On a theorem of Vignéras

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1 Introduction and statement of results

Let k be an integer, N a positive integer divisible by 4 and χ a Dirichlet character modulo N. Denote the space of modular forms of weight $k+1/2$ with respect to $\Gamma_1(N)$ by $M_{k+\frac{1}{2}}(\Gamma_1(N))$ and the subspace of modular forms with respect to $\Gamma_0(N)$ and Nebentypus character χ by $M_{k+\frac{1}{2}}(N,\chi)$. Further, denote the corresponding subspaces of cusp forms by $S_{k+\frac{1}{2}}(\Gamma_1(N))$ and $S_{k+\frac{1}{2}}(N,\chi)$. We abbreviate $e(z)=e^{2\pi iz}$ and $\chi^*=(\frac{-1}{\cdot})^k\chi$.

If m and r are positive integers and ψ is a primitive Dirichlet character modulo r, then the Shimura theta function $\theta_{\psi,m}(z)=\sum_{n=-\infty}^{\infty}\psi(n)n^\nu e(n^2mz)$ (where $\nu\in\{0,1\}$ is defined by $\psi(-1)=(-1)^\nu$) lies in $M_{\frac{1}{2}}(4r^2m,\left(\frac{m}{\cdot}\right)\psi)$ if $\nu=0$ resp. in $S_{\frac{3}{2}}(4r^2m,\left(\frac{-m}{\cdot}\right)\psi)$ if $\nu=1$ (cf. [Sh]).

Let $S^*_{\frac{3}{2}}(N,\chi)$ denote the orthogonal complement (in $S_{\frac{3}{2}}(N,\chi)$ with respect to the Petersson inner product) of the subspace of $S_{\frac{3}{2}}(N,\chi)$, which is spanned by theta series $\theta_{\psi,m}$ with odd character ψ. By the work of Shimura [Sh], Niwa [Ni], Cipra [Ci] and Sturm [St] it is known that all elements of $S^*_{\frac{3}{2}}(N,\chi)$ and $S_{k+\frac{1}{2}}(N,\chi)$ for $k\geq 2$ map to cusp forms under the Shimura lifting.

Vignéras proved that for every non-zero modular form $f=\sum_{n\geq 0}a(n)e(nz)$ in $M_{k+\frac{1}{2}}(\Gamma_1(N))$, which is not a linear combination of theta series of the above type, there exist infinitely many square-free integers d with $a(dm_d^2)\neq 0$ for an $m_d\in\mathbb{Z}$ (Thm. 3 in [Vi]).

The purpose of the present note is to give a simple new proof (in fact, a slight generalization) of her result.

We shall exploit the properties of various well known operators defined on modular forms to deduce the following fundamental

Lemma 1. *Let $f=\sum_{n\geq 0}a(n)e(nz)$ be a non-zero element of $M_{k+\frac{1}{2}}(N,\chi)$, p a prime not dividing N and $\varepsilon\in\{\pm 1\}$. Suppose that $a(n)=0$ whenever $\left(\frac{n}{p}\right)=-\varepsilon$. Then f is an eigenform of the Hecke operator $T(p^2)$ with corresponding eigenvalue $\varepsilon\chi^*(p)\left(p^k+p^{k-1}\right)$.*

Using a well known estimate for the Hecke eigenvalues, one obtains

Theorem 1. *Let p, ε, be defined as in Lemma 1 and let $f\in M_{k+\frac{1}{2}}(\Gamma_1(N))$ be a non-zero modular form with Fourier coefficients $a(n)$. i) If $k\geq 2$ or $k=1$*

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and $f \in \bigoplus_{\chi \bmod N} S^*_{\frac{3}{2}}(N,\chi)$*, then there exists an* $n \in \mathbb{N}$ *such that* $\left(\frac{n}{p}\right) = \varepsilon$ *and* $a(n) \neq 0$*. ii) Suppose that* $k = 1$ *and that* f *is not a cusp form. Assume that* $p \equiv \varepsilon \pmod{N^2}$*. Then there is an* $n \in \mathbb{N}$ *such that* $\left(\frac{n}{p}\right) = \varepsilon$ *and* $a(n) \neq 0$*.*

