ON TWO GEOMETRIC THETA LIFTS

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ABSTRACT. The theta correspondence has been an important tool in studying cycles in locally symmetric spaces of orthogonal type. In this paper, we establish for the orthogonal group $O(p, 2)$ an adjointness result between Borcherds' singular theta lift and the Kudla-Millson lift. We extend this result to arbitrary signature by introducing a new singular theta lift for $O(p, q)$. On the geometric side, this lift can be interpreted as a differential character for the cycles under consideration in the sense of Cheeger and Simons.

1. INTRODUCTION

Borcherds [Bo1] introduced a "singular" theta lift from modular forms of (typically) negative weight for $SL_2(\mathbb{R})$ to the orthogonal group $O(p, 2)$. This gave rise to remarkable product expansions of automorphic forms for $O(p, 2)$, but also to the realization of generating series of certain special divisors in locally symmetric spaces attached to $O(p, 2)$ as holomorphic modular forms of positive weight [Bo2]. The Borcherds lift was generalized by the first named author and studied in connection with the cohomology of the special divisors [Br1, Br2].

On the other hand, the Weil representation and theta series have been used in a more classical way by several people (e.g. [Sh, Od, RS, TW, W]) to show that generating series of special cycles in orthogonal and unitary locally symmetric spaces of arbitrary signature are modular forms. In greatest generality this was done by Kudla and Millson (see e.g. [KM3]).

The purpose of this paper is threefold:

(1) For $O(p, 2)$, we derive an adjointness result between the Kudla-Millson lift and the Borcherds lift.
(2) We also introduce a new singular theta lift of Borcherds' type for $O(p, q)$ and obtain a similar relationship to the Kudla-Millson lift.
(3) As a geometric application, we show that the exterior derivative of the current induced by this singular theta lift is closely related to the delta current of a special cycle.

We sketch the main results of the paper in more detail.

Let V be a quadratic space over $\mathbb{Q}$ of signature (p, q) and write D for the associated symmetric space. Let L be an even lattice in V, let Γ be a finite index subgroup of the group of units of L, and let $X = \Gamma \backslash D$ be the associated locally symmetric space. The

special cycles arise from subsymmetric spaces of codimension q induced by embeddings of orthogonal groups of signature $(p-1, q)$ into $\mathrm{O}(V)$.

To simplify the exposition we assume for now that L is unimodular. In the main body of the paper, we treat the case of arbitrary level by using the setting of Borcherds' vector-valued modular forms. Moreover, for q odd, this is essential in order to obtain a nonzero theory.

We first introduce a new space of automorphic forms of (typically negative) weight k. Its importance lies in the fact that on one hand we will systematically treat this space as the input space for both the Borcherds lift for $\mathrm{O}(p, 2)$ and its extension to $\mathrm{O}(p, q)$, while on the other hand it satisfies a duality with the space of holomorphic cusp forms of (positive) weight $2 - k$. This duality, interesting in its own right, will be crucial for us.

Namely, we let H_k be the space of *weak Maass forms*. This space consists of the smooth functions f on the upper half plane $\mathbb{H}$ that transform with weight $k = 2 - (p + q)/2$ under $\mathrm{SL}_2(\mathbb{Z})$, are annihilated by the weight k Laplacian, and satisfy $f(\tau) = O(e^{Cv})$ as $\tau = u + iv \to i\infty$ for some constant $C > 0$. For $f \in H_k$, put $\xi_k(f) = R_{-k}(v^k \bar{f})$, where R_{-k} is the standard raising operator for modular forms of weight $-k$. We prove that ξ_k defines an antilinear map $\xi_k : H_k \to M_{2-k}^!$, where $M_{2-k}^!$ is the space of meromorphic modular forms of weight $2 - k$ that are holomorphic on $\mathbb{H}$ (see Proposition 3.2). It is easily checked that $M_k^!$ is the kernel of ξ_k. We let H_k^+ be the preimage of S_{2-k}, the space of cusp forms of weight $2 - k$.

Theorem 1.1. *The bilinear pairing between S_{2-k} and H_k^+, defined by*

$$\{g, f\} = (g, \xi_k(f))_{2-k},$$

for $g \in S_{2-k}$ and $f \in H_k^+$, induces a non-degenerate pairing of S_{2-k} and $H_k^+/M_k^!$. Here $(\, , \,)_{2-k}$ is the Petersson scalar product for modular forms of weight $2 - k$.

The main point here is to show that the map ξ_k is surjective. The proof uses methods from complex geometry and is ultimately based on Serre duality. The pairing $\{g, f\}$ can be explicitly evaluated in terms of the Fourier coefficients of g and the singular part of f (Proposition 3.5). Note that Borcherds [Bo2] established a similar duality statement (in terms of formal power series).

For signature $(p, 2)$, when D is Hermitian, we then introduce the Borcherds lift as a map on H_k^+; i.e, for any $f \in H_k^+$, its lift is given by integrating $f \in H_k^+$ against $\Theta(\tau, z, \varphi_0)$, $(\tau = u + iv \in \mathbb{H}, z \in D)$, the Siegel theta series attached to the standard Gaussian φ_0 on $V(\mathbb{R})$ (see also [Bo1], [Br1]):

$$\Phi(z, f) = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}}^{reg} f(\tau) \Theta(\tau, z, \varphi_0) \frac{du\, dv}{v^2}. \tag{1.1}$$

The integral is usually divergent, and a suitable regularization was found by Harvey and Moore [HM]. The regularization process leads to logarithmic singularities along certain special cycles $Z(f)$ in D that in this case are divisors. Moreover, $\Lambda_B(z, f) := dd^c \Phi(z, f)$ actually extends to a closed smooth $(1, 1)$-form on X. Here d and d^c are the standard (exterior) differential operators on D. Hence we have a map

$$\Lambda_B : H_k^+ \longrightarrow \mathcal{Z}^2(X) \tag{1.2}$$

from H_k^+ to the space of closed 2-forms on X. For $f \in M_k^!$ one has $\Lambda_B(f) = a^+(0)\Omega$, where $a^+(0)$ is the constant coefficient of f, and Ω is the suitably normalized Kähler form on D.

On the other hand, Kudla and Millson [KM3] construct for general signature (p,q) a theta series $\Theta(\tau, z, \varphi_{KM})$ associated to a certain Schwartz function φ_{KM} on $V(\mathbb{R})$ taking values in $\mathcal{Z}^q(X)$. Then for $\eta \in \mathcal{Z}_c^{(p-1)q}(X)$, the compactly supported closed $(p-1)q$-forms on X, the Kudla-Millson lift is defined by

$$\Lambda_{KM}(\tau, \eta) = \int_X \eta \wedge \Theta(\tau, z, \varphi_{KM}).$$

It turns out that $\Lambda_{KM}(\tau, \eta)$ is actually a holomorphic modular form of weight $2 - k$, so that we have a map

$$(1.3) \qquad \Lambda_{KM} : \mathcal{Z}_c^{(p-1)q}(X) \longrightarrow M_{2-k},$$

which also factors through cohomology. Moreover, the Fourier coefficients of Λ_{KM} are given by periods of η over the special cycles.

Theorem 1.2. *Assume D be Hermitian, i.e., $q = 2$, and let $f \in H_k^+$ with constant coefficient $a^+(0)$. We then have the following identity of closed 2-forms on X:*

$$\Lambda_B(z, f) = (\Theta(\tau, z, \varphi_{KM}), \xi_k(f))_{2-k} + a^+(0)\Omega.$$

Therefore the maps Λ_B and Λ_{KM} are naturally adjoint via the standard pairing $(,)_X$ of $\mathcal{Z}_c^{2p-2}(X)$ with $\mathcal{Z}^2(X)$ and the pairing $\{ , \}$ of M_{2-k} with H_k^+, that is,

$$\left(\eta, \Lambda_B(f)\right)_X = \{\Lambda_{KM}(\eta), f\} + a^+(0)\left(\eta, \Omega\right)_X.$$

Furthermore, this duality factors through cohomology, and $H_k^+/M_k^!$, respectively. (See also Theorem 6.3.)

