and $\Gamma(s - \frac{n}{2} + \frac{1}{2})^{-1}$ for $m < 0$, appears in loc. cit. In our case this factor is included in the normalization of the $B(m, \mu, s)$, since this leads to slightly cleaner formulas in Theorem 3.4.2.

The integral representation (2.5.4) can be used to describe the asymptotic behavior of the Whittaker function as $v \to \infty$, which is given by

$$\mathcal{W}_m(s, v) = \begin{cases} 1 + O(v^{-1}) & \text{if } m > 0 \\ O(e^{-4\pi|m|v}) & \text{if } m < 0, \end{cases}$$

locally uniformly in s and m.

We will also require the following lemma on the value at $s_0 = (n-1)/2$ of the Whittaker function and its derivative $\mathcal{W}'_m(s_0, v) = \frac{\partial}{\partial s} \mathcal{W}_m(s, v) \mid_{s=s_0}$.

Lemma 2.5.2. *The special value at s_0 of the Whittaker function is*

$$\mathcal{W}_m(s_0, v) = \begin{cases} 1 & \textit{if } m > 0 \\ \Gamma(1-n, 4\pi|m|v) & \textit{if } m < 0. \end{cases}$$

For $m > 0$ we have

$$\mathcal{W}_m'(s_0, v) = \sum_{j=1}^{n-1} \binom{n-1}{j} \frac{\Gamma(j)}{(4\pi m v)^j}.$$

In particular,

$$\mathcal{W}_m'(s_0, v) = O(v^{-1}), \quad \text{for } v \to \infty.$$

Proof. We only prove this for $m > 0$ and leave the similar case when $m < 0$ to the reader. According to (2.5.3) and (2.5.4), we have for $m > 0$ that

$$\begin{aligned}\mathcal{W}_m(s, v) &= \frac{(4\pi m v)^{s-s_0}}{\Gamma(s - s_0)} \cdot \int_0^\infty e^{-4\pi m v t} t^{s-s_0-1} (1 + t)^{s+s_0} \, dt \\ &= 1 + \frac{(4\pi m v)^{s-s_0}}{\Gamma(s - s_0)} \cdot \int_0^\infty e^{-4\pi m v t} t^{s-s_0-1} ((1 + t)^{s+s_0} - 1) \, dt,\end{aligned}$$

where the latter integral converges absolutely for $\Re(s) > s_0 - 1$. We immediately see that $\mathcal{W}_m(s_0, v) = 1$, and in addition that

$$\begin{aligned}\mathcal{W}_m'(s_0, v) &= \int_0^\infty e^{-4\pi m v t} ((1 + t)^{n-1} - 1) \frac{dt}{t} \\ &= \sum_{j=1}^{n-1} \binom{n-1}{j} \int_0^\infty e^{-4\pi m v t} t^j \frac{dt}{t} \\ &= \sum_{j=1}^{n-1} \binom{n-1}{j} \frac{\Gamma(j)}{(4\pi m v)^j}.\end{aligned}$$

This gives the claimed formula and the bound for $v \to \infty$. □

We now turn to the explicit calculation of the factor $B(m, \mu, s)$ in (2.5.2), using formulas from [BrKu03] and [KY10]. Fix $\mu \in L'/L$, and let

$$d_\mu = \min\{b \in \mathbb{Z}^+ : b\mu = 0\}$$

be the order of μ. For $m \in \mathbb{Z} + Q(\mu)$ and $a \in \mathbb{Z}$, we denote by $N_{m,\mu}(a)$ the modulo a representation number

$$N_{m,\mu}(a) = \#\{r \in L/aL : \; Q(r + \mu) \equiv m \pmod{a\mathbb{Z}}\}.$$

For a prime p, set

$$L_{m,\mu}^{(p)}(p^{-s}) = (1 - p^{-s+2n-1}) \sum_{\nu=0}^\infty N_{m,\mu}(p^\nu) p^{-\nu s} \tag{2.5.6}$$

so that

$$\sum_{a=1}^\infty N_{m,\mu}(a) a^{-s} = \zeta(s - 2n + 1) \prod_p L_{m,\mu}^{(p)}(p^{-s}).$$

Proposition 2.5.3. *For nonzero $m \in \mathbb{Z} + Q(\mu)$ we have*

$$B(m,\mu,s) = -\frac{2^n \pi^{s+\frac{n}{2}+\frac{1}{2}}}{\sqrt{|L'/L|}} \cdot \begin{cases} \dfrac{|m|^{s+\frac{n}{2}-\frac{1}{2}}}{\Gamma(s+\frac{n}{2}+\frac{1}{2})} \prod_p L^{(p)}_{m,\mu}(p^{-2s-n}) & \text{if } m>0 \\ \dfrac{|m|^{s+\frac{n}{2}-\frac{1}{2}}}{\Gamma(s-\frac{n}{2}+\frac{1}{2})} \prod_p L^{(p)}_{m,\mu}(p^{-2s-n}) & \text{if } m<0. \end{cases}$$

Proof. This is obtained from Proposition 3.2 of [BrKu03] by noticing that our Eisenstein series $E_L(\tau,s,n)$ is 1/2 the Eisenstein series $E_0(\tau, s-\frac{n}{2}+\frac{1}{2})$ of [BrKu03] with $\kappa = n$ and $r = 2n$. □

According to equation (3.20) of [BrKu03], the Euler factors $L^{(p)}_{m,\mu}(p^{-s})$ are polynomials in p^{-s}. Moreover, by (3.23) of [BrKu03], if p does not divide $d_\mu^2 m|L'/L|$ then

$$L^{(p)}_{m,\mu}(p^{-s}) = 1 - \chi_F(p)p^{n-1-s},$$

where χ_F is the quadratic Dirichlet character associated to the quadratic extension $\mathbb{Q}(\sqrt{(-1)^n|L'/L|})$. This implies that the Euler product in Proposition 2.5.3 converges absolutely for $\mathrm{Re}(s) > 0$ and that $B(m,\mu,s_0)$ is rational if $n > 1$ (so that $s_0 = \frac{n-1}{2}$ lies in the region of convergence). In this case it is nonpositive for positive m and vanishes for negative m.

Later we will require the following lemma on the behavior of the coefficients of the Eisenstein series in the limit $v \to \infty$.

Lemma 2.5.4. *Assume that $n > 2$, or that V is anisotropic. The coefficients $B'(m,\mu;s_0,v)$ of $E'_L(\tau,s_0,n)$ satisfy*

$$\lim_{v\to\infty} \big(B'(m,\mu;s_0,v) - \kappa\delta_{m,0}\delta_{\mu,0}\log v\big) = \begin{cases} B'(m,\mu,s_0) & \text{if } m>0 \\ B'(0,\mu,s_0) & \text{if } m=0 \text{ and } n=1 \\ 0 & \text{if } m=0 \text{ and } n>1 \\ 0 & \text{if } m<0. \end{cases}$$

Proof. For $m \neq 0$ the assertion follows from Proposition 2.5.3 together with Lemma 2.5.2 on the asymptotic behavior of the Whittaker function $\mathcal{W}_m(s,v)$ and its derivative at $s = s_0$.

For $m = 0$, we consider the Laurent expansion of $B(0,\mu;s,v)$ at $s = s_0$, which is given by

$$\begin{aligned} B(0,\mu;s,v) &= \delta_{\mu,0} + B(0,\mu,s_0)v^{-2s_0} \\ &\quad + \big(\delta_{\mu,0}\log(v) - B(0,\mu,s_0)\log(v)v^{-2s_0} + B'(0,\mu,s_0)v^{-2s_0}\big)(s-s_0) \\ &\quad + O((s-s_0)^2). \end{aligned}$$

If $n > 1$, then the holomorphicity of $E_L(\tau,s_0,n)$ implies that $B(0,\mu,s_0) = 0$. Since $2s_0 > 0$, the term involving v^{-2s_0} vanishes in the limit $v \to \infty$, and we obtain the assertion.

If $n = 1$ then $s_0 = 0$ is the center of symmetry of the functional equation of the incoherent Eisenstein series, and hence $E_L(\tau, 0, n)$ vanishes. The vanishing of the constant Fourier coefficient implies

$$B(0, \mu, 0) = -\delta_{\mu,0},$$

and consequently

$$B'(0, \mu; 0, v) = 2\delta_{\mu,0} \log(v) + B'(0, \mu, s_0).$$

This concludes the proof of the lemma. □

2.6. **Self-dual lattices.** In this subsection we assume the $\mathcal{O}_{\boldsymbol{k}}$-lattice $L \subset V$ is self-dual (1.3.1) under the hermitian form, and that $D = -\mathrm{disc}(\boldsymbol{k})$ is odd. The first of these assumptions implies that (2.4.1) satisfies $L' = \mathfrak{d}_{\boldsymbol{k}}^{-1} L$, where $\mathfrak{d}_{\boldsymbol{k}} \subset \mathcal{O}_{\boldsymbol{k}}$ is the different. We will determine the coefficient $B(m, \mu, s)$ from (2.5.2) more explicitly.

To compute the Euler factors $L^{(p)}_{m,\mu}(p^{-s})$ in Proposition 2.5.3, we need to determine the representation numbers $N_{m,\mu}(p^\nu)$. We now derive explicit formulas for these quantities using finite Fourier transforms and formulas for certain lattice Gauss sums. As we will only compute the coefficients for $\mu = 0$, we abbreviate $N_m(p^\nu) = N_{m,0}(p^\nu)$.

Fix an $a \in \mathbb{Z}$. For $c \in \mathbb{Z}^+$, consider the Gauss sum

$$G(a, c) = \sum_{x \in \mathcal{O}_{\boldsymbol{k}}/c\mathcal{O}_{\boldsymbol{k}}} e\left(\frac{a\,\mathrm{N}(x)}{c}\right).$$

We are mainly interested in the case when $c = p^\nu$ is a prime power.

Lemma 2.6.1. *Let p^ν be a prime power. If we write $\gcd(a, p^\nu) = p^\alpha$ with $0 \le \alpha \le \nu$ and $a = p^\alpha a'$ with $a' \in \mathbb{Z}$, then*

$$G(a, p^\nu) = \begin{cases} p^{2\nu} & \text{if } \alpha = \nu \\ p^{\alpha+\nu} \left(\frac{-D}{p}\right)^{\nu-\alpha} & \text{if } \alpha < \nu \text{ and } p \nmid D \\ \epsilon_p p^{\alpha+\nu+\frac{1}{2}} \left(\frac{a'}{p}\right)\left(\frac{D'}{p}\right)^{\nu-\alpha-1} & \text{if } \alpha < \nu \text{ and } p \mid D. \end{cases}$$

Here, in the final case, we have set $D' = D/p$ and

$$\epsilon_r = \begin{cases} 1 & \text{if } r \equiv 1 \pmod 4 \\ i & \text{if } r \equiv 3 \pmod 4). \end{cases}$$

for any odd positive integer r.

Proof. It is easily seen that

$$G(a, p^\nu) = p^{2\alpha} G(a', p^{\nu-\alpha}).$$

Hence we may assume that a is coprime to p. Now, if p is odd, the assertion can be easily deduced from the classical Gauss sum

$$\sum_{x \in \mathbb{Z}/c\mathbb{Z}} e\left(\frac{ax^2}{c}\right) = \epsilon_c \sqrt{c}\left(\frac{a}{c}\right)$$

for odd positive c with $\gcd(c,a)=1$.

For $p=2$ and a odd we have

$$G(a,2^\nu)=\sum_{x,y\in\mathbb{Z}/2^\nu\mathbb{Z}} e\left(a\frac{x^2+xy+\frac{1+D}{4}y^2}{2^\nu}\right), \tag{2.6.1}$$

where we have used that D is odd. If $\nu=1$ the asserted formula is easily checked. Hence, assume that $\nu\geq 2$. Then we use the classical Gauss sum

$$\sum_{x\in\mathbb{Z}/2^\nu\mathbb{Z}} e\left(a\frac{x^2+xy}{2^\nu}\right)=\begin{cases}0, & 2\nmid y,\\ e\left(-a\frac{y^2}{2^{\nu+2}}\right)\sum_{x\in\mathbb{Z}/2^\nu\mathbb{Z}} e\left(a\frac{x^2}{2^\nu}\right), & 2\mid y,\end{cases}$$

to rewrite (2.6.1) as follows:

$$G(a,2^\nu)=\frac{1}{2}\sum_{x,y\in\mathbb{Z}/2^\nu\mathbb{Z}} e\left(a\frac{x^2+Dy^2}{2^\nu}\right).$$

Inserting the evaluation of the Gauss sum

$$\sum_{x\in\mathbb{Z}/2^\nu\mathbb{Z}} e\left(a\frac{x^2}{2^\nu}\right)=(1+i)\epsilon_a^{-1}2^{\nu/2}\left(\frac{2^\nu}{a}\right),$$

we finally find that

$$G(a,2^\nu)=2^\nu\left(\frac{2^\nu}{D}\right)=2^\nu\left(\frac{-D}{2}\right)^\nu,$$

concluding the proof of the lemma. □

Lemma 2.6.2. *Let p^ν be a prime power. If we write $\gcd(a,p^\nu)=p^\alpha$ with $0\leq\alpha\leq\nu$ and $a=p^\alpha a'$ with $a'\in\mathbb{Z}$, then the Gauss sum*

$$G_L(a,p^\nu)=\sum_{x\in L/p^\nu L} e\left(\frac{a\langle x,x\rangle}{p^\nu}\right)$$

is given by

$$G_L(a,p^\nu)=\begin{cases}p^{2n\nu} & \text{if } \alpha=\nu\\ p^{n\alpha+n\nu}\left(\frac{-D}{p}\right)^{n(\nu-\alpha)} & \text{if } \alpha<\nu \text{ and } p\nmid D\\ \epsilon_p^n \operatorname{inv}_p(V)p^{n\alpha+n\nu+\frac{n}{2}}\left(\frac{a'}{p}\right)^n\left(\frac{D'}{p}\right)^{n(\nu-\alpha-1)} & \text{if } \alpha<\nu \text{ and } p\mid D,\end{cases}$$

where ϵ_p and D' have the same meaning as in Lemma 2.6.1.

Proof. The Gauss sum $G_L(a,p^\nu)$ only depends on $L_p=L\otimes_\mathbb{Z}\mathbb{Z}_p$. If we let $b_1,\dots,b_n$ be an orthogonal $\mathcal{O}_{\boldsymbol{k},p}$-module basis of L_p (which exists because of our hypothesis that D is odd), then

$$G_L(a,p^\nu)=G(a\langle b_1,b_1\rangle,p^\nu)\cdots G(a\langle b_n,b_n\rangle,p^\nu),$$

and the claim follows from Lemma 2.6.1. Note that in the case $p \mid D$, standard formulas for the Hilbert symbol imply

$$\mathrm{inv}_p(V) = \left(\frac{\langle b_1, b_1\rangle}{p}\right) \cdots \left(\frac{\langle b_n, b_n\rangle}{p}\right). \qquad \square$$

Proposition 2.6.3. *Assume that n is even, and $m \in \mathbb{Z}$ is nonzero. Fix a prime p, and factor $m = p^\beta m'$ with $\gcd(m', p) = 1$. If $p \nmid D$ then*

$$L_m^{(p)}(p^{-s}) = (1 - p^{n-s-1}) \sum_{\gamma=0}^{\beta} p^{(n-s)\gamma}.$$

If $p \mid D$ then

$$\begin{aligned} L_m^{(p)}(p^{-s}) = 1 &- \left(\frac{-1}{p}\right)^{\frac{n}{2}} \mathrm{inv}_p(V) p^{\frac{n}{2}} p^{n-1-s} \\ &+ \left(\frac{-1}{p}\right)^{\frac{n}{2}} \mathrm{inv}_p(V) p^{\frac{n}{2}} \left(1 - p^{n-1-s}\right) \sum_{\gamma=1}^{\beta} p^{(n-s)\gamma}. \end{aligned}$$

Proof. We compute the representation number $N_m(p^\nu)$ using the identity

$$\begin{aligned} N_m(p^\nu) &= \frac{1}{p^\nu} \sum_{a \in \mathbb{Z}/p^\nu\mathbb{Z}} G_L(a, p^\nu) e\Big(-\frac{am}{p^\nu}\Big) \\ &= \frac{1}{p^\nu} \sum_{\alpha=0}^{\nu} \sum_{a' \in (\mathbb{Z}/p^{\nu-\alpha}\mathbb{Z})^\times} G_L(p^\alpha a', p^\nu) e\Big(-\frac{a'm}{p^{\nu-\alpha}}\Big). \end{aligned}$$

As n is even, Lemma 2.6.2 implies

$$G_L(a, p^\nu) = \begin{cases} p^{2n\nu} & \text{if } \alpha = \nu \\ p^{n\alpha + n\nu} & \text{if } \alpha < \nu \text{ and } p \nmid D \\ \left(\frac{-1}{p}\right)^{n/2} \mathrm{inv}_p(V) p^{n\alpha + n\nu + \frac{n}{2}} & \text{if } \alpha < \nu \text{ and } p \mid D. \end{cases}$$

If $p \nmid D$ we obtain

$$\begin{aligned} N_m(p^\nu) &= p^{n\nu-\nu} \sum_{\alpha=0}^{\nu} p^{n\alpha} \sum_{a' \in (\mathbb{Z}/p^{\nu-\alpha}\mathbb{Z})^\times} e\Big(-\frac{a'm}{p^{\nu-\alpha}}\Big) \\ &= p^{n\nu-\nu} \sum_{\alpha=0}^{\nu} p^{n\alpha} \sum_{d \mid \gcd(p^{\nu-\alpha}, m)} \mu\Big(\frac{p^{\nu-\alpha}}{d}\Big) d \\ &= p^{n\nu-\nu} \sum_{\alpha=0}^{\nu} p^{n\alpha} \sum_{\gamma=0}^{\min(\nu-\alpha,\beta)} \mu\big(p^{\nu-\alpha-\gamma}\big) p^\gamma. \end{aligned}$$

Here we have used the evaluation of the Ramanujan sum. Consequently, the Euler factor (2.5.6) is

$$\begin{aligned} L_m^{(p)}(p^{-s}) &= (1-p^{2n-s-1})\sum_{\nu=0}^{\infty} N_m(p^\nu)p^{-\nu s} \\ &= (1-p^{2n-s-1})\sum_{\gamma=0}^{\beta}\sum_{\alpha=0}^{\infty}\sum_{\nu=\alpha+\gamma}^{\infty} p^{n\alpha+\gamma}\mu(p^{\nu-\alpha-\gamma})p^{(n-1-s)\nu} \\ &= (1-p^{n-s-1})\sum_{\gamma=0}^{\beta} p^{(n-s)\gamma}. \end{aligned}$$

In the case $p \mid D$ we find

$$\begin{aligned} N_m(p^\nu) &= p^{2n\nu-\nu} + \left(\frac{-1}{p}\right)^{n/2} \mathrm{inv}_p(V) p^{n\nu-\nu+\frac{n}{2}} \sum_{\alpha=0}^{\nu-1} p^{n\alpha} \sum_{a'\in(\mathbb{Z}/p^{\nu-\alpha}\mathbb{Z})^\times} e\Big(-\frac{a'm}{p^{\nu-\alpha}}\Big) \\ &= p^{2n\nu-\nu} + \left(\frac{-1}{p}\right)^{n/2} \mathrm{inv}_p(V) p^{n\nu-\nu+\frac{n}{2}} \sum_{\alpha=0}^{\nu-1} p^{n\alpha} \sum_{d\mid\gcd(p^{\nu-\alpha},m)} \mu\Big(\frac{p^{\nu-\alpha}}{d}\Big)d \\ &= p^{2n\nu-\nu} + \left(\frac{-1}{p}\right)^{n/2} \mathrm{inv}_p(V) p^{n\nu-\nu+\frac{n}{2}} \sum_{\alpha=0}^{\nu-1} p^{n\alpha} \sum_{\gamma=0}^{\min(\nu-\alpha,\beta)} \mu\big(p^{\nu-\alpha-\gamma}\big)p^{\gamma}. \end{aligned}$$

Consequently, the Euler factor $L_m^{(p)}(p^{-s})$ of (2.5.6) is

$$\begin{aligned} &(1-p^{2n-s-1})\sum_{\nu=0}^{\infty} N_m(p^\nu)p^{-\nu s} \\ &= 1 + \left(\frac{-1}{p}\right)^{\frac{n}{2}} \mathrm{inv}_p(V)p^{\frac{n}{2}}(1-p^{2n-s-1})\sum_{\nu=1}^{\infty} p^{(n-1-s)\nu}\sum_{\alpha=0}^{\nu-1}\sum_{\gamma=0}^{\min(\nu-\alpha,\beta)} p^{n\alpha+\gamma}\mu(p^{\nu-\alpha-\gamma}) \\ &= 1 + \left(\frac{-1}{p}\right)^{\frac{n}{2}} \mathrm{inv}_p(V)p^{\frac{n}{2}}(1-p^{2n-s-1})\sum_{\gamma=0}^{\beta}\sum_{\alpha=0}^{\infty}\sum_{\substack{\nu\geq\alpha+\gamma\\ \nu\geq\alpha+1}} p^{(n-1-s)\nu}p^{n\alpha+\gamma}\mu(p^{\nu-\alpha-\gamma}). \end{aligned}$$

The contribution of $\gamma = 0$ to the latter sum is given by

$$(1-p^{2n-s-1})\sum_{\alpha=0}^{\infty}\sum_{\nu\geq\alpha+1} p^{(n-1-s)\nu}p^{n\alpha}\mu(p^{\nu-\alpha}) = -p^{n-1-s}.$$

The contribution of $\gamma \geq 1$ is equal to

$$\begin{aligned}
&(1-p^{2n-s-1})\sum_{\gamma=1}^{\beta}\sum_{\alpha=0}^{\infty}\sum_{\nu\geq\alpha+\gamma} p^{n\alpha+\gamma}p^{(n-1-s)\nu}\mu(p^{\nu-\alpha-\gamma})\\
&=(1-p^{2n-s-1})\sum_{\gamma=1}^{\beta}\sum_{\alpha=0}^{\infty} p^{n\alpha+\gamma}\left(p^{(n-1-s)(\alpha+\gamma)}-p^{(n-1-s)(\alpha+\gamma+1)}\right)\\
&=\left(1-p^{n-1-s}\right)\sum_{\gamma=1}^{\beta} p^{(n-s)\gamma}.
\end{aligned}$$

Putting the terms together, we find

$$\begin{aligned}
L_m^{(p)}(p^{-s}) = 1 &- \left(\frac{-1}{p}\right)^{\frac{n}{2}} \mathrm{inv}_p(V)p^{\frac{n}{2}}p^{n-1-s}\\
&+\left(\frac{-1}{p}\right)^{\frac{n}{2}} \mathrm{inv}_p(V)p^{\frac{n}{2}}\left(1-p^{n-1-s}\right)\sum_{\gamma=1}^{\beta}p^{(n-s)\gamma}.
\end{aligned}$$

This concludes the proof of the proposition. □

Proposition 2.6.4. *Assume n is odd, and $m \in \mathbb{Z}$ is nonzero. Fix a prime p, and factor $m = p^{\beta}m'$ with $\gcd(m', p) = 1$. If $p \nmid D$ then*

$$L_m^{(p)}(p^{-s}) = \left(1-\left(\frac{-D}{p}\right)p^{n-s-1}\right)\sum_{\gamma=0}^{\beta}\left(\frac{-D}{p}\right)^{\gamma}p^{(n-s)\gamma}.$$

If $p \mid D$ then

$$L_m^{(p)}(p^{-s}) = 1+\mathrm{inv}_p(V)\left(\frac{-1}{p}\right)^{\frac{n-1}{2}}\left(\frac{D'}{p}\right)^{\beta}\left(\frac{m'}{p}\right)p^{(\beta+3/2)n-1/2-(\beta+1)s},$$

where we have set $D' = D/p$.

Proof. We argue in the same way as in the proof of Proposition 2.6.3, using the identity

$$N_m(p^{\nu}) = \frac{1}{p^{\nu}}\sum_{a\in\mathbb{Z}/p^{\nu}\mathbb{Z}} G_L(a,p^{\nu})e\Big(-\frac{am}{p^{\nu}}\Big)$$

and the value of the Gauss sum $G_L(a, p^{\nu})$ computed in Lemma 2.6.2. If we write $a = p^{\alpha}a'$ with a' coprime to p, we have

$$G_L(a,p^{\nu}) = \begin{cases} p^{2n\nu} & \text{if } \alpha=\nu\\ p^{n\alpha+n\nu}\left(\frac{-D}{p}\right)^{\nu-\alpha} & \text{if } \alpha<\nu \text{ and } p\nmid D\\ \epsilon_p^n\,\mathrm{inv}_p(V)p^{n\alpha+n\nu+\frac{n}{2}}\left(\frac{a'}{p}\right)\left(\frac{D'}{p}\right)^{\nu-\alpha-1} & \text{if } \alpha<\nu \text{ and } p\mid D.\end{cases}$$

If $p \nmid D$ we obtain

$$\begin{aligned} N_m(p^\nu) &= p^{n\nu-\nu}\sum_{\alpha=0}^{\nu} p^{n\alpha}\left(\frac{-D}{p}\right)^{\nu-\alpha} \sum_{a'\in(\mathbb{Z}/p^{\nu-\alpha}\mathbb{Z})^\times} e\Big(-\frac{a'm}{p^{\nu-\alpha}}\Big)\\ &= p^{n\nu-\nu}\sum_{\alpha=0}^{\nu} p^{n\alpha}\left(\frac{-D}{p}\right)^{\nu-\alpha} \sum_{d\mid\gcd(p^{\nu-\alpha},m)} \mu\Big(\frac{p^{\nu-\alpha}}{d}\Big)d\\ &= p^{n\nu-\nu}\sum_{\alpha=0}^{\nu} p^{n\alpha}\left(\frac{-D}{p}\right)^{\nu-\alpha} \sum_{\gamma=0}^{\min(\nu-\alpha,\beta)} \mu\big(p^{\nu-\alpha-\gamma}\big)p^{\gamma}. \end{aligned}$$

Here we have used the evaluation of the Ramanujan sum. Hence, the Euler factor (2.5.6) is

$$\begin{aligned} L_m^{(p)}(p^{-s}) &= (1-p^{2n-s-1})\sum_{\nu=0}^{\infty} N_m(p^\nu)p^{-\nu s}\\ &= (1-p^{2n-s-1})\sum_{\gamma=0}^{\beta}\sum_{\alpha=0}^{\infty}\sum_{\nu=\alpha+\gamma}^{\infty} p^{n\alpha+\gamma}\left(\frac{-D}{p}\right)^{\nu-\alpha}\mu(p^{\nu-\alpha-\gamma})p^{(n-1-s)\nu}\\ &= \left(1-\left(\frac{-D}{p}\right)p^{n-s-1}\right)\sum_{\gamma=0}^{\beta}\left(\frac{-D}{p}\right)^{\gamma}p^{(n-s)\gamma}. \end{aligned}$$

When $p \mid D$ we find

$$\begin{aligned} N_m(p^\nu) = p^{2n\nu-\nu} &+ \epsilon_p^n \operatorname{inv}_p(V)p^{n\nu-\nu+\frac{n}{2}}\\ &\times \sum_{\alpha=0}^{\nu-1}\left(\frac{D'}{p}\right)^{\nu-\alpha-1} p^{n\alpha} \sum_{a'\in(\mathbb{Z}/p^{\nu-\alpha}\mathbb{Z})^\times}\left(\frac{a'}{p}\right) e\Big(-\frac{a'm}{p^{\nu-\alpha}}\Big). \end{aligned}$$

The latter Gauss sum is equal to $p^{\nu-\alpha-\frac12}\epsilon_p\left(\frac{-m'}{p}\right)$ if $\beta=\nu-\alpha-1$ and is zero otherwise. Inserting this we find that $N_m(p^\nu)$ is equal to

$$p^{2n\nu-\nu}\cdot\begin{cases} 1+\operatorname{inv}_p(V)\left(\frac{-1}{p}\right)^{\frac{n+1}{2}}\left(\frac{D'}{p}\right)^{\beta}\left(\frac{-m'}{p}\right)p^{(1-n)(\beta+1/2)} & \text{if } \nu>\beta\\ 1 & \text{if } \nu\le\beta, \end{cases}$$

and hence $L_m^{(p)}(p^{-s})$ is equal to

$$\begin{aligned} &(1-p^{2n-s-1})\sum_{\nu=0}^{\infty}N_m(p^\nu)p^{-\nu s}\\ &= 1+\operatorname{inv}_p(V)\left(\frac{-1}{p}\right)^{\frac{n+1}{2}}\left(\frac{D'}{p}\right)^{\beta}\left(\frac{-m'}{p}\right)p^{(1-n)(\beta+1/2)}p^{(2n-1-s)(\beta+1)}\\ &= 1+\operatorname{inv}_p(V)\left(\frac{-1}{p}\right)^{\frac{n-1}{2}}\left(\frac{D'}{p}\right)^{\beta}\left(\frac{m'}{p}\right)p^{(\beta+3/2)n-1/2-(\beta+1)s}. \end{aligned}$$

This concludes the proof of the proposition. □

Recall the function $\mathbf{a}_n(s)$ of (1.1.4), and abbreviate $s_0 = (n-1)/2$. We now express the coefficients $B(m,0,s)$ in terms of

$$\mathbf{a}_n(s-s_0) = \frac{D^{n/2}\Gamma\left(s+\frac{n}{2}+\frac{1}{2}\right) L\left(2s+1,\varepsilon^n\right)}{2^n \pi^{s+\frac{n}{2}+\frac{1}{2}}}.$$

Corollary 2.6.5. *Assume that $n \geq 2$ is even. If $m > 0$ then*

$$B(m,0,s) = -\frac{m^{s+\frac{n}{2}-\frac{1}{2}}}{\mathbf{a}_n(s-s_0)} \cdot \prod_{p\nmid D} \sum_{\gamma=0}^{v_p(m)} p^{-2s\gamma} \cdot \prod_{p\mid D} \frac{L_m^{(p)}(p^{-2s-n})}{1-p^{-2s-1}}.$$

If $m > 0$ is prime to D then

$$B(m,0,s) = -\frac{m^{s+\frac{n}{2}-\frac{1}{2}}\sigma_{-2s}(m)}{\mathbf{a}_n(s-s_0)} \cdot \prod_{p\mid D} \frac{1-\left(\frac{-1}{p}\right)^{\frac{n}{2}} \mathrm{inv}_p(V) p^{\frac{n}{2}-1-2s}}{1-p^{-2s-1}},$$

where $\sigma_s(m) = \sum_{d\mid m} d^s$ is the usual divisor sum.

Proof. Combine Propositions 2.5.3 and 2.6.3. □

Corollary 2.6.6. *Assume that $n \geq 1$ is odd. If $m > 0$ then*

$$B(m,0,s) = -\frac{m^{s+\frac{n}{2}-\frac{1}{2}}}{\mathbf{a}_n(s-s_0)} \cdot \prod_{p\nmid D} \sum_{\gamma=0}^{v_p(m)} \varepsilon(p)^\gamma p^{-2s\gamma} \cdot \prod_{p\mid D} L_m^{(p)}(p^{-2s-n}).$$

If $m > 0$ is prime to D, then

$$B(m,0,s) = -\frac{m^{s+\frac{n}{2}-\frac{1}{2}}\sigma_{-2s,\varepsilon}(m)}{\mathbf{a}_n(s-s_0)} \cdot \prod_{p\mid D} \left(1+\left(\frac{-1}{p}\right)^{\frac{n-1}{2}} \left(\frac{m}{p}\right) \mathrm{inv}_p(V) p^{\frac{n-1}{2}-2s}\right),$$

where $\sigma_{s,\varepsilon}(m) = \sum_{d\mid m} \varepsilon(d) d^s$ is the usual divisor sum.

Proof. Combine Propositions 2.5.3 and 2.6.4. □

Remark 2.6.7. If $n = 1$, then $s_0 = 0$ is the center of symmetry of the functional equation of the incoherent Eisenstein series $E_L(\tau, s, 1)$. Hence $E_L(\tau, s, 1)$ vanishes identically (to odd order) at $s = 0$. For all positive m this implies that the coefficients $B(m, \mu, s_0)$ also vanish. Consequently, in view of Proposition 2.5.3, there must be at least one prime $p < \infty$ for which the local factor $L_m^{(p)}(p^{-1})$ is zero. According to Proposition 2.6.4, this happens exactly when

$$\mathrm{inv}_p(V) = -\begin{cases} \left(\frac{-D}{p}\right)^\beta, & \text{if } p \nmid D, \\ \left(\frac{D'}{p}\right)^\beta \left(\frac{m'}{p}\right), & \text{if } p \mid D, \end{cases}$$

where we have used the notation of the cited proposition. In terms of the local Hilbert symbol this is equivalent to the condition $\mathrm{inv}_p(V) = -(-D, m)_p$, which means that V_p does not represent m.

The derivatives $B'(m, 0, s_0)$ of the coefficients can be computed as follows. Let $\mathbb{V}$ be the incoherent hermitian space over $\mathbb{A}_{\boldsymbol{k}}$ whose non-archimedian contribution is given by $V \otimes_{\mathbb{Q}} \mathbb{A}_f$ and whose archimedian contribution is $V \otimes_{\mathbb{Q}} \mathbb{R}$ with the positive definite hermitian form $-\langle \cdot, \cdot \rangle$. Following [Ku97] we define the 'Diff set' associated with $\mathbb{V}$ and $m \in \mathbb{Q}^\times$ by

$$\mathrm{Diff}(\mathbb{V}, m) = \{p \leq \infty \mid \ \mathbb{V}_p \text{ does not represent } m\}.$$

This is a finite set of places of $\mathbb{Q}$ of odd cardinality. When $m > 0$ it only consists of finite primes. The above argument shows that $L_m^{(p)}(p^{-1}) = 0$ if and only if $p \in \mathrm{Diff}(\mathbb{V}, m)$. Hence $B'(m, 0, s_0) \neq 0$ if and only if $\mathrm{Diff}(\mathbb{V}, m)$ consists of a single prime q. By Proposition 2.5.3 we find for $m \in \mathbb{Z}^+$ in this case that

$$B'(m, 0, 0) = -\frac{2\pi}{\sqrt{D}} \cdot \prod_{\substack{p \text{ prime} \\ p \neq q}} L_m^{(p)}(p^{-1}) \cdot \frac{d}{ds} L_m^{(q)}(q^{-2s-1})\big|_{s=0}.$$

This expression can be evaluated explicitly by means of Proposition 2.6.4. For instance, when $(m, D) = 1$ and $m = q^\beta m'$ with m' coprime to q, we see that $\mathrm{inv}_q(V) = 1$, the prime q must be inert in $\boldsymbol{k}$, and β must be odd. Under these conditions we find that

$$B'(m, 0, 0) = -\frac{2^{o(D)+1}\pi\, \sigma_{0,\varepsilon}(m')}{\sqrt{D} L(\varepsilon, 1)} (\beta + 1) \cdot \log(q),$$

where $o(D)$ denotes the number of prime factors of D.

Regardless of whether n is even or odd, we have the following lower bound for the coefficients of the Eisenstein series $E_L(\tau, s_0, n)$.

Corollary 2.6.8. *Assume that $n \geq 2$. There is a constant $C > 0$, depending only on L, such that*

$$-B(m, 0, s_0) > C \cdot m^{n-1}$$

for all $m \in \mathbb{Z}^+$.

Proof. This follows easily from Corollary 2.6.5 and Corollary 2.6.6. □

3. Automorphic Green functions

Keep V as in §2. We recall the family of special divisors on the complex Shimura variety associated to the unitary group $\mathrm{U}(V)$, and derive a geometric variant of the Siegel-Weil formula. Using this, we compute the integrals of automorphic Green functions for the special divisors.

3.1. **A complex Shimura variety.** The fixed embedding $\boldsymbol{k} \subset \mathbb{C}$ identifies $\boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$, and allows us to distinguish between the two orthogonal idempotents $\epsilon, \overline{\epsilon} \in \boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{C}$, which we label in such a way that

$$(\alpha \otimes 1)\epsilon = (1 \otimes \alpha)\epsilon, \qquad (\alpha \otimes 1)\overline{\epsilon} = (1 \otimes \overline{\alpha})\overline{\epsilon}$$

for all $\alpha \in \boldsymbol{k}$.