By the Serre-Stark theorem [SeSt] every modular form in $M_{\frac{1}{2}}(\Gamma_1(N))$ can be written as a linear combination of suitable theta series $\theta_{\psi,m}$ with even character ψ. Hence, if $f \in M_{k+\frac{1}{2}}(N,\chi,)$ is not a linear combination of Shimura theta series then $k \geq 1$ and the above corollary in particular implies the result of Vignéras.

Finally we consider a Hecke eigenform f. Here we can use the multiplicative properties of the coefficients combined with an inductive argument to find

Theorem 2. *Let* $f = \sum_{n \geq 0} a(n)e(nz)$ *be an element of* $S^*_{\frac{3}{2}}(N,\chi)$ *or an element of* $M_{k+\frac{1}{2}}(N,\chi)$ *with* $k \geq 2$*. Suppose that* f *is a common eigenform of all Hecke operators* $T(q^2)$*. Let* $p_1, \dots, p_r$ *be distinct primes not dividing* N *and* $\varepsilon_1, \dots, \varepsilon_r \in \{\pm 1\}$*. Then there exist infinitely many square-free integers* d *with* $a(d) \neq 0$ *and* $\left(\frac{d}{p_j}\right) = \varepsilon_j$ *for* $j = 1, \dots, r$*.*

By the work of Waldspurger [Wa], Kohnen and Zagier [KoZa, Koh2], Theorem 2 implies a non-vanishing result for the central critical values of twisted L-series attached to newforms of weight $2k$. However, this also easily follows from the more general theorem of Friedberg and Hoffstein [FrHo] (see also [BFH], [MuMu] for related results).

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2 Proofs

Let f be a modular form in $M_{k+\frac{1}{2}}(N,\chi)$ with Fourier coefficients $a(n)$. We will need the following well known operators (cf. [Sh, SeSt]):

(i) The Fricke involution W_N: $(f \mid W_N)(z) := N^{-k/2-1/4}(-iz)^{-k-1/2} f(\frac{-1}{Nz})$, $f \mid W_N \in M_{k+\frac{1}{2}}(N, \left(\frac{N}{\cdot}\right)\bar{\chi})$.

(ii) The shift V_m ($m \in \mathbb{N}$): $(f \mid V_m)(z) := f(mz)$, $f \mid V_m \in M_{k+\frac{1}{2}}(Nm, \left(\frac{m}{\cdot}\right)\chi)$.

(iii) The projection B_m ($m \in \mathbb{N}$): $(f \mid B_m)(z) := \sum_{n \geq 0} a(nm)e(nmz)$, $f \mid B_m \in M_{k+\frac{1}{2}}(Nm^2, \chi)$.

(iv) The twist with a Dirichlet character ψ modulo m:

$$f_\psi(z) := \sum_{n \geq 0} \psi(n)a(n)e(nz), \qquad f_\psi \in M_{k+\frac{1}{2}}(Nm^2, \chi\psi^2).$$

All these operators are linear and take cusp forms to cusp forms. Moreover, they can be defined as operation of certain elements of the group algebra $\mathbb{C}[G]$ of the metaplectic covering G of $GL_2^+(\mathbb{R})$. Therefore one can easily determine their commutation relations. For a prime p not dividing N we will in particular use the identity

$$f_\varphi \,|\, W_{Np^2} = \chi^*(p) \left(p^{1/2} f \,|\, W_N \,|\, B_p - p^{-1/2} f \,|\, W_N \right), \tag{1}$$

where φ denotes the primitive Dirichlet character defined by $\varphi(x) = \left(\frac{x}{p}\right)$ (cf. [Sh] §5).