This result is based on the fundamental relationship between the two theta series involved:

Theorem 1.3. *Let L_{2-k} be the lowering Maass operator of weight $2 - k$ on $\mathbb{H}$. Then*

$$(1.4) \qquad L_{2-k}\Theta(\tau, z, \varphi_{KM}) = -dd^c\Theta(\tau, z, \varphi_0).$$

We show this by switching to the Fock model of the Weil representation. Then the idea for the proof of Theorem 1.2 is given by the following formal (!) calculation:

$$(\Theta(\varphi_{KM}), \xi_k(f))_{2-k} \text{ "=" } -\left(L_{2-k}\Theta(\varphi_{KM}), v^k \bar{f}\right)_{-k}^{reg}$$

$$\text{ "=" } \left(dd^c\Theta(\varphi_0), v^k \bar{f}\right)_{-k}^{reg} \text{ "=" } dd^c \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}}^{reg} \Theta(\varphi_0) f \frac{du\,dv}{v^2}.$$

The first equality would follow from the adjointness of the raising and lowering operators, except that the second scalar product no longer converges and needs to be regularized. At this point, one also obtains an error term involving the Kähler form. The second equality follows from the key fact provided by Theorem 1.3, and the last equality by interchanging the integration and differentiation (which also needs careful consideration). Additional difficulties occur in the intermediate steps, where one must deal with singularities along the special divisors.

As an application, we are able to recover several of the geometric properties of the Borcherds lift in [Br1], often with simpler proofs.

For general signature (p, q), the work of Kudla and Millson [KM3] shows that there exists a theta function $\Theta(\tau, z, \psi)$ taking values in the $(q-1)$-forms of X such that

$$L_{2-k}\Theta(\tau, z, \varphi_{KM}) = d\Theta(\tau, z, \psi).$$

This is the exact analogue of the identity (1.4) in the Hermitian case. Hence for $f \in H_k^+$ we define a singular theta lift $\Phi(z, f, \psi)$ by replacing $\Theta(\tau, z, \varphi_0)$ with $\Theta(\tau, z, \psi)$ in (1.1). Note that the image is no longer scalar-valued but instead is a $(q-1)$-form with higher order singularities along the special cycle $Z(f)$ (see Proposition 5.6). The point is that although Borcherds introduces his lift for arbitrary signature and for general (scalar-valued) theta kernels, he focuses on the Hermitian case. In particular, the geometric interpretation given in [Bo2, Br1] only applies to this case. Hence from this aspect the lift $\Phi(z, f, \psi)$ and its features are new. We put $\Lambda_\psi(z, f) = -d\Phi(z, f, \psi)$ and one sees that this again extends to a closed smooth q-form on M. Moreover, it essentially vanishes for $f \in M_k^!$. One obtains as above (for a precise statement, see Theorem 6.3):

Theorem 1.4. *In the case of signature (p, q), the lifts Λ_{KM} and Λ_ψ are adjoint.*

This construction of a singular theta lift for general signature does indeed lead to new information:

Theorem 1.5. *Let $f \in H_k^+$. Then $\Lambda_\psi(z, f)$ is a harmonic representative of the Poincaré dual class of the cycle $Z(f)$. Moreover, we have the following equality of currents*

$$d[\Phi(z, f, \psi)] + \delta_{Z(f)} = [\Lambda_\psi(z, f)].$$

In particular, the pair $(Z(f), \Phi(z, f, \psi))$ defines a differential character in the sense of Cheeger and Simons (see [C, CS]). For $q = 2$, we have in addition for the Borcherds lift

$$dd^c[\Phi(z, f)] + \delta_{Z(f)} = [\Lambda_B(z, f)],$$

i.e., $\Phi(z, f)$ is a Green's function for the divisor $Z(f)$.

For $q = 2$ the latter result already follows from [Br1, Br2]. We also briefly discuss the relationship of $\Phi(z, f)$ to the Green's functions for the special divisors constructed by Oda and Tsuzuki [OT] and Kudla [Ku1, Ku2].

The results of this paper are subject to several extensions and generalizations. On one hand, one should be able to introduce suitable singular theta lifts for unitary groups, since [KM3] covers these groups as well. This is of particular interest, since in this case the symmetric spaces are Hermitian. In particular, such a lift should lead to the explicit construction of Green's currents for cycles on these spaces of complex codimension q. On the other hand, Funke and Millson [FM1] are currently developing a theory for special cycles with coefficients analogous to the Kudla-Millson lift. This also should give rise to new singular theta lifts with geometric importance. On a more speculative note, it is a very interesting problem to generalize the present results to special cycles of higher codimension. In the framework of this paper, one would need to define a suitable singular theta lift for the symplectic group $\mathrm{Sp}_n(\mathbb{R})$, and for this one would need analogues of the space H_k^+ and of Theorem 1.1. We hope to come back to these issues in the near future.

The paper is organized as follows. After setting up the basic notions in section 2, we discuss in section 3 the space of weak Maass forms H_k in detail and prove Theorem 1.1. In section 4, we consider the Fock model of the Weil representation and the Schwartz forms φ_{KM} and ψ in detail and derive Theorem 1.3. We discuss the Kudla-Millson lift and the Borcherds lift in section 5, introducing for general signature the lift $\Phi(z, f, \psi)$. Two of our main results, Theorems 1.2 and 1.4, are proved in section 6. Finally, Theorem 1.5 is considered in section 7.

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2. Basic notations

Let V be a rational vector space over $\mathbb{Q}$ with a non-degenerate bilinear form $(\,,\,)$ of signature (p, q); we assume $\dim V \geq 3$. Let L be an even lattice in V (that is $q(x) := \frac{1}{2}(x, x) \in \mathbb{Z}$ for $x \in L$) and write $L^{\#}$ for its dual. We will write V^- (respectively L^-) for the vector space V ($\mathbb{Z}$-module L) together with the bilinear form $-(\,,\,)$.

We pick an orthogonal basis $\{v_i\}$ of $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$ such that $(v_\alpha, v_\alpha) = 1$ for $\alpha = 1, \dots, p$ and $(v_\mu, v_\mu) = -1$ for $\mu = p + 1, \dots, p + q$. We denote the corresponding coordinates by x_i. Throughout the paper we will use the subscript α for the 'positive' variables and μ for the 'negatives' ones. We realize the symmetric space associated to V as the set of negative q-planes in $V(\mathbb{R})$:

$$D \simeq \{z \subset V(\mathbb{R}); \ \dim z = q \quad \text{and} \quad (\,,\,)|_z < 0\}.$$

Occasionally, we will write $D_{p,q}$ to emphasize the signature. The assignment $z \mapsto z^\perp$ gives the identification $D_{p,q} \simeq D_{q,p}$.

We have $D \simeq G/K$ with $G = \mathrm{SO}_0(V(\mathbb{R}))$, the connected component of the orthogonal group, and K the maximal compact subgroup of G stabilizing $z_0 = \mathrm{span}\{v_\mu; \ p + 1 \leq \mu \leq p + q\}$.