Viewing $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$ as $\mathbb{C}$-hermitian space of signature $(n-1,1)$, define a hermitian symmetric domain

$$\mathcal{D} = \{\text{negative definite complex lines } z \subset V(\mathbb{R})\}. \tag{3.1.1}$$

The natural map

$$V(\mathbb{R}) \hookrightarrow V(\mathbb{C}) = \epsilon V(\mathbb{C}) \oplus \overline{\epsilon} V(\mathbb{C}) \xrightarrow{\text{proj}} \epsilon V(\mathbb{C})$$

is a $\mathbb{C}$-linear isomorphism, and identifies

$$\mathcal{D} = \{z \in \mathbb{P}(\epsilon V(\mathbb{C})) : [z, \overline{z}] < 0\}. \tag{3.1.2}$$

Here we have endowed $V(\mathbb{C}) = V \otimes_{\mathbb{Q}} \mathbb{C}$ with the $\mathbb{C}$-bilinear extension of the $\mathbb{Q}$-bilinear form $[-,-]$ of (2.1.1), and $z \mapsto \overline{z}$ is the complex conjugation on the second factor in the tensor product. Under either interpretation, there is an evident action of the real points of

$$H = \mathrm{U}(V)$$

on $\mathcal{D}$ by holomorphic automorphisms.

Fix an $\mathcal{O}_{\boldsymbol{k}}$-lattice $L \subset V$ on which the hermitian form is $\mathcal{O}_{\boldsymbol{k}}$-valued. Choose a compact open subgroup $K \subset H(\mathbb{A}_f)$ that stabilizes L, and hence acts on the finite dimensional vector space $S_L = \mathbb{C}[L'/L]$ of §2.4. This choice determines a complex Shimura variety

$$\mathrm{Sh}_K(H, \mathcal{D}) = H(\mathbb{Q}) \backslash \mathcal{D} \times H(\mathbb{A}_f)/K, \tag{3.1.3}$$

which we view as a complex orbifold of dimension $n-1$. If we write $H(\mathbb{A}_f) = \bigsqcup_j H(\mathbb{Q}) h_j K$ as a finite disjoint union, then

$$\mathrm{Sh}_K(H, \mathcal{D}) = \bigsqcup_j (H(\mathbb{Q}) \cap K_j) \backslash \mathcal{D}, \tag{3.1.4}$$

where $K_j = h_j K h_j^{-1}$ stabilizes the lattice $L_j = h_j L$.

The Shimura variety carries a family of special divisors

$$Z(m) \in \mathrm{Div}(\mathrm{Sh}_K(H, \mathcal{D}))$$

indexed by positive integers m, which we describe in terms of the decomposition (3.1.4). For each $x \in V$ with $\langle x, x \rangle > 0$, define an analytic divisor

$$\mathcal{D}(x) = \{z \in \mathcal{D} : x \perp z\}$$

where $\perp$ means orthogonal with respect to the hermitian or bilinear form, depending on whether the reader prefers the model (3.1.1) or (3.1.2). The pullback of $Z(m)$ via the uniformization

$$\mathcal{D} \to (H(\mathbb{Q}) \cap K_j) \backslash \mathcal{D} \subset \mathrm{Sh}_K(H, \mathcal{D})$$

is the locally finite analytic divisor

$$\sum_{\substack{x \in L_j \\ \langle x, x \rangle = m}} \mathcal{D}(x) \in \mathrm{Div}(\mathcal{D}).$$

More generally, given a positive $m \in \mathbb{Q}$ and a K-fixed function $\varphi \in S_L$, there is a special divisor

$$Z(m, \varphi) \in \mathrm{Div}_{\mathbb{C}}(\mathrm{Sh}_K(H, \mathcal{D}))$$

whose pullback via the above uniformization is

$$\sum_{\mu \in L'/L} \varphi(\mu) \sum_{\substack{x \in \mu_j + L_j \\ \langle x, x \rangle = m}} \mathcal{D}(x) \in \mathrm{Div}_{\mathbb{C}}(\mathcal{D}),$$

where μ_j is the image of μ under $h_j : L'/L \to L'_j/L_j$. Taking $\varphi = \varphi_0$ to be the characteristic function of $0 \in L'/L$ recovers $Z(m)$.

Let $\mathcal{L}$ be the tautological line bundle over $\mathcal{D}$ whose fiber over a point $z \in \mathcal{D}$ is the complex line $\mathcal{L}_z = z$. Denote by $\widehat{\mathcal{L}}$ the tautological line bundle endowed with the hermitian metric $\| \cdot \|$ defined as follows: if we use the interpretation (3.1.1) to view $z \in \mathcal{D}$ as a line in $V(\mathbb{R})$, then

$$\|s\|^2 = -\frac{\langle s, s \rangle}{4\pi e^{\gamma}}$$

for every $s \in \mathcal{L}_z$. If we instead use (3.1.2) to view z as a line in $\epsilon V(\mathbb{C})$, the metric is the same, but with $\langle -, - \rangle$ replaced by $[-, -]$. The Chern form $\mathrm{ch}(\widehat{\mathcal{L}})$ is an $H(\mathbb{R})$-invariant Kähler form on $\mathcal{D}$. The metrized tautological bundle descends to $\mathrm{Sh}_K(H, \mathcal{D})$, and its holomorphic sections are hermitian modular forms of weight 1. As in the introduction, we abbreviate

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}) = \int_{\mathrm{Sh}_K(H, \mathcal{D})} \mathrm{ch}(\widehat{\mathcal{L}})^{n-1}.$$

In the subsections below we will recall the construction of Green functions for certain linear combinations of special divisors, and compute their integrals with respect to the volume form $\mathrm{ch}(\widehat{\mathcal{L}})^{n-1}$.

Remark 3.1.1. Although we will not need it, we note that

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}})^{-1} \int_{Z(m,\varphi)} \mathrm{ch}(\widehat{\mathcal{L}})^{n-2} = - \sum_{\mu \in L'/L} \varphi(\mu) B(m, \mu, s_0)$$

exactly as in the case of orthogonal Shimura varieties [Ku03]. On the right hand side the notation is that of §2.5.

3.2. **A geometric variant of the Siegel-Weil formula.** For $z \in \mathcal{D}$ and $x \in V(\mathbb{R})$, let

$$\varphi_\infty(z, x) = \exp\big(- 2\pi \langle x_{z^\perp}, x_{z^\perp} \rangle + 2\pi \langle x_z, x_z \rangle \big)$$

be the Gaussian. Here we are using the model (3.1.1), so that z determines an orthogonal decomposition $V(\mathbb{R}) = z^\perp \oplus z$, and the vectors $x_{z^\perp}$ and x_z are the projections of x to the two summands.

As in §2.1, let $G = \mathrm{U}(1,1)$ be the quasi-split unitary group over $\mathbb{Q}$, so that $\mathrm{SL}_2 \subset G$. Fix a base point $z_0 \in \mathcal{D}$ and consider the Schwartz function

$$\varphi_\infty(x) \stackrel{\mathrm{def}}{=} \varphi_\infty(z_0, x) \in S(V(\mathbb{R})).$$

It has weight $n-2$ under the action of the Weil representation (§2.3) of $\mathrm{SL}_2(\mathbb{R})$. In other words, using the notation of (2.2.1),

$$\omega(k)(\varphi_\infty) = \underline{k}^{n-2}\varphi_\infty$$

for all $k \in \mathrm{SO}(2,\mathbb{R})$.

A choice of $\varphi_f \in S(V(\mathbb{A}_f))$ determines a Schwartz function $\varphi_\infty \otimes \varphi_f \in S(V(\mathbb{A}))$. Applying the construction (2.3.1) yields a corresponding theta function $\theta(g,h,\varphi_\infty\otimes\varphi_f)$ of the variables $(g,h)\in G(\mathbb{A})\times H(\mathbb{A})$. Restricting the first variable to $g \in \mathrm{SL}_2(\mathbb{R}) \subset G(\mathbb{A})$, we obtain the Siegel theta function

$$\theta(g,h,\varphi_\infty\otimes\varphi_f) = \sum_{x\in V(\mathbb{Q})} \omega(g,1)\varphi_\infty(z_0,h_\infty^{-1}x)\cdot\varphi_f(h_f^{-1}x),$$

where $h = h_\infty h_f \in H(\mathbb{A})$.

We may view this theta function as a function of the variables

$$(\tau, z, h_f) \in \mathcal{H}\times\mathcal{D}\times H(\mathbb{A}_f)$$

as follows. Let $g_\tau \in \mathrm{SL}_2(\mathbb{R})$ be as in (2.1.3), and choose an $h_\infty \in H(\mathbb{R})$ with $h_\infty z_0 = z$, so that $\varphi_\infty(z_0,h_\infty^{-1}x) = \varphi_\infty(z,x)$. Now set

$$\begin{aligned}(3.2.1)\qquad \theta(\tau,z,h_f,\varphi_f) &= v^{1-\frac{n}{2}}\theta(g_\tau, h_\infty h_f, \varphi_\infty\otimes\varphi_f)\\ &= v\sum_{x\in V(\mathbb{Q})} e\big(\langle x_{z^\perp}, x_{z^\perp}\rangle\tau - \langle x_z,x_z\rangle\bar{\tau}\big)\cdot\varphi_f(h_f^{-1}x).\end{aligned}$$

In the variable τ, we have the transformation law

$$\theta(\gamma\tau,z,h_f,\varphi_f) = (c\tau+d)^{n-2}\theta(\tau,z,h_f,\omega(\gamma)^{-1}\varphi_f) \tag{3.2.2}$$

for $\gamma\in\Gamma$. In the variable z, we have the transformation law

$$\theta(\tau,\delta z,\delta h_f,\varphi_f) = \theta(\tau,z,h_f,\varphi_f). \tag{3.2.3}$$

for $\delta \in H(\mathbb{Q})$,

Now suppose that $\varphi_f \in S_L = \mathbb{C}[L'/L]$ is fixed by the action of K. Using (3.2.1) and (3.2.3), we see that the Siegel theta function (3.2.1) descends to a function on $\mathcal{H}\times\mathrm{Sh}_K(H,\mathcal{D})$. We are interested in the geometric theta integral

$$\mathcal{I}(\tau,\varphi_f) = \mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}})^{-1}\int_{\mathrm{Sh}_K(H,\mathcal{D})}\theta(\tau,z,h_f,\varphi_f)\ \mathrm{ch}(\widehat{\mathcal{L}})^{n-1} \tag{3.2.4}$$

as a function of $\tau\in\mathcal{H}$.

Proposition 3.2.1. *Assume that $n>2$ or that V is anisotropic. Recalling the notation of* (2.3.2) *and* (2.4.4)*, we have*

$$\kappa\cdot\mathcal{I}(\tau,\varphi_f) = \langle E_L(\tau,s_0,n-2),\varphi_f\rangle\,,$$

where the pairing on the right is (2.4.3).

Proof. We rewrite the integral in (3.2.4) as an integral over the quotient $H(\mathbb{Q})\backslash H(\mathbb{A})$. To this end, we normalize the Haar measure on the compact subgroup $H(\mathbb{R})_{z_0} \subset H(\mathbb{R})$ so that the volume is 1. Then we normalize the Haar measure dh_∞ on $H(\mathbb{R})$ so that the induced measure on the quotient $H(\mathbb{R})/H(\mathbb{R})_{z_0} \cong \mathcal{D}$ agrees with the measure defined by $\mathrm{ch}(\widehat{\mathcal{L}})^{n-1}$. Finally, we normalize the Haar measure dh_f on $H(\mathbb{A}_f)$ so that $\mathrm{vol}(H(\mathbb{Q})\backslash H(\mathbb{A})) = 1$. Setting

$$\mathrm{vol}(K) = \int_K dh_f,$$

we then have the identities

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})/H(\mathbb{R})_{z_0}K} dh_\infty dh_f = \mathrm{vol}(K)^{-1}.$$

Using (3.2.1) we obtain

$$\begin{aligned}\mathcal{I}(\tau,\varphi) &= \mathrm{vol}(K) \int_{\mathrm{Sh}_K(H,\mathcal{D})} \theta(\tau, z, h_f, \varphi_f)\ \mathrm{ch}(\widehat{\mathcal{L}})^{n-1}\\ &= v^{1-\frac{n}{2}} \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g_\tau, h_\infty h_f, \varphi_\infty \otimes \varphi_f)\, dh_\infty dh_f,\end{aligned}$$

and applying the Siegel-Weil formula of Theorem 2.3.1 shows

$$\kappa \cdot \mathcal{I}(\tau,\varphi) = v^{1-\frac{n}{2}} E(g_\tau, \lambda(\varphi_\infty \otimes \varphi_f), s_0).$$

As φ_∞ has weight $n-2$, Lemma 2.2.1 implies

$$\begin{aligned}&E(g_\tau, \lambda(\varphi_\infty \otimes \varphi_f), s_0)\\ &= v^{\frac{n}{2}-1} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma\tau)^{s_0+\frac{1}{2}-\frac{n-2}{2}} (c\tau+d)^{2-n} \cdot (\omega_L(\gamma)\varphi_f)(0)\\ &= v^{\frac{n}{2}-1} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma\tau)^{s_0+\frac{1}{2}-\frac{n-2}{2}} (c\tau+d)^{2-n} \cdot (\omega_L^\vee(\gamma)^{-1}\delta_0)(\varphi_f).\end{aligned}$$

Here $\delta_0 \in S_L^\vee$ denotes the functional $\varphi_f \mapsto \varphi_f(0)$. Since the Weil representation is unitary, we have

$$(\omega_L^\vee(\gamma)^{-1}\delta_0)(\varphi_f) = \langle \bar{\omega}_L(\gamma)^{-1}\varphi_0, \varphi_f\rangle.$$

To complete the proof, substitute this equality into the final expression above, and recall the definition of $E_L(\tau, s, n-2)$ from (2.4.4). ☐

3.3. **Automorphic Green functions.** Let σ be a finite dimensional representation of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on a complex vector space V_σ, and assume that σ factors through a finite quotient. In our applications, σ will be the Weil representation

$$\omega_L : \Gamma \to \mathrm{Aut}(S_L)$$

of (2.4.2), or its complex conjugate, or its dual.

For $k \in \mathbb{Z}$, denote by $H_k(\sigma)$ the vector space of (weak) harmonic Maass forms of weight k for the group Γ with representation σ as in [BY09]. We write

$$S_k(\sigma) \subset M_k(\sigma) \subset M_k^!(\sigma)$$

for the subspaces of cusp forms, holomorphic modular forms, and weakly holomorphic modular forms, respectively.

A harmonic Maass form $f \in H_k(\sigma)$ has a Fourier expansion

$$f(\tau) = \sum_{\substack{m\in\mathbb{Q} \\ m\gg-\infty}} c^+(m)q^m + \sum_{\substack{m\in\mathbb{Q} \\ m<0}} c^-(m)\Gamma(1-k, 4\pi|m|v)q^m \tag{3.3.1}$$

with coefficients $c^{\pm}(m) \in V_\sigma$. Here $\tau = i + iv \in \mathcal{H}$, $q = e^{2\pi i\tau}$, and

$$\Gamma(s, x) = \int_x^\infty e^{-t}t^{s-1}dt$$

is the incomplete gamma function. The coefficients are supported on rational numbers with uniformly bounded denominators. The first summand on the right hand side of (3.3.1) is denoted by f^+ and is called the holomorphic part of f, the second summand is denoted by f^- and is called the non-holomorphic part.

As in [BF04], there is a surjective conjugate-linear differential operator

$$\xi_k : H_k(\omega_L) \to S_{2-k}(\bar{\omega}_L) \tag{3.3.2}$$

defined by

$$\xi_k(f)(\tau) = 2iv^k\overline{\frac{\partial f}{\partial\bar{\tau}}}.$$

Its kernel is $M_k^!(\omega_L)$.

Suppose $f \in H_{2-n}(\omega_L)$ is K-fixed with Fourier coefficients $c^{\pm}(m) \in S_L$. Define the special divisor associated to f by

$$Z(f) = \sum_{m>0} \sum_{\mu\in L'/L} c^+(-m, \mu)Z(m, \varphi_\mu) \in \mathrm{Div}_{\mathbb{C}}(\mathrm{Sh}_K(H, \mathcal{D})). \tag{3.3.3}$$

Here $c^{\pm}(m, \mu) \in \mathbb{C}$ is the value of $c^{\pm}(m)$ at $\mu \in L'/L$, and φ_μ is the characteristic function of the coset $\mu + \hat{L}$.

We recall from §3 of [BHY15] the construction of an automorphic Green function for $Z(f)$. Using (3.2.1), define an S_L-valued Siegel theta function

$$\theta_L(\tau, z, h) = \sum_{\mu\in L'/L} \theta(\tau, z, h, \varphi_\mu)\varphi_\mu \tag{3.3.4}$$

for $\tau \in \mathcal{H}$, $z \in \mathcal{D}$, and $h \in H(\mathbb{A}_f)$. The transformation law (3.2.2) implies that it defines a non-holomorphic modular form for Γ of weight $n-2$ with representation $\bar{\omega}_L$, and so, using the notation (2.4.3), the pairing $\langle f, \theta_L(\tau, z, h)\rangle$ is Γ-invariant as a function of $\tau \in \mathcal{H}$.

Following [Bo98] and [BF04], we consider the regularized theta lift

$$\Phi(z,h,f) = \int_{\Gamma\backslash\mathcal{H}}^{\mathrm{reg}} \langle f, \theta_L(\tau,z,h)\rangle \, d\mu(\tau) \tag{3.3.5}$$

of f, where $d\mu(\tau) = \frac{du\,dv}{v^2}$ is the invariant measure. Here the integral is regularized as in [Bo98] and [BHY15]. The main properties of

$$\Phi(f) = \Phi(z,h,f),$$

viewed as a function on $\mathrm{Sh}_K(H,\mathcal{D})$, are summarized in the following theorem, see Theorem 3.3.1 and Proposition 3.3.4 of [BHY15].

Theorem 3.3.1. *The Green function $\Phi(z,h,f)$ is smooth on $\mathrm{Sh}_K(H,\mathcal{D}) \smallsetminus Z(f)$. It has logarithmic singularities along $Z(f)$, and it is integrable over $\mathrm{Sh}_K(H,\mathcal{D})$ if $n>2$ or if V is anisotropic. The corresponding current on smooth compactly supported $(n-1,n-1)$-forms on $\mathrm{Sh}_K(H,\mathcal{D})$ satisfies the Green equation*

$$dd^c[\Phi(f)] + \delta_{Z(f)} = [dd^c\Phi(f)].$$

3.4. **Integrals of Green functions.** In this subsection we assume that either $n>2$, or that $n\geq 1$ and V is anisotropic. Choose an S_L-valued harmonic Maass form

$$f \in H_{2-n}(\omega_L)$$

fixed by the action of the compact open subgroup $K \subset H(\mathbb{A}_f)$ on S_L.

The main result of this subsection expresses the integral

$$\mathcal{I}(f) = \mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}})^{-1} \int_{\mathrm{Sh}_K(H,\mathcal{D})} \Phi(f)\,\mathrm{ch}(\widehat{\mathcal{L}})^{n-1} \tag{3.4.1}$$

in terms of the derivatives $B'(m,\mu,s_0)$ at $s_0=(n-1)/2$ of the coefficients (2.5.2) of $E_L(\tau,s,n)$. The proof follows the argument of [Ku03] in the orthogonal case, which is based on the Siegel-Weil formula. In fact, we slightly generalize this argument by allowing harmonic Maass forms for the inputs of the regularized theta lift. Let $\kappa\in\{1,2\}$ be as in (2.3.2).

Lemma 3.4.1. *We have*

$$\mathcal{I}(f) = \lim_{T\to\infty}\left(\frac{1}{\kappa}\int_{\mathcal{F}_T} \langle f,\, E_L(\tau,s_0,n-2)\rangle\, d\mu(\tau) - c^+(0,0)\log(T)\right).$$

Here $\mathcal{F}_T = \{\tau\in\mathcal{F} : \mathrm{Im}(\tau)\leq T\}$ is the truncation at height T of the standard fundamental domain $\mathcal{F}$ for Γ.

Proof. Exactly as in Proposition 2.5 of [Ku03], on $\mathrm{Sh}_K(H,\mathcal{D})\smallsetminus Z(f)$ we have the equality

$$\Phi(z,h,f) = \lim_{T\to\infty}\left(\int_{\mathcal{F}_T}\langle f,\theta_L(\tau,z,h)\rangle\, d\mu(\tau) - c^+(0,0)\log(T)\right).$$

The integral $\mathcal{I}(f)$ is therefore given by the limit as $T\to\infty$ of

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}})^{-1}\int_{\mathrm{Sh}_K(H,\mathcal{D})}\int_{\mathcal{F}_T}\langle f,\theta_L(\tau,z,h)\rangle\, d\mu(\tau)\,\mathrm{ch}(\widehat{\mathcal{L}})^{n-1} - c^+(0,0)\log(T).$$

As $\mathcal{F}_T$ is compact, we may interchange the order of integration. Inserting (3.3.4) and (3.2.4), we find that $\mathcal{I}(f)$ is the limit as $T \to \infty$ of

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}})^{-1} \int_{\mathcal{F}_T} \left\langle f, \int_{\mathrm{Sh}_K(H,\mathcal{D})} \theta_L(\tau, z, h)\, \mathrm{ch}(\widehat{\mathcal{L}})^{n-1} \right\rangle d\mu(\tau) - c^+(0,0)\log(T),$$

which we rewrite as

$$\int_{\mathcal{F}_T} \sum_{\mu \in L'/L} f(\tau)(\mu) \cdot \mathcal{I}(\tau, \varphi_\mu)\, d\mu(\tau) - c^+(0,0)\log(T).$$

Applying Proposition 3.2.1 completes the proof. □

Theorem 3.4.2. *If $n > 2$, or if $n = 2$ and V is anisotropic, the integral* (3.4.1) *satisfies*

$$\mathcal{I}(f) = \sum_{\mu \in L'/L} \sum_{\substack{m \in \mathbb{Q} \\ m > 0}} c^+(-m, \mu) B'(m, \mu, s_0).$$

If $n = 1$, the integral (3.4.1) *satisfies*

$$\mathcal{I}(f) = \frac{1}{2} \sum_{\mu \in L'/L} \sum_{\substack{m \in \mathbb{Q} \\ m \geq 0}} c^+(-m, \mu) B'(m, \mu, s_0).$$

Proof. Abbreviate

$$A_T(f) = \int_{\mathcal{F}_T} \langle f,\, E_L(\tau, s_0, n-2) \rangle \, d\mu(\tau),$$

so that Lemma 3.4.1 becomes

$$\mathcal{I}(f) = \lim_{T \to \infty} \left(\frac{A_T(f)}{\kappa} - c^+(0,0)\log(T) \right). \tag{3.4.2}$$

Using Lemma 2.4.1 we see that

$$E_L(\tau, s_0, n-2)\, d\mu(\tau) = \big(L_n E_L'(\tau, s_0, n)\big)\, d\mu(\tau) = -\bar{\partial} E_L'(\tau, s_0, n)\, d\tau,$$

and so

$$\begin{aligned} A_T(f) &= -\int_{\mathcal{F}_T} \big\langle f,\, \bar{\partial} E_L'(\tau, s_0, n) \big\rangle \, d\tau \\ &= -\int_{\mathcal{F}_T} d\left(\big\langle f,\, E_L'(\tau, s_0, n) \big\rangle \, d\tau \right) + \int_{\mathcal{F}_T} \big\langle (\bar{\partial} f),\, E_L'(\tau, s_0, n) \big\rangle \, d\tau. \end{aligned}$$

Applying Stokes' theorem to the first term, and

$$(\bar{\partial} f) d\tau = -(L_{2-n} f)\, d\mu(\tau)$$

to the second, we obtain

$$\begin{aligned} A_T(f) &= -\int_{\partial\mathcal{F}_T} \big\langle f,\, E_L'(\tau,s_0,n)\,\big\rangle\, d\tau - \int_{\mathcal{F}_T} \big\langle L_{2-n}f,\, E_L'(\tau,s_0,n)\big\rangle\, d\mu(\tau) \\ &= \int_{u=0}^{1} \big\langle f(u+iT),\, E_L'(u+iT,s_0,n)\,\big\rangle\, du \\ &\quad - \int_{\mathcal{F}_T} \Big\langle \overline{\xi_{2-n}(f)},\, E_L'(\tau,s_0,n)\Big\rangle\, v^n d\mu(\tau), \end{aligned}$$

where ξ_{2-n} is the differential operator (3.3.2). Inserting the Fourier expansions (3.3.1) and (2.5.1) of f and the Eisenstein series yields

$$\begin{aligned} A_T(f) = &\sum_{\substack{\mu\in L'/L\\ m\in\mathbb{Q}}} c^+(-m,\mu)B'(m,\mu;s_0,T) \\ &+ \sum_{\substack{\mu\in L'/L\\ m\in\mathbb{Q}^+}} c^-(-m,\mu)\Gamma(n-1,4\pi|m|T)B'(m,\mu;s_0,T) \\ &- \int_{\mathcal{F}_T} \Big\langle \overline{\xi_{2-n}(f)},\, E_L'(\tau,s_0,n)\Big\rangle\, v^n d\mu(\tau). \end{aligned}$$

The exponential decay of the incomplete gamma function and the polynomial growth of the coefficients $c^-(m,\mu)$ imply that the second term goes to zero in the limit $T\to\infty$. The third term converges to the Petersson scalar product of $\xi_{2-n}(f)$ and $E_L'(\tau,s_0,n)$. But since $\xi_{2-n}(f)$ is cuspidal and hence orthogonal to Eisenstein series, this Petersson scalar product vanishes. Inserting this into (3.4.2), we find

$$\mathcal{I}(f) = \lim_{T\to\infty}\Bigg(\frac{1}{\kappa}\sum_{\substack{\mu\in L'/L\\ m\in\mathbb{Q}}} c^+(-m,\mu)B'(m,\mu;s_0,T) - c^+(0,0)\log(T)\Bigg).$$

The exponential decay of the coefficients $B'(m,\mu;s_0,T)$ in (2.5.2) for $m<0$ and the subexponential growth of the coefficients $c^+(-m,\mu)$ imply that the contribution of all $m<0$ in the above term vanishes. If $n>1$ we obtain by virtue of Lemma 2.5.4 that

$$\mathcal{I}(f) = \frac{1}{\kappa}\sum_{\substack{\mu\in L'/L\\ m>0}} c^+(-m,\mu)B'(m,\mu,s_0),$$

as desired. When $n=1$, again by Lemma 2.5.4, there is an additional contribution to the sum from $m=0$. □

4. Integral models and arithmetic intersection theory

We recall the integral models of $\mathrm{GU}(n-1,1)$ Shimura varieties constructed by Pappas and Krämer, and the arithmetic intersection theory of Gillet-Soulé and Burgos-Kramer-Kühn on their toroidal compactifications.

From here until the end of the paper, we assume $D=-\mathrm{disc}(\boldsymbol{k})$ is odd.

4.1. **Moduli problems.** For any $n \geq 1$, work of Pappas [Pap00] and Krämer [Kra03], as summarized in §2.3 of [BHK$^+$a], provides us with a regular and flat $\mathcal{O}_{\boldsymbol{k}}$-stack

$$\mathcal{M}_{(n-1,1)} \to \operatorname{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

with reduced fibers. For an $\mathcal{O}_{\boldsymbol{k}}$-scheme S, the objects of the groupoid $\mathcal{M}_{(n-1,1)}(S)$ are quadruples $(A, \iota, \psi, \mathcal{F})$ in which

- $A \to S$ is an abelian scheme of dimension n,
- $\iota : \mathcal{O}_{\boldsymbol{k}} \to \operatorname{End}(A)$ is an $\mathcal{O}_{\boldsymbol{k}}$-action,
- $\psi : A \to A^\vee$ is a principal polarization whose induceded Rosati involution restricts to complex conjugation on the image of ι,
- $\mathcal{F} \subset \operatorname{Lie}(A)$ is an $\mathcal{O}_{\boldsymbol{k}}$-stable hyperplane[1] satisfying the signature $(n-1,1)$ condition of Krämer [Kra03]: the actions of $\mathcal{O}_{\boldsymbol{k}}$ on $\mathcal{F}$ and on $\operatorname{Lie}(A)/\mathcal{F}$ are through the structure morphism $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$ and its complex conjugate, respectively.

Remark 4.1.1. The stack $\mathcal{M}_{(n-1,1)}$ is denoted $\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)}$ in [BHK$^+$a].

Remark 4.1.2. One does not know a good theory of integral models $\mathcal{M}_{(n-1,1)}$ if 2 is ramified in $\boldsymbol{k}$, and this lack of knowledge is the main reason for restricting to quadratic imaginary fields of odd discriminant.

Definition 4.1.3. A finite dimensional $\boldsymbol{k}$-hermitian space W is *relevant* if it admits a self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice $\mathfrak{a} \subset W$. Two relevant hermitian spaces W and W' are *strictly similar* if there is a $\boldsymbol{k}$-linear isomorphism $W \cong W'$ identifying the hermitian forms up to scaling by a positive rational number.

Proposition 4.1.4. *Let $o(D)$ denote the number of prime divisors of D.*

1. *Any two self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattices in a relevant $\boldsymbol{k}$-hermitian space are isometric everywhere locally.*
2. *There are $2^{o(D)-1}$ isomorphism classes of relevant hermitian spaces of any fixed signature (r,s). If $r+s$ is odd all lie in the same strict similarity class, but if $r+s$ is even all lie in different strict similarity classes.*

Proof. The first claim is a theorem of Jacobowitz [Jac62], which uses our assumption that D is odd. The second claim is Lemma 2.11 of [KR14]. □

As in Proposition 2.12(i) of [KR14], there is a decomposition

$$\mathcal{M}_{(n-1,1)} = \bigsqcup_{W} \mathcal{M}_W, \tag{4.1.1}$$

where the disjoint union is over the strict similarity classes of relevant hermitian spaces W of signature $(n-1,1)$, and the generic fiber of each $\mathcal{M}_W$ is a Shimura variety of type $\mathrm{GU}(W)$. When n is odd, Proposition 4.1.4 implies that the disjoint union has a single term. When n is even, the disjoint union is over the $2^{o(D)-1}$ isomorphism classes of relevant $\boldsymbol{k}$-hermitian spaces.

[1] That is to say, an $\mathcal{O}_S$-module local direct summand of rank $n-1$

Denote by

$$\mathcal{M}_{(1,0)} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

the moduli stack of elliptic curves with CM by $\mathcal{O}_{\boldsymbol{k}}$. More precisely, the functor of points assigns to an $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid $\mathcal{M}_{(1,0)}(S)$ whose objects are pairs (A_0, ι_0) in which

- $A_0 \to S$ is an elliptic curve,
- $\iota_0 : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A_0)$ is an $\mathcal{O}_{\boldsymbol{k}}$-action satisfying the signature $(1,0)$ condition: the induced action of $\mathcal{O}_{\boldsymbol{k}}$ on $\mathrm{Lie}(A_0)$ is through the structure morphism $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$.

As A_0 is an elliptic curve, there is a unique principal polarization

- $\psi_0 : A_0 \to A_0^\vee$,

and the induced Rosati involution on $\mathrm{End}(A_0)$ restricts to complex conjugation on the image of ι_0. By Proposition 5.1 of [KRY99] or Proposition 2.1.2 of [How15], $\mathcal{M}_{(1,0)}$ is finite étale over $\mathcal{O}_{\boldsymbol{k}}$.

Remark 4.1.5. For ease of notation, we usually write objects of $\mathcal{M}_{(1,0)}$ simply as A_0, suppressing the remaining data from the notation. Similarly, objects of $\mathcal{M}_{(n-1,1)}$ will usually be written simply as A.

When $n = 1$, the data of ψ and $\mathcal{F}$ in the moduli problem $\mathcal{M}_{(n-1,1)}$ are uniquely determined, and $\mathcal{M}_{(0,1)}$ classifies pairs (A, ι) over $\mathcal{O}_{\boldsymbol{k}}$-schemes S in which $A \to S$ is an elliptic curve and $\iota : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action such that the induced action of $\mathcal{O}_{\boldsymbol{k}}$ on the $\mathcal{O}_S$-module $\mathrm{Lie}(A)$ is through the conjugate of the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$. Replacing ι by its conjugate therefore defines a canonical isomorphism

$$\mathcal{M}_{(0,1)} \cong \mathcal{M}_{(1,0)}.$$

In particular, there is a decomposition of $\mathcal{M}_{(1,0)}$ analogous to (4.1.1), but this statement is vacuous, as the decomposition has a single term by Proposition 4.1.4. If W_0 is the unique, up to strict similarly, relevant $\boldsymbol{k}$-hermitian space of signature $(1,0)$, it will be convenient to set

$$\mathcal{M}_{W_0} = \mathcal{M}_{(1,0)}.$$

Fix relevant $\boldsymbol{k}$-hermitian spaces (W_0, h_0) and (W, h) of signatures $(1,0)$ and $(n-1,1)$, and self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattices $\mathfrak{a}_0 \subset W_0$ and $\mathfrak{a} \subset W$. The hermitian forms induce alternating $\mathbb{Q}$-bilinear forms

$$e_0(a,b) = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\left(\frac{h_0(a,b)}{\sqrt{-D}}\right), \qquad e(a,b) = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\left(\frac{h(a,b)}{\sqrt{-D}}\right) \tag{4.1.2}$$

on W_0 and W, and $\mathfrak{a}_0$ and $\mathfrak{a}$ are self-dual with respect to these forms.

Definition 4.1.6. Suppose we are given an integer $N \geq 1$ and an $\mathcal{O}_{\boldsymbol{k}}$-scheme S with $N \in \mathcal{O}_S^\times$. A *level N-structure* on $A_0 \in \mathcal{M}_{W_0}(S)$ is a pair (η_0, ζ_0) consisting of isomorphisms

$$\eta_0 : A_0[N] \cong \underline{\mathfrak{a}_0/N\mathfrak{a}_0}, \qquad \xi_0 : \mu_N \cong \underline{\mathbb{Z}/N\mathbb{Z}}$$

of étale sheaves on S that identify the Weil pairing on $A_0[N]$ induced by the principal polarization with the pairing on $\mathfrak{a}_0/N\mathfrak{a}_0$ induced by e_0. Similarly, a *level N-structure* on $A \in \mathcal{M}_W(S)$ is a pair (η, ζ) consisting of isomorphisms

$$\eta : A[N] \cong \underline{\mathfrak{a}/N\mathfrak{a}}, \qquad \xi : \mu_N \cong \underline{\mathbb{Z}/N\mathbb{Z}}$$

that identify the Weil pairing on $A[N]$ with the alternating pairing on $\mathfrak{a}/N\mathfrak{a}$.

Over $\mathcal{O}_{\boldsymbol{k}}[1/N]$, we obtain finite étale covers

$$\mathcal{M}_{W_0}(N) \to \mathcal{M}_{W_0/\mathcal{O}_{\boldsymbol{k}}[1/N]}, \qquad \mathcal{M}_W(N) \to \mathcal{M}_{W/\mathcal{O}_{\boldsymbol{k}}[1/N]} \tag{4.1.3}$$

by adding level structures to the moduli problems. By the first claim of Proposition 4.1.4, these are independent of the choices of $\mathfrak{a}_0$ and $\mathfrak{a}$.

4.2. **The exceptional divisor.** Now assume $n \geq 2$. We recall some more constructions from §2.3 of [BHK$^+$a], following work of Pappas [Pap00] and Krämer [Kra03].

There is a normal and flat $\mathcal{O}_{\boldsymbol{k}}$-stack

$$\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

defined as the moduli space of triples (A, ι, ψ) whose components are as in the definition of $\mathcal{M}_{(n-1,1)}$. However, instead of including a hyperplane $\mathcal{F} \subset \mathrm{Lie}(A)$ as part of the moduli problem, we demand that the action of $\mathcal{O}_{\boldsymbol{k}}$ on $\mathrm{Lie}(A)$ satisfy a Kottwitz-style signature $(n-1,1)$ condition, and the wedge conditions of Pappas. We refer the reader to §2.3 of [BHK$^+$a] for the precise definitions.

Forgetting the hyperplane $\mathcal{F} \subset \mathrm{Lie}(A)$ defines a morphism

$$\mathcal{M}_{(n-1,1)} \to \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}, \tag{4.2.1}$$

which can be realized as a blow-up. To interpret it as such, define the *singular locus*

$$\mathrm{Sing}_{(n-1,1)} \subset \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}, \tag{4.2.2}$$

as the reduced locus of nonsmooth points. It is a proper $\mathcal{O}_{\boldsymbol{k}}$-stack of dimension 0, supported in characteristics dividing D.