Proof of Lemma 1. Since f can be considered as an element of $M_{k+\frac{1}{2}}(N^2,\chi)$, we may assume that N is a square. Put $\varphi = \left(\frac{\cdot}{p}\right)$ and $g = f \,|\, W_N \in M_{k+\frac{1}{2}}(N,\bar{\chi})$. By the assumption on f we have

$$f \,|\, B_p = f - \varepsilon f_\varphi.$$

We consider the twist of g with φ and use (1):

$$\begin{aligned} g_\varphi &= \bar{\chi}^*(p) \left(p^{1/2} f \,|\, B_p - p^{-1/2} f \right) \,|\, W_{Np^2} \\ &= \bar{\chi}^*(p) \left(p^{1/2} (f - \varepsilon f_\varphi) - p^{-1/2} f \right) \,|\, W_{Np^2} \\ &= \bar{\chi}^*(p) \left(p^{1/2} - p^{-1/2} \right) f \,|\, W_{Np^2} - \varepsilon \bar{\chi}^*(p) p^{1/2} f_\varphi \,|\, W_{Np^2} \end{aligned}$$

Using $W_{Np^2} = p^{k+1/2} W_N V_{p^2}$ and (1) we find:

$$g_\varphi = \bar{\chi}^*(p) \left(p^{k+1} - p^k \right) g \,|\, V_{p^2} + \varepsilon g - \varepsilon p g \,|\, B_p$$

If we denote the Fourier coefficients of g by $b(n)$, we get an identity of power series

$$\begin{aligned} \sum_{n\geq 0} \left(\frac{n}{p}\right) b(n) e(nz) &= \bar{\chi}^*(p) \left(p^{k+1} - p^k \right) \sum_{n \geq 0} b(n/p^2) e(nz) \\ &\quad - \varepsilon (p-1) \sum_{n\geq 0} b(n) e(nz) + \varepsilon p \sum_{\substack{n \geq 0 \\ (n,p)=1}} b(n) e(nz). \end{aligned} \tag{2}$$

Comparing Fourier coefficients we obtain

$$b(n) = \begin{cases} \varepsilon \left(\frac{n}{p}\right) b(n) & \text{if } (p,n) = 1, \\ 0 & \text{if } (p^2,n) = p, \\ \varepsilon \bar{\chi}^*(p) p^k b(n/p^2) & \text{if } p^2 | n. \end{cases} \tag{3}$$

Now put $g \,|\, T(p^2) = \sum_{n \geq 0} c(n) e(nz)$. It is known (see [Sh]) that

$$c(n) = b(p^2 n) + \bar{\chi}^*(p) \left(\frac{n}{p}\right) p^{k-1} b(n) + \bar{\chi}^*(p^2) p^{2k-1} b(n/p^2),$$

and by (3) we find

$$c(n) = \varepsilon \bar{\chi}^*(p) \left(p^k + p^{k-1} \right) b(n).$$

This shows that $g \,|\, T(p^2) = \varepsilon \bar{\chi}^*(p) \left(p^k + p^{k-1} \right) g$. By (3) we have $b(n) = 0$ for all n with $\left(\frac{n}{p}\right) = -\varepsilon$. Thus we may apply the same argument with f replaced by g to prove the assertion. □

Proof of Theorem 1. Let $k \geq 1$ and assume that $f \in M_{k+\frac{1}{2}}(\Gamma_1(N))$ is a non-zero modular form with coefficients $a(n)$ such that $a(n) = 0$ whenever $\left(\frac{n}{p}\right) = \varepsilon$. Further assume that $p \equiv \varepsilon \pmod{N^2}$ if $k = 1$. First, we claim that f is a cusp form. Since

$$M_{k+\frac{1}{2}}(\Gamma_1(N)) = \bigoplus_{\chi \bmod N} M_{k+\frac{1}{2}}(N, \chi),$$

f can be uniquely written as a linear combination of modular forms with respect to $\Gamma_0(N)$ with Nebentypus character, and one easily sees that each form in this decomposition also has the property that its n-th coefficient vanishes if $\left(\frac{n}{p}\right) = \varepsilon$. Thus we may assume that $f \in M_{k+\frac{1}{2}}(N, \chi)$. (Recall that $f \neq 0$ implies that χ is even.)