For $z \in D$, we associate the standard majorant $(\,,\,)_z$ given by

$$(x, x)_z = (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z),$$

where $x = x_z + x_{z^\perp} \in V(\mathbb{R})$ is given by the orthogonal decomposition $V(\mathbb{R}) = z \oplus z^\perp$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ its Cartan decomposition. Then $\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{k}$ is isomorphic to the tangent space at the base point of D, and with respect to the above basis of $V(\mathbb{R})$ we have

$$(2.1) \qquad \mathfrak{p} \simeq \left\{ \begin{pmatrix} 0 & X \\ {}^t X & 0 \end{pmatrix}; \ X \in M_{p,q}(\mathbb{R}) \right\} \simeq M_{p,q}(\mathbb{R}).$$

For $q = 2$, it is well known that D is Hermitian, and we assume that the complex structure on $\mathfrak{p}$ is given by right multiplication with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let $\Gamma \subset G$ be a congruence subgroup of the orthogonal group of L fixing the discriminant lattice $L^{\#}/L$ and let $X = \Gamma \backslash D$ be the associated locally symmetric space. Throughout we assume that Γ is torsion free such that Γ acts freely on D, and X is a real analytic manifold of dimension pq. Observe that for instance the principal congruence subgroup $\Gamma(N)$ of level N (of the discriminant kernel of $\mathrm{O}(L)$), i.e., the kernel of the natural homomorphism $\mathrm{O}(L) \to \mathrm{O}\left((\frac{1}{N}L^{\#})/L\right)$, is torsion free for $N \geq 3$ (which is seen similarly as Hilfssatz 6.5 in [Fr] chapter II.6).

For $x \in V(\mathbb{R})$ with $q(x) > 0$, we let

$$D_x = \{z \in D;\ z \perp x\}.$$

Note that D_x is a subsymmetric space of type $D_{p-1,q}$ attached to the orthogonal group G_x, the stabilizer of x in G. Put $\Gamma_x = \Gamma \cap G_x$. Then for $x \in L^{\#}$, the quotient

$$Z(x) = \Gamma_x \backslash D_x \longrightarrow X$$

defines a (in general relative) cycle in X. For $h \in L^{\#}/L$ and $n \in \mathbb{Q}$, the group Γ acts on $L_{h,n} = \{x \in L + h;\ q(x) = n\}$ with finitely many orbits, and we define the composite cycle

$$Z(h,n) = \sum_{x \in \Gamma \backslash L_{h,n}} Z(x).$$

Occasionally, we will identify $Z(h,n)$ with its preimage in D.

Borcherds [Bo2] and Bruinier [Br1] (for $\mathrm{O}(2,p)$) use vectors of *negative* length to define special cycles (which are divisors in that case); by switching to the space V^- these are the same as the divisors for $D_{p,2}$ defined above by vectors of *positive* length.

We orient D and the cycles D_x as in [KM3], page 130/131. Note that for q even, D_x and D_{-x} have the same orientation while for q odd, the opposite. Moreover, for $q = 2$, this orientation coincides with the orientation given by the complex structure on D and D_x.

Let $G' = \mathrm{Mp}_2(\mathbb{R})$ be the two-fold cover of $\mathrm{SL}_2(\mathbb{R})$, realized by the two choices of holomorphic square roots of $\tau \mapsto j(g,\tau) = c\tau + d$; here $\tau \in \mathbb{H} = \{w \in \mathbb{C};\ \Im(w) > 0\}$, the upper half plane and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Therefore elements of $\mathrm{Mp}_2(\mathbb{R})$ are of the form $(g, \phi(\tau))$ with $g \in \mathrm{SL}_2(\mathbb{R})$ and $\phi(\tau)$ a holomorphic function such that $\phi(\tau)^2 = j(g,\tau)$. The multiplication is given by $(g_1, \phi_1(\tau))(g_2, \phi_2(\tau)) = (g_1 g_2, \phi_1(g_2\tau)\phi_2(\tau))$, where $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by linear fractional transformations. For $z = re^{i\theta} \in \mathbb{C}^\times$ with $\theta \in (-\pi, \pi]$ and r positive, we take $\sqrt{z} = z^{1/2} = r^{1/2}e^{i\theta/2}$. Occasionally we just write g for $(g, \sqrt{j(g,\tau)}) \in G'$.

We write K' for the inverse image of $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \longrightarrow \mathrm{SL}_2(\mathbb{R})$. Note that for $k_\theta \in \mathrm{SO}(2)$ with $k_\theta = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$, $(\theta \in (-\pi, \pi])$, we obtain a character of K' by the assignment

$$\chi_{1/2} : (k_\theta, \pm\sqrt{j(k_\theta, \tau)}) \mapsto \pm\sqrt{j(k_\theta, i)}^{-1} = \pm e^{i\theta/2}.$$

We denote by $\omega = \omega_V$ the Schrödinger model of the (restriction of the) Weil representation of $G' \times \mathrm{O}(V(\mathbb{R}))$ acting on $\mathcal{S}(V(\mathbb{R}))$, the space of Schwartz functions on $V(\mathbb{R})$. We have

$$\omega(g)\varphi(x) = \varphi(g^{-1}x)$$

for $\varphi \in \mathcal{S}(V(\mathbb{R}))$ and $g \in \mathrm{O}(V(\mathbb{R}))$. The action of G' is given as follows:

$$\omega(m(a))\varphi(x) = a^{(p+q)/2}\varphi(ax)$$

for $a > 0$ and with $m(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$;

$$\omega(n(b))\varphi(x) = e^{\pi i b(x,x)}\varphi(x)$$

with $n(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$; and

$$\omega(S)\varphi(x) = \sqrt{i}^{p-q}\hat{\varphi}(-x)$$

with $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$, and where $\hat{\varphi}(y) = \int_{V(\mathbb{R})} \varphi(x)e^{2\pi i(x,y)}dx$ is the Fourier transform. If $p + q$ is even, $(1, t) \in G'$ acts trivially, otherwise by multiplication by t.

For $\varphi \in \mathcal{S}(V(\mathbb{R}))$ and $h \in L^{\#}/L$, we define the theta function

$$\theta(g', \varphi, h) = \sum_{\lambda \in L+h} \omega(g')\varphi(\lambda) \qquad (g' \in G').$$

We write Γ' for the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_2(\mathbb{R})$. For the generators S and $T = (n(1), 1)$ of Γ', we then have

$$(2.2) \qquad \theta(Tg', \varphi, h) = e^{\pi i(h,h)}\theta(g', \varphi, h),$$

and by Poisson summation

$$(2.3) \qquad \theta(Sg', \varphi, h) = \frac{\sqrt{i}^{p-q}}{\sqrt{|L^{\#}/L|}} \sum_{h' \in L^{\#}/L} e^{-2\pi i(h,h')}\theta(g', \varphi, h').$$

The equations (2.2) and (2.3) define a representation ϱ_L of Γ' acting on the group algebra $\mathbb{C}[L^{\#}/L]$, whose standard basis elements we denote by $\mathfrak{e}_h$, $h \in L^{\#}/L$. We introduce the vector

$$\Theta(g', \varphi, L) := (\theta(g', \varphi, h))_{h \in L^{\#}/L} = \sum_{h \in L^{\#}/L} \theta(g', \varphi, h)\mathfrak{e}_h$$

and see

$$\Theta(\gamma g', \varphi, L) = \varrho_L(\gamma)\Theta(g', \varphi, L)$$

for all $\gamma \in \Gamma'$. Note that with respect to the standard scalar product $\langle \, , \, \rangle$ on $\mathbb{C}[L^{\#}/L]$ (linear in the first and anti-linear in the second variable), we have that $\varrho_{L^-} = \varrho_L^* = \bar{\varrho}_L$; i.e., L and L^- give rise to dual representations. From (2.2) and (2.3) it is clear that ϱ_L coincides with the representation ϱ_L considered in [Bo1] and [Br1]. Ultimately, ϱ_L goes back to Shintani [Sh], which is also a good reference for the above discussion.