The morphism (4.2.1) can be identified with the blow-up of the singular locus, and so there is a canonical cartesian diagram

$$\begin{array}{ccc} \mathrm{Exc}_{(n-1,1)} & \longrightarrow & \mathcal{M}_{(n-1,1)} \\ \downarrow & & \downarrow \\ \mathrm{Sing}_{(n-1,1)} & \longrightarrow & \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \end{array} \tag{4.2.3}$$

in which the upper left corner is the *exceptional divisor*. In particular, (4.2.1) is relatively representable and projective, and restricts to an isomorphism

$$\mathcal{M}_{(n-1,1)} \smallsetminus \mathrm{Exc}_{(n-1,1)} \cong \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \smallsetminus \mathrm{Sing}_{(n-1,1)}.$$

Remark 4.2.1. If $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ is a prime above $p \mid D$ with residue field $\mathbb{F}_{\mathfrak{p}}$, the $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$-points of the singular locus are those

$$A \in \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})$$

for which the action of $\mathcal{O}_{\boldsymbol{k}}$ on the $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$-vector space $\mathrm{Lie}(A)$ is through the reduction map $\mathcal{O}_{\boldsymbol{k}} \to \mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$. At such a point, *any* hyperplane $\mathcal{F} \subset \mathrm{Lie}(A)$ is $\mathcal{O}_{\boldsymbol{k}}$-stable and satisfies Krämer's signature condition, and the fiber over A of the left vertical morphism in (4.2.3) is the projective space parametrizing all such $\mathcal{F}$.

4.3. **Toroidal compactification.** We turn to the study of compactifications of $\mathcal{M}_{(n-1,1)}$. When $n = 1$ this stack is finite (hence proper) over $\mathcal{O}_{\boldsymbol{k}}$, and so we set

$$\bar{\mathcal{M}}_{(0,1)} = \mathcal{M}_{(0,1)}.$$

For the rest of this subsection, assume $n \geq 2$.

One finds in §2 of [How15] the construction of a toroidal compactification

$$\mathcal{M}_{(n-1,1)} \subset \bar{\mathcal{M}}_{(n-1,1)},$$

obtained by imitating the constructions of Faltings and Chai [FC90]. If one works over $\mathcal{O}_{\boldsymbol{k}}[1/D]$, these are special cases of the constructions of Lan [Lan13]. The compactification is regular, and is smooth outside the exceptional divisor of (4.2.3).

The universal abelian scheme $A \to \mathcal{M}_{(n-1,1)}$ extends uniquely to a semi-abelian scheme

$$\bar{A} \to \bar{\mathcal{M}}_{(n-1,1)}$$

with $\mathcal{O}_{\boldsymbol{k}}$-action. At a geometric point of the boundary, this semi-abelian scheme is an extension of an abelian variety by a torus of rank two. The universal hyperplane $\mathcal{F} \subset \mathrm{Lie}(A)$ over $\mathcal{M}_{(n-1,1)}$ extends uniquely to an $\mathcal{O}_{\boldsymbol{k}}$-stable hyperplane

$$\bar{\mathcal{F}} \subset \mathrm{Lie}(\bar{A}), \tag{4.3.1}$$

which again satisfies Krämer's signature $(n-1,1)$ condition.

There is a similar toroidal compactification of $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ (now only normal instead of regular), and a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Exc}_{(n-1,1)} & \longrightarrow & \bar{\mathcal{M}}_{(n-1,1)} \\ \downarrow & & \downarrow \\ \mathrm{Sing}_{(n-1,1)} & \longrightarrow & \bar{\mathcal{M}}^{\mathrm{Pap}}_{(n-1,1)} \end{array} \tag{4.3.2}$$

extending (4.2.3). The vertical arrow on the right is the blow-up along the closed immersion in the bottom row.

Remark 4.3.1. For general Shimura varieties, a toroidal compactification depends on a choice of rational polyhedral cone decomposition of a convex cone sitting inside the real points of a $\mathbb{Q}$-vector space. For our $\mathrm{GU}(n-1,1)$ Shimura varieties, the vector space in question is one-dimensional. Hence the rational polyhedral cone decomposition is uniquely determined, and the toroidal compactifications are canonical.

Let $\mathcal{A}_n$ be the moduli stack of principally polarized abelian schemes of dimension n. There are canonical morphisms

$$\mathcal{M}_{(n-1,1)} \xrightarrow{(4.2.1)} \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \to \mathcal{A}_{n/\mathcal{O}_{\boldsymbol{k}}}, \tag{4.3.3}$$

in which the first arrow is projective, and the second is finite.

Lemma 4.3.2. *If $\bar{\mathcal{A}}_n$ is any choice of smooth Faltings-Chai toroidal compactification, $\bar{\mathcal{M}}^{\mathrm{Pap}}_{(n-1,1)}$ is canonically identified with the normalization of*

$$\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \to \bar{\mathcal{A}}_{n/\mathcal{O}_{\boldsymbol{k}}}. \tag{4.3.4}$$

See §29.53 of [Sta18] *for normalization.*

Proof. By [How15], the universal abelian scheme $A \to \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ extends (necessarily uniquely) to a semi-abelian scheme $\bar{A}$ over the compactification. This extension satisfies a universal property: Suppose S is an irreducible normal scheme with generic point $\eta \to S$, and we are given a morphism

$$\eta \to \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}.$$

If the pullback A_η extends to a semi-abelian scheme over S, then this extension is the pullback of $\bar{A}$ via a (necessarily unique) morphism

$$S \to \bar{\mathcal{M}}^{\mathrm{Pap}}_{(n-1,1)}$$

restricting to the given morphism at the generic point.

This extension property is analogous to Theorem IV.5.7(5) of [FC90], but the statement is simplified by the uniqueness of the rational polyhedral cone decomposition of Remark 4.3.1. If one works over $\mathcal{O}_{\boldsymbol{k}}[1/D]$, the extension property is a special case of Theorem 6.4.1.1(6) of [Lan13]. The proof over $\mathcal{O}_{\boldsymbol{k}}$ is exactly the same.

On the other hand, Theorem 11.4 of [Lan16] shows that the compactification of $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ defined by normalization also carries a semi-abelian scheme satisfying the same universal property, and the lemma follows. □

Fix an $\mathcal{M}_W$ as in (4.1.1). We need compactifications of the étale covers

$$\mathcal{M}_W(N) \to \mathcal{M}_{W/\mathcal{O}_{\boldsymbol{k}}[1/N]}$$

of (4.1.3). Rather than repeat the constructions of [How15] with level structure, we will appeal to the results of [Lan16] and [MP19], in which compactifications of quite general Shimura varieties are constructed as normalizations of Faltings-Chai compactifications of Siegel moduli spaces. These results cannot be applied directly to the covers above, as the failure of the first arrow in (4.3.3) to be finite (it contracts the entire exceptional divisor to a 0-dimensional substack) implies the failure of Lemma 4.3.2 if $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ is replaced by $\mathcal{M}_{(n-1,1)}$. Thus the proof of Proposition 4.3.3 below requires the roundabout step of first compactifying covers of $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$.

For any $N \geq 1$, define $\bar{\mathcal{M}}_W(N)$ as the normalization of

$$\mathcal{M}_W(N) \to \bar{\mathcal{M}}_{(n-1,1)/\mathcal{O}_{\boldsymbol{k}}[1/N]}.$$

Proposition 4.3.3. *The stack $\bar{\mathcal{M}}_W(N)$ is regular, flat, and proper over $\mathcal{O}_{\boldsymbol{k}}[1/N]$, and is a projective scheme if $N \geq 3$. It is smooth away from its exceptional divisor*

$$\mathrm{Exc}_{(n-1,1)} \times_{\bar{\mathcal{M}}_{(n-1,1)}} \bar{\mathcal{M}}_W(N).$$

In particular, it is smooth in a neighborhood of its boundary, which is a Cartier divisor smooth over $\mathcal{O}_{\boldsymbol{k}}[1/N]$.

Proof. There is a decomposition

$$\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} = \bigsqcup_W \mathcal{M}^{\mathrm{Pap}}_W,$$

exactly as in (4.1.1). We may add $N \geq 1$ level structure, exactly as in Definition 4.1.6, to obtain a finite étale cover

$$\mathcal{M}^{\mathrm{Pap}}_W(N) \to \mathcal{M}^{\mathrm{Pap}}_{W/\mathcal{O}_{\boldsymbol{k}}[1/N]}.$$

This cover has a compactification $\bar{\mathcal{M}}^{\mathrm{Pap}}_W(N)$ defined as the normalization of

$$\mathcal{M}^{\mathrm{Pap}}_W(N) \to \bar{\mathcal{M}}^{\mathrm{Pap}}_{(n-1,1)/\mathcal{O}_{\boldsymbol{k}}[1/N]},$$

which, by Lemma 4.3.2, is the same as the normalization of

$$\mathcal{M}^{\mathrm{Pap}}_W(N) \to \bar{\mathcal{A}}_{n/\mathcal{O}_{\boldsymbol{k}}[1/N]}. \tag{4.3.5}$$

The properness, flatness, and normality of $\bar{\mathcal{M}}^{\mathrm{Pap}}_W(N)$ follow from its construction as the normalization of a proper, flat, and smooth $\mathcal{O}_{\boldsymbol{k}}[1/N]$-stack. It is not regular (as this is not true even of its interior). However, its singular locus

$$\mathrm{Sing}_W(N) = \mathrm{Sing}_{(n-1,1)} \times_{\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}} \mathcal{M}^{\mathrm{Pap}}_W(N)$$

is proper over $\mathcal{O}_{\boldsymbol{k}}[1/N]$, and the smoothness of

$$\mathcal{M}^{\mathrm{Pap}}_W(N) \smallsetminus \mathrm{Sing}_W(N)$$

implies the smoothness of

$$\bar{\mathcal{M}}^{\mathrm{Pap}}_W(N) \smallsetminus \mathrm{Sing}_W(N).$$

This is a consequence of the following principle, found in §14 of [Lan16] and Theorem 1 of [MP19]: if $U \subset \bar{\mathcal{M}}^{\mathrm{Pap}}_W(N)$ is an open neighborhood of the boundary $\partial\bar{\mathcal{M}}_W^{\mathrm{Pap}}(N)$, then the singularities of U are no worse than the singularities of $U \smallsetminus \partial\bar{\mathcal{M}}_W^{\mathrm{Pap}}(N)$.

By the same principle (and its proof), the smoothness of $\bar{\mathcal{M}}_W^{\mathrm{Pap}}(N)$ near the boundary implies the smoothness of the boundary divisor itself. This is actually a particular feature of our $\mathrm{GU}(W)$ Shimura varieties. For more general Shimura varieties the boundary has a canonical stratification by locally closed substacks (see Theorem 4.1.5 of [MP19] or §9 of [Lan16]), and the above cited principle tells us that each *stratum* is smooth. In our special case each boundary stratum is a divisor, and is simply a union of connected components of $\partial\bar{\mathcal{M}}_W^{\mathrm{Pap}}(N)$. Hence the entire boundary is a Cartier divisor smooth over $\mathcal{O}_{\boldsymbol{k}}[1/N]$.

Using the characterization of (4.3.2) as a blow-up, one may identify $\bar{\mathcal{M}}_W(N)$ with the blow-up of $\bar{\mathcal{M}}_W^{\mathrm{Pap}}(N)$ along its singular locus. As the singular locus does not meet the boundary, it follows that $\bar{\mathcal{M}}_W(N)$ is itself proper and flat with boundary a smooth Cartier divisor. Moreover, this shows that $\bar{\mathcal{M}}_W(N)$ is smooth away from the exceptional divisor of the blow-up, and in particular in an open neighborhood of its boundary. As the interior is regular, being étale over the regular stack $\mathcal{M}_W$, we find that $\bar{\mathcal{M}}_W(N)$ is regular.

Finally, when $N \geq 3$ we claim that both the source and target of

$$\bar{\mathcal{M}}_W(N) \to \bar{\mathcal{M}}_W^{\mathrm{Pap}}(N)$$

are projective schemes. As this morphism is a blow-up, it suffices to prove this for the target. For this, we use the results of Chapter V.5 of [FC90]. Adding principal level structure yields a finite étale cover

$$\mathcal{A}_n(N) \to \mathcal{A}_{n/\mathbb{Z}[1/N]},$$

and the compactification $\bar{\mathcal{A}}_n$ may be chosen so that (4.3.5) factors as

$$\mathcal{M}_W^{\mathrm{Pap}}(N) \to \bar{\mathcal{A}}_n(N) \to \bar{\mathcal{A}}_{n/\mathcal{O}_{\boldsymbol{k}}[1/N]},$$

where $\bar{\mathcal{A}}_n(N)$ is the smooth projective scheme obtained by normalizing

$$\mathcal{A}_n(N) \to \bar{\mathcal{A}}_{n/\mathcal{O}_{\boldsymbol{k}}[1/N]}.$$

This realizes $\bar{\mathcal{M}}_W^{\mathrm{Pap}}(N)$ as the normalization of the first arrow in the composition, and hence provides us with a finite and relatively representable morphism

$$\bar{\mathcal{M}}_W^{\mathrm{Pap}}(N) \to \bar{\mathcal{A}}_n(N).$$

As the target is a projective scheme, so is the source. □

Remark 4.3.4. One could avoid the constructions of [How15] and [Lan13] altogether, and define $\bar{\mathcal{M}}_{(n-1,1)}$ from the start as a blow-up of the normalization of (4.3.4). All properties that we need to know about this compactification could then be deduced from [Lan16] and [MP19], except for the extension (4.3.1) of the universal hyperplane across the boundary (which is needed in the proof of Proposition 5.2.1). For this one needs detailed information about the local charts near the boundary, as found in [How15].

4.4. **Arithmetic Chow groups.** We summarize the arithmetic intersection theory of Gillet-Soulé [GS90, SABK] and Burgos-Kramer-Kühn [BBK07, BKK07] for the Shimura varieties $\mathcal{M}_W$ of (4.1.1), mostly for the purposes of fixing normalizations of arithmetic degrees, heights, and volumes.

As usual, denote by $\widehat{\mathrm{Pic}}(\mathcal{M}_W)$ the group of isomorphism classes of hermitian line bundles on $\mathcal{M}_W$, and similarly for the toroidal compactification $\bar{\mathcal{M}}_W$. In order to account for metrics that do not extend smoothly across the boundary, when we work on the compactification $\bar{\mathcal{M}}_W$ we must relax the notion of a hermitian metric.

Accordingly, denote by $\widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}})$ the group of isomorphism classes of pre-log singular hermitian line bundles on $\bar{\mathcal{M}}_W$, in the sense of Definition 1.20 of [BBK07]. Exactly as in Proposition 5.2.1 of [How20], the natural restriction maps

$$\widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W) \to \widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}) \to \widehat{\mathrm{Pic}}(\mathcal{M}_W) \tag{4.4.1}$$

are injective.

Fix an integer $N \geq 3$, so that the toroidal compactification $\bar{\mathcal{M}}_W(N)$ of Proposition 4.3.3 is a regular scheme of dimension n, projective and flat over $\mathcal{O}_{\boldsymbol{k}}[1/N]$. For $0 \leq d \leq n$, we have a homomorphism of arithmetic Chow groups

$$\widehat{\mathrm{CH}}^d_{\mathrm{GS}}(\bar{\mathcal{M}}_W(N)) \to \widehat{\mathrm{CH}}^d(\bar{\mathcal{M}}_W(N), \mathscr{D}_{\mathrm{pre}})$$

from Theorem 6.23 and (7.49) of [BKK07]. The domain is the arithmetic Chow group of Gillet-Soulé [GS90, SABK], while the codomain is the arithmetic Chow group of Burgos-Kramer-Kühn [BBK07, BKK07]. In both groups elements are rational equivalence classes of pairs $(\mathcal{Z}, g)$ in which $\mathcal{Z}$ is a codimension d cycle on $\bar{\mathcal{M}}_W(N)$. In the Gillet-Soulé construction g is a $(d-1, d-1)$ current on $\bar{\mathcal{M}}_W(N)(\mathbb{C})$ satisfying the Green equation

$$dd^c g + \delta_{\mathcal{Z}} = [\omega]$$

of currents for some smooth (d, d) form ω. In the Burgos-Kramer-Kühn construction g is a more elaborate *Green object*, allowed to have pre-log-log singularities along the boundary of the compactification.

Whenever $N \mid N'$ the map $\bar{\mathcal{M}}_W(N') \to \bar{\mathcal{M}}_W(N)$ induces a pullback

$$\widehat{\mathrm{CH}}^d(\bar{\mathcal{M}}_W(N), \mathscr{D}_{\mathrm{pre}}) \to \widehat{\mathrm{CH}}^d(\bar{\mathcal{M}}_W(N'), \mathscr{D}_{\mathrm{pre}}),$$

allowing us to define, exactly as in §6.3 of [BKK07],

$$\widehat{\mathrm{CH}}^d(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}) = \varprojlim_{N \geq 3} \widehat{\mathrm{CH}}^d(\bar{\mathcal{M}}_W(N), \mathscr{D}_{\mathrm{pre}}).$$

There is an *arithmetic intersection pairing*

$$\widehat{\mathrm{CH}}^{d_1}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}) \otimes \widehat{\mathrm{CH}}^{d_2}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}) \to \widehat{\mathrm{CH}}^{d_1+d_2}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}})_{\mathbb{Q}},$$

an *arithmetic Chern class map*

$$\widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}) \to \widehat{\mathrm{CH}}^1(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}),$$

and an *arithmetic degree*

$$\widehat{\deg} : \widehat{\mathrm{CH}}^n(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}) \to \mathbb{R}$$

induced by the analogous structures on the arithmetic Chow group of each scheme $\bar{\mathcal{M}}_W(N)$ with $N \geq 3$.

Remark 4.4.1. The conventions for delta currents, currents associated to smooth forms, and Green forms used in [BKK07] and [BBK07] differ from those of Gillet-Soulé [GS90, SABK] by powers of 2 and $2\pi i$. We will always use the conventions of Gillet-Soulé. For example, the arithmetic Chern class map sends a pre-log singular hermitian line bundle $\widehat{\mathcal{L}}$ to the arithmetic divisor

$$\widehat{\mathrm{div}}(s) = (\mathrm{div}(s), -\log \|s\|^2)$$

for any nonzero rational section s.

To make explicit the normalization of the arithmetic degree, first note that for each $N \geq 3$, the finite étale cover $\mathcal{M}_W(N) \to \mathcal{M}_{W/\mathcal{O}_{\boldsymbol{k}}[1/N]}$ has constant fiber degree, in the sense that there is a $d_N \in \mathbb{Z}$ satisfying

$$\frac{d_N}{\#\,\mathrm{Aut}(x)} = \#\{\text{geometric points } y \to \mathcal{M}_W(N) \text{ above } x\} \tag{4.4.2}$$

for every geometric point $x \to \mathcal{M}_{W/\mathcal{O}_{\boldsymbol{k}}[1/N]}$. Now suppose we have a class

$$\widehat{\mathcal{Z}} = (\mathcal{Z}_N, g_N)_{N\geq 3} \in \widehat{\mathrm{CH}}^n(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}).$$

If we write each $\mathcal{Z}_N = \sum_i m_{N,i}\mathcal{Z}_{N,i}$ as a $\mathbb{Z}$-linear combination of 0-dimensional irreducible closed subschemes on $\bar{\mathcal{M}}_W(N)$, then the real number

$$\widehat{\deg}_N(\mathcal{Z}_N, g_N) = \sum_i m_{N,i} \cdot \#\mathcal{Z}_{N,i}(\mathbb{F}_i^{\mathrm{alg}}) \cdot \log(\#\mathbb{F}_i) + \int_{\mathcal{M}_W(N)(\mathbb{C})} g_N,$$

where $\mathbb{F}_i$ is the residue field of the unique prime of $\mathcal{O}_{\boldsymbol{k}}[1/N]$ below $\mathcal{Z}_{N,i}$, is well-defined up to adding a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid N\}$, and

$$\widehat{\deg}(\widehat{\mathcal{Z}}) = \frac{1}{d_N} \cdot \widehat{\deg}_N(\mathcal{Z}_N, g_N) \tag{4.4.3}$$

up to the same ambiguity.

Remark 4.4.2. There is no 1/2 in front of the integral, in apparent disagreement with §3.4.3 of [GS90]. In fact there is no disagreement. For us $\mathcal{M}_W(N)(\mathbb{C})$ means the set of all morphisms $\mathrm{Spec}(\mathbb{C}) \to \mathcal{M}_W(N)$ as $\boldsymbol{k}$-schemes, whereas Gillet-Soulé would take morphisms as $\mathbb{Q}$-schemes. Thus for Gillet-Soulé the domain of integration would be our $\mathcal{M}_W(N)(\mathbb{C})$ *together with its complex conjugate*, yielding an integral twice as big as ours.

4.5. **Volumes and heights.** For a pre-log singular hermitian line bundle
$$\widehat{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}),$$
define the *arithmetic volume* by
$$\widehat{\mathrm{vol}}(\widehat{\mathcal{L}}) = \widehat{\deg}(\widehat{\mathcal{L}}^n).$$
Here $\widehat{\mathcal{L}}^n$ is the n-fold iterated intersection of the arithmetic Chern class
$$\widehat{\mathcal{L}} \in \widehat{\mathrm{CH}}^1(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}).$$

Let $\mathcal{Z}$ be a divisor on $\bar{\mathcal{M}}_W$, and assume that $\mathcal{Z}$ intersects the boundary $\partial\bar{\mathcal{M}}_W$ properly in the generic fiber. As in the discussion following Theorem 7.58 of [BKK07], one can then define the *arithmetic height* (or *Faltings height*) *of $\mathcal{Z}$ with respect to* $\widehat{\mathcal{L}}$, denoted
$$\mathrm{ht}_{\widehat{\mathcal{L}}}(\mathcal{Z}) \in \mathbb{R}.$$
To make it explicit, fix $N \geq 3$. Choose a pair $(\mathcal{Y}, g_{\mathcal{Y}})$ representing the iterated intersection
$$\widehat{\mathcal{L}}^{n-1} \in \widehat{\mathrm{CH}}^{n-1}(\bar{\mathcal{M}}_W(N), \mathscr{D}_{\mathrm{pre}}),$$
and do this in such a way that the supports of $\mathcal{Z}_N$ and $\mathcal{Y}$ do not intersect in the generic fiber, where $\mathcal{Z}_N$ is the pullback of $\mathcal{Z}$ via
$$\bar{\mathcal{M}}_W(N) \to \bar{\mathcal{M}}_{W/\mathcal{O}_{\boldsymbol{k}}[1/N]}.$$
Now form the intersection
$$\mathcal{Z}_N \cdot \mathcal{Y} \in \mathrm{CH}^n_{\mathcal{Z}_N \cap \mathcal{Y}}(\bar{\mathcal{M}}_W(N))_{\mathbb{Q}}$$
in the Chow group with support along the closed subset $\mathrm{Sppt}(\mathcal{Z}_N) \cap \mathrm{Sppt}(\mathcal{Y})$. As this closed subset is supported in finitely many nonzero characteristics, there is a natural change-of-support map
$$\mathrm{CH}^n_{\mathcal{Z}\cap\mathcal{Y}}(\bar{\mathcal{M}}_W(N)) \to \bigoplus_{\substack{\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}} \\ \mathfrak{p} \nmid N\mathcal{O}_{\boldsymbol{k}}}} \mathrm{CH}^n_{\mathfrak{p}}(\bar{\mathcal{M}}_W(N))$$
where $\mathrm{CH}^n_{\mathfrak{p}}(\bar{\mathcal{M}}_W(N))$ is the Chow group with support in the mod $\mathfrak{p}$ fiber of $\bar{\mathcal{M}}_W(N)$. For each $\mathfrak{p}$ there is a degree map
$$\deg_{\mathfrak{p}} : \mathrm{CH}^n_{\mathfrak{p}}(\bar{\mathcal{M}}_W(N)) \to \mathbb{Z}$$
sending a reduced and irreducible codimension n subscheme (a.k.a., a closed point) $\mathcal{C} \subset \bar{\mathcal{M}}_W(N)$ supported at $\mathfrak{p}$ to
$$\deg_{\mathfrak{p}}(\mathcal{C}) = \#\mathcal{C}(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})$$
where $\mathbb{F}_{\mathfrak{p}} = \mathcal{O}_{\boldsymbol{k}}/\mathfrak{p}$. The arithmetic height satisfies
$$d_N \cdot \mathrm{ht}_{\widehat{\mathcal{L}}}(\mathcal{Z}) = \sum_{\substack{\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}} \\ \mathfrak{p} \nmid N\mathcal{O}_{\boldsymbol{k}}}} \deg_{\mathfrak{p}}(\mathcal{Z}_N \cdot \mathcal{Y}) \cdot \log(\#\mathbb{F}_{\mathfrak{p}}) + \int_{\mathcal{Z}_N(\mathbb{C})} g_{\mathcal{Y}}$$
up to a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid N\}$, where d_N is as in (4.4.2).

A choice of pre-log singular hermitian metric on the line bundle $\mathcal{O}(\mathcal{Z})$ determines a class

$$\widehat{\mathcal{Z}} \in \widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}).$$

If we view the constant function 1 as a global section of $\mathcal{O}(\mathcal{Z})$ and set $g_{\mathcal{Z}} = -\log \|1\|^2$, the arithmetic Chern class of this hermitian line bundle is

$$(\mathcal{Z}, g_{\mathcal{Z}}) \in \widehat{\mathrm{CH}}^1(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}),$$

and Proposition 7.56 of [BKK07] gives the relation

$$\widehat{\deg}(\widehat{\mathcal{Z}} \cdot \widehat{\mathcal{L}}^{n-1}) = \mathrm{ht}_{\widehat{\mathcal{L}}}(\mathcal{Z}) + \int_{\mathcal{M}_W(\mathbb{C})} g_{\mathcal{Z}} \cdot \mathrm{ch}(\widehat{\mathcal{L}})^{n-1}. \tag{4.5.1}$$

From the point of view of arithmetic volumes and heights, some hermitian line bundles are essentially indistinguishable.

Definition 4.5.1. We say that

$$\widehat{\mathcal{L}}_1, \widehat{\mathcal{L}}_2 \in \widehat{\mathrm{Pic}}(\mathcal{M}_W)$$

are *numerically equivalent* if the difference $\widehat{\mathcal{E}} = \widehat{\mathcal{L}}_1 - \widehat{\mathcal{L}}_2$ lies in the subgroup

$$\widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}) \subset \widehat{\mathrm{Pic}}(\mathcal{M}_W)$$

of (4.4.1) and satisfies $\widehat{\deg}(\widehat{\mathcal{Z}} \cdot \widehat{\mathcal{E}}) = 0$ for all $\widehat{\mathcal{Z}} \in \widehat{\mathrm{CH}}^{n-1}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}})$.

Lemma 4.5.2. *If $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ are numerically equivalent, then their Chern forms are equal. Moreover,*

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{L}_1}) = \widehat{\mathrm{vol}}(\widehat{\mathcal{L}}_2) \quad \textit{and} \quad \mathrm{ht}_{\widehat{\mathcal{L}}_1}(\mathcal{Z}) = \mathrm{ht}_{\widehat{\mathcal{L}}_2}(\mathcal{Z}) \tag{4.5.2}$$

for any divisor $\mathcal{Z}$ as above.

Proof. Set $\widehat{\mathcal{E}} = \widehat{\mathcal{L}}_1 - \widehat{\mathcal{L}}_2$. If g is any smooth $(n-2, n-2)$ form on $\bar{\mathcal{M}}_W(\mathbb{C})$, the arithmetic cycle class

$$\widehat{\mathcal{Z}} = (0, g) \in \widehat{\mathrm{CH}}^{n-1}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}})$$

satisfies

$$0 = \widehat{\deg}(\widehat{\mathcal{Z}} \cdot \widehat{\mathcal{E}}) = \int_{\mathcal{M}_W(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{E}}) \wedge g.$$

As this holds for all choices of g, we must have $\mathrm{ch}(\widehat{\mathcal{E}}) = 0$. This proves the desired equality of Chern forms. The first equality of (4.5.2) is clear from

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{L}_1}) = \sum_{i=0}^{n} \widehat{\deg}(\widehat{\mathcal{L}}_2^{n-i} \cdot \widehat{\mathcal{E}}^i),$$

as every term with $i > 0$ vanishes. The second equality follows from (4.5.1) and the equality of Chern forms already proved. □

If $\mathcal{Z}$ is a Cartier divisor on $\bar{\mathcal{M}}_W$ supported in nonzero characteristics, the line bundle $\mathcal{O}(\mathcal{Z})$ is canonically trivialized in the generic fiber by the constant function 1. For any real number c, denote by

$$(\mathcal{Z}, c) \in \widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W) \tag{4.5.3}$$

the line bundle $\mathcal{O}(\mathcal{Z})$ endowed with the constant metric characterized by $-\log \|1\|^2 = c$. In particular, adding $(0, c)$ to a hermitian line bundle just rescales $\|\cdot\|^2$ by e^{-c}. The following lemma describes the effect of this on arithmetic volumes.

Lemma 4.5.3. *For any* $\widehat{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}})$ *and any constant* $c \in \mathbb{R}$*, we have*

$$\widehat{\mathrm{vol}}\big(\widehat{\mathcal{L}} + (0,c)\big) = \widehat{\mathrm{vol}}(\widehat{\mathcal{L}}) + nc \int_{\mathcal{M}_W(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{L}})^{n-1},$$

where $\mathrm{ch}(\widehat{\mathcal{L}})$ *is the Chern form* (1.1.2)*.*

Proof. A trivial induction argument shows that

$$\widehat{\mathcal{L}}^{k-1} \cdot (0,c) = (0, c \cdot \mathrm{ch}(\widehat{\mathcal{L}})^{k-1}) \in \widehat{\mathrm{CH}}^k(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}})$$

for all $k > 0$. As $(0,c)\cdot(0,c) = 0$ we deduce

$$\begin{aligned}\big(\widehat{\mathcal{L}} + (0,c)\big)^k &= \widehat{\mathcal{L}}^k + k\widehat{\mathcal{L}}^{k-1} \cdot (0,c) \\ &= \widehat{\mathcal{L}}^k + k(0, c \cdot \mathrm{ch}(\widehat{\mathcal{L}})^{k-1}).\end{aligned}$$

Setting $k = n$ and taking the arithmetic degree completes the proof. □

5. A Shimura variety and its line bundles

For the rest of this paper we fix a $\boldsymbol{k}$-hermitian space V of signature $(n-1, 1)$, with $n \geq 1$, and assume that V admits a self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice (1.3.1). We will associate to V an n-dimensional open and closed substack

$$\mathcal{S}_V \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-1,1)},$$

construct some interesting hermitian line bundles on it, and explain the relations between them.

5.1. A special Shimura variety.

To attach a Shimura variety to our fixed V, choose relevant (Definition 4.1.3) hermitian spaces (W_0, h_0) and (W, h) of signatures $(1,0)$ and $(n-1,1)$, respectively, in such a way that

$$V \cong \mathrm{Hom}_{\boldsymbol{k}}(W_0, W) \tag{5.1.1}$$

as $\boldsymbol{k}$-hermitian spaces, where the hermitian form $\langle -, - \rangle$ on the right satisfies

$$\langle x, y \rangle \cdot h_0(w_0, w_0') = h(x(w_0), y(w_0')) \tag{5.1.2}$$

for all $w_0, w_0' \in W_0$ and $x, y \in V$.

Remark 5.1.1. Such W_0 and W always exist: take $W_0 = \boldsymbol{k}$ with its norm form, and $W = V$. They are not uniquely determined by V, but their strict similarity classes are.

As in §2.2 of [KR14], if S is a connected $\mathcal{O}_{\boldsymbol{k}}$-scheme and

$$(A_0, A) \in \mathcal{M}_{W_0}(S) \times \mathcal{M}_W(S),$$

then $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ carries a positive definite hermitian form

$$\langle x, y\rangle = \psi_0^{-1} \circ y^\vee \circ \psi \circ x \in \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(A_0) \cong \mathcal{O}_{\boldsymbol{k}}, \tag{5.1.3}$$

where $\psi_0 : A_0 \to A_0^\vee$ and $\psi : A \to A^\vee$ are the principal polarizations.

As in §2.3 of [BHK+a], there is an open and closed substack

$$\mathcal{S}_V \subset \mathcal{M}_{W_0} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_W \tag{5.1.4}$$

characterized by its points valued in algebraically closed fields, which are those pairs

$$(A_0, A) \in \mathcal{M}_{W_0}(F) \times \mathcal{M}_W(F)$$

for which there exists an isometry

$$V \otimes \mathbb{Q}_\ell \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(T_\ell(A_0), T_\ell(A)\big) \otimes \mathbb{Q}_\ell$$

of $\boldsymbol{k}_\ell$-hermitian spaces for every $\ell \neq \mathrm{char}(F)$. Here the hermitian form on the right is defined as in (5.1.3). When $F = \mathbb{C}$ this is equivalent to the existence of an isometry of $\boldsymbol{k}$-hermitian spaces

$$V \cong \mathrm{Hom}_{\boldsymbol{k}}\big(H_1(A_0, \mathbb{Q}), H_1(A, \mathbb{Q})\big).$$

Remark 5.1.2. When n is even the inclusion (5.1.4) is an isomorphism.

Remark 5.1.3. The projection $\mathcal{S}_V \to \mathcal{M}_W$ is a finite étale surjection, and the fiber over a geometric point $x \in \mathcal{M}_W(\mathbb{F})$ satisfies

$$\sum_{y \in \mathcal{S}_{V,x}(\mathbb{F})} \frac{1}{|\mathrm{Aut}(y)|} = \frac{|\mathrm{CL}(\boldsymbol{k})|}{|\mathcal{O}_{\boldsymbol{k}}^\times|} \cdot \begin{cases} 1 & \text{if } n \text{ is even} \\ 2^{1-o(D)} & \text{if } n \text{ is odd.} \end{cases}$$

Remark 5.1.4. The stack $\mathcal{S}_V$ is denoted $\mathcal{S}_{\mathrm{Kra}}$ in [BHK+a]. Later we will want to vary V, and so we have included it in the notation to avoid confusion. As explained in [loc. cit.], the generic fiber of $\mathcal{S}_V$ is a Shimura variety for the subgroup $G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W)$ of pairs (g_0, g) for which the similitude factors of the two components are equal.

Definition 5.1.5. If $n \geq 2$, define the *exceptional divisor*

$$\mathrm{Exc}_V \subset \mathcal{S}_V$$

as the pullback of the exceptional divisor (4.2.3) via $\mathcal{S}_V \to \mathcal{M}_{(n-1,1)}$. Equivalently, it is defined by the cartesian diagram

$$\begin{array}{ccc} \mathrm{Exc}_V & \longrightarrow & \mathcal{S}_V \\ \downarrow & & \downarrow \\ \mathrm{Sing}_{(n-1,1)} & \longrightarrow & \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}. \end{array} \tag{5.1.5}$$

Definition 5.1.6 ([KR14])**.** For any positive $m \in \mathbb{Z}$, the *Kudla-Rapoport divisor* $\mathcal{Z}_V(m)$ is the $\mathcal{O}_{\boldsymbol{k}}$-stack classifying triples (A_0, A, x) consisting of a pair

$$(A_0, A) \in \mathcal{S}_V(S)$$

and an $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ satisfying $\langle x, x\rangle = m$.

The natural forgetful morphism

$$\mathcal{Z}_V(m) \to \mathcal{S}_V$$

is finite and unramified, with image of codimension 1. We denote again by $\mathcal{Z}_V(m)$ the image of this morphism, viewed as a divisor on $\mathcal{S}_V$ in the usual way (that is to say, each irreducible component of $\mathcal{Z}_V(m)$ contributes an irreducible component of the image, counted with multiplicity equal to the length of the local ring of its generic point). Denote by

$$\bar{\mathcal{Z}}_V(m) \to \bar{\mathcal{S}}_V$$

the normalization of $\mathcal{Z}_V(m) \to \bar{\mathcal{S}}_V$, and denote in the same way its image, viewed as a divisor on $\bar{\mathcal{S}}_V$. In other words, take the Zariski closure of $\mathcal{Z}_V(m)$.

Loosely speaking, each Kudla-Rapoport divisor is a union of unitary Shimura varieties associated to $\boldsymbol{k}$-hermitian spaces of signature $(n-2, 1)$. In §6 we will make this more precise, at least when m is a prime split in $\boldsymbol{k}$.