According to Lemma 1 one has $f \,|\, T(p^2) = -\varepsilon\chi^*(p)\left(p^k + p^{k-1}\right)$. Comparing the constant terms we find

$$a(0)\left(\varepsilon\chi^*(p)p^k + 1\right)\left(\varepsilon\chi^*(p)p^{k-1} + 1\right) = 0.$$

Hence, one immediately deduces $a(0) = 0$ (note that $p \equiv \varepsilon \pmod{N^2}$ if $k = 1$), i.e. f vanishes at the cusp ∞.

The proof of Lemma 1 (equation (3)) shows that $f \,|\, W_N$ also has the property that its n-th coefficient $b(n)$ vanishes if $\left(\frac{n}{p}\right) = \varepsilon$. Thus we may apply the same argument to infer $b(0) = 0$, i.e. f vanishes at the cusp 0.

If $u \in \mathbb{Z}$ then $f(z + u/N)$ is a modular form in $M_{k+\frac{1}{2}}(\Gamma_1(N^2))$. The n-th coefficient of $f(z + u/N)$ obviously also vanishes if $\left(\frac{n}{p}\right) = \varepsilon$, and the above argument shows that $f(z + u/N)$ is zero at the cusp 0. Hence, f vanishes at u/N. This proves the claim, since all cusps of $\Gamma_0(N)$ are known to be equivalent to a cusp of the form v/N. In particular this proves (ii).

For the proof of (i) we may now assume that $f \in S^*_{\frac{3}{2}}(N, \chi)$ or $k \geq 2$ and $f \in S_{k+\frac{1}{2}}(N, \chi)$. In view of Lemma 1 it suffices to show that the eigenvalues λ_p of the restriction of $T(p^2)$ on these spaces satisfy $|\lambda_p| < p^k + p^{k-1}$.

By the work of Shimura, Niwa, Cipra and Sturm, via Shimura-lifting λ_p is also an eigenvalue of the Hecke operator $T(p)$ on $S_{2k}(N/2, \chi^2)$. Here, we may for instance apply a simple argument due to Kohnen [Koh1] to infer the desired estimate. □

Proof of Theorem 2. We use induction on r. Let $r = 0$ and $\mathcal{D}$ be the set of square-free d with $a(d) \neq 0$. Suppose that $\mathcal{D}$ is finite. Then, using the multiplicative properties of the coefficients, one deduces that $a(dm^2) = 0$ for all square-free d which are not in $\mathcal{D}$ and all $m \in \mathbb{N}$. Let p_0 be a prime with $(p_0, N) = 1$ and $\left(\frac{d}{p_0}\right) = 1$ for all $d \in \mathcal{D}$ (such a p_0 clearely exists by Dirichlets theorem on arithmetic progressions). Then we have $a(m) = 0$ for all $m \in \mathbb{N}$ with $\left(\frac{m}{p_0}\right) = -1$, a contradiction to Theorem 1.

Now, let $r \geq 1$, $\varphi_r = \left(\frac{\cdot}{p_r}\right)$, and consider

$$h := f - f \,|\, B_{p_r} + \varepsilon_r f_{\varphi_r} = 2 \sum_{\substack{n \geq 0 \\ \varphi_r(n) = \varepsilon_r}} a(n)e(nz).$$

According to Theorem 1, h is a non-zero element of $M_{k+\frac{1}{2}}(Np_r^2,\chi)$ for $k\geq 2$ resp. $S^*_{\frac{3}{2}}(Np_r^2,\chi)$ for $k=1$. Moreover, h is an eigenform of all $T(q^2)$. Hence, the assertion follows by induction. $\square$

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