Let φ now be an eigenfunction under the action of K'; i.e., $\omega(k_\theta)\varphi = \chi_{1/2}^r(k_\theta)\varphi$ for some $r \in \mathbb{Z}$, so that

$$\Theta(g'k_\theta, \varphi, L) = \chi_{1/2}^r(k_\theta)\Theta(g', \varphi, L).$$

Then we can associate to $\Theta(g', \varphi, L)$ a (vector-valued) function on the upper half plane in the usual way: We let $g'_\tau = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{smallmatrix}\right)$ with $\tau = u + iv \in \mathbb{H}$ be the standard element moving the base point $i \in \mathbb{H}$ to τ and define

$$\Theta(\tau, \varphi, L) := j(g'_\tau, i)^{r/2}\Theta(g'_\tau, \varphi, L) = \sum_{h \in L^{\#}/L} \sum_{\lambda \in L+h} \varphi(\lambda, \tau, z)\mathfrak{e}_h.$$

with

$$\varphi(\lambda, \tau, z) = j(g'_\tau, i)^{r/2} \omega(g'_\tau) \varphi(\lambda).$$

Hence, for $(\gamma, \phi) \in \Gamma'$,

(2.4) $$\Theta(\gamma\tau, \varphi, L) = \phi(\tau)^r \varrho_L(\gamma, \phi) \Theta(\tau, \varphi, L);$$

i.e., $\Theta(\tau, \varphi, L)$ is a C^∞-automorphic form of weight $r/2$ with respect to the representation ϱ_L. (Note however, that it is usually not an eigenfunction of the Laplacian).

We denote the real analytic functions on $\mathbb{H}$ satisfying the transformation property (2.4) with weight k by $A_{k,L}$.

From now on we will frequently drop the lattice L from the argument of $\Theta(g', \varphi, L)$.

The space of K-invariant Schwartz functions $\mathcal{S}(V(\mathbb{R}))^K$ is of particular interest as we have

$$\mathcal{S}(V(\mathbb{R}))^K \simeq [\mathcal{S}(V(\mathbb{R})) \otimes C^\infty(D)]^G,$$

where the isomorphism is giving by evaluation at the base point z_0 of D. Note that for such $\varphi(x, z) \in [\mathcal{S}(V(\mathbb{R})) \otimes C^\infty(D)]^G$, the theta function $\Theta(g', z, \varphi)$ is Γ-invariant as a function of z; hence, it descends to a function on X.

The Gaussian on $V(\mathbb{R})$ is given by $\varphi_0(x) = e^{-\pi \sum_{i=1}^{p+q} x_i^2}$. Certainly $\varphi_0 \in \mathcal{S}(V(\mathbb{R}))^K$, and the corresponding function in $[\mathcal{S}(V(\mathbb{R})) \otimes C^\infty(D)]^G$ is given by

$$\varphi_0(x, z) = e^{-\pi(x,x)_z}.$$

Occasionally we will write $\varphi_0^{p,q}$ to emphasize the signature. We have

$$\Theta(\tau, z, \varphi_0) = \sum_{h \in L^\#/L} \sum_{\lambda \in L+h} \varphi_0(\lambda, \tau, z) \mathfrak{e}_h \in A_{(p-q)/2, L} \otimes C^\infty(X),$$

and

$$\varphi_0(\lambda, \tau, z) = v^{q/2} \exp\left(\pi i((\lambda, \lambda)u + (\lambda, \lambda)_z iv)\right) = \exp\left(\pi i((\lambda_{z^\perp}, \lambda_{z^\perp})\tau + (\lambda_z, \lambda_z)\bar\tau)\right).$$

3. Weak Maass forms

In this section we introduce a new space of Maass wave forms. In particular, we establish a pairing with holomorphic modular forms and obtain a duality theorem for this pairing. The results of this section can be either viewed as an analytic version of the Serre duality result in [Bo2] section 3, or as an algebraic approach to [Br1] chapter 1.

Let $k \in \frac{1}{2}\mathbb{Z}$ with $k \neq 1$. (In our later applications k will be smaller than 1. However, we do not need that here.) Moreover, let $\Gamma'' \leq \Gamma'$ be a subgroup of finite index.

We write $H_{k,L}(\Gamma'')$ for the space of *weak Maass forms* of weight k with representation ϱ_L for the group Γ''. By definition, this is the space of twice continuously differentiable functions $f : \mathbb{H} \to \mathbb{C}[L^\#/L]$ satisfying:

   i) $f(\gamma\tau) = \phi(\tau)^{2k} \varrho_L(\gamma, \phi) f(\tau)$ for all $(\gamma, \phi) \in \Gamma''$;
   ii) there is a $C > 0$ such that for any cusp $s \in \mathbb{Q} \cup \{\infty\}$ of Γ'' and $(\delta, \phi) \in \Gamma'$ with $\delta\infty = s$ the function $f_s(\tau) = \phi(\tau)^{-2k} \varrho_L^{-1}(\delta, \phi) f(\delta\tau)$ satisfies $f_s(\tau) = O(e^{Cv})$ as $v \to \infty$ (uniformly in u, where $\tau = u + iv$);

iii) $\Delta_k f(\tau) = 0$, where

$$(3.1) \qquad \Delta_k = -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)$$

denotes the usual hyperbolic Laplace operator in weight k.

Since Δ_k is an elliptic differential operator, such a function f will automatically be real analytic. We will mainly work with the full group Γ' and therefore abbreviate $H_{k,L} = H_{k,L}(\Gamma')$. The transformation property (i) implies that any $f \in H_{k,L}$ has a Fourier expansion

$$f(\tau) = \sum_{h \in L^\#/L} \sum_{n \in \mathbb{Q}} a(h, n; v) e(nu) \mathfrak{e}_h,$$

where $e(u) = e^{2\pi i u}$, as usual. The coefficients $a(h, n; v)$ vanish unless $n - q(h) \in \mathbb{Z}$. In particular, the denominators of the indices n of all non-zero coefficients $a(h, n; v)$ are bounded by the level N of the lattice L. Since the element $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right), i \right)$ of Γ' acts by $\varrho_L(Z)\mathfrak{e}_h = i^{q-p}\mathfrak{e}_{-h}$, we have the relation $a(h, n; v) = (-1)^{k+\frac{q-p}{2}} a(-h, n; v)$. This implies that any weak Maass form in $H_{k,L}$ vanishes, if $2k \not\equiv p - q \pmod{2}$.

Because of property (iii), the coefficients $a(h, n; v)$ satisfy the second order differential equation $\Delta_k a(h, n; v) e(nu) = 0$ as functions in v. If $n = 0$, one finds that $a(h, 0; v)$ is a linear combination of 1 and v^{1-k}. If $n \neq 0$, then, writing $b(2\pi n v)$ for $a(h, n; v)$, it is easily seen that $b(w)$ is a solution of the second order linear differential equation

$$\frac{\partial^2}{\partial w^2} b(w) - b(w) + \frac{k}{w} \left(\frac{\partial}{\partial w} b(w) + b(w) \right) = 0,$$

which is independent of h and n. It is immediately checked that e^{-w} is a solution. A second, linearly independent solution is found by reduction of the order. Here we choose the function

$$H(w) = e^{-w} \int_{-2w}^{\infty} e^{-t} t^{-k} \, dt.$$

The integral converges for $k < 1$ and can be holomorphically continued in k (for $w \neq 0$) in the same way as the Gamma function. If $w < 0$, then $H(w) = e^{-w}\Gamma(1 - k, -2w)$, where $\Gamma(a, x)$ denotes the incomplete Gamma function as in [AbSt] (6.5.3). The function $H(w)$ has the asymptotic behavior

$$H(w) \sim \begin{cases} (2|w|)^{-k} e^{-|w|}, & \text{for } w \to -\infty, \\ (-2w)^{-k} e^{w}, & \text{for } w \to +\infty. \end{cases}$$