Definition 5.1.7. Let N be a positive integer, and let S be an $\mathcal{O}_{\boldsymbol{k}}$-scheme with $N \in \mathcal{O}_S^\times$. A *level N-structure* on a pair $(A_0, A) \in \mathcal{S}_V(S)$ consists of level N-structures (η_0, ξ_0) and (η, ξ) on A_0 and A, in the sense of Definition 4.1.6, such that $\xi_0 = \xi$.

Adding level structure to pairs (A_0, A) defines a finite étale cover

$$\mathcal{S}_V(N) \to \mathcal{S}_{V/\mathcal{O}_{\boldsymbol{k}}[1/N]},$$

and (5.1.4) lifts to a canonical open and closed immersion

$$\mathcal{S}_V(N) \subset \mathcal{M}_{W_0}(N) \times_{\mathcal{O}_{\boldsymbol{k}}[1/N]} \mathcal{M}_W(N).$$

Define a toroidal compactification

$$\bar{\mathcal{S}}_V(N) \subset \mathcal{M}_{W_0}(N) \times_{\mathcal{O}_{\boldsymbol{k}}[1/N]} \bar{\mathcal{M}}_W(N) \tag{5.1.6}$$

as the Zariski closure of $\mathcal{S}_V(N)$, or, equivalently, as the normalization of

$$\mathcal{S}_V(N) \to \bar{\mathcal{M}}_{W/\mathcal{O}_{\boldsymbol{k}}[1/N]}.$$

When $N = 1$ we abbreviate this to $\bar{\mathcal{S}}_V$.

Remark 5.1.8. As (5.1.6) is an open and closed immersion, and $\mathcal{M}_{W_0}(N)$ is finite étale over $\mathcal{O}_{\boldsymbol{k}}[1/N]$, the compactification $\bar{\mathcal{S}}_V(N)$ inherits all the nice properties of $\bar{\mathcal{M}}_W(N)$. In particular, Proposition 4.3.3 holds word-for-word with $\mathcal{M}_W$ replaced by $\mathcal{S}_V$, and the same is true of the entire discussion of arithmetic intersection theory in §4.4 and §4.5.

5.2. **Construction of hermitian line bundles.** The construction (1.1.1) associates to the universal CM elliptic curve $A_0 \to \mathcal{M}_{W_0} = \mathcal{M}_{(1,0)}$ its metrized Hodge bundle

$$\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{M}_{W_0}} \in \widehat{\mathrm{Pic}}(\mathcal{M}_{W_0}). \tag{5.2.1}$$

Similarly, the universal $A \to \mathcal{M}_W$ determines a metrized Hodge bundle

$$\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W} \in \widehat{\mathrm{Pic}}(\mathcal{M}_W). \tag{5.2.2}$$

Pulling back the universal objects via projection to the two factors in (5.1.4) yields a universal pair (A_0, A) of polarized abelian schemes (of relative dimensions 1 and n) over $\mathcal{S}_V$. As in §2.4 of [BHK$^+$a] there is a *metrized line bundle of modular forms*

$$\widehat{\mathcal{L}}_V \in \widehat{\mathrm{Pic}}(\mathcal{S}_V). \tag{5.2.3}$$

The line bundle underlying (5.2.3) has inverse

$$\mathcal{L}_V^{-1} = \mathrm{Lie}(A_0) \otimes \mathrm{Lie}(A)/\mathcal{F}, \tag{5.2.4}$$

where $\mathcal{F} \subset \mathrm{Lie}(A)$ is the universal hyperplane satisfying Krämer's signature condition (§4.1). The hermitian metric is defined as in §7.2 of [BHK$^+$a]: if we use Proposition 2.4.2 of [BHK$^+$a] to identify

$$\mathcal{L}_{V,z} \subset \mathrm{Hom}_{\boldsymbol{k}}(H_1(A_{0,z}, \mathbb{Q}), H_1(A_z, \mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{C} \cong V \otimes_{\mathbb{Q}} \mathbb{C} \tag{5.2.5}$$

at a complex point $z \in \mathcal{S}_V(\mathbb{C})$, the line $\mathcal{L}_{V,z}$ is isotropic with respect to the $\mathbb{C}$-bilinear extension of the $\mathbb{Q}$-bilinear form $[x, y] = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\langle x, y\rangle$ on V, and

$$\|s\|^2 = -\frac{[s, \overline{s}]}{4\pi e^{\gamma}} \tag{5.2.6}$$

for any $s \in \mathcal{L}_{V,z}$. Note that our $\widehat{\mathcal{L}}_V$ is denoted $\widehat{\boldsymbol{\omega}}$ in [BHK$^+$a, BHK$^+$b].

Proposition 5.2.1. *The hermitian line bundles* (5.2.2) *and* (5.2.3) *lie in the subgroups*

$$\widehat{\mathrm{Pic}}(\bar{\mathcal{M}}_W, \mathscr{D}_{\mathrm{pre}}) \subset \widehat{\mathrm{Pic}}(\mathcal{M}_W) \quad \textit{and} \quad \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V, \mathscr{D}_{\mathrm{pre}}) \subset \widehat{\mathrm{Pic}}(\mathcal{S}_V),$$

respectively, of (4.4.1).

Proof. The extension to $\bar{\mathcal{M}}_W$ of the line bundle

$$\omega^{\mathrm{Hdg}}_{A/\mathcal{M}_W} \cong \det(\mathrm{Lie}(A))^{-1}$$

underlying (5.2.2) is part of (4.3.1). The fact that the hermitian metric has a pre-log singularity along the boundary is a special case of Theorem 6.16 of [BKK05]. Note that this also uses Proposition 3.2 of [Fr09], which shows that any rank one log-singular hermitian vector bundle in the sense of [BKK05] is also a pre-log-singular hermitian line bundle in the sense of Definition 1.20 of [BBK07]. This proves the claim for (5.2.2), and the proof for (5.2.3) is the same. □

Pulling back the hermitian line bundles (5.2.1) and (5.2.2) via projection to the two factors in (5.1.6), we obtain three hermitian line bundles

$$\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V},\, \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V},\, \widehat{\mathcal{L}}_V \in \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V, \mathscr{D}_{\mathrm{pre}}). \tag{5.2.7}$$

In the remaining subsections we will make explicit the relations between them. The reader may wish to skip directly to Theorem 5.5.1 for the main results.

5.3. **An application of the Chowla-Selberg formula.** We will prove that, up to numerical equivalence (Definition 4.5.1), the first line bundle in (5.2.7) is just the trivial line bundle with a constant metric. The constant defining the metric is an interesting quantity in its own right. Recall the Chowla-Selberg formula: the Faltings height, normalized as in §5.2 of [BHK+b], of any elliptic curve with CM by $\mathcal{O}_{\boldsymbol{k}}$ is

$$h^{\mathrm{Falt}}_{\boldsymbol{k}} = -\frac{1}{2}\frac{L'(0,\varepsilon)}{L(0,\varepsilon)} - \frac{1}{4}\log(4\pi^2 D). \tag{5.3.1}$$

Proposition 5.3.1. *The metrized line bundles*

$$\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V} \in \widehat{\mathrm{Pic}}(\mathcal{S}_V) \quad \textit{and} \quad (0, C_1) \in \widehat{\mathrm{Pic}}(\mathcal{S}_V)$$

are numerically equivalent, where $C_1 = \log(2\pi) + 2h^{\mathrm{Falt}}_{\boldsymbol{k}}$*, and we are using the notation of* (4.5.3).

Proof. Fix an $N \geq 3$. As $\mathcal{M}_{W_0}(N)$ is finite étale over $\mathcal{O}_{\boldsymbol{k}}[1/N]$, there is an isomorphism

$$\mathcal{M}_{W_0}(N) \cong \bigsqcup_i \mathcal{X}_i$$

with each $\mathcal{X}_i \cong \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}_i}[1/N])$ for a finite field extension $\boldsymbol{k}_i/\boldsymbol{k}$ unramified outside N. For each $\mathcal{X}_i$, consider the metrized Hodge bundle

$$\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{X}_i} \in \widehat{\mathrm{Pic}}(\mathcal{X}_i)$$

of the universal $A_0 \to \mathcal{X}_i$. The underlying line bundle can be identified with an element in the ideal class group of $\mathcal{O}_{\boldsymbol{k}_i}[1/N]$, and hence some power of it admits a trivializing section

$$s_i \in H^0\big(\mathcal{X}_i, (\omega^{\mathrm{Hdg}}_{A_0/\mathcal{X}_i})^{\otimes d_i}\big) \cong H^0(A_0, \Omega^{\otimes d_i}_{A_0/\mathcal{X}_i}).$$

Comparing (1.1.1) with the definition of the Faltings height, normalized as in §5.2 of [BHK+b], shows that

$$\frac{-1}{\#\mathcal{X}_i(\mathbb{C})} \sum_{x \in \mathcal{X}_i(\mathbb{C})} \log \|s_{i,x}\|^2 = d_i \cdot C_1,$$

up to a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid N\}$.

Letting d be the least common multiple of all d_i's, we see that

$$\big(\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V(N)}\big)^{\otimes d} \in \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V(N))$$

admits a trivializing section s with $-\log\|s\|^2$ constant on every connected component of $\bar{\mathcal{S}}_V(N)(\mathbb{C})$, and such that the average value of $-\log\|s\|^2$ over any $\mathrm{Aut}(\mathbb{C}/\boldsymbol{k})$-orbit of components is $d\cdot C_1$, up to a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid N\}$.

If we represent

$$d\cdot\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V(N)} \in \widehat{\mathrm{CH}}^1(\bar{\mathcal{S}}_V(N))$$

by the arithmetic divisor $\widehat{\mathrm{div}}(s) = (0, -\log\|s\|^2)$, then for any

$$(\mathcal{Z}_N, g_N) \in \widehat{\mathrm{CH}}^{n-1}(\bar{\mathcal{S}}_V(N))$$

the vanishing of the Chern form of $-\log\|s\|^2$ implies the $*$-product formula

$$[-\log\|s\|^2] * g_N = -\log\|s\|^2 \wedge \delta_{\mathcal{Z}_N},$$

which implies the intersection formula

$$d\cdot\widehat{\deg}_N\big((\mathcal{Z}_N, g_N)\cdot\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V(N)}\big) = -\sum_{x\in\mathcal{Z}_N(\mathbb{C})}\log\|s_x\|^2$$

up to a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid N\}$.

Let $L \subset \mathbb{C}$ be a finite Galois extension of $\boldsymbol{k}$ large enough that all complex points of $\mathcal{Z}_N$ are defined over L, and rewrite the equality above as

$$d\cdot\widehat{\deg}_N\big((\mathcal{Z}_N, g_N)\cdot\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V(N)}\big) = \frac{-1}{[L:\boldsymbol{k}]}\sum_{\substack{x\in\mathcal{Z}_N(L)\\ \sigma\in\mathrm{Gal}(L/\boldsymbol{k})}}\log\|s_{x^\sigma}\|^2.$$

The right hand side is $dC_1\cdot\#\mathcal{Z}_N(\mathbb{C})$, and hence

$$\widehat{\deg}_N\big((\mathcal{Z}_N, g_N)\cdot\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V(N)}\big) = C_1\cdot\#\mathcal{Z}_N(\mathbb{C}) = \widehat{\deg}_N\big((\mathcal{Z}_N, g_N)\cdot(0, C_1)\big)$$

up to a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid N\}$. Varying N shows that

$$\widehat{\deg}\big(\widehat{\mathcal{Z}}\cdot\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V}\big) = \widehat{\deg}\big(\widehat{\mathcal{Z}}\cdot(0, C_1)\big)$$

for every $\widehat{\mathcal{Z}} \in \widehat{\mathrm{CH}}^{n-1}(\bar{\mathcal{S}}_V)$, and the claim follows. □

5.4. **Another hermitian line bundle.** In order to relate the hermitian line bundles of (5.2.7), we recall from §5.1 of [BHK+b] a fourth hermitian line bundle. Denote by

$$H^1_{\mathrm{dR}}(A) = \mathbb{R}^1\pi_*\Omega^\bullet_{A/\mathcal{S}_V}$$

the first relative algebraic deRham cohomology of $\pi : A \to \mathcal{S}_V$, a rank $2n$ vector bundle on $\mathcal{S}_V$. The action of $\mathcal{O}_{\boldsymbol{k}}$ on A induces an action on $H_1^{\mathrm{dR}}(A) = H^1_{\mathrm{dR}}(A)^\vee$, which is locally free of rank n over $\mathcal{O}_{\boldsymbol{k}}\otimes_{\mathbb{Z}}\mathcal{O}_S$. If

$$H_1^{\mathrm{dR}}(A) \to \mathcal{V}$$

denotes the largest quotient on which the action of $\mathcal{O}_{\boldsymbol{k}}$ is through the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_{\mathcal{S}_V}$, then $\mathcal{V}$ is a rank n vector bundle on $\mathcal{S}_V$, equipped with a morphism

$$\det(\mathcal{V})^{-1} \to \bigwedge^n H^1_{\mathrm{dR}}(A) \to H^n_{\mathrm{dR}}(A).$$

Given a complex point $z \in \mathcal{S}_V(\mathbb{C})$ and a vector $s_z \in \det(\mathcal{V}_z)^{-1}$, we view s_z as an element of $H^n(A_z, \mathbb{C})$, and define

$$\|s_z\|^2 = \left| \int_{A_z(\mathbb{C})} s_z \wedge \overline{s}_z \right|.$$

This defines a hermitian metric on $\det(\mathcal{V})^{-1}$, and hence we obtain a hermitian line bundle

$$\det(\mathcal{V}) \in \widehat{\mathrm{Pic}}(\mathcal{S}_V). \tag{5.4.1}$$

Proposition 5.4.1. *Assume $n \geq 2$. Recalling the exceptional divisor of Definition 5.1.5 and the notation* (4.5.3)*, we have the equality*

$$2\widehat{\mathcal{L}}_V = \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V} + 2\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V} + \det(\mathcal{V}) + (\mathrm{Exc}_V, C_2)$$

in $\widehat{\mathrm{Pic}}(\mathcal{S}_V)$, where

$$C_2 = 2\log\left(\frac{2e^\gamma}{D}\right) + (2-n)\log(2\pi).$$

In particular, $\det(\mathcal{V}) \in \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V, \mathscr{D}_{\mathrm{pre}})$ *by* (5.2.7)*.*

Proof. This is a restatement of Proposition 5.1.2 of [BHK$^+$b], keeping in mind that the metric on

$$\det(\mathrm{Lie}(A))^{-1} = \omega^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$$

used in [loc. cit.] differs from (1.1.1) by a power of 2π. □

It is an observation of Gross [Gro] that the hermitian line bundle $\det(\mathcal{V})$ behaves in the generic fiber, for all arithmetic purposes, like the trivial bundle with a constant metric. This observation was extended to integral models in §5.3 of [BHK$^+$b], whose results are the basis of the following proposition.

Proposition 5.4.2. *The Chern form of* $\det(\mathcal{V})$ *is identically* 0*. If $n > 2$ then, up to numerical equivalence,*

$$\det(\mathcal{V}) = (0, C_3),$$

where

$$C_3 = (4-2n)h^{\mathrm{Falt}}_{\boldsymbol{k}} + \log(4\pi^2 D).$$

Proof. Fix an integer $N \geq 3$. As in Theorem 1 of [Gro], the $\mathbb{C}$-vector space $H^0(\mathcal{S}_V(N)_{/\mathbb{C}}, \det(\mathcal{V}))$ has dimension 1, and the norm of any global section is a locally constant function on $\mathcal{S}_V(N)(\mathbb{C})$. The vanishing of the Chern form follows. Moreover, one can choose a nonzero global section t defined over a finite Galois extension $\boldsymbol{k}'/\boldsymbol{k}$, and then for every $\boldsymbol{k}$-algebra embedding $\sigma : \boldsymbol{k}' \to \mathbb{C}$ the norm $\|t^\sigma\|$ must again be locally constant.

If $n > 2$ then[2] Theorem 5.3.1 of [BHK$^+$b] allows us to choose t so that it extends to a nowhere vanishing section

$$t \in H^0\big(\mathcal{S}_V(N)_{/\mathcal{O}_{\boldsymbol{k}'}[1/N]}, \det(\mathcal{V})\big).$$

Setting $d = [\boldsymbol{k}' : \boldsymbol{k}]$ and taking the tensor product of all $\mathrm{Gal}(\boldsymbol{k}'/\boldsymbol{k})$-conjugates of t, we obtain a section

$$s \in H^0(\mathcal{S}_V(N), \det(\mathcal{V})^{\otimes d})$$

such that $\mathrm{div}(s) = 0$, and such that $-\log\|s\|$ is locally constant. Let

$$c_X = -\log\|s\|^2$$

denote its value on the connected component $X \subset \mathcal{S}_V(N)_{/\mathbb{C}}$.

Fix a finite Galois extension $L/\boldsymbol{k}$ contained $\mathbb{C}$ large enough that every component $X \subset \mathcal{S}_V(N)_{/\mathbb{C}}$ is defined over L. We may further enlarge L to assume that each X admits an L-point

$$x \in X(L) \subset \mathcal{S}_V(N)(L)$$

that extends to

$$\underline{x} : \mathrm{Spec}(\mathcal{O}_L[1/N]) \to \mathcal{S}_V(N).$$

For example, start by fixing a complex point $x \in X(\mathbb{C})$ corresponding to a pair (A_0, A) for which A has complex multiplication. Then enlarge L so that both A_0 and A (along with their level N-structures) are defined over L and have everywhere good reduction.

Applying Corollary 5.3.2 and Proposition 5.3.3 of [BHK$^+$b] to $\underline{x}$, we find

$$\frac{-1}{[L:\boldsymbol{k}]} \sum_{\sigma \in \mathrm{Gal}(L/\boldsymbol{k})} \log\|s_{x^\sigma}\|^2 = dC_3$$

up to a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid N\}$, and hence

$$\frac{1}{[L:\boldsymbol{k}]} \sum_{\sigma \in \mathrm{Gal}(L/\boldsymbol{k})} c_{X^\sigma} = dC_3$$

up to the same ambiguity.

We have now shown that the average value of $-\log\|s\|^2$ over any $\mathrm{Aut}(\mathbb{C}/\boldsymbol{k})$ orbit of connected components in $\mathcal{S}_V(N)(\mathbb{C})$ is dC_3, up to a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid N\}$. The proposition follows from this, by the same argument used in the proof of Proposition 5.3.1. □

[2]The assumption $n > 2$ is mistakenly omitted in Theorem 5.3.1 of [BHK$^+$b], whose proof requires that $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ has geometrically normal fibers. The normality is a theorem of Pappas when $n > 2$, but is false when $n = 2$.

5.5. **Comparison of hermitian line bundles.** We now come to the main results of §5.

The exceptional divisor of Definition 5.1.5 is a recurring nuisance, in part because it has nontrivial arithmetic intersection with $\widehat{\mathcal{L}}_V$. This will be explored more fully in §5.6 below. The following proposition allows us to avoid this nuisance by slightly modifiying $\widehat{\mathcal{L}}_V$, and also clarifies the relation between the second and third line bundles in (5.2.7).

Theorem 5.5.1. *Assume $n \geq 2$. Recalling the notation* (4.5.3)*, the hermitian line bundle*

$$\widehat{\mathcal{K}}_V \stackrel{\mathrm{def}}{=} 2\widehat{\mathcal{L}}_V - (\mathrm{Exc}_V, 0) \in \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V, \mathscr{D}_{\mathrm{pre}})$$

enjoys the following properties.

(1) *Every irreducible component $E \subset \mathrm{Exc}_V$ satisfies*

$$(E, 0) \cdot \widehat{\mathcal{K}}_V = 0$$

in $\widehat{\mathrm{CH}}^2(\bar{\mathcal{S}}_V, \mathscr{D}_{\mathrm{pre}})_{\mathbb{Q}}$, as well as the height relation

$$\mathrm{ht}_{\widehat{\mathcal{K}}_V}(E) = 0.$$

The same equalities hold if we replace $\widehat{\mathcal{K}}_V$ with $\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$ or $\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V}$.

(2) *We have the equality of Chern forms*

$$\mathrm{ch}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) = 2\,\mathrm{ch}(\widehat{\mathcal{L}}_V) = \mathrm{ch}(\widehat{\mathcal{K}}_V).$$

(3) *If $n > 2$ then, up to numerical equivalence,*

$$\widehat{\mathcal{K}}_V = \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V} + (0, C_0(n)) \in \widehat{\mathrm{Pic}}(\mathcal{S}_V),$$

where $C_0(n)$ is the constant of Theorem A. In particular, up to numerical equivalence,

$$2\widehat{\mathcal{L}}_V = \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V} + (\mathrm{Exc}_V, C_0(n)).$$

Proof. For (1), they key observation is that the abelian scheme $A \to \mathcal{S}_V$ is a pullback via the vertical arrow on the right in (5.1.5). In particular, $\omega^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$ is isomorphic to the pullback of

$$\omega^{\mathrm{Hdg}}_{A/\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}} \in \mathrm{Pic}(\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}).$$

If $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ were a scheme we could trivialize this latter line bundle over a Zariski open neighborhood of the (0-dimensional) singular locus. This would pull-back to a trivialization of $\omega^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$ over an open neighborhood of Exc_V, and in particular over an open neighborhood of any irreducible component $E \subset \mathrm{Exc}_V$.

To account for the stackiness, simply fix an integer $N \geq 3$ and apply the same reasoning with level N-structure to see that $\omega^{\mathrm{Hdg}}_{A/\mathcal{S}_V(N)}$ is trivial in some Zariski open neighborhood of

$$E(N) = E \times_{\mathcal{S}_V} \mathcal{S}_V(N).$$

This implies the arithmetic intersection formula $(E(N), 0) \cdot \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V(N)} = 0$ and varying N proves

$$(E, 0) \cdot \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V} = 0.$$

The line bundle (5.4.1) is also a pullback via the vertical arrow on the right in (5.1.5), hence the same argument shows

$$(E, 0) \cdot \det(\mathcal{V}) = 0.$$

The proof of Proposition 5.3.1 shows that, for any $N \geq 3$, there is a positive multiple of

$$\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V(N)} \in \widehat{\mathrm{CH}}^1(\bar{\mathcal{S}}_V(N), \mathscr{D}_{\mathrm{pre}})$$

that can be represented by a purely archimedean arithmetic divisor $(0, g)$. Any such arithmetic divisor satisfies $(E(N), 0) \cdot (0, g) = 0$, and varying N shows that

$$(E, 0) \cdot \widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V} = 0.$$

Rewriting the relation of Proposition 5.4.1 as

$$\widehat{\mathcal{K}}_V = \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V} + 2\widehat{\omega}^{\mathrm{Hdg}}_{A_0/\mathcal{S}_V} + \det(\mathcal{V}) + (0, C_2), \tag{5.5.1}$$

we have shown that the right hand side has trivial arithmetic intersection with $(E, 0)$, and hence so does the left hand side. This proves the first equality of (1). The second is a formal consequence of this and (4.5.1).

The first equality of (2) follows from (5.5.1), as the Chern forms of the final three terms on the right vanish by Lemma 4.5.2, Proposition 5.3.1, and Proposition 5.4.2. The second equality of (2) is clear from the definitions.

Claim (3) also follows from (5.5.1), using Proposition 5.3.1, Proposition 5.4.2, and the equality $C_0(n) = 2C_1 + C_2 + C_3$. □

Remark 5.5.2. If $n > 2$ then part (3) of Theorem 5.5.1 implies

$$\begin{aligned} 2^n \cdot \widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) &= \widehat{\mathrm{vol}}\big(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V} + (0, C_0(n))\big) \\ &= \widehat{\mathrm{vol}}\big(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}\big) + nC_0(n) \int_{\mathcal{S}_V(\mathbb{C})} \mathrm{ch}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V})^{n-1} \end{aligned}$$

where we have used Lemma 4.5.2 for the first equality, and Lemma 4.5.3 for the second. In Proposition 8.1.1 we will show that this equality also holds when $n = 2$.

5.6. **Volume of the exceptional divisor.** In this subsection we assume $n \geq 2$, and fix an irreducible (= connected) component E of the exceptional divisor of Definition 5.1.5. Recalling the notation (4.5.3), our goal is to compute the arithmetic volume of

$$(E, 0) \in \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V).$$

By definition of the exceptional divisor, there is a commutative diagram with cartesian squares

$$\begin{array}{ccc} E & \longrightarrow & e \\ \downarrow & & \downarrow \\ \mathrm{Exc}_V & \longrightarrow & \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathrm{Sing}_{(n-1,1)} \\ \downarrow & & \downarrow \\ \mathcal{S}_V & \longrightarrow & \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \end{array}$$

in which e is a connected component of the 0-dimensional reduced and irreducible $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathrm{Sing}_{(n-1,1)}$. In particular, e is supported in a single characteristic $p \mid D$, and admits a presentation as a stack quotient

$$e \cong \Delta \backslash \operatorname{Spec}(\mathbb{F}'_{\mathfrak{p}}),$$

in which $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ is the prime above p, $\mathbb{F}'_{\mathfrak{p}}$ is a finite extension of $\mathbb{F}_{\mathfrak{p}} = \mathcal{O}_{\boldsymbol{k}}/\mathfrak{p}$, and Δ is a finite group acting on $\mathbb{F}'_{\mathfrak{p}}$. Define a rational number

$$m_E \stackrel{\text{def}}{=} \sum_{z \in e(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{p}})} \frac{1}{|\operatorname{Aut}(z)|} = \frac{[\mathbb{F}'_{\mathfrak{p}} : \mathbb{F}_{\mathfrak{p}}]}{|\Delta|}.$$

Proposition 5.6.1. *The iterated intersection*

$$\widehat{\mathcal{L}}_V^{-1} \cdots \widehat{\mathcal{L}}_V^{-1} \cdot (E, 0) \in \widehat{\mathrm{CH}}^n(\bar{\mathcal{S}}_V, \mathscr{D}_{\mathrm{pre}})$$

has arithmetic degree $\widehat{m}_E = m_E \log(p)$

Proof. After Remark 4.2.1, one might expect that E is isomorphic to the projective space of hyperplanes in an n-dimensional vector space. In particular, there should be a universal hyperplane $\mathcal{F} \subset \mathcal{O}_E^n$, with the property that the restriction of $\mathcal{L}_V^{-1}$ to E is isomorphic to $\mathcal{O}_E^n/\mathcal{F}$.

The obstruction to this being true is due to stacky issues, which can be removed by fixing an integer $N \geq 3$ prime to p and adding level N-structure to the universal pair (A_0, A) over e. Let $e(N) \to e$ be the scheme classifying such level structures, and consider the cartesian diagram (this is the definition of the upper left corner)

$$\begin{array}{ccc} E(N) & \longrightarrow & e(N) \\ \downarrow & & \downarrow \\ E & \longrightarrow & e. \end{array}$$

The scheme $e(N)$, being a reduced scheme finite over $\mathbb{F}_\mathfrak{p}$, is a disjoint union of finitely many spectra of finite extensions of $\mathbb{F}_\mathfrak{p}$.

Fix a connected component $E' \subset E(N)$, and let

$$\mathrm{Spec}(\mathbb{F}'_\mathfrak{p}) = e' \subset e(N)$$

be the connected component below it. As in Remark 4.2.1, E' is precisely the projective space over e' classifying hyperplanes $\mathcal{F} \subset \mathrm{Lie}(A|_{e'})$. Moreover, after fixing a trivialization $\mathrm{Lie}(A_0|_{e'}) \cong \mathbb{F}'_\mathfrak{p}$, the isomorphism (5.2.4) identifies

$$\mathcal{L}_V^{-1}|_{E'} = \mathrm{Lie}(A|_{E'})/\mathcal{F} \in \mathrm{Pic}(E') \cong \mathrm{CH}^1(E')$$

where $\mathcal{F} \subset \mathrm{Lie}(A|_{E'})$ is the universal hyperplane. A routine exercise then shows that the iterated intersection

$$(\mathcal{L}_V^{-1}|_{E'}) \cdots (\mathcal{L}_V^{-1}|_{E'}) \in \mathrm{CH}^{n-1}(E') \cong \mathbb{Z}$$

is represented by the cycle class of any $\mathbb{F}'_\mathfrak{p}$-valued point of E'. In other words, the cycle class of any section to $E' \to e'$.

Now allow the connected component $E' \subset E(N)$ to vary. If we fix any section to $E(N) \to e(N)$, and use it to view $e(N)$ as a 0-cycle on $E(N)$, then

$$e(N) = (\mathcal{L}_V^{-1}|_{E(N)}) \cdots (\mathcal{L}_V^{-1}|_{E(N)}) \in \mathrm{CH}^{n-1}(E(N)).$$

This implies the arithmetic intersection formula

$$(e(N), 0) = \widehat{\mathcal{L}}_V^{-1} \cdots \widehat{\mathcal{L}}_V^{-1} \cdot (E(N), 0) \in \widehat{\mathrm{CH}}^n(\bar{\mathcal{S}}_V(N), \mathscr{D}_{\mathrm{pre}})_\mathbb{Q}.$$

Finally, recalling (4.4.3), we deduce

$$\widehat{\deg}\big(\widehat{\mathcal{L}}_V^{-1} \cdots \widehat{\mathcal{L}}_V^{-1} \cdot (E, 0)\big) = \frac{\# e(N)(\mathbb{F}_\mathfrak{p}^{\mathrm{alg}})}{d_N} \log(p) = \sum_{z \in e(\mathbb{F}_\mathfrak{p}^{\mathrm{alg}})} \frac{\log(p)}{|\mathrm{Aut}(z)|}$$

up to a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid N\}$. Varying N completes the proof. □

Corollary 5.6.2. *The hermitian line bundle* $(E, 0) \in \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V)$ *has arithmetic volume*

$$\widehat{\mathrm{vol}}(E, 0) = (-2)^{n-1} \cdot \widehat{m}_E.$$

Proof. Part (1) of Theorem 5.5.1 implies the second equality in

$$(E, 0) \cdot (E, 0) = (\mathrm{Exc}_V, 0) \cdot (E, 0) = 2\widehat{\mathcal{L}}_V \cdot (E, 0) \in \widehat{\mathrm{CH}}^2(\bar{\mathcal{S}}_V, \mathscr{D}_{\mathrm{pre}})_\mathbb{Q}.$$

This implies the iterated intersection formula

$$(E, 0) \cdots (E, 0) = 2^{n-1} \widehat{\mathcal{L}}_V \cdots \widehat{\mathcal{L}}_V \cdot (E, 0) \in \widehat{\mathrm{CH}}^n(\bar{\mathcal{S}}_V, \mathscr{D}_{\mathrm{pre}})_\mathbb{Q},$$

and claim follows using Proposition 5.6.1. □

6. Kudla-Rapoport divisors at split primes

Assume $n > 2$, and let $\mathcal{S}_V$ be the Shimura variety (5.1.4) associated to a $\boldsymbol{k}$-hermitian space V of signature $(n-1, 1)$ containing a self-dual lattice. Fix a prime p split in $\boldsymbol{k}$, and factor

$$p\mathcal{O}_{\boldsymbol{k}} = \mathfrak{p}\bar{\mathfrak{p}}.$$

We explain how the Kudla-Rapoport divisor $\mathcal{Z}_V(p) \to \mathcal{S}_V$ of Definition 5.1.6 is related to a Shimura variety $\mathcal{S}_{V'}$ with V' of signature $(n-2, 1)$.

6.1. **Statement of the results.** Let V' denote the $\boldsymbol{k}$-hermitian space of signature $(n-2, 1)$ whose local invariants satisfy

$$\mathrm{inv}_\ell(V') = (p, -D)_\ell \cdot \mathrm{inv}_\ell(V) \tag{6.1.1}$$

for all places $\ell \le \infty$. Equivalently, V' is the orthogonal complement to the $\boldsymbol{k}$-span of any $x \in V$ of hermitian norm $\langle x, x \rangle = p$.

Lemma 6.1.1. *The hermitian space V' admits a self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice.*

Proof. Suppose ℓ is a rational prime unramified in $\boldsymbol{k}$. If $\ell \neq p$ then $p \in \mathbb{Z}_\ell^\times$, and hence is a norm from the unramified extension $\boldsymbol{k}_\ell$. If $\ell = p$ then p is again a norm from $\boldsymbol{k}_\ell \cong \mathbb{Q}_p \times \mathbb{Q}_p$. In either case, $(p, -D)_\ell = 1$, and so the local invariants of V and V' agree at all unramified primes.

In general, a $\boldsymbol{k}$-hermitian space over $\boldsymbol{k}$ admits a self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice if and only if it has local invariant 1 at every prime unramified in $\boldsymbol{k}$. As V satisfies this condition by hypothesis, so does V'. □

Lemma 6.1.1 allows us to form the $\mathcal{O}_{\boldsymbol{k}}$-stacks

$$\mathcal{S}_{V'} \subset \bar{\mathcal{S}}_{V'} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \bar{\mathcal{M}}_{(n-2,1)}$$

analogous to

$$\mathcal{S}_V \subset \bar{\mathcal{S}}_V \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \bar{\mathcal{M}}_{(n-1,1)},$$

but in one dimension lower. The stack $\mathcal{S}_{V'}$ is endowed with its own hermitian line bundle $\widehat{\mathcal{L}}_{V'}$ as in (5.2.3), its own exceptional divisor $\mathrm{Exc}_{V'}$ as in Definition 5.1.5, and its own universal pair (A_0', A') of polarized abelian schemes with $\mathcal{O}_{\boldsymbol{k}}$-actions.

The following theorems, whose proofs will occupy the remainder of §6, lie at the core of the inductive arguments of §8.

Theorem 6.1.2. *If we set*

$$\widehat{\mathcal{K}}_V = 2\widehat{\mathcal{L}}_V - (\mathrm{Exc}_V, 0) \in \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V, \mathscr{D}_{\mathrm{pre}})$$

as in Theorem 5.5.1, and similarly with V replaced by V', then

$$\int_{\mathcal{Z}_V(p)(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{K}}_V)^{n-2} = (p^{n-1} + 1)\, \mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_{V'}).$$

Moreover, there is a rational number $a(p)$ such that

$$\frac{\mathrm{ht}_{\widehat{\mathcal{K}}_V}(\bar{\mathcal{Z}}_V(p))}{p^{n-1} + 1} = \widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_{V'}) + a(p) \log(p).$$

Theorem 6.1.3. *There is a rational number $b(p)$ such that*

$$\frac{\mathrm{ht}_{\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}}(\bar{\mathcal{Z}}_V(p))}{p^{n-1}+1} = \widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}}) + b(p)\log(p) + (1-n)\left(\frac{L'(0,\varepsilon)}{L(0,\varepsilon)} + \frac{\log(D)}{2}\right)\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}}).$$

The proofs are rather long, so we summarize now the key steps. The central idea is to make precise the impressionistic relation

$$\mathcal{Z}_V(p) \text{“} = \text{”} (p^{n-1}+1)\cdot \mathcal{S}_{V'},$$

by decomposing

$$\mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]} = \mathcal{U}_0 \sqcup \mathcal{U}_{\mathfrak{p}} \sqcup \mathcal{U}_{\overline{\mathfrak{p}}} \tag{6.1.2}$$

as a disjoint union of open and closed substacks (Proposition 6.2.2 below). The stacks on the right hand side are related to $\mathcal{S}_{V'}$ by closed immersions

$$i_{\mathfrak{p}} : \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \to \mathcal{U}_{\mathfrak{p}}, \qquad i_{\overline{\mathfrak{p}}} : \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \to \mathcal{U}_{\overline{\mathfrak{p}}},$$

and by a diagram

$$\mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \longleftarrow \mathcal{T}_{V'} \xrightarrow{i_0} \mathcal{U}_0$$

in which the leftward arrow is a finite étale surjection of degree $p^{n-1}-1$, and the rightward arrow is a closed immersion. See (6.5.11) for the definition of the stack $\mathcal{T}_{V'}$.

Recalling the exceptional locus of Definition 5.1.5, denote by

$$\mathcal{S}_V^{\mathrm{nexc}} = \mathcal{S}_V \smallsetminus \mathrm{Exc}_V$$

the (open) nonexceptional locus of $\mathcal{S}_V$, set

$$\mathcal{Z}_V^{\mathrm{nexc}}(p) = \mathcal{Z}_V(p) \times_{\mathcal{S}_V} \mathcal{S}_V^{\mathrm{nexc}},$$

and make the same definitions with V replaced by V'. Set

$$\mathcal{U}_{\Box}^{\mathrm{nexc}} = \mathcal{U}_{\Box} \cap \mathcal{Z}_V^{\mathrm{nexc}}(p)$$

for $\Box \in \{0, \mathfrak{p}, \overline{\mathfrak{p}}\}$, and

$$\mathcal{T}_{V'}^{\mathrm{nexc}} = \mathcal{T}_{V'} \times_{\mathcal{S}_{V'}} \mathcal{S}_{V'}^{\mathrm{nexc}}.$$

We will show that the closed immersions i_0, $i_{\mathfrak{p}}$, and $i_{\overline{\mathfrak{p}}}$ are close to being isomorphisms. More precisely, i_0 restricts to an isomorphism

$$i_0 : \mathcal{T}_{V'}^{\mathrm{nexc}} \cong \mathcal{U}_0^{\mathrm{nexc}},$$

while $i_{\mathfrak{p}}$ and $i_{\overline{\mathfrak{p}}}$ restrict to isomorphisms

$$i_{\mathfrak{p}} : \mathcal{S}^{\mathrm{nexc}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \cong \mathcal{U}_{\mathfrak{p}}^{\mathrm{nexc}} \quad \text{and} \quad i_{\overline{\mathfrak{p}}} : \mathcal{S}^{\mathrm{nexc}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \cong \mathcal{U}_{\overline{\mathfrak{p}}}^{\mathrm{nexc}}.$$

After taking compactifications into account, both theorems above will follow easily from these isomorphisms. Note that it suffices to work only over the nonexceptional locus, as part (1) of Theorem 5.5.1 guarantees that the hermitian line bundles in questions have trivial arithmetic intersection with all components of the exceptional divisor (which is why we work with $\widehat{\mathcal{K}}_V$ in Theorem 6.1.2 instead of $\widehat{\mathcal{L}}_V$).

6.2. **Decomposing the Kudla-Rapoport divisor.** The decomposition (6.1.2) is a geometric reflection of a result of linear algebra.

As in (5.1.1), choose an isomorphism

$$V \cong \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$$

in which (W_0, h_0) and (W, h) are relevant hermitian spaces of signatures $(1,0)$ and $(n-1,1)$. Fix self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattices $\mathfrak{a}_0 \subset W_0$ and $\mathfrak{a} \subset W$. Any vector

$$x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}) \subset V$$

with $\langle x, x\rangle = p$ determines an orthogonal decomposition

$$W = \tilde{W}_0 \oplus W'$$

with $\tilde{W}_0 = x(W_0)$, and a corresponding decomposition

$$V = \boldsymbol{k}x \oplus V'$$

with $V' = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W')$.

Lemma 6.2.1. *The $\mathcal{O}_{\boldsymbol{k}}$-lattice $\tilde{\mathfrak{a}}_0 = \mathfrak{a} \cap \tilde{W}_0$ satisfies exactly one of*

$$\tilde{\mathfrak{a}}_0 = x(\mathfrak{a}_0), \qquad \tilde{\mathfrak{a}}_0 = \mathfrak{p}^{-1}x(\mathfrak{a}_0), \qquad \tilde{\mathfrak{a}}_0 = \overline{\mathfrak{p}}^{-1}x(\mathfrak{a}_0).$$

Proof. Use x to identify $W_0 = \tilde{W}_0$, and hence $\mathfrak{a}_0 \subset \tilde{\mathfrak{a}}_0 \subset \mathfrak{a}$. Recalling the symplectic forms

$$e_0 : W_0 \times W_0 \to \mathbb{Q}, \qquad e : W \times W \to \mathbb{Q}$$

of (4.1.2), the relation (5.1.2) implies that $e|_{W_0} = p \cdot e_0$. The inclusion

$$e_0(p\tilde{\mathfrak{a}}_0, \mathfrak{a}_0) = e(\tilde{\mathfrak{a}}_0, \mathfrak{a}_0) \subset e(\mathfrak{a}, \mathfrak{a}) = \mathbb{Z},$$

together with the self-duality of $\mathfrak{a}_0$ under e_0, shows that $p\tilde{\mathfrak{a}}_0 \subset \mathfrak{a}_0$. If equality held we would have

$$e_0(\mathfrak{a}_0, \mathfrak{a}_0) = p^{-1}e(\mathfrak{a}_0, \mathfrak{a}_0) \subset pe(\tilde{\mathfrak{a}}_0, \tilde{\mathfrak{a}}_0) \subset pe(\mathfrak{a}, \mathfrak{a}) \subset p\mathbb{Z},$$

contradicting $\mathfrak{a}_0$ being self-dual under e_0.

As $\tilde{\mathfrak{a}}_0$ is $\mathcal{O}_{\boldsymbol{k}}$-stable with $\mathfrak{a}_0 \subset \tilde{\mathfrak{a}}_0 \subsetneq p^{-1}\mathfrak{a}_0$, it is $\mathfrak{a}_0$, $\mathfrak{p}^{-1}\mathfrak{a}_0$, or $\overline{\mathfrak{p}}^{-1}\mathfrak{a}_0$. $\square$

Proposition 6.2.2. *Let (A_0, A, x) be the universal triple over $\mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]}$, so that $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ satisfies $\langle x, x\rangle = p$. There is a decomposition* (6.1.2) *into open and closed substacks in which*

- *$\mathcal{U}_0$ is the locus of points where $\ker(x) = 0$,*
- *$\mathcal{U}_{\mathfrak{p}}$ is the locus of points where $\ker(x) = A_0[\mathfrak{p}]$,*
- *$\mathcal{U}_{\overline{\mathfrak{p}}}$ is the locus of points where $\ker(x) = A_0[\overline{\mathfrak{p}}]$.*

Proof. Recalling (5.1.3), the relation $\langle x, x\rangle = p$ implies that multiplication-by-p factors as

$$A_0 \xrightarrow{x} A \cong A^\vee \xrightarrow{x^\vee} A_0^\vee \cong A_0,$$

and so $\ker(x) \subset A_0[p]$. Both of these group schemes are finite étale over $\mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]}$, which implies that each of $\mathcal{U}_0$, $\mathcal{U}_{\mathfrak{p}}$, and $\mathcal{U}_{\overline{\mathfrak{p}}}$ is open and closed. It only remains to prove that every geometric point $s \to \mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]}$ is contained in one of them.

Abbreviate $T_0 = T_p(A_{0s})$ and $T = T_p(A_s)$. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_0 & \longrightarrow & T_0 \otimes \mathbb{Q}_p & \longrightarrow & A_{0s}[p^\infty] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_s} & & \downarrow{\scriptstyle x_s} & & \downarrow{\scriptstyle x_s} & & \\ 0 & \longrightarrow & T & \longrightarrow & T \otimes \mathbb{Q}_p & \longrightarrow & A_s[p^\infty] & \longrightarrow & 0, \end{array}$$

and using the vertical arrow on the left to identify $T_0 \subset T$, we find that

$$\ker(x_s) \cong \tilde{T}_0 / T_0,$$

where $\tilde{T}_0 = T \cap T_{0\mathbb{Q}_p}$.

After fixing an isomorphism $\mathbb{Z}_p \cong \mathbb{Z}_p(1)$ of étale sheaves on s, there are unique $\mathcal{O}_{\boldsymbol{k},p}$-valued hermitian forms h_0 and h on T_0 and T, respectively, related to the Weil pairings e_0 and e by (4.1.2). Thus we may apply Lemma 6.2.1 with $\mathfrak{a}_0$ and $\mathfrak{a}$ replaced by T_0 and T, to see that $\tilde{T}_0$ must be one of T_0, $\mathfrak{p}^{-1}T_0$, or $\overline{\mathfrak{p}}^{-1}T_0$. These three cases correspond to $\ker(x_s)$ being trivial, $A_{0s}[\mathfrak{p}]$, or $A_{0s}[\overline{\mathfrak{p}}]$. □

6.3. **Analysis of $\mathcal{U}_{\mathfrak{p}}$.** In this subsection we study the structure of the substack $\mathcal{U}_{\mathfrak{p}}$ of (6.1.2), and make explicit its relation to $\mathcal{S}_{V'}$. The analogous analysis of $\mathcal{U}_{\overline{\mathfrak{p}}}$ is obtained by replacing $\mathfrak{p}$ by $\overline{\mathfrak{p}}$ everywhere.

Return to the situation of Lemma 6.2.1, so that $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a})$ with $\langle x, x\rangle = p$ determines an orthogonal decomposition

$$W = \tilde{W}_0 \oplus W'.$$

Set $\tilde{\mathfrak{a}}_0 = \mathfrak{a} \cap \tilde{W}_0$.

Lemma 6.3.1. *If $\tilde{\mathfrak{a}}_0 = \mathfrak{p}^{-1}x(\mathfrak{a}_0)$, there is an orthogonal decomposition*

$$\mathfrak{a} = \tilde{\mathfrak{a}}_0 \oplus \mathfrak{a}' \subset W$$

in which $\mathfrak{a}' = \mathfrak{a} \cap W'$.

Proof. If we use x to identify $W_0 = \tilde{W}_0$, the assumption $\tilde{\mathfrak{a}}_0 = \mathfrak{p}^{-1}\mathfrak{a}_0$ implies that $\tilde{\mathfrak{a}}_0$ is self-dual with respect to the hermitian form $h|_{\tilde{W}_0} = ph_0$. The desired decomposition then follows by elementary linear algebra. □

The lemma suggests that if $(A_0, A, x) \in \mathcal{U}_{\mathfrak{p}}(S)$ for an $\mathcal{O}_{\boldsymbol{k}}[1/p]$-scheme S, then $x : A_0 \to A$ should determine an $\mathcal{O}_{\boldsymbol{k}}$-linear splitting

$$A = \tilde{A}_0 \times A' \qquad (6.3.1)$$

of principally polarized abelian schemes. Indeed, this is the case. If we set

$$\tilde{A}_0 = A_0/A_0[\mathfrak{p}]$$

and recall that $\ker(x) = A_0[\mathfrak{p}]$, the morphism $x : A_0 \to A$ factors as

$$A_0 \to \tilde{A}_0 \xrightarrow{y} A$$

for some $y \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\tilde{A}_0, A)$ satisfying $\langle y, y\rangle = 1$. In other words, the composition

$$\tilde{A}_0 \xrightarrow{y} A \cong A^\vee \xrightarrow{y^\vee} \tilde{A}_0^\vee \cong \tilde{A}_0$$

is the identity. This implies that the composition

$$A \cong A^\vee \xrightarrow{y^\vee} \tilde{A}_0^\vee \cong \tilde{A}_0 \xrightarrow{y} A$$

is a Rosati-fixed idempotent in $\mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(A)$, and A admits a unique splitting (6.3.1) of principally polarized abelian schemes over S such that this idempotent is the projection to the first factor.

Apply the above construction to the universal triple (A_0, A, x) over $\mathcal{U}_\mathfrak{p}$, and recall that A comes equipped with an $\mathcal{O}_{\boldsymbol{k}}$-stable hyperplane $\mathcal{F} \subset \mathrm{Lie}(A)$ satisfying Krämer's signature condition. Using the decomposition

$$\mathrm{Lie}(A) = \mathrm{Lie}(\tilde{A}_0) \oplus \mathrm{Lie}(A')$$

of vector bundles on S, denote by $\mathcal{U}_\mathfrak{p}^\dagger \subset \mathcal{U}_\mathfrak{p}$ the largest closed substack over which $\mathrm{Lie}(\tilde{A}_0) \subset \mathcal{F}$.

Proposition 6.3.2. *There is a canonical isomorphism*

$$i_\mathfrak{p} : \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \cong \mathcal{U}_\mathfrak{p}^\dagger.$$

Proof. As above, let S be an $\mathcal{O}_{\boldsymbol{k}}[1/p]$-scheme. Given a point $(A_0, A, x) \in \mathcal{U}_\mathfrak{p}^\dagger(S)$, the splitting (6.3.1) determines

$$A' \in \mathcal{M}_{(n-2,1)}(S),$$

where we have endowed A' with the $\mathcal{O}_{\boldsymbol{k}}$-stable hyperplane

$$\mathcal{F}' = \mathcal{F}/\,\mathrm{Lie}(\tilde{A}_0) \subset \mathrm{Lie}(A)/\,\mathrm{Lie}(\tilde{A}_0) \cong \mathrm{Lie}(A').$$

satisfying Krämer's condition. The pair (A_0, A') defines an S-point of

$$\mathcal{S}_{V'} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-2,1)},$$

and we have now constructed a morphism

$$\mathcal{U}_\mathfrak{p}^\dagger \xrightarrow{(A_0,A,x)\to(A_0,A')} \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]}. \tag{6.3.2}$$

Conversely, start with an S-point

$$(A_0', A') \in \mathcal{S}_{V'}(S).$$

First define elliptic curves $A_0 = A_0'$ and $\tilde{A}_0 = A_0/A_0[\mathfrak{p}]$. Then define an abelian scheme A by (6.3.1), and endow A with its product principal polarization and product $\mathcal{O}_{\boldsymbol{k}}$-action. Recalling that A' comes equipped with a hyperplane $\mathcal{F}' \subset \mathrm{Lie}(A')$ satisfying Krämer's condition, we endow A with the hyperplane

$$\mathcal{F} = \mathrm{Lie}(\tilde{A}_0) \oplus \mathcal{F}' \subset \mathrm{Lie}(A).$$

It is easy to check that A, with its extra data, defines an S-point of $\mathcal{M}_{(n-1,1)}$, and that (A_0, A) defines an S-point of the open and closed substack

$$\mathcal{S}_V \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-2,1)}.$$

If we define $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ as the composition

$$A_0 \to \tilde{A}_0 \hookrightarrow \tilde{A}_0 \times A' = A,$$

where the first arrow is the quotient map, then $\langle x, x\rangle = p$ and $\ker(x) = A_0[\mathfrak{p}]$. The triple (A_0, A, x) defines an S-point of $\mathcal{U}_{\mathfrak{p}}^{\dagger}$, and the morphism

$$\mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \xrightarrow{(A_0',A')\to(A_0,A,x)} \mathcal{U}_{\mathfrak{p}}^{\dagger}$$

is inverse to (6.3.2). □

Proposition 6.3.2 gives us a commutative diagram

(6.3.3)

$$\begin{array}{ccc} \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} & \longrightarrow & \mathcal{S}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]} \\ \downarrow & & \downarrow \\ \big(\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}^{\mathrm{Pap}}_{(n-2,1)}\big)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]} & \longrightarrow & \big(\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}\big)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]} \end{array}$$

in which the top horizontal arrow is the composition

$$\mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \xrightarrow{i_{\mathfrak{p}}} \mathcal{U}_{\mathfrak{p}}^{\dagger} \hookrightarrow \mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]} \to \mathcal{S}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]},$$

and the bottom horizontal arrow sends $(A_0', A') \mapsto (A_0, A)$, where

$$A_0 = A_0', \quad \tilde{A}_0 = A_0/A_0[\mathfrak{p}], \quad A = \tilde{A}_0 \times A'. \tag{6.3.4}$$

Both horizontal arrows are finite and unramified.

Proposition 6.3.3. *The homomorphism*

$$\widehat{\mathrm{Pic}}(\mathcal{S}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]})_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(\mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]})_{\mathbb{Q}}$$

induced by (6.3.3) *sends*

$$\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V} \mapsto \widehat{\omega}^{\mathrm{Hdg}}_{A_0'/\mathcal{S}_{V'}} + \widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}},$$

where (A_0', A') *is the universal pair over* $\mathcal{S}_{V'}$*. The same map also sends*

$$\widehat{\mathcal{L}}_V \mapsto \widehat{\mathcal{L}}_{V'} \quad \textit{and} \quad (\mathrm{Exc}_V, 0) \mapsto (\mathrm{Exc}_{V'}, 0).$$

Proof. It follows from (6.3.4) that the top horizontal arrow in (6.3.3) sends

$$\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V} \mapsto \widehat{\omega}^{\mathrm{Hdg}}_{\tilde{A}_0/\mathcal{S}_{V'}} + \widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}}.$$

As we have inverted p on the base, the degree p quotient map $A'_0 \to \tilde{A}_0$ induces an isomorphism on Lie algebras, and hence an isomorphism

$$\omega^{\mathrm{Hdg}}_{\tilde{A}_0/\mathcal{S}_{V'}} \cong \omega^{\mathrm{Hdg}}_{A'_0/\mathcal{S}_{V'}}.$$

This isomorphism does not respect the metrics (1.1.1), but one can easily check that

$$\widehat{\omega}^{\mathrm{Hdg}}_{\tilde{A}_0/\mathcal{S}_{V'}} = \widehat{\omega}^{\mathrm{Hdg}}_{A'_0/\mathcal{S}_{V'}} + (0, -\log(p))$$

as elements of $\widehat{\mathrm{Pic}}(\mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]})$. The correction term $(0, -\log(p))$ is torsion in the arithmetic Picard group, as

$$2(0, -\log(p)) = (\mathrm{div}(p), -\log(p^2)) = 0,$$

and the first claim follows.

For the second claim, let S be an $\mathcal{O}_{\boldsymbol{k}}[1/p]$-scheme, fix an S-point $(A'_0, A') \in \mathcal{S}_{V'}(S)$, and let $(A_0, A) \in \mathcal{S}_V(S)$ be its image under the top horizontal arrow in (6.3.3). Tracing through the construction of $i_{\mathfrak{p}}$ in Proposition 6.3.2, we see that

$$\mathrm{Lie}(A)/\mathcal{F} \cong \mathrm{Lie}(A')/\mathcal{F}'.$$

It follows immediately from this and (5.2.4) that $\mathcal{L}_V|_S \cong \mathcal{L}_{V'}|_S$. At a complex point $s \in S(\mathbb{C})$ we have a decomposition

$$H_1(A_s, \mathbb{Q}) = H_1(\tilde{A}_{0s}, \mathbb{Q}) \oplus H_1(A'_s, \mathbb{Q}),$$

which induces an isometric embedding

$$\mathrm{Hom}_{\boldsymbol{k}}(H_1(A_{0s}, \mathbb{Q}), H_1(A'_s, \mathbb{Q})) \subset \mathrm{Hom}_{\boldsymbol{k}}(H_1(A_{0s}, \mathbb{Q}), H_1(A_s, \mathbb{Q}))$$

of $\boldsymbol{k}$-hermitian spaces. Recalling (5.2.5), this isometric embedding restricts to the isomorphism $\mathcal{L}_{V',s} \cong \mathcal{L}_{V,s}$ just constructed, which therefore preserves the metrics defined by (5.2.6).

Finally, we show that under the top horizontal arrow in (6.3.3), the exceptional divisor $\mathrm{Exc}_V \subset \mathcal{S}_V$ pulls back to the exceptional divisor $\mathrm{Exc}_{V'} \subset \mathcal{S}_{V'}$. These divisors are, by Definition 5.1.5, the pullbacks of the singular loci

$$\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathrm{Sing}_{(n-1,1)} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \tag{6.3.5}$$

$$\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathrm{Sing}_{(n-2,1)} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}^{\mathrm{Pap}}_{(n-2,1)} \tag{6.3.6}$$

of (4.2.2) under the vertical arrows in (6.3.3), and so it suffices to prove that the first singular locus (6.3.5) pulls back to the second singular locus (6.3.6) under the bottom horizontal arrow in (6.3.3). Moreover, as each of (6.3.5) and (6.3.6) is a reduced $\mathcal{O}_{\boldsymbol{k}}$-stack of dimension 0, and as the bottom horizontal arrow in (6.3.3) is finite and unramified, it suffices to check this on the level of geometric points. This is an easy exercise in linear algebra, using the characterization of the singular locus found in Remark 4.2.1. □

Proposition 6.3.4. *The nonexceptional locus* $\mathcal{U}^{\mathrm{nexc}}_{\mathfrak{p}} \subset \mathcal{U}_{\mathfrak{p}}$ *satisfies*

$$\mathcal{U}^{\mathrm{nexc}}_{\mathfrak{p}} \subset \mathcal{U}^{\dagger}_{\mathfrak{p}},$$

and the isomorphism of Proposition 6.3.2 restricts to an isomorphism

$$\mathcal{S}^{\mathrm{nexc}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \cong \mathcal{U}^{\mathrm{nexc}}_{\mathfrak{p}}.$$

Proof. Suppose S is an $\mathcal{O}_{\boldsymbol{k}}[1/p]$-scheme,

$$(A_0, A, x) \in \mathcal{U}^{\mathrm{nexc}}_{\mathfrak{p}}(S),$$

and let A' be the abelian scheme of (6.3.1). In particular,

$$\mathrm{Lie}(A) = \mathrm{Lie}(\tilde{A}_0) \oplus \mathrm{Lie}(A').$$

The natural map

$$\mathcal{U}^{\mathrm{nexc}}_{\mathfrak{p}} \to \mathcal{M}_{(n-1,1)} \smallsetminus \mathrm{Exc}_{(n-1,1)}$$

endows A with a hyperplane $\mathcal{F} \subset \mathrm{Lie}(A)$, and results of Krämer, as summarized in Theorem 2.3.3 of [BHK$^+$a], shows that this hyperplane is actually determined by the $\mathcal{O}_{\boldsymbol{k}}$-action on A using the following recipe: If we fix a $\pi \in \mathcal{O}_{\boldsymbol{k}}$ such that $\mathcal{O}_{\boldsymbol{k}} = \mathbb{Z}[\pi]$, and let

$$\overline{\epsilon}_S = \overline{\pi} \otimes 1 - 1 \otimes i_S(\overline{\pi}) \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S,$$

where $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$ is the structure map, then

$$\mathcal{F} = \ker(\overline{\epsilon}_S : \mathrm{Lie}(A) \to \mathrm{Lie}(A)).$$

On the other hand, the action of $\mathcal{O}_{\boldsymbol{k}}$ on the Lie algebra of A_0 is through the structure morphism i_S, and so the same is true of $\tilde{A}_0 = A_0/A_0[\mathfrak{p}]$. This implies that $\overline{\epsilon}_S$ annihilates $\mathrm{Lie}(\tilde{A}_0)$, and hence $\mathrm{Lie}(\tilde{A}_0) \subset \mathcal{F}$. Having shown

$$(A_0, A, x) \in \mathcal{U}^{\dagger}_{\mathfrak{p}}(S),$$

the first claim of the proposition is proved. The second claim follows from the first and Proposition 6.3.3. □

Proposition 6.3.5. *The closed immersion* $i_{\mathfrak{p}} : \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \to \mathcal{U}_{\mathfrak{p}}$ *of Proposition 6.3.2 extends uniquely to a proper morphism*

$$\bar{\mathcal{S}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \to \bar{\mathcal{U}}_{\mathfrak{p}},$$

where the codomain is defined as the Zariski closure of

$$\mathcal{U}_{\mathfrak{p}} \subset \bar{\mathcal{Z}}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]}.$$

or, equivalently, as the normalization of $\mathcal{U}_{\mathfrak{p}} \to \bar{\mathcal{S}}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]}$.

Proof. For ease of notation, we omit all subscripts $\mathcal{O}_{\boldsymbol{k}}[1/p]$ in the proof.

As the exceptional locus of $\bar{\mathcal{S}}_{V'}$ does not meet the boundary, it suffices to construct the extension over its complement

$$\bar{\mathcal{S}}^{\mathrm{nexc}}_{V'} = \bar{\mathcal{S}}_{V'} \smallsetminus \mathrm{Exc}_{V'}.$$

Consider the universal pair (A_0, A') over $\mathcal{S}^{\mathrm{nexc}}_{V'}$, and let

$$x : A_0 \to A = \tilde{A}_0 \times A'$$

be as in the proof of Proposition 6.3.2. In other words, the triple (A_0, A, x) over $\mathcal{S}_{V'}^{\mathrm{nexc}}$ determines the isomorphism $i_{\mathfrak{p}} : \mathcal{S}_{V'}^{\mathrm{nexc}} \cong \mathcal{U}_{\mathfrak{p}}^{\mathrm{nexc}}$. of Proposition 6.3.4.

The discussion of §4.3 shows that A' extends to a semi-abelian scheme over $\bar{\mathcal{S}}_{V'}^{\mathrm{nexc}}$. The elliptic curve A_0 extends to an elliptic curve $\bar{\mathcal{S}}_{V'}^{\mathrm{nexc}}$, and so the same is true of its quotient $\tilde{A}_0$. It follows that A extends to a semi-abelian scheme over $\bar{\mathcal{S}}_{V'}^{\mathrm{nexc}}$, and the extension property used in the proof of Lemma 4.3.2 provides us with a morphism

$$i_{\mathfrak{p}} : \bar{\mathcal{S}}_{V'}^{\mathrm{nexc}} \to \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \bar{\mathcal{M}}_{(n-1,1)}^{\mathrm{Pap}}$$

taking values in the open subscheme

$$\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \big(\bar{\mathcal{M}}_{(n-1,1)}^{\mathrm{Pap}} \smallsetminus \mathrm{Sing}_{(n-1,1)}\big) \cong \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \big(\bar{\mathcal{M}}_{(n-1,1)} \smallsetminus \mathrm{Exc}_{(n-1,1)}\big).$$

This provides us with a morphism

$$i_{\mathfrak{p}} : \bar{\mathcal{S}}_{V'}^{\mathrm{nexc}} \to \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \bar{\mathcal{M}}_{(n-1,1)},$$

taking values in the open and closed substack $\bar{\mathcal{S}}_V$.

We now have a commutative diagram of solid arrows

$$\begin{array}{ccc} \mathcal{S}_{V'}^{\mathrm{nexc}} & \xrightarrow{i_{\mathfrak{p}}} & \bar{\mathcal{U}}_{\mathfrak{p}} \\ \downarrow & \nearrow & \downarrow \\ \bar{\mathcal{S}}_{V'}^{\mathrm{nexc}} & \longrightarrow & \bar{\mathcal{S}}_V \end{array}$$

in which the vertical arrow on the right is integral, and hence affine, by its construction as a normalization; see Lemma 29.53.4 of [Sta18]. To complete the proof of the proposition, it suffices to show that there is a unique dotted arrow making the diagram commute. This immediately reduces to the corresponding claim for affine schemes, in which we are given homomorphisms of rings

$$\begin{array}{ccc} R' & \longleftarrow & B \\ \uparrow & \swarrow & \uparrow \\ \bar{R}' & \longleftarrow & A \end{array}$$

with B integral over A, and $\bar{R}' \subset R'$ is an inclusion of normal domains with the same field of fractions. The image of any $b \in B$ under $B \to R'$ is integral over $\bar{R}'$, hence contained in $\bar{R}'$. Thus there is a unique dotted arrow making the diagram commute. □

6.4. **Two lemmas on abelian schemes.** For use in the next subsection, we prove two lemmas on abelian schemes. The first is a criterion for determining when a polarization descends to a quotient. The second is a criterion for the existence of an extension as a semi-abelian scheme.

Lemma 6.4.1. *Suppose $f : X \to Y$ is an isogeny of abelian schemes over a Noetherian scheme S, and $\psi_X : X \to X^\vee$ is a polarization whose degree d is invertible in $\mathcal{O}_S$. The following are equivalent.*

(1) *The kernel of f is contained in $\ker(\psi_X)$, and is totally isotropic with respect to the Weil pairing*

$$e : \ker(\psi_X) \times \ker(\psi_X) \to \mu_d.$$

(2) *There exists a (necessarily unique) polarization $\psi_Y : Y \to Y^\vee$ making the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\psi_X} & X^\vee \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^\vee} \\ Y & \xrightarrow[\psi_Y]{} & Y^\vee \end{array}$$

commute.

Proof. This is routine. See, for example, Proposition 10.4 of [Pol03]. □

Lemma 6.4.2. *Suppose S is a normal Noetherian scheme, $U \subset S$ is a dense open subscheme, and $f : X \to Y$ is an isogeny of abelian schemes over U whose degree d is invertible in $\mathcal{O}_S$. If X extends to a semi-abelian scheme over S then so does Y, and both extensions are unique.*

Proof. The uniqueness claim follows from Proposition I.2.7 of [FC90], so we only need to prove the existence of a semi-abelian extension of Y.

Consider the morphisms $\ker(f) \to X[d] \to U$. The second arrow and the composition are both étale, as $d \in \mathcal{O}_U^\times$, and so the first arrow is also étale by [Sta18, Tag 02GW]. As the first arrow is also a closed immersion, we deduce that $\ker(f) \subset X[d]$ is a union of connected components.

If $X^* \to S$ denotes the semi-abelian extension of X, the multiplication map $[d] : X^* \to X^*$ is quasi-finite and flat. Indeed, by Lemma 37.16.4 of [Sta18] flatness can be checked fiberwise on S, where it follows from Proposition 3.11 of Exposé VI.B of [SGA70], and the surjectivity of $[d]$ on any extension of an abelian variety by a torus. The group scheme $X^*[d] \to S$ is therefore quasi-finite and flat. Using our assumption that $d \in \mathcal{O}_S^\times$, and Lemma 29.36.8 of [Sta18], we deduce that it is étale.

As we assume that S is normal, so is $X^*[d]$. As the connected components of a normal scheme are the same as its irreducible components, it follows that the Zariski closure of $\ker(f)$ in $X^*[d]$ is an open and closed subscheme $\ker(f)^* \subset X^*[d]$. In particular $\ker(f)^* \to S$ is quasi-finite étale.

By Lemma IV.7.1.2 of [MB85], the fppf quotient $Y^* = X^*/\ker(f)^*$ provides a semi-abelian extension of Y to S. □

6.5. **Analysis of $\mathcal{U}_0$.** In this subsection we study the substack $\mathcal{U}_0$ of (6.1.2), and make explicit its relation with $\mathcal{S}_{V'}$.

Return to the situation of Lemma 6.2.1, so that $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a})$ with $\langle x, x\rangle = p$ determines an orthogonal decomposition

$$W = \tilde{W}_0 \oplus W'. \tag{6.5.1}$$

Set $\tilde{\mathfrak{a}}_0 = \mathfrak{a} \cap \tilde{W}_0$.

Lemma 6.5.1. *Assume that $\tilde{\mathfrak{a}}_0 = x(\mathfrak{a}_0)$. If we set $\mathfrak{b} = \mathfrak{a} \cap W'$ and let $\mathfrak{b}^\vee \subset W'$ be its dual lattice with respect to $h|_{W'}$, the $\mathcal{O}_{\boldsymbol{k}}$-lattice*

$$\mathfrak{a}' = \mathfrak{p} \cdot \mathfrak{b}^\vee + \mathfrak{b},$$

is self-dual with respect to $h|_{W'}$. Moreover, there are inclusions of $\mathcal{O}_{\boldsymbol{k}}$-lattices

$$\tilde{\mathfrak{a}}_0 \oplus \mathfrak{a}' \overset{p}{\supset} \tilde{\mathfrak{a}}_0 \oplus \mathfrak{b} \overset{p^2}{\subset} \mathfrak{a} \tag{6.5.2}$$

in W of the indicated indices, and a canonical $\mathcal{O}_{\boldsymbol{k}}$-linear injection

$$\mathfrak{p}^{-1}\tilde{\mathfrak{a}}_0/\tilde{\mathfrak{a}}_0 \cong \mathfrak{p}^{-1}\mathfrak{b}/\mathfrak{b} \xrightarrow{t} \mathfrak{p}^{-1}\mathfrak{a}'/\mathfrak{a}'. \tag{6.5.3}$$

Proof. Throughout the proof, we use x to identify $W_0 = \tilde{W}_0$, so that $h|_{W_0} = ph_0$. The projections to the two factors in (6.5.1) induce isomorphisms

$$\mathfrak{a}/(\mathfrak{a}_0 \oplus \mathfrak{b}) \to \mathfrak{a}_0^\vee/\mathfrak{a}_0, \qquad \mathfrak{a}/(\mathfrak{a}_0 \oplus \mathfrak{b}) \to \mathfrak{b}^\vee/\mathfrak{b},$$

where $\mathfrak{a}_0^\vee = p^{-1}\mathfrak{a}_0$ is the dual lattice of $\mathfrak{a}_0$ with respect to $h|_{W_0}$. Composing these isomorphisms yields an $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism

$$p^{-1}\mathfrak{a}_0/\mathfrak{a}_0 \cong \mathfrak{b}^\vee/\mathfrak{b} \tag{6.5.4}$$

respecting the $p^{-1}\mathcal{O}_{\boldsymbol{k}}/\mathcal{O}_{\boldsymbol{k}}$-valued hermitian forms induced by $h|_{W_0} = ph_0$ and $h|_{W'}$. It follows that $\mathfrak{p} \cdot (\mathfrak{b}^\vee/\mathfrak{b}) \subset \mathfrak{b}^\vee/\mathfrak{b}$ is maximal isotropic with respect to $h|_{W'}$, and so $\mathfrak{a}'$ is self-dual under $h|_{W'}$.

Consider the inclusions $\mathfrak{b} \subset \mathfrak{a}' \subset \mathfrak{b}^\vee$ of $\mathcal{O}_{\boldsymbol{k}}$-modules. The first is an isomorphism everywhere locally except at $\overline{\mathfrak{p}}$, while the second is an isomorphism everywhere locally except at $\mathfrak{p}$. In particular, the first induces an isomorphism

$$\mathfrak{p}^{-1}\mathfrak{b}/\mathfrak{b} \cong \mathfrak{p}^{-1}\mathfrak{a}'/\mathfrak{a}'.$$

Restricting (6.5.4) to an isomorphism of $\mathfrak{p}$-torsion submodules defines (6.5.3). The inclusions of (6.5.2), with the indicated indices, are clear from what we have said. ☐

Just as Lemma 6.5.1 is more complicated than Lemma 6.3.1, the analysis of $\mathcal{U}_0$ is more complicated than that of $\mathcal{U}_{\mathfrak{p}}$.

Let S be an $\mathcal{O}_{\boldsymbol{k}}[1/p]$-scheme. Given Lemma 6.5.1, one expects that any S-point $(A_0, A, x) \in \mathcal{U}_0(S)$ should determine a diagram of abelian schemes with $\mathcal{O}_{\boldsymbol{k}}$-actions

$$\tilde{A}_0 \times A' \xleftarrow{\deg=p} \tilde{A}_0 \times B \xrightarrow{\deg=p^2} A \tag{6.5.5}$$

in which the arrows are $\mathcal{O}_{\boldsymbol{k}}$-linear isogenies of the indicated degrees. There should also be a canonical $\mathcal{O}_{\boldsymbol{k}}$-linear closed immersion

$$\tilde{A}_0[\mathfrak{p}] \cong B[\mathfrak{p}] \xrightarrow{t} A'[\mathfrak{p}], \tag{6.5.6}$$

and the morphism $x : A_0 \to A$ should factor as

$$A_0 \cong \tilde{A}_0 \hookrightarrow \tilde{A}_0 \times B \to A.$$

Moreover, these abelian schemes should come with polarizations with the following properties. The elliptic curve $\tilde{A}_0$ is equipped with its unique polarization of degree p^2, B is equipped with a polarization of degree p^2, and A' is equipped with a principal polarization. The pullback of the product polarization on $\tilde{A}_0 \times A'$ via the leftwards arrow in (6.5.5) is the product polarization on $\tilde{A}_0 \times B$, and similarly for the the pullback of the principal polarization on A via the rightward arrow.