(Clearly the functions $|w|^{k/2} e^{-w/2}$ and $|w|^{k/2} H(w/2)$ are the special values of the standard Whittaker functions $W_{\nu,\mu}(|w|)$ and $M_{\nu,\mu}(|w|)$ for $\nu = \mathrm{sgn}(n) k/2$, $\mu = k/2 - 1/2$ as in [AbSt] chapter 13.) We find that

$$a(h, n; v) = \begin{cases} a^+(h, 0) + a^-(h, 0) v^{1-k}, & \text{if } n = 0, \\ a^+(h, n) e^{-2\pi n v} + a^-(h, n) H(2\pi n v), & \text{if } n \neq 0, \end{cases}$$

with complex coefficients $a^{\pm}(h,n)$. Thus any weak Maass form f of weight k has a unique decomposition $f = f^{+} + f^{-}$, where

$$(3.2\text{a}) \qquad f^{+}(\tau) = \sum_{h \in L^{\#}/L} \sum_{n \in \mathbb{Q}} a^{+}(h,n) e(n\tau) \mathfrak{e}_h,$$

$$(3.2\text{b}) \qquad f^{-}(\tau) = \sum_{h \in L^{\#}/L} \left(a^{-}(h,0) v^{1-k} + \sum_{\substack{n \in \mathbb{Q} \\ n \neq 0}} a^{-}(h,n) H(2\pi nv) e(nu) \right) \mathfrak{e}_h.$$

Note that if f satisfies condition (ii) above, then all but finitely many $a^{+}(h,n)$ (respectively $a^{-}(h,n)$) with negative (respectively positive) index n vanish.

Let us briefly recall the Maass raising and lowering operators on non-holomorphic modular forms of weight k. They are defined as the differential operators

$$R_k = 2i\frac{\partial}{\partial \tau} + kv^{-1} \qquad \text{and} \qquad L_k = -2iv^2 \frac{\partial}{\partial \bar{\tau}}.$$

The raising operator R_k maps $A_{k,L}$ to $A_{k+2,L}$, and the lowering operator L_k maps $A_{k,L}$ to $A_{k-2,L}$. The Laplacian Δ_k can be expressed in terms of R_k and L_k by

$$(3.3) \qquad -\Delta_k = L_{k+2} R_k + k = R_{k-2} L_k.$$

The following lemma is proved by a straightforward computation.

Lemma 3.1. *Let $f \in H_{k,L}$ be a weak Maass form of weight k and write $f = f^{+} + f^{-}$ as in (3.2). Then*

$$L_k f = L_k f^{-} = -2v^{2-k} \sum_{h \in L^{\#}/L} \left((k-1) a^{-}(h,0) + \sum_{\substack{n \in \mathbb{Q} \\ n \neq 0}} a^{-}(h,n)(-4\pi n)^{1-k} e(n\bar{\tau}) \right) \mathfrak{e}_h.$$

We write $M^{!}_{k,L}(\Gamma'')$ for the space of holomorphic $\mathbb{C}[L^{\#}/L]$-valued functions on $\mathbb{H}$ satisfying the transformation property (i) above and being meromorphic at the cusps of Γ''. We call such modular forms *weakly holomorphic*. Identity (3.3) implies that $M^{!}_{k,L}(\Gamma'') \subset H_{k,L}(\Gamma'')$. If we also require holomorphicity (vanishing) at the cusp, we obtain the space of holomorphic modular forms $M_{k,L}(\Gamma'')$ (cusp forms $S_{k,L}(\Gamma'')$). Finally, if Γ'' is the full modular group Γ', then we will briefly write $M^{!}_{k,L}$, $M_{k,L}$, and $S_{k,L}$ for these spaces of modular forms. They were first considered by Borcherds [Bo1] and later in [Br1].

The lattice L^{-} gives rise to the dual representation $\varrho^{*}_{L} = \bar{\varrho}_L$. Hence, we can consider $A_{k,L^{-}}$, etc. also as the space of modular forms with respect to the dual representation ϱ^{*}_{L} of ϱ_L. We will make frequent use of this fact.

Recall that the Petersson scalar product on $M_{k,L}$ is given by

$$(3.4) \qquad (f,g)_{k,L} = \int_{\Gamma' \backslash \mathbb{H}} \langle f, g \rangle v^k \, d\mu$$

for $f, g \in M_{k,L}$, whenever the integral converges absolutely. Here $d\mu = \frac{du\,dv}{v^2}$ denotes the usual invariant volume form on $\mathbb{H}$.

Proposition 3.2. *The assignment $f(\tau) \mapsto \xi_k(f)(\tau) := v^{k-2}\overline{L_k f(\tau)} = R_{-k}v^k\overline{f(\tau)}$ defines an anti-linear mapping*

$$(3.5) \qquad \qquad \xi_k : H_{k,L} \longrightarrow M^!_{2-k,L^-}.$$

Its kernel is $M^!_{k,L} \subset H_{k,L}$.

Proof. By Lemma 3.1, the function $\xi_k(f)(\tau)$ is holomorphic on $\mathbb{H}$. It vanishes if and only if f^- vanishes. It is meromorphic at the cusp via the growth condition on f. The transformation behavior of $\xi_k(f)(\tau)$ is easily checked. $\qquad \square$

We let $H^+_{k,L}$ denote the inverse image of the space of holomorphic cusp forms S_{2-k,L^-} under the mapping ξ_k. Hence, if $f \in H^+_{k,L}$, then the Fourier coefficients $a^-(h,n)$ with non-negative index n vanish, so f^- is rapidly decreasing for $v \to \infty$. Clearly $M^!_{k,L} \subset H^+_{k,L}$. Moreover, if $k \geq 2$, then f^- vanishes for all $f \in H^+_{k,L}$.

Now let $f \in H_{k,L}$ and write its Fourier expansion as in (3.2). Then we call the Fourier polynomial

$$(3.6) \qquad \qquad P(f)(\tau) = \sum_{h \in L^\#/L} \sum_{\substack{n \in \mathbb{Q} \\ n \leq 0}} a^+(h,n)e(n\tau)\mathfrak{e}_h$$

the *principal part* of f. Observe that if $f \in H^+_{k,L}$, then $f - P(f)$ is exponentially decreasing as $v \to \infty$. (This property could actually be used to define the space $H^+_{k,L}$ in an alternative way).

Lemma 3.3. *If $f \in H_{k,L}$, then there is a constant $C > 0$ such that $f(\tau) = O(e^{C/v})$ as $v \to 0$, uniformly in u.*

Proof. This follows from the transformation behavior and the growth of f in the same way as the analogous statement for holomorphic modular forms. $\qquad \square$

Later, we will need the following growth estimate for the Fourier coefficients of weak Maass forms.

Lemma 3.4. *Let $f \in H_{k,L}$ and write its Fourier expansion as in (3.2). Then there is a constant $C > 0$ such that the Fourier coefficients satisfy*

$$a^+(h,n) = O\left(e^{C\sqrt{|n|}}\right), \quad n \to +\infty,$$
$$a^-(h,n) = O\left(e^{C\sqrt{|n|}}\right), \quad n \to -\infty.$$

If $f \in H^+_{k,L}$, then the $a^-(h,n)$ actually satisfy the stronger bound $a^-(h,n) = O(|n|^{k/2})$ as $n \to -\infty$.