Here is how to construct the data above from (A_0, A, x). As in the proof of Proposition 6.2.2, at any geometric point $s \to S$ we may apply Lemma 6.5.1 with $\mathfrak{a}_0$ and $\mathfrak{a}$ replaced by the p-adic Tate modules of A_{0s} and A_s to obtain $\mathcal{O}_{\boldsymbol{k}}$-lattices

$$T_p(\tilde{A}_{0s}) \oplus T_p(A'_s) \supset T_p(\tilde{A}_{0s}) \oplus T_p(B_s) \subset T_p(A_s). \tag{6.5.7}$$

analogous to (6.5.2), along with an $\mathcal{O}_{\boldsymbol{k}}$-linear injection

$$\mathfrak{p}^{-1}T_p(\tilde{A}_{0s})/T_p(\tilde{A}_{0s}) \cong \mathfrak{p}^{-1}T_p(B_s)/T_p(B_s) \xrightarrow{t} \mathfrak{p}^{-1}T_p(A'_s)/T_p(A'_s). \tag{6.5.8}$$

To be clear, we have not yet constructed the abelian schemes $\tilde{A}_0$, A', and B, only lattices in $T_p(A_s)[1/p]$ that we choose to call $T_p(\tilde{A}_s)$, $T_p(A'_s)$, and $T_p(B_s)$. However, as $p \in \mathcal{O}_S^\times$ and both inclusions in (6.5.7) have finite index, there are abelian schemes C and D over S with $\mathcal{O}_{\boldsymbol{k}}$-actions and $\mathcal{O}_{\boldsymbol{k}}$-linear isogenies

$$D \xleftarrow{p} C \xrightarrow{p^2} A$$

of the indicated degrees that identify

$$T_p(C_s) = T_p(\tilde{A}_{0s}) \oplus T_p(B_s), \qquad T_p(D_s) = T_p(\tilde{A}_{0s}) \oplus T_p(A'_s). \tag{6.5.9}$$

The condition that $\langle x, x\rangle = p$ implies that

$$A_0 \xrightarrow{x} A \cong A^\vee \xrightarrow{x^\vee} A_0^\vee \cong A_0$$

is multiplication by p, which implies that the composition

$$A \cong A^\vee \xrightarrow{x^\vee} A_0^\vee \cong A_0 \xrightarrow{x} A$$

is a Rosati-fixed element $\alpha \in \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(A)[1/p]$ such that $\alpha \circ \alpha = p\alpha$. In particular, we obtain an idempotent

$$p^{-1}\alpha \in \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(A)[1/p] \cong \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(C)[1/p] \cong \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(D)[1/p],$$

whose induced actions on $T_p(C_s)[1/p]$ and $T_p(D_s)[1/p]$ are just the projections to the first summands in (6.5.9). It follows that we in fact have

$$p^{-1}\alpha \in \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(C) \cong \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(D).$$

These idempotents are the projections to the first factors for unique decompositions

$$C = \tilde{A}_0 \times B, \qquad D = \tilde{A}_0 \times A'$$

of abelian schemes with $\mathcal{O}_{\boldsymbol{k}}$-actions, and the image of x under

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)[1/p] \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, C)[1/p] \to \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, \tilde{A}_0)[1/p]$$

is an isomorphism $A_0 \cong \tilde{A}_0$. In particular, we now have $\mathcal{O}_{\boldsymbol{k}}$-linear isogenies (6.5.5) such that taking p-adic Tate modules recovers (6.5.7), and (6.5.8) induces the closed immersion (6.5.6).

If we pullback the principal polarization via $\tilde{A}_0 \times B \to A$, we obtain a polarization on $\tilde{A}_0 \times B$ of degree p^4. As the idempotent $p^{-1}\alpha$ constructed above is Rosati-fixed, this polarization of $\tilde{A}_0 \times B$ splits as the product of polarizations $\tilde{A}_0 \to \tilde{A}_0^\vee$ and $B \to B^\vee$. By examining the induced Weil pairing on p-adic Tate modules, one can show that each polarization has kernel isomorphic, étale locally on S, to $\mathcal{O}_{\boldsymbol{k}}/(p)$, and that

$$\ker(B \to A') = \ker(B \to B^\vee)[\overline{\mathfrak{p}}]$$

is totally isotropic under the Weil pairing on $\ker(B \to B^\vee)$. Hence, by Lemma 6.4.1, $B \to B^\vee$ descends to a principal polarization on A'. This completes the construction of the abelian schemes (6.5.5) with all of their expected extra structure.

Now apply this construction to the universal triple (A_0, A, x) over $\mathcal{U}_0$. As $p \in \mathcal{O}_{\mathcal{U}_0}^\times$, the p-power isogenies (6.5.5) induce an isomorphism

$$\mathrm{Lie}(A) \cong \mathrm{Lie}(\tilde{A}_0) \times \mathrm{Lie}(A') \tag{6.5.10}$$

of rank n vector bundles on $\mathcal{U}_0$. Recalling that A comes equipped with an $\mathcal{O}_{\boldsymbol{k}}$-stable hyperplane $\mathcal{F} \subset \mathrm{Lie}(A)$, denote by

$$\mathcal{U}_0^\dagger \subset \mathcal{U}_0$$

the largest closed substack over which $\mathrm{Lie}(\tilde{A}_0) \subset \mathcal{F}$

Define a finite étale cover

$$\mathcal{T}_{V'} \to \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \tag{6.5.11}$$

of degree $p^{n-1} - 1$ as follows: for any $\mathcal{O}_{\boldsymbol{k}}[1/p]$-scheme S, let $\mathcal{T}_{V'}(S)$ be the groupoid of triples (A_0', A', t) in which

$$(A_0', A') \in \mathcal{S}_{V'}(S) \quad \text{and} \quad t : A_0'[\mathfrak{p}] \to A'[\mathfrak{p}]$$

is an $\mathcal{O}_{\boldsymbol{k}}$-linear closed immersion of group schemes over S.

Proposition 6.5.2. *There is a canonical isomorphism* $i_0 : \mathcal{T}_{V'} \cong \mathcal{U}_0^\dagger$.

Proof. The morphism $\mathcal{U}_0^\dagger \to \mathcal{T}_{V'}$ is essentially described above. Suppose S is an $\mathcal{O}_{\boldsymbol{k}}[1/p]$-scheme and

$$(A_0, A, x) \in \mathcal{U}_0^\dagger(S).$$

We can use the isomorphism (6.5.10) to endow the abelian scheme A' of (6.5.5) with the rank one local direct summand

$$\mathcal{F}' = \mathcal{F}/\,\mathrm{Lie}(\tilde{A}_0) \subset \mathrm{Lie}(A)/\,\mathrm{Lie}(\tilde{A}_0) = \mathrm{Lie}(A')$$

to obtain a point $A' \in \mathcal{M}_{(n-2,1)}(S)$. Setting $A'_0 = \tilde{A}_0$, this defines a morphism

$$\mathcal{U}_0^\dagger \xrightarrow{(A_0,A,x)\mapsto(A'_0,A')} \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]},$$

and the closed immersion (6.5.6) defines the desired lift to $\mathcal{U}_0^\dagger \to \mathcal{T}_{V'}$.

Now we construct the inverse. Start with a point $(A'_0, A', t) \in \mathcal{T}_{V'}(S)$. Denote by B the abelian scheme dual to

$$B^\vee = A'/\mathrm{Im}(t).$$

Using the principal polarization to identify A' with its dual, the quotient map $A' \to B^\vee$ dualizes to a morphism $B \to A'$, and the composition

$$B \to A' \to B^\vee$$

is a degree p^2 polarization (it is the pullback via $B \to A'$ of the principal polarization on A'). It has the property that each factor on the right hand side of

$$\ker(B \to B^\vee) = \ker(B \to B^\vee)[\mathfrak{p}] \times \ker(B \to B^\vee)[\overline{\mathfrak{p}}]$$

has order p, and the second factor is the kernel of $B \to A'$. In particular, the induced map $B[\mathfrak{p}] \to A'[\mathfrak{p}]$ is an isomorphism, and the closed immersion

$$A'_0[\mathfrak{p}] \xrightarrow{t} A'[\mathfrak{p}] \cong B[\mathfrak{p}]$$

has image $\ker(B \to B^\vee)[\mathfrak{p}]$. Setting $\tilde{A}_0 = A'_0$, the resulting isomorphism

$$\tilde{A}_0[\mathfrak{p}] \cong \ker(B \to B^\vee)[\mathfrak{p}]$$

admits a unique $\mathcal{O}_{\boldsymbol{k}}$-linear extension to an isomorphism

$$t : \tilde{A}_0[p] \cong \ker(B \to B^\vee)$$

identifying the Weil pairings on source and target.

The antidiagonal subgroup

$$\Delta \stackrel{\mathrm{def}}{=} \tilde{A}_0[p] \xrightarrow{a_0 \mapsto (-a_0, t(a_0))} \tilde{A}_0 \times B.$$

is totally isotropic under the Weil pairing induced by the product polarization (where $\tilde{A}_0$ is given the polarization of degree p^2), which therefore descends, by Lemma 6.4.1, to a principal polarization on the quotient

$$A = (\tilde{A}_0 \times B)/\Delta.$$

As $p \in \mathcal{O}_S^\times$, there is an induced isomorphism of Lie algebras (6.5.10), which allows us to define a corank one local direct summand $\mathcal{F} \subset \mathrm{Lie}(A)$ as the product

$$\mathcal{F} = \mathrm{Lie}(\tilde{A}_0) \times \mathcal{F}'.$$

This defines a point $A \in \mathcal{M}_{(n-1,1)}(S)$. Setting $A_0 = \tilde{A}_0$, the composition

$$A_0 = \tilde{A}_0 \hookrightarrow \tilde{A}_0 \times B \to A$$

defines $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ with $\langle x, x \rangle = p$

The above construction determines a morphism

$$\mathcal{T}_{V'} \xrightarrow{(A_0',A',t)\to(A_0,A,x)} \mathcal{Z}_V(p)$$

taking values in the open substack $\mathcal{U}_0^\dagger$, which is inverse to the map $\mathcal{U}_0^\dagger \to \mathcal{T}_{V'}$ constructed above. □

Proposition 6.5.2 gives us morphisms

$$\mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \xleftarrow{(6.5.11)} \mathcal{T}_{V'} \longrightarrow \mathcal{S}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]} \tag{6.5.12}$$

in which the arrow on the right is the (finite and unramified) composition

$$\mathcal{T}_{V'} \xrightarrow{i_0} \mathcal{U}_0^\dagger \hookrightarrow \mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]} \to \mathcal{S}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]},$$

sending $(A_0', A', t) \mapsto (A_0, A)$, where $A_0 = A_0'$, and A is related to A' by a diagram (6.5.5).

Proposition 6.5.3. *The hermitian line bundles*

$$\widehat{\omega}^{\mathrm{Hdg}}_{A_0'/\mathcal{S}_{V'}} + \widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}} \in \widehat{\mathrm{Pic}}(\mathcal{S}_{V'})$$

and $\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V} \in \widehat{\mathrm{Pic}}(\mathcal{S}_V)$ *have the same images under the homomorphisms*

$$\widehat{\mathrm{Pic}}(\mathcal{S}_{V'}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{T}_{V'})_{\mathbb{Q}} \longleftarrow \widehat{\mathrm{Pic}}(\mathcal{S}_V)$$

induced by (6.5.12)*. The same is true of* $(\mathrm{Exc}_{V'}, 0)$ *and* $(\mathrm{Exc}_V, 0)$*, and of* $\widehat{\mathcal{L}}_{V'}$ *and* $\widehat{\mathcal{L}}_V$*.*

Proof. If (A_0', A') denotes the pullback of the universal object via the left arrow in (6.5.12), and (A_0, A) denotes the pullback of the universal object via the right arrow in (6.5.12), examination of the proof of Proposition 6.5.2 shows that there is a quasi-isogeny

$$f \in \mathrm{Hom}(A_0' \times A', A)[1/p]$$

of degree $\deg(f) = p$. As in the proof of Proposition 6.3.3, this implies that

$$\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{T}_{V'}} = \widehat{\omega}^{\mathrm{Hdg}}_{A_0'/\mathcal{T}_{V'}} + \widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{T}_{V'}} + (0, -\log(p))$$

as elements of $\widehat{\mathrm{Pic}}(\mathcal{T}_{V'})$, and the term $(0, -\log(p))$ is torsion. The first claim follows from this. The remaining claims are essentially the same as in Proposition 6.3.3, and we leave the details to the reader. □

Proposition 6.5.4. *The nonexceptional locus* $\mathcal{U}_0^{\mathrm{nexc}} \subset \mathcal{U}_0$ *satisfies*

$$\mathcal{U}_0^{\mathrm{nexc}} \subset \mathcal{U}_0^\dagger.$$

Moreover, the morphism of Proposition 6.5.2 restricts to an isomorphism

$$\mathcal{S}^{\mathrm{nexc}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \cong \mathcal{U}_0^{\mathrm{nexc}}.$$

Proof. The proof is essentially the same as Proposition 6.3.4, and the details are left to the reader. □

Proposition 6.5.5. *Denote by $\bar{\mathcal{T}}_{V'}$ the normalization of $\mathcal{T}_{V'} \to \bar{\mathcal{S}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]}$. The closed immersion $i_0 : \mathcal{T}_{V'} \to \mathcal{U}_0$ of Proposition 6.5.2 extends to a proper morphism*

$$\bar{\mathcal{T}}_{V'} \to \bar{\mathcal{U}}_0,$$

where the codomain is defined as the Zariski closure of

$$\mathcal{U}_0 \subset \bar{\mathcal{Z}}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]},$$

or, equivalently, as the normalization of $\mathcal{U}_0 \to \bar{\mathcal{S}}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]}$.

Proof. Consider the universal triple (A_0, A', t) over $\mathcal{T}_{V'}$, and let (A_0, A) be the pullback of the universal pair over $\mathcal{S}_V$ via the composition

$$\mathcal{T}_{V'} \xrightarrow{i_0} \mathcal{U}_0 \to \mathcal{S}_V.$$

We know that A' extends to a semi-abelian scheme over $\bar{\mathcal{T}}_{V'}$, obtained as a pullback via

$$\bar{\mathcal{T}}_{V'} \to \bar{\mathcal{S}}_{V'} \to \bar{\mathcal{M}}_{(n-2,1)},$$

and that A_0 extends to an elliptic curve over $\bar{\mathcal{T}}_{V'}$, obtained as a pullback via

$$\bar{\mathcal{T}}_{V'} \to \bar{\mathcal{S}}_{V'} \to \mathcal{M}_{(1,0)},$$

Using Lemma 6.4.2 and the isogenies (6.5.5), we deduce that A also extends to a semi-abelian scheme over $\bar{\mathcal{T}}_{V'}$. With this extension in hand, the proof is essentially identical to that of Proposition 6.3.5. □

6.6. **Proof of Theorems 6.1.2 and 6.1.3.** Define an $\mathcal{O}_{\boldsymbol{k}}[1/p]$-stack

$$\mathcal{Z}_V^\dagger(p) = \mathcal{T}_{V'} \sqcup \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \sqcup \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]}.$$

Propositions 6.3.2 and 6.5.2 provide us with a canonical closed immersion

$$\mathcal{Z}_V^\dagger(p) \xrightarrow{i = i_0 \sqcup i_{\mathfrak{p}} \sqcup i_{\overline{\mathfrak{p}}}} \mathcal{U}_0 \sqcup \mathcal{U}_{\mathfrak{p}} \sqcup \mathcal{U}_{\overline{\mathfrak{p}}} = \mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]}$$

with image the closed substack $\mathcal{U}_0^\dagger \sqcup \mathcal{U}_{\mathfrak{p}}^\dagger \sqcup \mathcal{U}_{\overline{\mathfrak{p}}}^\dagger$. Hence we have a diagram

$$\begin{array}{ccc} \mathcal{Z}_V^\dagger(p) & \xrightarrow{\quad i \quad} & \mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \mathcal{S}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} & & \mathcal{S}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]} \end{array} \tag{6.6.1}$$

in which α is a finite étale surjection of degree $p^{n-1}+1$, and β is finite.

Lemma 6.6.1. *As divisors on $\mathcal{S}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]}$, we have*

$$\mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]} = \mathcal{Z}_V^\dagger(p) + E,$$

where $\mathcal{Z}_V^\dagger(p)$ has no irreducible components contained in the exceptional divisor $\mathrm{Exc}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]} \subset \mathcal{S}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]}$ of Definition 5.1.5, and E is supported entirely on the exceptional divisor.

Proof. The *exceptional locus* of $\mathcal{Z}^\dagger_V(p)$ is the preimage of the exceptional divisor under α, and an irreducible component of it is *exceptional* if it is contained in the exceptional locus. Similarly, the *exceptional locus* of $\mathcal{Z}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]}$ is the preimage of the exceptional divisor under β, and an irreducible component of it is *exceptional* if it is contained in the exceptional locus.

The final claims of Propositions 6.3.4 and 6.5.4 show that i restricts to an isomorphism between the non-exceptional loci of the source and target. In particular, it establishes a bijection between the generic points of non-exceptional irreducible components, and corresponding non-exceptional generic have local rings of the same (finite) length.

However, the morphism α is finite étale, so no irreducible component of the source can map into the exceptional divisor of the target. Thus the closed immersion i actually establishes a bijection between irreducible components of the source and non-exceptional irreducible components of the target, in a way that matches up their multiplicities. The claim follows immediately. $\square$

Propositions 6.3.5 and 6.5.5 imply that the diagram above extends to

$$\begin{array}{ccc} \bar{\mathcal{Z}}^\dagger_V(p) & \xrightarrow{\ i\ } & \bar{\mathcal{Z}}_V(p)_{/\mathcal{O}_{\boldsymbol{k}}[1/p]} \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\ \bar{\mathcal{S}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} & & \bar{\mathcal{S}}_{V/\mathcal{O}_{\boldsymbol{k}}[1/p]}, \end{array} \tag{6.6.2}$$

where we have defined

$$\bar{\mathcal{Z}}^\dagger_V(p) = \bar{\mathcal{T}}_{V'} \sqcup \bar{\mathcal{S}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]} \sqcup \bar{\mathcal{S}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]}. \tag{6.6.3}$$

Equivalently, this is the normalization of

$$\alpha : \mathcal{Z}^\dagger_V(p) \to \bar{\mathcal{S}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/p]}.$$

For any $N \geq 1$ prime to p, we can add level structure to $\bar{\mathcal{T}}_{V'}$ by defining

$$\mathcal{T}_{V'}(N) = \mathcal{T}_{V'} \times_{\mathcal{S}_{V'}} \mathcal{S}_{V'}(N),$$

and letting $\bar{\mathcal{T}}_{V'}(N)$ be the normalization of

$$\mathcal{T}_{V'}(N) \to \bar{\mathcal{T}}_{V'/\mathcal{O}_{\boldsymbol{k}}[1/N]}.$$

Equivalently, this is the normalization of

$$\mathcal{T}_{V'}(N) \to \bar{\mathcal{S}}_{V'}(N)_{/\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]}.$$

Lemma 6.6.2. *The stack $\bar{\mathcal{T}}_{V'}(N)$ is regular, flat, and proper over $\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]$, and is a projective scheme if $N \geq 3$. It is smooth in a neighborhood of its boundary*

$$\partial\bar{\mathcal{T}}_{V'}(N) = \bar{\mathcal{T}}_{V'}(N) \smallsetminus \mathcal{T}_{V'}(N),$$

which is a Cartier divisor smooth over $\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]$.

Proof. The proof is similar to that of Proposition 4.3.3, and is again based on the results of [Lan16] and [MP19].

Recall from Remark 5.1.4 that the generic fiber of $\mathcal{S}_{V'}$ is the Shimura variety associated to a reductive group G' and a certain compact open subgroup of $G'(\mathbb{A}_f)$, and one can easily check that the generic fiber of the finite étale cover

$$\mathcal{T}_{V'}(N) \to \mathcal{S}_{V'}(N)_{/\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]} \tag{6.6.4}$$

is obtained by shrinking compact open subgroup.

Recall from §5.1 that $\mathcal{S}_{V'}(N)$ is constructed as an open and closed substack

$$\mathcal{S}_{V'}(N) \subset \mathcal{M}_{W_0}(N) \times_{\mathcal{O}_{\boldsymbol{k}}[1/N]} \mathcal{M}_{W'}(N).$$

In this construction, we may replace $\mathcal{M}_{W'}(N)$ with the stack $\mathcal{M}^{\mathrm{Pap}}_{W'}(N)$ appearing in the proof of Proposition 4.3.3 to obtain a blow-down

$$\mathcal{S}^{\mathrm{Pap}}_{V'}(N) \subset \mathcal{M}_{W_0}(N) \times_{\mathcal{O}_{\boldsymbol{k}}[1/N]} \mathcal{M}^{\mathrm{Pap}}_{W'}(N).$$

Repeating the construction of (6.6.4) with $\mathcal{S}_{V'}(N)$ replaced by $\mathcal{S}^{\mathrm{Pap}}_{V'}(N)$ yields a finite étale cover

$$\mathcal{T}^{\mathrm{Pap}}_{V'}(N) \to \mathcal{S}^{\mathrm{Pap}}_{V'}(N)_{/\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]},$$

and $\mathcal{T}_{V'}(N)$ is the blow-up of the normal stack $\mathcal{T}^{\mathrm{Pap}}_{V'}(N)$ along its proper and 0-dimensional locus of nonsmooth points.

As in the proof of Proposition 4.3.3, we now have morphisms

$$\mathcal{T}^{\mathrm{Pap}}_{V'}(N) \to \mathcal{S}^{\mathrm{Pap}}_{V'}(N)_{/\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]} \to \mathcal{M}^{\mathrm{Pap}}_{W'}(N)_{/\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]} \to \bar{\mathcal{A}}_{/\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]},$$

and we may define $\bar{\mathcal{T}}^{\mathrm{Pap}}_{V'}(N)$ as the normalization of the composition. We may now apply the results of [Lan16] and [MP19] to deduce properties of $\bar{\mathcal{T}}^{\mathrm{Pap}}_{V'}(N)$ from properties of its interior. Arguing exactly as in the proof of Proposition 4.3.3, this compactification is normal and flat, and is a projective scheme if $N \geq 3$. Its boundary is a smooth Cartier divisor. The nonsmooth locus has dimension 0, and does not meet the boundary. As $\bar{\mathcal{T}}_{V'}(N)$ can be identified with the blow-up of $\bar{\mathcal{T}}^{\mathrm{Pap}}_{V'}(N)$ along its nonsmooth locus, it has all the same properties, and is also regular (being smooth near the boundary and regular in the interior). □

Lemma 6.6.2 and Remark 5.1.8 provide us with an $\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]$-stack

$$\bar{\mathcal{Z}}^{\dagger}_{V}(p,N) \stackrel{\mathrm{def}}{=} \bar{\mathcal{T}}_{V'}(N) \sqcup \bar{\mathcal{S}}_{V'}(N)_{/\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]} \sqcup \bar{\mathcal{S}}_{V'}(N)_{/\mathcal{O}_{\boldsymbol{k}}[\frac{1}{Np}]},$$

with all the nice properties enumerated in Proposition 4.3.3. Thus (6.6.3) has its own theory of pre-log singular hermitian line bundles and Burgos-Kramer-Kühn arithmetic Chow groups

$$\widehat{\mathrm{CH}}^d(\bar{\mathcal{Z}}_V^\dagger(p), \mathscr{D}_{\mathrm{pre}}) = \varprojlim_{\substack{N\ge 3 \\ p\nmid N}} \widehat{\mathrm{CH}}^d(\bar{\mathcal{Z}}_V^\dagger(p,N), \mathscr{D}_{\mathrm{pre}}),$$

exactly as in §4.4. These Chow groups include notions of arithmetic heights and volumes as in §4.5, taking values in the abelian group $\mathbb{R}/\mathbb{Q}\log(p)$.

One can repeat the construction of the diagram (6.6.2) with level structures, and so obtain pullbacks

$$\widehat{\mathrm{CH}}^d(\bar{\mathcal{S}}_{V'}, \mathscr{D}_{\mathrm{pre}}) \xrightarrow{\alpha^*} \widehat{\mathrm{CH}}^d(\bar{\mathcal{Z}}_V^\dagger(p), \mathscr{D}_{\mathrm{pre}}) \xleftarrow{(\beta\circ i)^*} \widehat{\mathrm{CH}}^d(\bar{\mathcal{S}}_{V}, \mathscr{D}_{\mathrm{pre}})$$

and

$$\widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_{V'}, \mathscr{D}_{\mathrm{pre}})_{\mathbb{Q}} \xrightarrow{\alpha^*} \widehat{\mathrm{Pic}}(\bar{\mathcal{Z}}_V^\dagger(p), \mathscr{D}_{\mathrm{pre}})_{\mathbb{Q}} \xleftarrow{(\beta\circ i)^*} \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_{V}, \mathscr{D}_{\mathrm{pre}})_{\mathbb{Q}}$$

Lemma 6.6.3. *If two pre-log singular hermitian line bundles*

$$\widehat{\mathcal{P}}' \in \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_{V'}, \mathscr{D}_{\mathrm{pre}}) \quad \text{and} \quad \widehat{\mathcal{P}} \in \widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_{V}, \mathscr{D}_{\mathrm{pre}})$$

have the same pullback to $\widehat{\mathrm{Pic}}(\bar{\mathcal{Z}}_V^\dagger(p), \mathscr{D}_{\mathrm{pre}})_{\mathbb{Q}}$ *then*

$$\int_{\mathcal{Z}_V(p)(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{P}})^{n-2} = (p^{n-1}+1) \int_{\mathcal{S}_{V'}(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{P}}')^{n-2}.$$

Moreover, there is an $a(p) \in \mathbb{Q}$ *such that*

$$\mathrm{ht}_{\widehat{\mathcal{P}}}(\bar{\mathcal{Z}}_V(p)) = (p^{n-1}+1)\cdot \widehat{\mathrm{vol}}(\widehat{\mathcal{P}}') + \mathrm{ht}_{\widehat{\mathcal{P}}}(E) + a(p)\log(p),$$

where E *is the divisor of Lemma 6.6.1.*

Proof. As in the proof of Lemma 6.6.1, the closed immersion i in (6.6.1) restricts to an isomorphism of non-exceptional loci, and hence induces an isomorphism in the generic fiber. Thus

$$\begin{aligned}\int_{\mathcal{Z}_V(p)(\mathbb{C})} \beta^*\,\mathrm{ch}(\widehat{\mathcal{P}})^{n-2} &= \int_{\mathcal{Z}_V^\dagger(p)(\mathbb{C})} (\beta\circ i)^*\,\mathrm{ch}(\widehat{\mathcal{P}})^{n-2} \\ &= \int_{\mathcal{Z}_V^\dagger(p)(\mathbb{C})} \alpha^*\,\mathrm{ch}(\widehat{\mathcal{P}}')^{n-2} \\ &= (p^{n-1}+1)\int_{\mathcal{S}_{V'}(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{P}}')^{n-2}.\end{aligned}$$

The second claim follows from Lemma 6.6.1 and the equalities

$$\mathrm{ht}_{\widehat{\mathcal{P}}}(\bar{\mathcal{Z}}_V^\dagger(p)) = \widehat{\mathrm{vol}}((\beta\circ i)^*\widehat{\mathcal{P}}) = \widehat{\mathrm{vol}}(\alpha^*\widehat{\mathcal{P}}') = (p^{n-1}+1)\cdot\widehat{\mathrm{vol}}(\widehat{\mathcal{P}}')$$

in $\mathbb{R}/\mathbb{Q}\log(p)$, where the first and last equalities are obtained directly by unpacking the definitions of pullbacks, heights, and arithmetic intersections in [BKK07], and using the fact that α is (away from the boundary) a finite étale surjection of degree $p^{n-1}+1$. □

Proof of Theorems 6.1.2 and 6.1.3. Propositions 6.3.3 and 6.5.3 imply that the hypotheses of Lemma 6.6.3 are satisfied by

$$\widehat{\mathcal{P}}' = \widehat{\mathcal{K}}_{V'} \quad \text{and} \quad \widehat{\mathcal{P}} = \widehat{\mathcal{K}}_V$$

which therefore gives the equality

$$\int_{\mathcal{Z}_V(p)(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{K}}_V)^{n-2} = (p^{n-1}+1) \int_{\mathcal{S}_{V'}(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{K}}_{V'})^{n-2},$$

and the equality

$$\mathrm{ht}_{\widehat{\mathcal{K}}_V}(\bar{\mathcal{Z}}_V(p)) = (p^{n-1}+1)\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_{V'}) + \mathrm{ht}_{\widehat{\mathcal{K}}_V}(E)$$

up to a rational multiple of $\log(p)$. The second term on the right vanishes by claim (1) of Theorem 5.5.1, completing the proof of Theorem 6.1.2.

Similarly, Propositions 6.3.3 and 6.5.3 show that the hypotheses of Lemma 6.6.3 are satisfied by

$$\widehat{\mathcal{P}}' = \widehat{\omega}^{\mathrm{Hdg}}_{A_0'/\mathcal{S}_{V'}} + \widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}} \quad \text{and} \quad \widehat{\mathcal{P}} = \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V},$$

and so

$$\mathrm{ht}_{\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}}(\bar{\mathcal{Z}}_V(p)) = (p^{n-1}+1)\cdot \widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A_0'/\mathcal{S}_{V'}} + \widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}}) + \mathrm{ht}_{\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}}(E)$$

up to a rational multiple of $\log(p)$. One again, the second term on the right vanishes by claim (1) of Theorem 5.5.1. For the first term on the right, Proposition 5.3.1 and Lemmas 4.5.2 and 4.5.3 imply

$$\begin{aligned}\widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A_0'/\mathcal{S}_{V'}} + \widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}}) &= \widehat{\mathrm{vol}}((0,C_1) + \widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}}) \\ &= \widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}}) + (n-1)C_1 \int_{\mathcal{S}_{V'}(\mathbb{C})} \mathrm{ch}(\widehat{\omega}^{\mathrm{Hdg}}_{A'/\mathcal{S}_{V'}})^{n-2},\end{aligned}$$

where

$$C_1 = \log(2\pi) + 2h^{\mathrm{Falt}}_{\boldsymbol{k}} = -\frac{L'(0,\varepsilon)}{L(0,\varepsilon)} - \frac{\log(D)}{2}.$$

This proves Theorem 6.1.3. □

7. Borcherds products

We continue to work with the Shimura variety $\mathcal{S}_V$ of (5.1.4) associated to a $\boldsymbol{k}$-hermitian space V of signature $(n-1,1)$, and now assume $n \geq 2$.

After explaining the connection between the complex orbifold $\mathcal{S}_V(\mathbb{C})$ and the Shimura variety (3.1.3) associated to the unitary group $\mathrm{U}(V)$, we will use the results of §3.3 to construct Green functions for certain linear combinations of the Kudla-Rapoport divisors $\mathcal{Z}_V(m) \to \mathcal{S}_V$.

We then recall the arithmetic theory of Borcherds products on $\mathcal{S}_V$ from [BHK+a], and show that one can produce Borcherds products whose divisors are linear combinations of only those $\mathcal{Z}_V(p)$ with p a prime congruent to 1 modulo D, up to a linear combination of vertical divisors which can be computed explicitly.

7.1. **Green functions and Borcherds products.** Write

$$V \cong \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$$

as in §5.1, fix self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattices $\mathfrak{a}_0 \subset W_0$ and $\mathfrak{a} \subset W$ as in §4.1, and define a self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice

$$L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}) \subset V. \tag{7.1.1}$$

The subgroup $G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W)$ of Remark 5.1.4 acts on both W_0 and W via unitary similitudes, and we denote by $K \subset G(\mathbb{A}_f)$ the largest compact open subgroup fixing the lattices $\mathfrak{a}_0$ and $\mathfrak{a}$. Recalling the hermitian symmetric domain $\mathcal{D}$ of §3.1, we identify

$$\mathcal{S}_V(\mathbb{C}) \cong G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K$$

as in §2 of [BHK$^+$a].

The group G also acts on V, defining a surjective homomorphism

$$G \to H = \mathrm{U}(V)$$

with kernel the diagonally embedded $\mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m$. Denoting again by K the image of the above compact open subgroup under $G(\mathbb{A}_f) \to H(\mathbb{A}_f)$, and recalling the complex Shimura variety (3.1.3), we obtain a finite cover

$$\mathcal{S}_V(\mathbb{C}) \to \mathrm{Sh}_K(H, \mathcal{D}). \tag{7.1.2}$$

Let $H^{\infty}_{2-n}(D, \varepsilon^n)$ denote the space of $\mathbb{C}$-valued harmonic Maass forms f of weight $2-n$, level $\Gamma_0(D)$, and character ε^n such that

- f is bounded at all cusps of $\Gamma_0(D)$ different from the cusp ∞,
- f has polynomial growth at ∞, in sense that there is a

  $$P_f = \sum_{m \leq 0} c^+(m) q^m \in \mathbb{C}[q^{-1}]$$

  such that $f - P_f = o(1)$ as q goes to 0.

Such a harmonic Maass form has a Fourier expansion analogous to (3.3.1) with Fourier coefficients $c^{\pm}(m) \in \mathbb{C}$.

Fix an $f \in H^{\infty}_{2-n}(D, \varepsilon^n)$ with Fourier coefficients $c^{\pm}(m)$. This form can be lifted to a vector valued harmonic Maass form, in the sense of §3.3, by setting

$$\tilde{f} = \sum_{\gamma \in \Gamma_0(D) \backslash \mathrm{SL}_2(\mathbb{Z})} (f|_{2-n}\gamma)(\omega_L(\gamma)^{-1}\varphi_0) \in H_{2-n}(\omega_L), \tag{7.1.3}$$

where $\varphi_0 \in S_L = \mathbb{C}[L'/L]$ is the characteristic function of $0 \in L'/L$. We denote the Fourier coefficients of $\tilde{f}$ by $\tilde{c}^{\pm}(m, \mu)$ for $\mu \in L'/L$ and $m \in \mathbb{Q}$. The coefficients of $\tilde{f}$ can be computed in terms of the coefficients of f, and for $m < 0$ we have

$$\tilde{c}^+(m, \mu) = \begin{cases} c^+(m) & \text{if } \mu = 0 \\ 0 & \text{if } \mu \neq 0 \end{cases} \tag{7.1.4}$$

as in Proposition 6.1.2 of [BHK$^+$a] or §5 of [Sch09].

Under the covering map (7.1.2), the divisors $Z(m)$ and the hermitian line bundle $\widehat{\mathcal{L}}$ of §3.1 pull pack to the divisors $\mathcal{Z}_V(m)$ and $\widehat{\mathcal{L}}_V$ of §5.1. This allows us to apply the construction (3.3.5) to the vector valued form (7.1.3) to obtain a Green function $\Phi(z,h,\tilde{f})$ for the analytic divisor

$$Z(f) = \sum_{m>0} c^+(-m) Z(m) \in \mathrm{Div}_{\mathbb{C}}(\mathrm{Sh}_K(H,\mathcal{D}))$$

of (3.3.3), which we pull back to a Green function $\Phi_V(f)$ for the divisor

$$\mathcal{Z}_V(f) = \sum_{m>0} c^+(-m)\mathcal{Z}_V(m) \in \mathrm{Div}_{\mathbb{C}}(\mathcal{S}_V).$$

If $n>2$, or if $n=2$ and V is anisotropic, then

$$\text{(7.1.5)} \qquad \mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_V)^{-1} \int_{\mathcal{S}_V(\mathbb{C})} \Phi_V(f)\,\mathrm{ch}(\widehat{\mathcal{L}}_V)^{n-1} = \sum_{m>0} c^+(-m) B'(m,0,s_0)$$

by Theorem 3.4.2 and (7.1.4). Here, as always, $s_0 = (n-1)/2$.

Now consider the subspace of weakly holomorphic forms

$$M^{!,\infty}_{2-n}(D,\varepsilon^n) \subset H^{\infty}_{2-n}(D,\varepsilon^n).$$

These are meromorphic modular forms of the indicated weight, level, and character that are holomorphic outside the cusp ∞ of $\Gamma_0(D)$. If we fix an

$$\text{(7.1.6)} \qquad f(\tau) = \sum_{m \gg -\infty} c(m) q^m \in M^{!,\infty}_{2-n}(D,\varepsilon^n)$$

in such a way that $c(m)\in\mathbb{Z}$ for all m then, after possibly replacing f by a nonzero integer multiple, Theorem 5.3.1 of [BHK$^+$a] provides us with a Borcherds product $\psi(f)$. This is a rational section of the hermitian line bundle $\mathcal{L}_V^{\otimes k(f)}$ on $\bar{\mathcal{S}}_V$ satisfying

$$\text{(7.1.7)} \qquad \Phi_V(f) = -\log \|\psi(f)\|^2,$$

whose divisor (at least if $n>2$) is

$$\text{(7.1.8)} \qquad \bar{\mathcal{Z}}_V(f) = \sum_{m>0} c(-m)\bar{\mathcal{Z}}_V(m)$$

plus an explicit (but complicated) linear combination of boundary components and vertical divisors in characteristics $p \mid D$.

Remark 7.1.1. The integer $k(f)$, called the *weight* of the Borcherds product, is given as follows: Applying the construction (7.1.3) yields a weakly holomorphic vector valued form

$$\tilde{f} = \sum_{\substack{m\in\mathbb{Q} \\ m\gg-\infty}} \tilde{c}(m) q^m \in M^{!}_{2-n}(\omega_L)$$

as in (3.3.1), with coefficients $\tilde{c}(m) \in S_L = \mathbb{C}[L'/L]$. The weight

$$\text{(7.1.9)} \qquad k(f) = \tilde{c}(0,0)$$

is the value of $\tilde{c}(0)$ at the trivial coset in L'/L.

The space of all forms (7.1.6) is an infinite dimensional $\mathbb{C}$-vector space. As evidenced by the following theorem, this gives us a great deal of freedom to choose a Borcherds product $\psi(f)$ with prescribed properties.

Theorem 7.1.2. *Assume $n > 2$. Given any infinite subset $\mathscr{A} \subset \mathbb{Z}^+$, there is a weakly holomorphic form* (7.1.6) *satisfying*

(1) $c(m) \in \mathbb{Z}$ *for all* $m \in \mathbb{Z}$,
(2) $c(-m) = 0$ *for all positive integers* $m \notin \mathscr{A}$,
(3) $k(f) \neq 0$,
(4) *up to a vertical divisor supported in characteristics $p \mid D$, the divisor of $\psi(f)$ is equal to* (7.1.8).

In particular, the divisor of $\psi(f)$ contains no irreducible components supported on the boundary.

The proof will occupy the entirety of the next subsection.

7.2. **Borcherds products with prescribed properties.** As $L' \subset V$ is the dual lattice of L with respect to the quadratic form 2.1.2, any $\mu \in L'/L$ determines a coset $\mathbb{Z} - Q(\mu) \subset \mathbb{Q}$. Denote by $P(\omega_L)$ the space of finite Fourier polynomials

$$\sum_{\mu \in L'/L} \sum_{\substack{m \in \mathbb{Z} - Q(\mu) \\ m \leq 0}} c(m,\mu)\varphi_\mu \cdot q^m$$

valued in $S_L = \mathbb{C}[L'/L]$, whose coefficients satisfy $c(m,\mu) = c(m,-\mu)$. As $n \geq 2$, we may view $H_{2-n}(\omega_L) \subset P(\omega_L)$ by sending a harmonic Maass form (3.3.1) to its principal part

$$\sum_{m \leq 0} c^+(m) \cdot q^m = \sum_{\mu \in L'/L} \sum_{m \leq 0} c^+(m,\mu)\varphi_\mu \cdot q^m.$$

In particular, this allows us to view $M^!_{2-n}(\omega_L) \subset P(\omega_L)$.

Denote by $S_L[[q^{1/D}]]$ the space of S_L-valued formal power series in the variable $q^{1/D}$, and denote by

$$\vartheta \stackrel{\text{def}}{=} q\frac{d}{dq} : S_L[[q^{1/D}]] \to S_L[[q^{1/D}]]$$

the Ramanujan theta operator. For any $k \in \mathbb{Z}$, taking q-expansions allows us to view $M_k(\bar{\omega}_L) \subset S_L[[q^{1/D}]]$.

Following [BF04] we consider the $\mathbb{C}$-bilinear pairing

$$\{-,-\} : P(\omega_L) \times S_L[[q^{1/D}]] \to \mathbb{C}$$

defined by

$$\{p,g\} = \sum_{\substack{m \geq 0 \\ \mu \in L'/L}} c(-m,\mu) b(m,\mu),$$

where $c(m,\mu)$ and $b(m,\mu)$ denote the coefficients of p and g, respectively.

Proposition 7.2.1. *For any*

$$p = \sum_{\mu \in L'/L} \sum_{m<0} c(m,\mu)\varphi_\mu \, q^m \in P(\omega_L),$$

the following are equivalent:

(1) *There exists a weakly holomorphic modular form $f \in M^!_{2-n}(\omega_L)$ whose principal part agrees with p up to a constant in S_L.*
(2) *We have $\{p, g\} = 0$ for every $g \in S_n(\bar{\omega}_L)$.*

When these conditions holds, the constant term $c(0,0)$ of f is related to the value of the Eisenstein series

$$E_L = E_L(\tau, s_0, n) \in M_n(\bar{\omega}_L) \tag{7.2.1}$$

of 2.5.1 at $s_0 = (n-1)/2$ by

$$-c(0,0) = \{p, E_L\} = \sum_{\mu \in L'/L} \sum_{m>0} c(-m,\mu)\, B(m,\mu,s_0). \tag{7.2.2}$$

Proof. See [Bo99], or Corollary 3.9 of [BF04]. □

Proposition 7.2.2. *Let $\mathscr{A} \subset \mathbb{Z}^+$ be any infinite subset. If $n > 2$, there exists a weakly holomorphic form $f \in M^!_{2-n}(\omega_L)$ whose Fourier coefficients $c(m,\mu)$ are integers satisfying the following properties:*

(i) *if $c(-m,\mu) \neq 0$ with $m > 0$, then $\mu = 0$ and $m \in \mathscr{A}$,*
(ii) *$c(0,0) \neq 0$,*
(iii) *$\{f, \vartheta(g)\} = 0$ for all $g \in M_{n-2}(\bar{\omega}_L)$.*

Proof. We generalize the argument of [Br17, Proposition 3.1]. It follows from the main result of [McG03] that the space $M^!_{2-n}(\omega_L)$ has a basis of weakly holomorphic modular forms with integral coefficients. Hence, it suffices to show the existence of an $f \in M^!_{2-n}(\omega_L)$ with *rational* coefficients satisfying the stated properties.

Write $M_n(\bar{\omega}_L, \mathbb{Q})$ for the $\mathbb{Q}$-vector space of modular forms in $M_\kappa(\bar{\omega}_L)$ with rational coefficients, and $S_n(\bar{\omega}_L, \mathbb{Q})$ for the subspace of cusp forms with rational coefficients. To lighten notation, throughout the proof we denote by M the finite dimensional $\mathbb{Q}$-vector space

$$M = M_n(\bar{\omega}_L, \mathbb{Q}) \oplus \vartheta M_{n-2}(\bar{\omega}_L, \mathbb{Q}) \subset S_L[[q^{1/D}]],$$

and by S the subspace

$$S = S_n(\bar{\omega}_L, \mathbb{Q}) \oplus \vartheta M_{n-2}(\bar{\omega}_L, \mathbb{Q}) \subset M.$$

The $\mathbb{Q}$-duals are denoted $M^\vee$ and $S^\vee$, and we denote by

$$\mathrm{pr} : M^\vee \to S^\vee.$$

the surjection induced by $S \subset M$.

For $\mu \in L'/L$ and $m \in \mathbb{Z} + Q(\mu)$, denote by $a_{m,\mu} \in M^\vee$ the linear functional sending

$$g = \sum_{\nu \in L'/L} \sum_{\ell \geq 0} b(\ell, \nu)\varphi_\nu \cdot q^\ell \in M$$

to the Fourier coefficient $a_{m,\mu}(g) = b(m, \mu)$. Let $M^\vee_{\mathscr{A}} \subset M^\vee$ be the subspace generated by all functionals $a_{m,0}$ with $m \in \mathscr{A}$, and fix $e_1, \dots, e_d \in M^\vee_{\mathscr{A}}$ such that $\mathrm{pr}(e_1), \dots, \mathrm{pr}(e_d)$ is a basis of the subspace $\mathrm{pr}(M^\vee_{\mathscr{A}}) \subset S^\vee$.

For every $m \in \mathscr{A}$ there is a unique tuple

$$r(m) = (r_1(m), \dots, r_d(m)) \in \mathbb{Q}^d$$

such that

$$\mathrm{pr}(a_{m,0}) = r_1(m) \cdot \mathrm{pr}(e_1) + \ldots + r_d(m) \cdot \mathrm{pr}(e_d).$$

The linear combination

$$\tilde{a}_{m,0} \stackrel{\text{def}}{=} a_{m,0} - (r_1(m) \cdot e_1 + \cdots + r_d(m) \cdot e_d) \in M^\vee_{\mathscr{A}}$$

clearly lies in the kernel of pr.

Lemma 7.2.3. *There is an $m \in \mathscr{A}$ such that the Eisenstein series* (7.2.1) *satisfies $\tilde{a}_{m,0}(E_L) \neq 0$.*

Proof. We assume on the contrary that $\tilde{a}_{m,0}(E_L) = 0$ for all $m \in \mathscr{A}$. In other words, that the coefficients (2.5.2) satisfy

$$B(m, 0, s_0) = r_1(m) \cdot e_1(E_L) + \ldots + r_d(m) \cdot e_d(E_L) \tag{7.2.3}$$

for all such m.

Let $\|r\|$ be the euclidian norm of a vector $r \in \mathbb{R}^d$, and denote by $\| \cdot \|$ a norm on $S^\vee \otimes_{\mathbb{Q}} \mathbb{R}$, say the operator norm with respect to a fixed norm on the finite dimensional vector space $S \otimes_{\mathbb{Q}} \mathbb{R}$. Since $\mathrm{pr}(e_1), \dots, \mathrm{pr}(e_d)$ are linearly independent, there exists an $\varepsilon > 0$ such that

$$\varepsilon \|r\| \leq \|r_1 \,\mathrm{pr}(e_1) + \ldots + r_d \,\mathrm{pr}(e_d)\|$$

for all $r = (r_1, \dots, r_d) \in \mathbb{R}^d$. Taking $r = r(m)$, we obtain

$$\varepsilon \cdot \|r(m)\| \leq \|\,\mathrm{pr}(a_{m,0})\|.$$

On the other hand, (7.2.3) implies that there is a constant $c > 0$ such that

$$|B(m, 0, s_0)| \leq c \cdot \|r(m)\|.$$

Combining these last two inequalities, we find

$$|B(m, 0, s_0)| \leq \frac{c}{\varepsilon} \cdot \|\,\mathrm{pr}(a_{m,0})\| \tag{7.2.4}$$

for all $m \in \mathscr{A}$.

The Hecke bound for the coefficients of (scalar valued) cusp forms of weight n for $\Gamma(D)$ implies that

$$|\,\mathrm{pr}(a_{m,0})(g)| = O(m^{n/2}),$$

as $m \to \infty$ for any $g \in S_n(\bar{\omega}_L)$. On the other hand, an elementary estimate shows that

$$|\operatorname{pr}(a_{m,0})(g)| = O(m^{n-2+\delta}),$$

as $m \to \infty$ for any $\delta > 0$ and any $g \in \vartheta(M_{n-2}(\bar{\omega}_L))$. As $n > 2$, these bounds imply $\|\operatorname{pr}(a_{m,0})\| = O(m^{n-3/2})$. Combining this with (7.2.4) shows that

$$|B(m, 0, s_0)| = O(m^{n-3/2})$$

for $m \in \mathscr{A}$ and $m \to \infty$, contradicting Corollary 2.6.8. □

We now complete the proof of Proposition 7.2.2. The lemma provides us with an $a = \tilde{a}_{m,0} \in M_{\mathscr{A}}^\vee$ satisfying $\operatorname{pr}(a) = 0$ and $a(E_L) \neq 0$. By definition of $M_{\mathscr{A}}^\vee$, we may expand a as a finite linear combination

$$a = \sum_{m \in \mathscr{A}} c(m, 0) a_{m,0}$$

with $c(m, 0) \in \mathbb{Q}$, and then form the Fourier polynomial

$$p = \sum_{m \in \mathscr{A}} c(m, 0) \varphi_0 \cdot q^m \in P(\omega_L).$$

The condition $\operatorname{pr}(a) = 0$ implies that $\{p, g\} = 0$ for all $g \in S_n(\bar{\omega}_L)$, and $\{p, \vartheta(g)\} = 0$ for all $g \in S_{n-2}(\bar{\omega}_L)$. In particular, Proposition 7.2.1 provides us with a form $f \in M^!_{2-n}(\omega_L)$ whose principal part agrees with p up to a constant, satisfies

$$\{f, \vartheta(g)\} = \{p, \vartheta(g)\} = 0$$

for all $g \in S_{n-2}(\bar{\omega}_L)$, and has constant term $\{p, E_L\} = a(E_L) \neq 0$. □

As a special case of the following proposition, the form f of Proposition 7.2.2 lies in the image of the lifting map

$$M^{!,\infty}_{2-n}(D, \varepsilon^n) \xrightarrow{h \mapsto \tilde{h}} M^!_{2-n}(\omega_L) \tag{7.2.5}$$

defined by (7.1.3).

Proposition 7.2.4. *Assume $n > 2$, and let $f \in M^!_{2-n}(\omega_L)$ be a form whose Fourier coefficients $c(m, \mu)$ satisfy $c(m, \mu) = 0$ for all (m, μ) with $m < 0$ and $\mu \neq 0$. In the notation of (7.1.3), there exists an $h \in M^{!,\infty}_{2-n}(D, \varepsilon^n)$ with principal part*

$$\sum_{m<0} c(m, 0) q^m + \text{constant}, \tag{7.2.6}$$

and such that $\tilde{h} = f$.

Proof. Using the basis $\{\varphi_\mu\}_{\mu \in L'/L}$ of S_L, any form $g(\tau) \in S_n(\bar{\omega}_L)$ can be written as $g(\tau) = \sum_{\mu \in L'/L} g_\mu(\tau) \varphi_\mu$. Taking the component corresponding to $\mu = 0$ defines a linear map

$$S_n(\bar{\omega}_L) \xrightarrow{g \mapsto g_0} S_n(D, \varepsilon^n).$$

We claim that the map $g \mapsto g_0$ is surjective. Indeed, this is equivalent to the injectivity of the adjoint map $S_n(D, \varepsilon^n) \to S_n(\bar{\omega}_L)$, which is just the map $g \mapsto \tilde{g}$ of (7.1.3). This injectivity follows from the explicit formula for the Fourier expansion of $\tilde{g}$ found in Proposition 3.3.2 of [BHK$^+$b], along with the fact that the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on L'/L represents all elements of $\frac{1}{D}\mathbb{Z}/\mathbb{Z}$ primitively.

Now suppose we are given a cusp form

$$\sum_{m>0} b(m)q^m \in S_n(D, \varepsilon^n). \tag{7.2.7}$$

If we choose a $g \in S_n(\bar{\omega}_L)$ such that $g_0 = \sum_m b(m)q^m$, then

$$\sum_{m>0} c(-m, 0)b(m) = \{f, g\} = 0,$$

where the first equality follows from our hypotheses on the coefficients of f, and the second follows from the residue theorem. As this vanishing holds for all forms (7.2.7), it follows from Serre duality on the modular curve $X_0(D)$ that there exists an $h \in M^{!,\infty}_{2-n}(D, \varepsilon^n)$ with principal part (7.2.6). In particular, it follows from (7.1.4) that $\tilde{h} - f$ is holomorphic at ∞. As $\tilde{h} - f$ has negative weight, it is identically 0. □

Proof of Theorem 7.1.2. First pick a vector-valued form $\tilde{f} \in M^!_{2-n}(\omega_L)$ as in Proposition 7.2.2, and then apply Proposition 7.2.4 to pick a scalar-valued form

$$f(\tau) = \sum_{m \gg -\infty} c(m)q^m \in M^{!,\infty}_{2-n}(D, \varepsilon^n)$$

that maps to it under (7.2.5). It follows from the relation (7.1.4) between the coefficients of f and $\tilde{f}$ that for all $m > 0$ the coefficient $c(-m)$ is an integer, and vanishes unless $m \in \mathscr{A}$.

Now we use the fact that $M^{!,\infty}_{2-n}(D, \varepsilon^n)$ has a basis of forms with integer coefficients[3]. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, it follows that the formal q-expansion f^σ again lies in $M^{!,\infty}_{2-n}(D, \varepsilon^n)$. The difference $f(\tau) - f^\sigma(\tau)$ is then holomorphic at every cusp with weight $n - 2 < 0$, and so vanishes identically. Thus all coefficients of $f(\tau)$ are rational, and we may replace $f(\tau)$ by a positive integer multiple to assume that $c(m) \in \mathbb{Z}$ for all $m \in \mathbb{Z}$.

As we chose $\tilde{f}$ so that its constant term $\tilde{c}(0, 0)$ is nonzero, the associated Borcherds product $\psi(f)$ has nonzero weight by Remark 7.1.1.

It only remains to verify property (4) in Theorem 7.1.2. For this we appeal to Theorem 5.3.3 of [BHK$^+$a], which tells us that

$$\mathrm{div}(\psi(f)) = \bar{\mathcal{Z}}_V(f) + \sum_{m>0} c(-m)\mathcal{B}(m), \tag{7.2.8}$$

[3] One can deduce this from the corresponding statement for holomorphic modular forms, by multiplying weakly holomorphic modular forms by powers of Ramanujan's discriminant to kill the poles at ∞

up to a linear combination of divisors supported in characteristics $p \mid D$. Here, as in (5.3.3) of [BHK$^+$a],

$$\mathcal{B}(m) = \frac{m}{n-2} \sum_{\Phi} \rho_{L_0}(m) \cdot \mathcal{S}_V(\Phi), \tag{7.2.9}$$

where the sum is over a finite set $\{\Phi\}$ indexing the irreducible boundary components $\mathcal{S}_V(\Phi) \subset \partial \bar{\mathcal{S}}_V$, each of which is connected and smooth over $\mathcal{O}_{\boldsymbol{k}}$. Inside the sum, each L_0 is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-module (which depends on Φ) of signature $(n-2, 0)$, and

$$\rho_{L_0}(m) = \#\{x \in L_0 : \langle x, x \rangle = m\}$$

is the number of times that lattice represents m.

As explained in §3.1 of [BHK$^+$a], each Φ is an equivalence class of pairs (I, g) in which $I \subset V$ is an isotropic $\boldsymbol{k}$-line, and $g \in G(\mathbb{A}_f)$. This data determines a filtration

$$\mathfrak{a} \subset \mathfrak{a}^\perp \subset gL$$

by $\mathcal{O}_{\boldsymbol{k}}$-module direct summands, with $\mathfrak{a} = I \cap gL$ isotropic of rank one and $\mathfrak{a}^\perp = \{x \in gL : \langle x, \mathfrak{a} \rangle = 0\}$. The quotient $L_0 = \mathfrak{a}^\perp / \mathfrak{a}$ inherits a self-dual hermitian form from that on $gL \subset V$, and the filtration admits a (non-canonical) splitting

$$gL = \mathfrak{a} \oplus L_0 \oplus \mathfrak{b}$$

in which $\mathfrak{b}$ is rank one and isotropic, and $\mathfrak{a} \oplus \mathfrak{b}$ is orthogonal to L_0.

As L_0 and gL are themselves self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules, we may form modular forms valued in the finite dimensional vector spaces $S_{L_0} = \mathbb{C}[L_0'/L_0]$ and $S_{gL} = \mathbb{C}[(gL)'/(gL)]$, exactly we did for L. The action of $g \in G(\mathbb{A}_f)$ defines a canonical bijection $L'/L \cong (gL)'/(gL)$, which induces an isomorphism $S_L \cong S_{gL}$ respecting the Weil representations on source and target.

To each Φ, we may attach the S_{L_0}-valued theta series

$$\Theta_\Phi(\tau) = \sum_{\mu \in L_0'/L_0} \Theta_{\Phi,\mu}(\tau) \varphi_\mu \in M_{n-2}(\bar{\omega}_{L_0})$$

where $\Theta_{\Phi,\mu}(\tau) = \sum_{x \in \mu + L_0} q^{\langle x, x \rangle}$. As in Theorem 4.1 of [Sch15], there is an induced S_{gL}-valued modular form

$$\mathrm{ind}_L(\Theta_\Phi) = \sum_{\substack{\mu \in L_0'/L_0 \\ \beta \in \mathfrak{d}_{\boldsymbol{k}}^{-1}\mathfrak{a}/\mathfrak{a}}} \Theta_{E,\mu}(\tau) \varphi_{\mu+\beta} \in M_{n-2}(\bar{\omega}_{gL}),$$

and we use $S_L \cong S_{gL}$ to view $\mathrm{ind}_L(\Theta_\Phi) \in M_{n-2}(\bar{\omega}_L)$.

Using the relation (7.1.4) between the coefficients of $\tilde{f}$ and f, we see that

$$\sum_{m>0} mc(-m) \rho_{L_0}(m) = \{\tilde{f}, \vartheta(\mathrm{ind}_L(\Theta_\Phi))\} = 0,$$

where the second equality follows from condition (iii) in Proposition 7.2.2. Comparing with (7.2.9) shows that $\sum_{m>0} c(-m)\mathcal{B}(m) = 0$, and property (4) of Theorem 7.1.2 follows by comparison with (7.2.8). □

7.3. **A carefully chosen Borcherds product.** Recalling the functions $\mathbf{a}_k(s)$ of (1.1.4), set $\mathbf{b}_{V,1}(s) = \mathbf{a}_1(s)$. For $k \geq 2$ even, set

$$\mathbf{b}_{V,k}(s) = \mathbf{a}_k(s) \prod_{\ell \mid D} \left(1 + \left(\frac{-1}{\ell} \right)^{\frac{k}{2}} \mathrm{inv}_\ell(V) \ell^{-s-\frac{k}{2}} \right).$$

For $k \geq 3$ odd, set

$$\mathbf{b}_{V,k}(s) = \mathbf{a}_k(s) \prod_{\ell \mid D} \left(1 + \left(\frac{-1}{\ell} \right)^{\frac{k-1}{2}} \mathrm{inv}_\ell(V) \ell^{-s+\frac{1-k}{2}} \right)^{-1}.$$

When $k > 1$ we have $\mathbf{b}_{V,k}(s)\mathbf{b}_{V,k+1}(s) = \mathbf{a}_k(s)\mathbf{a}_{k+1}(s)$, which implies that the function (1.1.5) factors as

$$\mathbf{A}_V(s) = \mathbf{b}_{V,1}(s) \cdots \mathbf{b}_{V,n}(s). \tag{7.3.1}$$

Now suppose $n > 2$, abbreviate

$$\mathscr{A} = \{\text{primes } p \equiv 1 \pmod{D}\},$$

and assume that the weakly modular form

$$f(\tau) = \sum_{m \gg -\infty} c(m) q^m \in M^{!,\infty}_{2-n}(D, \varepsilon^n)$$

of (7.1.6) is chosen as in Theorem 7.1.2. In particular, the divisor of the Borcherds product $\psi(f)$ contains no components of the boundary, and so

$$\mathrm{div}(\psi(f)) = \bar{\mathcal{Z}}_V(f) + \mathrm{Vert}(f) \tag{7.3.2}$$

in which

$$\bar{\mathcal{Z}}_V(f) = \sum_{p \in \mathscr{A}} c(-p) \bar{\mathcal{Z}}_V(p),$$

and $\mathrm{Vert}(f)$ is supported in characteristics dividing D.

Remark 7.3.1. The notation $\mathrm{Vert}(f)$ is slightly misleading, as $\bar{\mathcal{Z}}_V(f)$ may itself have vertical components. Any such components are supported on the exceptional divisor $\mathrm{Exc}_V \subset \bar{\mathcal{S}}_V$, by Corollary 3.7.3 of [BHK⁺a].

Proposition 7.3.2. *The Borcherds product $\psi(f)$ has weight*

$$k(f) = \frac{1}{\mathbf{b}_{V,n}(0)} \sum_{p \in \mathscr{A}} c(-p)(p^{n-1} + 1),$$

and, recalling the notation (4.5.3)*, satisfies*

$$(\mathrm{Vert}(f), 0) = (E, 0) - k(f) \sum_{\ell \mid D} \left(0, \frac{\log(\ell)}{1 + \beta_\ell} \right) \in \widehat{\mathrm{Pic}}(\mathcal{S}_V)_{\mathbb{Q}},$$

where E is a divisor (which may depend on f) supported on the exceptional divisor of Definition 5.1.5, and

$$\beta_\ell = (-1)^{n+1} \cdot \begin{cases} \left(\frac{-1}{\ell}\right)^{\frac{n}{2}} \operatorname{inv}_\ell(V)\ell^{\frac{n}{2}} & \text{if } n \text{ is even} \\ \left(\frac{-1}{\ell}\right)^{\frac{n-1}{2}} \operatorname{inv}_\ell(V)\ell^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases} \tag{7.3.3}$$

Proof. The proof requires a short digression on Eisenstein series.

For any divisor $r \mid D$ set $r' = D/r$. Our assumption that D is odd implies that the quadratic character $\varepsilon : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ determined by $\boldsymbol{k}$ is

$$\varepsilon(a) = \left(\frac{a}{D}\right).$$

Hence we may factor $\varepsilon = \varepsilon_r \cdot \varepsilon_{r'}$ with

$$\varepsilon_r(a) = \left(\frac{a}{r}\right) \quad \text{and} \quad \varepsilon_{r'}(a) = \left(\frac{a}{r'}\right).$$

Define the quadratic Gauss sum

$$\tau(\varepsilon_r) = \sum_{a \in (\mathbb{Z}/r\mathbb{Z})^\times} \varepsilon_r(a) e^{2\pi i a/r} = \begin{cases} \sqrt{r} & \text{if } r \equiv 1 \pmod 4 \\ i\sqrt{r} & \text{if } r \equiv 3 \pmod 4, \end{cases}$$

and similarly with r replaced by r'.

Lemma 7.3.3. *For every divisor $r \mid D$ there is an Eisenstein series*

$$E_r = \sum_{m \geq 0} e_r(m) \cdot q^m \in M_n(\Gamma_0(D), \varepsilon^n)$$

whose Fourier coefficients are as follows. The constant term is

$$e_r(0) = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If n is even the coefficients indexed by $m > 0$ are

$$e_r(m) = \frac{r^{n/2}(-2\pi i)^n}{D^n \Gamma(n) L_D(n, \varepsilon^n)} \sum_{\substack{c \mid m \\ c > 0 \\ \gcd(m/c, r) = 1}} c^{n-1} \sum_{d \mid \gcd(c, r')} d\mu(r'/d).$$

If n is odd the coefficients indexed by $m > 0$ are

$$e_r(m) = \varepsilon_r(r') \frac{r^{n/2}(-2\pi i)^n \tau(\varepsilon_{r'})}{D^n \Gamma(n) L_D(n, \varepsilon^n)} \sum_{\substack{c \mid m \\ c > 0 \\ \gcd(m/c, r) = 1}} \varepsilon_r(m/c) \varepsilon_{r'}(c) \cdot c^{n-1}.$$

In both formulas $L_D(s, \varepsilon^n)$ is the Dirichlet L-function with Euler factors at all $\ell \mid D$ removed.

Proof. For each $s, t \in \mathbb{Z}/D\mathbb{Z}$ define an Eisenstein series

$$G_{(s,t)}(z) = \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\neq(0,0) \\ (c,d)\equiv(s,t) \pmod D}} (cz+d)^{-n}.$$

Theorem 7.1.3 of [Miy89] shows that

$$E_r(z) = \frac{\varepsilon_r^n(r')}{2r^{n/2}L_D(n,\varepsilon^n)} \sum_{s,t\in\mathbb{Z}/D\mathbb{Z}} \varepsilon_r^n(s)\varepsilon_{r'}^n(t)G_{(s,t)}(r'z)$$

has the desired Fourier expansion. □

Suppose $r \mid D$, and set $r' = D/r$. If n is even, set

$$a = \frac{(-2\pi i)^n \mu(D)}{D^n\Gamma(n)L_D(n,\varepsilon^n)} \quad \text{and} \quad b_r = r^{n/2}\mu(r).$$

If n is odd, set

$$a = \frac{(-2\pi i)^n \tau(\varepsilon)}{D^n\Gamma(n)L_D(n,\varepsilon^n)} \quad \text{and} \quad b_r = \frac{r^{n/2}\varepsilon_r(r')\tau(\varepsilon_{r'})}{\tau(\varepsilon)}.$$

One may check directly that $b_r = \prod_{\ell\mid r} b_\ell$, where the product is over all primes $\ell \mid r$, and that when $p \in \mathscr{A}$ the formulas of Lemma 7.3.3 simplify to

$$e_r(p) = ab_r \cdot (p^{n-1}+1). \tag{7.3.4}$$

For any prime $\ell \mid D$, abbreviate

$$\gamma_\ell = \epsilon_\ell^{-n} \cdot (D,\ell)_\ell^n \cdot \mathrm{inv}_\ell(V),$$

where $\epsilon_\ell \in \{1, i\}$ is as in Lemma 2.6.1. For a divisor $r \mid D$, set $\gamma_r = \prod_{\ell\mid r}\gamma_\ell$.

Lemma 7.3.4. *If we set $\beta_r = \prod_{\ell\mid r}\beta_\ell$ for any divisor $r \mid D$, then*

$$\frac{1}{\mathbf{b}_{V,n}(0)} = -a\prod_{\ell\mid D}(1+\beta_\ell) = -a\sum_{r\mid D}\beta_r$$

and $\beta_r = \gamma_r b_r$. Here the product is over all prime divisors of D, while the sum is over all positive divisors of D.

Proof. This is an elementary calculation. For the reader interested in working out the details, we remark that one needs the equality

$$\prod_{\ell\mid D}\left(\frac{-1}{\ell}\right)\mathrm{inv}_\ell(V) = 1,$$

which follows from (1.1.3) and our hypothesis that D is odd, hence congruent to $3 \pmod 4$. □

For any divisor $r \mid D$, let $c_r(0)$ be the constant term of $f(\tau)$ at the cusp ∞_r, in the sense of Definition 4.1.1 of [BHK$^+$a]. According to §5.3 of [BHK$^+$a] the weight of $\psi(f)$ is given by

$$k(f) = \sum_{r|D} \gamma_r c_r(0).$$

On the other hand, according to Proposition 4.2.2 of [BHK$^+$a],

$$c_r(0) = -\sum_{p\in\mathscr{A}} c(-p) e_r(p). \tag{7.3.5}$$

Combining these with (7.3.4) and Lemma 7.3.4 shows

$$\begin{aligned} k(f) &= -\sum_{r|D}\sum_{p\in\mathscr{A}} \gamma_r c(-p) e_r(p) \\ &= -a \sum_{r|D} \beta_r \sum_{p\in\mathscr{A}} c(-p)(p^{n-1}+1) \\ &= \frac{1}{\mathbf{b}_{V,n}(0)} \sum_{p\in\mathscr{A}} c(-p)(p^{n-1}+1), \end{aligned}$$

proving the first claim of the proposition.

Now we turn to the second claim of the proposition. In $\widehat{\mathrm{Pic}}(\mathcal{S}_V)_{\mathbb{Q}}$ we have the relation

$$(\mathrm{div}(\sqrt{-D}), 0) = (0, \log(D)).$$

Similarly, if $\ell \mid D$ and $\mathfrak{l} \subset \mathcal{O}_{\boldsymbol{k}}$ is the unique prime above it, then

$$(\mathcal{S}_{V/\mathbb{F}_{\mathfrak{l}}}, 0) = (0, \log(\ell))$$

where $\mathbb{F}_{\mathfrak{l}} = \mathcal{O}_{\boldsymbol{k}}/\mathfrak{l}$. Theorem 5.3.3 of [BHK$^+$a] tells us that

$$\mathrm{Vert}(f) = E - k(f)\cdot \mathrm{div}(\sqrt{-D}) + \sum_{r|D} \gamma_r c_r(0) \sum_{\ell|r} \mathcal{S}_{V/\mathbb{F}_{\mathfrak{l}}},$$

where E is a divisor supported on Exc_V, and so

$$(\mathrm{Vert}(f), 0) = (E, 0) + \Big(0, -k(f)\cdot\log(D) + \sum_{r|D} \gamma_r c_r(0)\log(r)\Big). \tag{7.3.6}$$

Combining (7.3.4), (7.3.5), and the above formula for $k(f)$ shows that

$$\begin{aligned} \sum_{r|D} \gamma_r c_r(0)\log(r) &= -a\left(\sum_{p\in\mathscr{A}} c(-p)(p^{n-1}+1)\right)\left(\sum_{r|D}\beta_r\log(r)\right) \\ &= -a\mathbf{b}_{V,n}(0) k(f) \sum_{r|D} \beta_r \log(r). \end{aligned}$$

Using Lemma 7.3.4 we may rewrite this as

$$\begin{aligned}\sum_{r|D}\gamma_r c_r(0)\log(r) &= k(f)\left(\prod_{\ell|D}\frac{1}{1+\beta_\ell}\right)\left(\sum_{r|D}\beta_r\log(r)\right)\\ &= k(f)\sum_{\ell|D}\frac{\beta_\ell}{1+\beta_\ell}\log(\ell).\end{aligned}$$

It follows easily that

$$-k(f)\cdot\log(D)+\sum_{r|D}\gamma_r c_r(0)\log(r) = -k(f)\sum_{\ell|D}\frac{\log(\ell)}{1+\beta_\ell},$$

and plugging this into (7.3.6) completes the proof of Proposition 7.3.2. □

Remark 7.3.5. There is another proof the first claim of Proposition 7.3.2 that uses the Eisenstein series $E_L(\tau,s,n)$ of (2.4.4) in place of the Eisenstein series E_r of Lemma 7.3.3. Briefly, combining (7.1.9) with (7.2.2) and (7.1.4) gives

$$-k(f)=-\tilde{c}(0,0)=\sum_{\substack{m>0\\ \mu\in L'/L}}\tilde{c}(-m,\mu)B(m,\mu,s_0)=\sum_{p\in\mathscr{A}}c(-p)B(p,0,s_0),$$

where $s_0=(n-1)/2$. Corollaries 2.6.5 and 2.6.6 imply

$$B(p,0,s_0)=-\frac{p^{n-1}+1}{\mathbf{b}_{V,n}(0)}$$

for all $p\in\mathscr{A}$, yielding the desired formula for $k(f)$.

8. The volume calculations

We continue to work with the Shimura variety $\mathcal{S}_V$ of §5.1.4 defined by a $\boldsymbol{k}$-hermitian space V of signature $(n-1,1)$ containing a self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice, and maintain the assumption that D is odd imposed since §4.

We will use induction on n to compute the complex and arithmetic volumes of the hermitian line bundle $\widehat{\mathcal{L}}_V$ of (5.2.3), and from this deduce Theorem A.

8.1. **Comparing volumes.** Suppose $n\geq 2$. In what follows, our main interest lies in the hermitian line bundles

$$\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V},\widehat{\mathcal{L}}_V\in\widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V,\mathscr{D}_{\mathrm{pre}})$$

of (5.2.7), but we will make systematic use of the hermitian line bundle

$$\widehat{\mathcal{K}}_V=2\widehat{\mathcal{L}}_V-(\mathrm{Exc}_V,0)\in\widehat{\mathrm{Pic}}(\bar{\mathcal{S}}_V,\mathscr{D}_{\mathrm{pre}}) \tag{8.1.1}$$

of Theorem 5.5.1 and Theorem 6.1.2. Our first goal is to clarify the relation between their arithmetic volumes.

The intersection formula $\widehat{\mathcal{K}}_V \cdot (\mathrm{Exc}_V, 0) = 0$ of Theorem 5.5.1 implies the second equality in

$$\begin{aligned} 2^n \cdot \widehat{\mathrm{vol}}(\widehat{\mathcal{L}}_V) &= \widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V + (\mathrm{Exc}_V, 0)) \\ &= \widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) + \widehat{\mathrm{vol}}(\mathrm{Exc}_V, 0), \end{aligned} \tag{8.1.2}$$

and the arithmetic volume of $(\mathrm{Exc}_V, 0)$ was computed in §5.6. Thus it remains to compare the arithmetic volumes of $\widehat{\mathcal{K}}_V$ and $\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$.

Proposition 8.1.1. *We have the arithmetic volume relation*

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) = \widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) + nC_0(n)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}),$$

where $C_0(n)$ is the constant of Theorem A.

Proof. If $n > 2$, the stated formula is Remark 5.5.2. We will fill in the missing case $n = 2$ using descending induction, and so assume that $n = 2$ in all that follows. In other words, V has signature $(1,1)$.