Proof. To prove the asymptotic for $a^-(h,n)$ we consider the weakly holomorphic modular form $\xi_k(f) \in M^!_{2-k,L^-}$. By Lemma 3.1 and the formula for the Fourier coefficients we have

$$(3.7) \qquad \qquad 2a^-(h,n)(-4\pi n)^{1-k} = -\int_0^1 \left\langle v^{k-2}\overline{L_k f(\tau)}, \mathfrak{e}_h \right\rangle e(n\tau)\,du.$$

Thus, according to Lemma 3.3, we get

$$a^-(h,n) \ll |n|^{k-1} \int_0^1 e^{C/v} e^{-2\pi n v} \, du$$

for all positive $0 < v \leq 1$ with some positive constant C (independent of v and n). If we take v equal to $1/\sqrt{|n|}$, we see that

$$a^-(h,n) \ll |n|^{k-1} e^{C\sqrt{|n|}} e^{2\pi\sqrt{|n|}}$$

for all $n < 0$, proving the first assertion on the $a^-(h,n)$.

From $\xi_k(f) \in M_{2-k,L^-}^!$ it can be deduced that the individual functions f^+ and f^- in the splitting $f = f^+ + f^-$ also satisfy the estimate of Lemma 3.3. We may apply the above argument to

$$(3.8) \qquad\qquad a^+(h,n) = \int_0^1 \left\langle f^+(\tau), \mathfrak{e}_h \right\rangle e(-n\tau) \, du$$

to derive the estimate for the $a^+(n,h)$ as $n \to +\infty$.

If $f \in H_{k,L}^+$, then $\xi_k(f) \in S_{2-k,L^-}$ is a holomorphic cusp form. Hence the usual Hecke bound for the Fourier coefficients of cusp forms implies that the left hand side of (3.7) is bounded by some constant times $|n|^{1-k/2}$ for all $n < 0$. Thus $a^-(h,n) = O(|n|^{k/2})$ as $n \to -\infty$. $\qquad\square$

The estimates of Lemma 3.4 are far from being optimal. However, they are sufficient for our purposes. Observe that all the above results have obvious generalizations to finite index subgroups $\Gamma'' \leq \Gamma'$.

Put $\kappa = 2 - k$. We now define a bilinear pairing between the spaces M_{κ,L^-} and $H_{k,L}^+$ by putting

$$(3.9) \qquad\qquad \{g,f\} = \left(g, \xi_k(f)\right)_{\kappa,L^-}$$

for $g \in M_{\kappa,L^-}$ and $f \in H_{k,L}^+$.

Proposition 3.5. *Let $g \in M_{\kappa,L^-}$ with Fourier expansion $g = \sum_{h,n} b(h,n) e(n\tau) \mathfrak{e}_h$, and $f \in H_{k,L}^+$ with Fourier expansion as in (3.2). Then the pairing (3.9) of g and f is determined by the principal part of f. It is equal to*

$$(3.10) \qquad\qquad \{g,f\} = \sum_{h \in L^\#/L} \sum_{n \leq 0} a^+(h,n) b(h,-n).$$

Proof. We begin by noticing that $\langle g, \overline{f} \rangle d\tau$ is a Γ'-invariant 1-form on $\mathbb{H}$. We have

$$d\left(\langle g, \overline{f} \rangle d\tau\right) = \bar{\partial}\left(\langle g, \overline{f} \rangle d\tau\right) = \left\langle g, \overline{\frac{\partial}{\partial\bar{\tau}} f} \right\rangle d\bar{\tau} \, d\tau = -\langle g, \overline{L_k f} \rangle \, d\mu.$$

Hence, by Stokes' theorem we get

$$\int_{\mathcal{F}_t} \langle g, \overline{L_k f} \rangle \, d\mu = -\int_{\partial\mathcal{F}_t} \langle g, \overline{f} \rangle \, d\tau = \int_{-1/2}^{1/2} \langle g(u+it), \overline{f(u+it)} \rangle \, du,$$

where

(3.11) $$\mathcal{F}_t = \{\tau \in \mathbb{H};\ |\tau| \geq 1,\ |u| \leq 1/2\ ,\ \text{and } v \leq t\}$$

denotes the truncated fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. In the last line we have used the invariance of $\langle g, \overline{f}\rangle d\tau$. If we insert the Fourier expansions of f and g, we see that the integral over u picks out the 0-th Fourier coefficient of $\langle g, \overline{f}\rangle$. Thus

$$\int_{\mathcal{F}_t} \langle g, \overline{L_k f}\rangle\, d\mu = \sum_{h} \sum_{n \leq 0} a^+(h, n) b(h, -n) + O(e^{-\varepsilon t})$$

for some $\varepsilon > 0$. We finally obtain

$$\{g, f\} = \lim_{t \to \infty} \int_{\mathcal{F}_t} \langle g, \overline{L_k f}\rangle\, d\mu = \sum_{h \in L^\#/L} \sum_{n \leq 0} a^+(h, n) b(h, -n)$$

as asserted. $\square$

Observe that the definition of the pairing (3.9) immediately implies that $\{g, f\} = 0$ for all $f \in M_{k,L}^!$.

By (3.10) we get nontrivial relations among the coefficients of modular forms in M_{κ,L^-}. These are also easily obtained by means of the residue theorem on the Riemann sphere. Moreover, it is clear that $\{g, f\} = 0$ for all Eisenstein series $g \in M_{\kappa,L^-}$. This fact can be used to determine the constant term of a weak Maass form $f \in H_{k,L}^+$ in terms of the coefficients with negative index: For instance if $k < 0$, and $\mathfrak{a} \in \mathbb{C}[L^\#/L]$ satisfies $\varrho_L(T)\mathfrak{a} = \mathfrak{a}$ and $\varrho_L(Z)\mathfrak{a} = i^{-2k}\mathfrak{a}$, then there is a unique Eisenstein series $E_\mathfrak{a} \in M_{\kappa,L^-}$, whose constant term is equal to $\mathfrak{a}$. The pairing $\{E_\mathfrak{a}, f\}$ gives a formula for the scalar product in $\mathbb{C}[L^\#/L]$ of the constant term of f and $\bar{\mathfrak{a}}$.

Theorem 3.6. *The pairing between the quotient $H_{k,L}^+/M_{k,L}^!$ and S_{κ,L^-} induced by (3.9) is non-degenerate.*

It suffices to show that the mapping $\xi_k : H_{k,L}^+ \to S_{\kappa,L^-}$ is surjective. This is an immediate consequence of

Theorem 3.7. *The mapping $\xi_k : H_{k,L} \to M_{\kappa,L^-}^!$ defined in Proposition 3.2 is surjective.*

Before beginning with the proof, we need to introduce some notation.

Let $\Gamma'' \leq \Gamma'$ be a normal subgroup of finite index. We write $X(\Gamma'')$ for the compact modular curve corresponding to Γ'' and $\pi : \mathbb{H} \to X(\Gamma'')$ for the canonical map. The Poincaré metric $1/v^2$ on $\mathbb{H}$ induces a Kähler metric on $X(\Gamma'')$ (with logarithmic singularities at the cusps). We write $*$ for the corresponding Hodge star operator.

We denote the sheaf of holomorphic functions (respectively 1-forms) on $X(\Gamma'')$ by $\mathcal{O}$ (respectively Ω) and the sheaf of C^∞ differential forms of type (p, q) by $\mathcal{E}^{p,q}$. If D is a divisor on $X(\Gamma'')$, then we write $\mathcal{O}_D$ for the sheaf corresponding to D. The sections of $\mathcal{O}_D$ over an open subset $U \subset X(\Gamma'')$ are given by meromorphic functions f satisfying $\mathrm{div}(f) \geq -D$ on U.

If $\mathcal{L}$ is any $\mathcal{O}$-module on $X(\Gamma'')$, we let $\mathcal{E}^{p,q}(U, \mathcal{L})$ denote the $\mathcal{L}$-valued C^∞ differential forms on an open set $U \subset X(\Gamma'')$, that is, the sections of $\mathcal{E}^{p,q} \otimes_\mathcal{O} \mathcal{L}$ over U. Moreover, we write $\mathcal{L}_{k,L}$ for the $\mathcal{O}$-module sheaf of modular forms of weight k with representation ϱ_L on

$X(\Gamma'')$. If $U \subset X(\Gamma'')$ is open, then the sections $\mathcal{L}_{k,L}(U)$ are holomorphic $\mathbb{C}[L^\#/L]$-valued functions on the open subset $\pi^{-1}(U) \subset \mathbb{H}$ satisfying the transformation law of modular forms of weight k with representation ϱ_L for Γ'', and being holomorphic at the cusps. If Γ'' acts freely on $\mathbb{H}$, then $\mathcal{L}_{k,L}$ is a holomorphic vector bundle. A Hermitian metric (with logarithmic singularities at the cusps) on it is given by the Hermitian scalar product $(f,g)_\tau = \langle f, g \rangle v^k$ on the fiber over $\pi(\tau)$, where $\tau = u + iv \in \mathbb{H}$ and $\langle f, g \rangle$ denotes the standard scalar product on $\mathbb{C}[L^\#/L]$ as before.

The dual vector bundle $\mathcal{L}_{k,L}^*$ of $\mathcal{L}_{k,L}$ can be identified with the vector bundle $\mathcal{L}_{-k,L^-}$ of modular forms of weight $-k$ with representation ϱ_L^* on $X(\Gamma'')$. The mapping $f \mapsto v^k \bar{f}$ defines an anti-linear bundle isomorphism $\mathcal{L}_{k,L} \to \mathcal{L}_{k,L}^*$. It induces a Hodge star operator

$$\bar{*}_\mathcal{L} : \mathcal{E}^{p,q} \otimes_\mathcal{O} \mathcal{L}_{k,L} \longrightarrow \mathcal{E}^{1-p,1-q} \otimes_\mathcal{O} \mathcal{L}_{k,L}^*, \qquad \bar{*}_\mathcal{L}(\phi \otimes f) = (*\bar{\phi}) \otimes (v^k \bar{f}),$$

on $\mathcal{L}_{k,L}$-valued C^∞ differential forms on $X(\Gamma'')$ (see [We] chapter V.2). It is easily verified that

(3.12a) $$\bar{*}_\mathcal{L}(f) = v^{k-2} \bar{f} \, du \, dv, \qquad \text{for } f \in \mathcal{E}^{0,0}(U, \mathcal{L}_{k,L}),$$

(3.12b) $$\bar{*}_\mathcal{L}(f d\tau) = i v^k \bar{f} d\bar{\tau}, \qquad \text{for } f d\tau \in \mathcal{E}^{1,0}(U, \mathcal{L}_{k,L}),$$

(3.12c) $$\bar{*}_\mathcal{L}(\bar{f} d\bar{\tau}) = i v^k f d\tau, \qquad \text{for } \bar{f} d\bar{\tau} \in \mathcal{E}^{0,1}(U, \mathcal{L}_{k,L}).$$

The Laplace operator on differential forms in $\mathcal{E}^{p,q}(U, \mathcal{L}_{k,L})$ is given by

$$\Box = \bar{\partial}\bar{*}_\mathcal{L}\bar{\partial}\bar{*}_\mathcal{L} + \bar{*}_\mathcal{L}\bar{\partial}\bar{*}_\mathcal{L}\bar{\partial}.$$

By a straightforward computation using (3.12) it can be shown that

(3.13) $$\Box f = \bar{*}_\mathcal{L}\bar{\partial}\bar{*}_\mathcal{L}\bar{\partial} f = R_{k-2}L_k f = -\Delta_k f$$

for functions $f \in \mathcal{E}^{0,0}(U, \mathcal{L}_{k,L})$.

Proof of Theorem 3.7. Let $\Gamma'' \leq \Gamma'$ be a normal subgroup of finite index that acts freely on $\mathbb{H}$. We first prove the analogous result for the group Γ'', namely that the mapping $\xi_k : H_{k,L}(\Gamma'') \to M^!_{\kappa,L^-}(\Gamma'')$ is surjective.

We consider the Riemann surface $X = X(\Gamma'')$ using the above notation. Let $s_1, \ldots, s_r$ be the cusps of X and write $D = \sum_i s_i$ for the divisor on X given by the cusps. Let n be a positive integer. By tensoring the Dolbeault resolution of the structure sheaf $\mathcal{O}$ with the locally free $\mathcal{O}$-module $\mathcal{L}_{k,L} \otimes_\mathcal{O} \mathcal{O}_{nD}$, we get the exact sheaf sequence

$$0 \longrightarrow \mathcal{O} \otimes \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD} \longrightarrow \mathcal{E}^{0,0} \otimes \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD} \xrightarrow{\bar{\partial} \otimes 1 \otimes 1} \mathcal{E}^{0,1} \otimes \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD} \longrightarrow 0.$$

Since $\mathcal{E}^{p,q} \otimes \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD}$ is a fine sheaf, it is acyclic. Hence we obtain the following long exact cohomology sequence:

(3.14) $$0 \longrightarrow (\mathcal{L}_{k,L} \otimes \mathcal{O}_{nD})(X) \longrightarrow \mathcal{E}^{0,0}(X, \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD})$$
$$\longrightarrow \mathcal{E}^{0,1}(X, \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD}) \longrightarrow H^1(X, \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD}) \longrightarrow 0.$$

We claim that $H^1(X, \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD})$ vanishes, if n is large. In fact, using Serre duality (see [We] chapter V Theorem 2.7) we find

$$H^1(X, \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD}) \cong H^0(X, \Omega \otimes \mathcal{L}_{k,L}^* \otimes \mathcal{O}_{nD}^*) \cong H^0(X, \mathcal{L}_{\kappa,L^-} \otimes \mathcal{O}_{-(n+1)D}).$$

Because the number of zeros (counted with multiplicities) of holomorphic modular forms of fixed weight κ is bounded, the latter cohomology group vanishes if n is sufficiently large. This proves the claim.

Now let $g \in M^!_{\kappa,L^-}(\Gamma'')$. We want to show that there is an $f \in H_{k,L}(\Gamma'')$ such that $\xi_k(f) = g$. We choose a positive integer n greater or equal to the orders of the poles of g at the cusps and such that $H^1(X, \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD})$ vanishes. Then the 1-form $g\,d\tau$ defines a global holomorphic section in $\mathcal{E}^{1,0}(X, \mathcal{L}^*_{k,L} \otimes \mathcal{O}_{nD})$. Applying the Hodge star operator, we get the 1-form $iv^{-k}\bar{g}\,d\bar{\tau} \in \mathcal{E}^{0,1}(X, \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD})$. By virtue of the exact sequence (3.14) we find that there is a function $f \in \mathcal{E}^{0,0}(X, \mathcal{L}_{k,L} \otimes \mathcal{O}_{nD})$ satisfying $\bar{\partial}f = iv^{-k}\bar{g}\,d\bar{\tau}$. But this is equivalent to saying

$$-v^{k-2}L_k f\,d\bar{\tau} = \bar{g}\,d\bar{\tau}.$$

We are left with showing that $\Delta_k f = 0$. This follows from

$$-\Delta_k f = \Box f = \bar{*}_{\mathcal{L}}\bar{\partial}\bar{*}_{\mathcal{L}}\bar{\partial} f = \bar{*}_{\mathcal{L}}\bar{\partial}\bar{*}_{\mathcal{L}} \, iv^{-k}\bar{g}\,d\bar{\tau} = -\bar{*}_{\mathcal{L}}\bar{\partial} \, gd\tau = 0$$

completing the proof of Theorem 3.7 for the group Γ''.