Fix a prime p split in $\boldsymbol{k}$. Let $V^\flat$ be the $\boldsymbol{k}$-hermitian space of signature $(2,1)$ whose local invariants satisfy

$$\mathrm{inv}_\ell(V) = (p, -D)_\ell \cdot \mathrm{inv}_\ell(V^\flat)$$

for all places $\ell \le \infty$, so that the discussion of §6.1 applies with the pair $(V^\flat, V)$ replacing (V, V'). Define $H_p \in \mathbb{R}$ by the relation

$$(p+1)H_p = \mathrm{ht}_{\widehat{\mathcal{K}}_{V^\flat}}(\bar{\mathcal{Z}}_{V^\flat}(p)) - \mathrm{ht}_{\widehat{\omega}^{\mathrm{Hdg}}_{A^\flat/\mathcal{S}_{V^\flat}}}(\bar{\mathcal{Z}}_{V^\flat}(p)).$$

On the one hand, part (3) of Theorem 5.5.1 and Lemma 4.5.2 imply

$$\mathrm{ht}_{\widehat{\mathcal{K}}_{V^\flat}}(\bar{\mathcal{Z}}_{V^\flat}(p)) = \mathrm{ht}_{\widehat{\omega}^{\mathrm{Hdg}}_{A^\flat/\mathcal{S}_{V^\flat}} + (0, C_0(3))}(\bar{\mathcal{Z}}_{V^\flat}(p)). \tag{8.1.3}$$

In the codimension 2 arithmetic Chow group of $\bar{\mathcal{S}}_{V^\flat}$ we have the relation

$$\Big(\widehat{\omega}^{\mathrm{Hdg}}_{A^\flat/\mathcal{S}_{V^\flat}} + (0, C_0(3))\Big)^2 = \Big(\widehat{\omega}^{\mathrm{Hdg}}_{A^\flat/\mathcal{S}_{V^\flat}}\Big)^2 + \Big(0, 2C_0(3)\,\mathrm{ch}(\widehat{\omega}^{\mathrm{Hdg}}_{A^\flat/\mathcal{S}_{V^\flat}})\Big),$$

as in the proof of Lemma 4.5.3. Directly from the definition of arithmetic height in §4.5, it follows that the right hand side of (8.1.3) is equal to

$$\mathrm{ht}_{\widehat{\omega}^{\mathrm{Hdg}}_{A^\flat/\mathcal{S}_{V^\flat}}}(\bar{\mathcal{Z}}_{V^\flat}(p)) + 2C_0(3) \int_{\mathcal{Z}_{V^\flat}(p)(\mathbb{C})} \mathrm{ch}(\widehat{\omega}^{\mathrm{Hdg}}_{A^\flat/\mathcal{S}_{V^\flat}}).$$

Combining the first claim of Theorem 6.1.2 with the equality of Chern forms of Theorem 5.5.1, we find

$$\int_{\mathcal{Z}_{V^\flat}(p)(\mathbb{C})} \mathrm{ch}(\widehat{\omega}^{\mathrm{Hdg}}_{A^\flat/\mathcal{S}_{V^\flat}}) = (p+1)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}).$$

Putting all of this together allows us to rewrite (8.1.3) as

$$\mathrm{ht}_{\widehat{\mathcal{K}}_{V^\flat}}(\bar{\mathcal{Z}}_{V^\flat}(p)) = \mathrm{ht}_{\widehat{\omega}^{\mathrm{Hdg}}_{A^\flat/\mathcal{S}_{V^\flat}}}(\bar{\mathcal{Z}}_{V^\flat}(p)) + 2C_0(3)(p+1)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}),$$

which is equivalent to

$$H_p = 2C_0(3)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}). \tag{8.1.4}$$

On the other hand, the height formulas of Theorems 6.1.2 and 6.1.3 imply

$$\begin{aligned} H_p &= \widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) - \widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) + 2\left(\frac{L'(0,\varepsilon)}{L(0,\varepsilon)} + \frac{\log(D)}{2}\right)\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) \\ &= \widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) - \widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) + 2(C_0(3)-C_0(2))\,\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) \end{aligned}$$

up to a rational multiple of $\log(p)$, and combining this with (8.1.4) proves

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) - \widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) - 2C_0(2)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) \in \mathbb{Q}\log(p).$$

The only way this can hold for all primes p split in $\boldsymbol{k}$ is if this real number is 0, completing the proof when $n=2$. □

8.2. **Shimura curves.** The following proposition is a consequence of the main result of [How20], which is itself a consequence of the calculation of arithmetic volumes of modular curves and quaternion Shimura curves due to Kühn [Kuhn01], Bost (unpublished), and Kudla-Rapoport-Yang [KRY06].

Proposition 8.2.1. *Suppose $n=2$. The metrized Hodge bundle of $A \to \mathcal{S}_V$ has complex volume*

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) = \frac{|\mathrm{CL}(\boldsymbol{k})|^2}{2^{o(D)-1}\cdot 12\cdot |\mathcal{O}_{\boldsymbol{k}}^\times|^2}\prod_{\ell\mid D}(1+\ell^*)$$

and arithmetic volume

$$\widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) = \left(-2 - \frac{4\zeta'(-1)}{\zeta(-1)} - \sum_{\ell\mid D}\frac{1-\ell^*}{1+\ell^*}\cdot\log(\ell)\right)\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}),$$

where $\mathrm{CL}(\boldsymbol{k})$ is the class group of $\boldsymbol{k}$, $o(D)$ is the number of prime divisors of D, and we abbreviate

$$\ell^* = \left(\frac{-1}{\ell}\right)\mathrm{inv}_\ell(V)\ell.$$

Proof. As in (5.1.4), we realize

$$\mathcal{S}_V \subset \mathcal{M}_{W_0}\times_{\mathcal{O}_{\boldsymbol{k}}}\mathcal{M}_W$$

as a union of connected components, where W has signature $(1,1)$.

The $\mathrm{GU}(W)$ Shimura datum defining the generic fibers of $\mathcal{M}_W^{\mathrm{Pap}}$ and $\mathcal{M}_W$ actually has reflex field $\mathbb{Q}$ rather than $\boldsymbol{k}$, and in §4 of [How20] one finds the definition of a regular and flat Deligne-Mumford stack

$$\mathcal{X}_W \to \mathrm{Spec}(\mathbb{Z})$$

such that $\mathcal{M}_W^{\mathrm{Pap}} \cong \mathcal{X}_{W/\mathcal{O}_{\boldsymbol{k}}}$, and such that the universal abelian surface over $\mathcal{M}_W^{\mathrm{Pap}}$ descends to $A \to \mathcal{X}_W$. The stack $\mathcal{X}_W$ has its own canonical toroidal compactification and its own theory of arithmetic Chow groups, and Theorem A of [How20] implies

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{X}_W}) = \frac{|\mathrm{CL}(\boldsymbol{k})|}{2^{o(D)-1}\cdot 12\cdot|\mathcal{O}^{\times}_{\boldsymbol{k}}|}\prod_{\ell|D}(1+\ell^*) \tag{8.2.1}$$

and

$$\widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{X}_W}) = \left(-1-\frac{2\zeta'(-1)}{\zeta(-1)}-\frac{1}{2}\sum_{\ell|D}\frac{1-\ell^*}{1+\ell^*}\cdot\log(\ell)\right)\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{X}_W}). \tag{8.2.2}$$

The composition

$$\mathcal{S}_V \to \mathcal{M}_W \to \mathcal{M}^{\mathrm{Pap}}_W \cong \mathcal{X}_{W/\mathcal{O}_{\boldsymbol{k}}}$$

extends to a proper morphism

$$\varphi : \bar{\mathcal{S}}_V \to \bar{\mathcal{X}}_{V/\mathcal{O}_{\boldsymbol{k}}},$$

which, using Remark 5.1.3, restricts to a finite étale surjection of degree

$$\deg(\varphi) = |\mathrm{CL}(\boldsymbol{k})|/|\mathcal{O}^{\times}_{\boldsymbol{k}}|$$

from $\bar{\mathcal{S}}_V \smallsetminus \mathrm{Exc}_V$ to smooth locus of $\bar{\mathcal{X}}_{V/\mathcal{O}_{\boldsymbol{k}}}$.

If $\widehat{\mathcal{L}}$ is any pre-log singular hermitian line bundle on $\bar{\mathcal{X}}_V$, and $\widehat{\mathcal{L}}_{/\mathcal{O}_{\boldsymbol{k}}}$ is its base change to $\bar{\mathcal{X}}_{V/\mathcal{O}_{\boldsymbol{k}}}$, then unpacking the definitions shows that[4]

$$\mathrm{vol}_{\mathbb{C}}(\varphi^*\widehat{\mathcal{L}}_{/\mathcal{O}_{\boldsymbol{k}}}) = \deg(\varphi)\cdot\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}})$$

and

$$\widehat{\mathrm{vol}}(\varphi^*\widehat{\mathcal{L}}_{/\mathcal{O}_{\boldsymbol{k}}}) = 2\cdot\deg(\varphi)\cdot\widehat{\mathrm{vol}}(\widehat{\mathcal{L}}).$$

The proposition follows by applying these to the hermitian line bundles

$$\widehat{\mathcal{L}} = \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{X}_V}, \qquad \varphi^*\widehat{\mathcal{L}}_{/\mathcal{O}_{\boldsymbol{k}}} = \widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}$$

and using (8.2.1) and (8.2.2). □

We now express the complex and arithmetic volumes of $\widehat{\mathcal{K}}_V$ in terms of the entire function $\mathbf{A}_V(s)$ defined by (1.1.5). This will form the base case of the inductive proofs of Theorems 8.3.2 and 8.4.1 below.

Corollary 8.2.2. *Suppose $n=2$. The hermitian line bundle* (8.1.1) *has complex volume*

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V) = \frac{|\mathrm{CL}(\boldsymbol{k})|}{2^{o(D)-2}|\mathcal{O}^{\times}_{\boldsymbol{k}}|}\cdot\mathbf{A}_V(0),$$

and arithmetic volume

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) = \left(2\frac{\mathbf{A}'_V(0)}{\mathbf{A}_V(0)}+\log(D)\right)\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V).$$

---

[4]The factor of 2 appears in the second equality because it is the degree of the finite flat morphism $\mathcal{X}_{W/\mathcal{O}_{\boldsymbol{k}}} \to \mathcal{X}_W$. It does not appear in the first equality because, under the conventions of Remark 4.4.2, this morphism induces an isomorphism on complex points.

Proof. Replacing W by V in (1.1.3), and using $D \equiv 3 \pmod 4$, we find

$$\prod_{\ell | D} \left(\frac{-1}{\ell} \right) \mathrm{inv}_\ell(V) = 1.$$

Using this and (1.1.5) proves the first equality in

$$\mathbf{A}_V(0) = \frac{D^{\frac{1}{2}} L(1,\varepsilon)\zeta(2)}{8\pi^3} \prod_{\ell | D} (1+\ell^*) = \frac{|\mathrm{CL}(\boldsymbol{k})|}{24 \cdot |\mathcal{O}_{\boldsymbol{k}}^\times|} \prod_{\ell | D} (1+\ell^*) \, .$$

Comparing this with Proposition 8.2.1, and using the equality of Chern forms of Theorem (5.5.1), shows

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V) = \mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) = \frac{|\mathrm{CL}(\boldsymbol{k})|}{2^{o(D)-2}|\mathcal{O}_{\boldsymbol{k}}^\times|} \cdot \mathbf{A}_V(0).$$

For the second claim, taking the logarithmic derivative of (1.1.5) yields

$$\begin{aligned} \frac{\mathbf{A}'_V(0)}{\mathbf{A}_V(0)} &= \frac{\mathbf{a}'_1(0)}{\mathbf{a}_1(0)} + \frac{\mathbf{a}'_2(0)}{\mathbf{a}_2(0)} - \sum_{\ell | D} \frac{\log(\ell)}{1+\ell^*} \\ &= \frac{\mathbf{a}'_1(0)}{\mathbf{a}_1(0)} + \frac{\mathbf{a}'_2(0)}{\mathbf{a}_2(0)} - \frac{1}{2}\log(D) - \frac{1}{2} \sum_{\ell | D} \left(\frac{1-\ell^*}{1+\ell^*} \right) \log(\ell). \end{aligned}$$

One can use the functional equation of $L(s, \varepsilon^k)$ to see that

$$\frac{\mathbf{a}'_k(0)}{\mathbf{a}_k(0)} = -2\frac{L'(1-k,\varepsilon^k)}{L(1-k,\varepsilon^k)} - \frac{\Gamma'(k)}{\Gamma(k)} + \log\left(\frac{4\pi}{D}\right) + (-1)^k \log(D) \tag{8.2.3}$$

for all $k \ge 1$. Recalling the constant $C_0(2)$ of Theorem A, and using the well-known formulas $\Gamma'(1)/\Gamma(1) = -\gamma$ and $\Gamma'(2)/\Gamma(2) = 1-\gamma$, we find

$$\frac{\mathbf{a}'_1(0)}{\mathbf{a}_1(0)} + \frac{\mathbf{a}'_2(0)}{\mathbf{a}_2(0)} = -1 + C_0(2) - 2\frac{\zeta'(-1)}{\zeta(-1)}.$$

Putting this all together gives

$$2\frac{\mathbf{A}'_V(0)}{\mathbf{A}_V(0)} - 2C_0(2) + \log(D) = -2 - 4\frac{\zeta'(-1)}{\zeta(-1)} - \sum_{\ell | D} \left(\frac{1-\ell^*}{1+\ell^*} \right) \log(\ell),$$

and comparing with Proposition 8.2.1 shows that

$$\widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) = \left(2\frac{\mathbf{A}'_V(0)}{\mathbf{A}_V(0)} - 2C_0(2) + \log(D) \right) \mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}).$$

The second claim follows from this using Proposition 8.1.1 and the equality of Chern forms of Theorem 5.5.1. ◻

8.3. **Complex volumes.** We can now compute the complex volumes of $\widehat{\mathcal{L}}_V$ and $\widehat{\mathcal{K}}_V$ for all $n \geq 2$ using induction. The calculation requires the following lemma, in which

$$\mathscr{A} = \{\text{primes } p \equiv 1 \pmod{D}\}.$$

Lemma 8.3.1. *Assume $n > 2$. For any weakly holomorphic form*

$$f(\tau) = \sum_{m \gg -\infty} c(m) q^m \in M^{!,\infty}_{2-n}(D, \varepsilon^n)$$

as in Theorem 7.1.2, we have

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_V) = \frac{1}{k(f)} \sum_{p \in \mathscr{A}} c(-p) \int_{\mathcal{Z}_V(p)(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{L}}_V)^{n-2}.$$

Proof. Recall from §7 that the Borcherds product $\psi(f)$ has nonzero weight $k(f)$, divisor (7.3.2), and satisfies (7.1.7). It follows that

$$dd^c \Phi_V(f) = k(f)\, \mathrm{ch}(\widehat{\mathcal{L}}_V)$$

and, by Lemma 1.21 of [BBK07], we have the equality of currents

$$k(f)[\mathrm{ch}(\widehat{\mathcal{L}}_V)] = dd^c[\Phi_V(f)] + \sum_{p \in \mathscr{A}} c(-p) \delta_{\bar{\mathcal{Z}}_V(p)}$$

on $\bar{\mathcal{S}}_V(\mathbb{C})$. In particular

$$\int_{\mathcal{S}_V(\mathbb{C})} \varphi \wedge \mathrm{ch}(\widehat{\mathcal{L}}_V) = \frac{1}{k(f)} \sum_{p \in \mathscr{A}} c(-p) \int_{\mathcal{Z}_V(p)} \varphi$$

for any closed form φ on $\bar{\mathcal{S}}_V(\mathbb{C})$ of type $(n-2, n-2)$.

We would like to apply this last equality with $\varphi = \mathrm{ch}(\widehat{\mathcal{L}}_V)^{n-2}$, but cannot do so directly because the Chern form does not extend smoothly across the boundary. However, one can find a closed form φ, smooth on all of $\bar{\mathcal{S}}_V(\mathbb{C})$, and a smooth form α on the interior $\mathcal{S}_V(\mathbb{C})$ satisfying

$$\varphi = \mathrm{ch}(\widehat{\mathcal{L}}_V)^{n-2} + d\alpha,$$

and such that α is a pre-log-log form, in the sense of Definition 7.3 of [BBK07], with respect to the boundary $\partial \bar{\mathcal{S}}_V(\mathbb{C})$. Indeed, this follows from Proposition 5.2.1: simply choose any hermitian metric on $\mathcal{L}_V$, smooth on all of $\bar{\mathcal{S}}_V(\mathbb{C})$, and let φ be the $(n-2)$-fold exterior power of its Chern form.

If β is any pre-log-log form on $\bar{\mathcal{S}}_V(\mathbb{C})$ of type $(n-2, n-2)$, the current on $\bar{\mathcal{S}}_V(\mathbb{C})$ associated to β satisfies $[d\beta] = d[\beta]$ by Proposition 7.6 of [BKK07], and evaluating both sides of this equality on the constant function 1 shows that the integral of $d\beta$ vanishes. Applying this to $\beta = \alpha \wedge \mathrm{ch}(\widehat{\mathcal{L}}_V)$ shows that

$$\int_{\mathcal{S}_V(\mathbb{C})} d\alpha \wedge \mathrm{ch}(\widehat{\mathcal{L}}_V) = 0.$$

The pullback of α to $\bar{\mathcal{Z}}_V(p)(\mathbb{C})$ is again a pre-log-log form, by Proposition 7.12 of [BKK07], and so the same reasoning shows that

$$\int_{\mathcal{Z}_V(p)(\mathbb{C})} d\alpha = 0.$$

Putting everything together, we find

$$\begin{aligned}\int_{\mathcal{S}_V(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{L}}_V)^{n-1} &= \int_{\mathcal{S}_V(\mathbb{C})} \varphi \wedge \mathrm{ch}(\widehat{\mathcal{L}}_V) \\ &= \frac{1}{k(f)} \sum_{p\in\mathscr{A}} c(-p) \int_{\mathcal{Z}_V(p)} \varphi \\ &= \frac{1}{k(f)} \sum_{p\in\mathscr{A}} c(-p) \int_{\mathcal{Z}_V(p)(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{L}}_V)^{n-2}. \qquad \square\end{aligned}$$

Theorem 8.3.2. *If $n \geq 2$, the complex volumes of* (5.2.3) *and* (8.1.1) *are*

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_V) = \frac{|\mathrm{CL}(\boldsymbol{k})|}{2^{o(D)-1}|\mathcal{O}_{\boldsymbol{k}}^\times|} \cdot \mathbf{A}_V(0)$$

and

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V) = \frac{|\mathrm{CL}(\boldsymbol{k})|}{2^{o(D)-n}|\mathcal{O}_{\boldsymbol{k}}^\times|} \cdot \mathbf{A}_V(0).$$

In particular, both are positive rational numbers.

Proof. The two claims are equivalent, by the equality of Chern forms of Theorem 5.5.1, and both hold when $n = 2$ by Corollary 8.2.2. Hence we assume $n > 2$.

Let V' be the hermitian space over $\boldsymbol{k}$ of signature $(n-2, 1)$ with the same local invariants as V. If $p \in \mathscr{A}$ the Hilbert symbol $(p, -D)_\ell$ is trivial for all finite primes ℓ, and so this V' agrees with (6.1.1). By the first claim of Theorem 6.1.2, and the equality of Chern forms of Theorem 5.5.1,

$$\int_{\mathcal{Z}_V(p)(\mathbb{C})} \mathrm{ch}(\widehat{\mathcal{L}}_V)^{n-2} = (p^{n-1}+1)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_{V'}).$$

Plugging this into the equality of Lemma 8.3.1, and recalling the formula for $k(f)$ from Proposition 7.3.2, we find

$$\begin{aligned}\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_V) &= \frac{1}{k(f)}\Bigg(\sum_{p\in\mathscr{A}} c(-p)(p^{n-1}+1)\Bigg)\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_{V'}) \\ &= \mathbf{b}_{V,n}(0)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_{V'}).\end{aligned}$$

To complete the proof, apply the induction hypothesis

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_{V'}) = \frac{|\mathrm{CL}(\boldsymbol{k})|}{2^{o(D)-1}|\mathcal{O}_{\boldsymbol{k}}^\times|} \cdot \mathbf{A}_{V'}(0),$$

and note that for our particular choice of V' the factorization (7.3.1) implies

$$\begin{aligned}\mathbf{A}_V(s) &= \mathbf{b}_{V,1}(s)\cdots\mathbf{b}_{V,n-1}(s)\mathbf{b}_{V,n}(s)\\ &= \mathbf{b}_{V',1}(s)\cdots\mathbf{b}_{V',n-1}(s)\mathbf{b}_{V,n}(s)\\ &= \mathbf{A}_{V'}(s)\mathbf{b}_{V,n}(s).\end{aligned} \tag{8.3.1}$$

□

8.4. **The key volume calculation.** At last we come to the technical core of this work: the calculation of the arithmetic volume of $\widehat{\mathcal{K}}_V$ via induction.

Theorem 8.4.1. *Assume $n \geq 2$. The hermitian line bundle* (8.1.1) *satisfies*

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) = \left(2\frac{\mathbf{A}'_V(0)}{\mathbf{A}_V(0)} + \log(D)\right)\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V).$$

Proof. Define $C(V)$ by the relation

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) = C(V)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V). \tag{8.4.1}$$

We will prove by induction on n that

$$C(V) = 2\frac{\mathbf{A}'_V(0)}{\mathbf{A}_V(0)} + \log(D). \tag{8.4.2}$$

The base case $n = 2$ is Corollary 8.2.2, so from now on assume $n > 2$.

Once again, abbreviate $\mathscr{A} = \{\text{primes } p \equiv 1 \pmod{D}\}$ and let

$$f(\tau) = \sum_{m \gg -\infty} c(m)q^m \in M^{!,\infty}_{2-n}(D, \varepsilon^n)$$

be as in Theorem 7.1.2. In particular, the Borcherds product $\psi(f)$, a rational section of $\mathcal{L}_V^{\otimes k(f)}$, has nonzero weight $k(f)$.

It follows from (7.1.7) and (7.3.2) that the relations

$$\begin{aligned}k(f)\widehat{\mathcal{L}}_V &= \big(\mathrm{div}(\psi(f)), -\log\|\psi(f)\|^2\big)\\ &= \big(\bar{\mathcal{Z}}_V(f) + \mathrm{Vert}(f), \Phi_V(f)\big)\end{aligned}$$

hold in the codimension one arithmetic Chow group of $\bar{\mathcal{S}}_V$. Recalling (8.1.1), claim (1) of Theorem 5.5.1 implies

$$\widehat{\mathcal{K}}_V \cdot \widehat{\mathcal{K}}_V = 2\widehat{\mathcal{L}}_V \cdot \widehat{\mathcal{K}}_V$$

in the codimension 2 arithmetic Chow group, and therefore

$$k(f)\widehat{\mathcal{K}}_V^n = 2\big(\bar{\mathcal{Z}}_V(f) + \mathrm{Vert}(f), \Phi_V(f)\big) \cdot \widehat{\mathcal{K}}_V^{n-1}$$

holds in the codimension n arithmetic Chow group. Taking the arithmetic degree of both sides and using (4.5.1) yields

$$\begin{aligned}\frac{k(f)}{2}\cdot\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) &= \mathrm{ht}_{\widehat{\mathcal{K}}_V}(\bar{\mathcal{Z}}_V(f)) + \mathrm{ht}_{\widehat{\mathcal{K}}_V}(\mathrm{Vert}(f))\\ &\quad + \int_{\mathcal{S}_V(\mathbb{C})} \Phi_V(f)\,\mathrm{ch}(\widehat{\mathcal{K}}_V)^{n-1}.\end{aligned} \tag{8.4.3}$$

We will compute each term on the right hand side separately.

Consider the finite set

$$\mathscr{A}_f = \{p \in \mathscr{A} : c(-p) \neq 0\}.$$

Lemma 8.4.2. *If V' denotes the $\boldsymbol{k}$-hermitian space of signature $(n-2,1)$ with the same local invariants as V, then*

$$\mathrm{ht}_{\widehat{\mathcal{K}}_V}(\bar{\mathcal{Z}}_V(f)) = \frac{k(f)C(V')}{2}\,\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V),$$

up to a $\mathbb{Q}$-linear combination $\{\log(p) : p \in \mathscr{A}_f\}$.

Proof. For any $p \in \mathscr{A}$, the hermitian space V' satisfies (6.1.1). Thus we may apply Theorem 6.1.2 to see that

$$\begin{aligned}\mathrm{ht}_{\widehat{\mathcal{K}}_V}(\bar{\mathcal{Z}}_V(f)) &= \sum_{p\in\mathscr{A}_f} c(-p)\mathrm{ht}_{\widehat{\mathcal{K}}_V}(\bar{\mathcal{Z}}_V(p)) \\ &= \widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_{V'}) \sum_{p\in\mathscr{A}_f} c(-p)(p^{n-1}+1),\end{aligned}$$

up to a $\mathbb{Q}$-linear combination $\{\log(p) : p \in \mathscr{A}_f\}$. Combining this with Proposition 7.3.2 shows that

$$\begin{aligned}\mathrm{ht}_{\widehat{\mathcal{K}}_V}(\bar{\mathcal{Z}}_V(f)) &= k(f)\mathbf{b}_{V,n}(0)\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_{V'}) \\ &= k(f)\mathbf{b}_{V,n}(0)C(V')\,\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_{V'}),\end{aligned}$$

up to the same ambiguity. Using the factorization $\mathbf{b}_{V,n}(s)\mathbf{A}_{V'}(s) = \mathbf{A}_V(s)$ of (8.3.1), Theorem 8.3.2 implies

$$\mathbf{b}_{V,n}(0)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_{V'}) = \frac{1}{2}\,\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V),$$

completing the proof. □

Lemma 8.4.3. *We have*

$$\mathrm{ht}_{\widehat{\mathcal{K}}_V}(\mathrm{Vert}(f)) = -k(f)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V)\sum_{\ell\mid D}\frac{\log(\ell)}{1+\beta_\ell},$$

where β_ℓ is defined by (7.3.3).

Proof. Combining the second claim of Proposition 7.3.2 with the intersection formula

$$\widehat{\mathcal{K}}_V \cdot (E,0) = 0$$

of Theorem 5.5.1 shows that

$$(\mathrm{Vert}(f),0)\cdot\widehat{\mathcal{K}}_V = -k(f)\sum_{\ell\mid D}\Big(0,\frac{\log(\ell)}{1+\beta_\ell}\Big)\cdot\widehat{\mathcal{K}}_V \in \widehat{\mathrm{CH}}^2(\bar{\mathcal{S}}_V,\mathscr{D}_{\mathrm{pre}})_{\mathbb{Q}}.$$

As in the proof of Lemma 4.5.3, this implies the equality

$$(\mathrm{Vert}(f),0)\cdot\widehat{\mathcal{K}}_V^{n-1} = -k(f)\Big(\sum_{\ell\mid D}\frac{\log(\ell)}{1+\beta_\ell}\Big)(0,\mathrm{ch}(\widehat{\mathcal{K}}_V)^{n-1})$$

in the codimension n arithmetic Chow group. Taking the arithmetic degree of both sides and using (4.5.1) yields the desired equality. □

Lemma 8.4.4. *We have the equality*

$$\int_{\mathcal{S}_V(\mathbb{C})} \Phi_V(f)\,\mathrm{ch}(\widehat{\mathcal{K}}_V)^{n-1} = k(f)\,\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V)\left(\frac{\mathbf{b}'_{V,n}(0)}{\mathbf{b}_{V,n}(0)} + \sum_{\ell \mid D} \frac{\log(\ell)}{1+\beta_\ell}\right) + \mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V) \sum_{p \in \mathscr{A}_f} \frac{c(-p)(p^{n-1}-1)\log(p)}{\mathbf{b}_{V,n}(0)}$$

where β_ℓ is defined by (7.3.3)*, and $\mathbf{b}_{V,n}(s)$ is the function of* (7.3.1)*.*

Proof. Consider the Eisenstein series

$$E_L(\tau, s_0, n) \in M_n(\bar{\omega}_L)$$

of (2.4.4) at $s_0 = (n-1)/2$ associated to the self-dual lattice (7.1.1), and its coefficients $B(m,\mu,s_0)$ defined by (2.5.2). The equality (7.1.5) and the equality of Chern forms of Theorem 5.5.1 imply

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V)^{-1} \int_{\mathcal{S}_V(\mathbb{C})} \Phi_V(f)\,\mathrm{ch}(\widehat{\mathcal{K}}_V)^{n-1} = \sum_{p\in\mathscr{A}_f} c(-p)B'(p,0,s_0) \tag{8.4.4}$$

For any $p \in \mathscr{A}$, Corollaries 2.6.5 and 2.6.6 imply

$$B(p,0,s_0) = -\frac{p^{n-1}+1}{\mathbf{b}_{V,n}(0)}$$

and

$$\frac{B'(p,0,s_0)}{B(p,0,s_0)} = -\frac{\mathbf{b}'_{V,n}(0)}{\mathbf{b}_{V,n}(0)} - \sum_{\ell\mid D}\frac{\log(\ell)}{1+\beta_\ell} + \frac{1-p^{n-1}}{1+p^{n-1}}\log(p).$$

In particular

$$\begin{aligned} B'(p,0,s_0) &= -\frac{p^{n-1}+1}{\mathbf{b}_{V,n}(0)}\frac{B'(p,0,s_0)}{B(p,0,s_0)} \\ &= \frac{p^{n-1}+1}{\mathbf{b}_{V,n}(0)}\left(\frac{\mathbf{b}'_{V,n}(0)}{\mathbf{b}_{V,n}(0)} + \sum_{\ell\mid D}\frac{\log(\ell)}{1+\beta_\ell}\right) + \frac{(p^{n-1}-1)\log(p)}{\mathbf{b}_{V,n}(0)}. \end{aligned}$$

The lemma follows by plugging this last expression into (8.4.4) and using the formula

$$k(f) = \frac{1}{\mathbf{b}_{V,n}(0)} \sum_{p\in\mathscr{A}} c(-p)(p^{n-1}+1)$$

of Proposition 7.3.2. □

Substituting the equalities of Lemmas 8.4.2, 8.4.3, and 8.4.4 into (8.4.3) shows that

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) = e(f) + \left(C(V') + 2\frac{\mathbf{b}'_{V,n}(0)}{\mathbf{b}_{V,n}(0)}\right)\mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{K}}_V), \tag{8.4.5}$$

in which $e(f)$ is a $\mathbb{Q}$-linear combination of $\{\log(p) : p \in \mathscr{A}_f\}$. Here we are using rationality of $\mathbf{b}_{V,n}(0)$, and of the integrals in Theorem 8.3.2. Every term in (8.4.5) except $e(f)$ is a priori independent of the choice of f, and hence so is $e(f)$. Using Theorem 7.1.2, we may choose a second weakly holomorphic form f' as above in such a way that $\mathscr{A}_f \cap \mathscr{A}_{f'} = \emptyset$. The equality $e(f) = e(f')$ and the $\mathbb{Q}$-linear independence of $\{\log(p) : p \in \mathscr{A}_f \cup \mathscr{A}_{f'}\}$ then imply $e(f) = 0$.

Comparing (8.4.1) with (8.4.5) now shows that

$$C(V) = C(V') + 2\frac{\mathbf{b}'_{V,n}(0)}{\mathbf{b}_{V,n}(0)},$$

and the equality (8.4.2) follows from the induction hypothesis

$$C(V') = 2\frac{\mathbf{A}'_{V'}(0)}{\mathbf{A}_{V'}(0)} + \log(D)$$

and the factorization $\mathbf{A}_{V'}(s)\mathbf{b}_{V,n}(s) = \mathbf{A}_V(s)$ of (8.3.1). □

Corollary 8.4.5. *We have the arithmetic volume formulas*

$$\widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}) = \left(2\frac{\mathbf{A}'_V(0)}{\mathbf{A}_V(0)} - nC_0(n) + \log(D)\right) \mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V})$$

and

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{L}}_V) = \left(\frac{\mathbf{A}'_V(0)}{\mathbf{A}_V(0)} + \frac{\log(D)}{2}\right) \mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_V) + \frac{(-1)^{n-1}}{2} \sum_{E \subset \mathrm{Exc}_V} \widehat{m}_E,$$

where the sum is over all connected components $E \subset \mathrm{Exc}_V$ and, if E is supported in characteristic p, $\widehat{m}_E = m_E \log(p)$ is the constant of §5.6.

Proof. The first equality follows from Theorem 8.4.1, using Proposition 8.1.1 and the equality of Chern forms of Theorem 5.5.1.

For the second equality, combine Theorem 8.4.1 with the equality

$$2^n \cdot \widehat{\mathrm{vol}}(\widehat{\mathcal{L}}_V) = \widehat{\mathrm{vol}}(\widehat{\mathcal{K}}_V) + \widehat{\mathrm{vol}}(\mathrm{Exc}_V, 0)$$

of (8.1.2) and the equality of Chern forms of Theorem 5.5.1 to obtain

$$\widehat{\mathrm{vol}}(\widehat{\mathcal{L}}_V) = \left(\frac{\mathbf{A}'_V(0)}{\mathbf{A}_V(0)} + \frac{\log(D)}{2}\right) \mathrm{vol}_{\mathbb{C}}(\widehat{\mathcal{L}}_V) + \frac{\widehat{\mathrm{vol}}(\mathrm{Exc}_V, 0)}{2^n},$$

and then use Corollary 5.6.2. □

8.5. **Volume of the Hodge bundle.** We now compute the complex and arithmetic volumes of the metrized Hodge bundle of the universal abelian scheme

$$A \to \mathcal{M}_{(n-1,1)} \cong \bigsqcup_W \mathcal{M}_W,$$

where, as in (4.1.1), the disjoint union is over the strict similarity classes of relevant $\boldsymbol{k}$-hermitian spaces of signature $(n-1,1)$. Recall that if n is odd there is a unique such class, and if n is even there are $2^{o(D)-1}$ such classes.

Theorem 8.5.1. *Fix $n \geq 1$. For any $\mathcal{M}_W$ in the decomposition above, we have*

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}) = \mathbf{A}_W(0) \times \begin{cases} 2^{n-1} & \text{if } n \text{ is odd} \\ 2^{n-o(D)} & \text{if } n \text{ is even} \end{cases}$$

and

$$\widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}) = \left(2\frac{\mathbf{A}'_W(0)}{\mathbf{A}_W(0)} - nC_0(n) + \log(D) \right) \mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}),$$

where $C_0(n)$ is the constant of Theorem A.

Proof. First assume $n \geq 2$. Let W_0 be any $\boldsymbol{k}$-hermitian space of signature $(1, 0)$, and define a hermitian form on $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$ as in (5.1.2). It is easy to see that V and W lie in the same strict similarity class. In particular, if n is even they are isomorphic. Recalling (1.1.5), we therefore have $\mathbf{A}_W(s) = \mathbf{A}_V(s)$ both for even and odd n.

As in Remark 5.1.3, there is a finite étale surjection $\mathcal{S}_V \to \mathcal{M}_W$ of degree

$$t_n = \frac{|\mathrm{CL}(\boldsymbol{k})|}{|\mathcal{O}_{\boldsymbol{k}}^\times|} \times \begin{cases} 1 & \text{if } n \text{ is even} \\ 2^{1-o(D)} & \text{if } n \text{ is odd.} \end{cases}$$

Denote again by $A \to \mathcal{S}_V$ the pullback of $A \to \mathcal{M}_W$ via this morphism, so that

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}) = \frac{1}{t_n} \mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V})$$

and

$$\widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_W}) = \frac{1}{t_n} \widehat{\mathrm{vol}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{S}_V}).$$

Using these formulas, the theorem for $n > 2$ follows from Theorem 8.3.2 and Corollary 8.4.5, along with the equality of Chern forms of Theorem 5.5.1.

It remains to treat the case $n = 1$, so that

$$\mathcal{M}_W \cong \mathcal{M}_{(0,1)} \cong \mathcal{M}_{(1,0)}.$$

In this case $\mathbf{A}_W(s) = \mathbf{a}_1(s)$. Dirichlet's class number formula implies

$$\mathbf{a}_1(0) = \frac{|\mathrm{CL}(\boldsymbol{k})|}{|\mathcal{O}_{\boldsymbol{k}}^\times|},$$

while (8.2.3) and (5.3.1) imply

$$2\frac{\mathbf{a}'_1(0)}{\mathbf{a}_1(0)} - C_0(1) + \log(D) = -\frac{L'(0,\varepsilon)}{L(0,\varepsilon)} - \frac{\log(D)}{2} = \log(2\pi) + 2h^{\mathrm{Falt}}_{\boldsymbol{k}}.$$

The theory of complex multiplication implies

$$\mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_{(1,0)}}) = \int_{\mathcal{M}_{(1,0)}(\mathbb{C})} 1 = \sum_{x \in \mathcal{M}_{(1,0)}(\mathbb{C})} \frac{1}{|\operatorname{Aut}(x)|} = \frac{|\mathrm{CL}(\boldsymbol{k})|}{|\mathcal{O}_{\boldsymbol{k}}^\times|} = \mathbf{A}_W(0),$$

while the argument of Proposition 5.3.1, which is essentially the Chowla-Selberg formula, implies the first equality in

$$\begin{aligned}\widehat{\deg}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_{(1,0)}}) &= \widehat{\deg}(0, \log(2\pi) + 2h^{\mathrm{Falt}}_{\boldsymbol{k}}) \\ &= (\log(2\pi) + 2h^{\mathrm{Falt}}_{\boldsymbol{k}}) \int_{\mathcal{M}_{(1,0)}(\mathbb{C})} 1 \\ &= \left(2\frac{\mathbf{A}'_W(0)}{\mathbf{A}_W(0)} - C_0(1) + \log(D)\right) \mathrm{vol}_{\mathbb{C}}(\widehat{\omega}^{\mathrm{Hdg}}_{A/\mathcal{M}_{(1,0)}}).\end{aligned}$$

This completes the proof when $n = 1$. □

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