For the full modular group Γ', we use the fact that the cohomology of a finite group acting on a $\mathbb{C}$-vector space vanishes. This implies that $H^j(X(\Gamma'), \mathcal{L}) \cong H^j(X(\Gamma''), \mathcal{L})^{\Gamma'/\Gamma''}$ for any coherent $\mathcal{O}$-module $\mathcal{L}$. Hence the assertion follows by considering the Γ'/Γ''-invariant subspaces in (3.14). $\qquad\Box$

Corollary 3.8. *The following sequences are exact:*

$$0 \longrightarrow M^!_{k,L} \longrightarrow H_{k,L} \overset{\xi_k}{\longrightarrow} M^!_{\kappa,L^-} \longrightarrow 0,$$

$$0 \longrightarrow M^!_{k,L} \longrightarrow H^+_{k,L} \overset{\xi_k}{\longrightarrow} S_{\kappa,L^-} \longrightarrow 0.$$

We will also need a slight variation of the above duality result. To formulate it, we introduce a second pairing between the spaces $H^+_{k,L}$ and M_{κ,L^-}. If $g \in M_{\kappa,L^-}$ with Fourier expansion $g = \sum_{h,n} b(h,n)e(n\tau)\mathfrak{e}_h$, and $f \in H^+_{k,L}$ with Fourier expansion as in (3.2), then we put

$$(3.15) \qquad \{g,f\}' = \sum_{h \in L^{\#}/L} \sum_{n<0} a^+(h,n)b(h,-n).$$

This definition differs from (3.10) by the fact that we only sum over negative n. We let $M^{!!}_{k,L}$ be the subspace of $M^!_{k,L}$ consisting of those weakly holomorphic modular forms f whose constant term $\sum_h a^+(h,0)\mathfrak{e}_h$ in the Fourier expansion is orthogonal to the constant terms of all $g \in M_{\kappa,L^-}$ with respect to $\langle \cdot, \cdot \rangle$. If $k < 0$ or if $k = 0$ and ϱ_L does not contain the trivial representation, then, using Eisenstein series in M_{κ,L^-}, one sees that $M^{!!}_{k,L}$ simply consists of those $f \in M^!_{k,L}$ with vanishing constant term. However, otherwise the fact that Eisenstein series of weight ≤ 2 may be non-holomorphic implies that some $f \in M^{!!}_{k,L}$ may have non-vanishing constant term. If $f \in M^{!!}_{k,L}$ or $g \in S_{\kappa,L^-}$, then $\{g,f\} = \{g,f\}'$. In particular the pairing $\{g,f\}'$ vanishes, if $f \in M^{!!}_{k,L}$.

Corollary 3.9. *The pairings between $H^+_{k,L}/M^{!!}_{k,L}$ and M_{κ,L^-} (respectively $H^+_{k,L}/M^!_{k,L}$ and S_{κ,L^-}) induced by (3.15) are non-degenerate.*

This can be proved using Theorem 3.6 and Eisenstein series in M_{κ,L^-}. The details are left to the reader.

Remark 3.10. If $h \in L^\#/L$ and $m \in \mathbb{Z} + q(h)$ is negative, the Hejhal Poincaré series $F_{h,m}(\tau, \kappa/2)$ defined in [Br1] Definition 1.8 and Proposition 1.10 (see also [He] and [Ni]) are examples of weak Maass forms in $H_{k,L}^+$. If $k < 0$, it can be shown that they span the whole space $H_{k,L}^+$ (see [Br1] Proposition 1.12). Moreover, one can check that $\xi_k(F_{h,m}(\tau, \kappa/2))$ equals up to a constant factor the usual holomorphic Poincaré series $P_{h,-m}(\tau) \in S_{\kappa,L^-}$ (as e.g. defined in [Br1] chapter 1.2.1). Therefore, for $\kappa > 2$ (and $k < 0$), the above duality statement also follows from chapter 1 of [Br1]. However, the approach of the present paper is more conceptual while also covering the low weights $\kappa = 3/2$ and 2 (where the $P_{h,-m}$ would need to be defined by Hecke summation).

The principal part of the Hejhal Poincaré series $F_{h,m}(\tau, \kappa/2)$ is equal to

$$e(m\tau)\mathfrak{e}_h + (-1)^{k + \frac{q-p}{2}} e(m\tau)\mathfrak{e}_{-h} + \text{constant term.}$$

This shows that, up to the constant term, any Fourier polynomial as in (3.6) occurs as the principal part $P(f)$ of some $f \in H_{k,L}^+$. This can also be deduced from Theorem 3.7 as follows:

Proposition 3.11. *For every Fourier polynomial of the form*

$$Q(\tau) = \sum_{h \in L^\#/L} \sum_{\substack{n \in \mathbb{Z} + q(h) \\ n < 0}} a^+(h,n) e(n\tau) \mathfrak{e}_h$$

with $a^+(h,n) = (-1)^{k + \frac{q-p}{2}} a^+(-h,n)$, *there exists an* $f \in H_{k,L}^+$ *with principal part* $P(f) = Q + \mathfrak{c}$ *for some* T-*invariant constant* $\mathfrak{c} \in \mathbb{C}[L^\#/L]$. *The function* f *is uniquely determined, if* $k < 0$ *(and in certain cases also for* $k = 0$).

Proof. Given Q as above, we define a linear functional λ_Q on S_{κ,L^-} via the right hand side of (3.15). By virtue of Corollary 3.9 this functional is represented by a weak Maass form $h \in H_{k,L}^+$, i.e., $\{g, h\} = \lambda_Q(g)$ for all $g \in S_{\kappa,L^-}$. The functional $\lambda_{Q-P(h)}$ then vanishes identically on S_{κ,L^-}. Hence, by a variant of Theorem 3.1 of [Bo2] (see [Br1] Theorem 1.17), there exists a weakly holomorphic $h' \in M_{k,L}^!$ with principal part $Q - P(h) + \mathfrak{c}$ for some T-invariant constant $\mathfrak{c} \in \mathbb{C}[L^\#/L]$. Then $f := h + h' \in H_{k,L}^+$ has principal part $Q + \mathfrak{c}$. □

Remark 3.12. Borcherds [Bo2] uses the equation (3.10) to introduce a pairing between formal Laurent series and formal power series. Via Serre duality he also obtains that $M_{k,L}^!$ is the orthogonal complement of S_{κ,L^-}, but he does not consider weak Maass forms.

4. Special Schwartz functions

For the following discussion the reader should consult [KM3], sections 5-8.

Kudla and Millson constructed (in more generality) Schwartz forms φ_{KM} on $V(\mathbb{R})$ taking values in $\mathcal{A}^q(D)$, the differential q-forms on D. More precisely,

$$\varphi_{KM} \in [\mathcal{S}(V(\mathbb{R})) \otimes \mathcal{A}^q(D)]^G \simeq [\mathcal{S}(V(\mathbb{R})) \otimes \bigwedge^q(\mathfrak{p}^*)]^K,$$

where the isomorphism is again given by evaluation at the base point of D. In fact, $\varphi_{KM}(x)$ is closed for all $x \in V(\mathbb{R})$.

We let $X_{\alpha\mu}$ ($1 \le \alpha \le p$, $p+1 \le \mu \le p+q$) denote the elements of the obvious basis of $\mathfrak{p}$ in (2.1), and let $\omega_{\alpha\mu}$ be the elements of the dual basis which pick out the $\alpha\mu$-th coordinate of $\mathfrak{p}$. Then φ_{KM} is given by applying the operator

$$\mathcal{D} = \frac{1}{2^{q/2}} \prod_{\mu=p+1}^{p+q} \left[\sum_{\alpha=1}^{p} \left(x_\alpha - \frac{1}{2\pi} \frac{\partial}{\partial x_\alpha} \right) \otimes A_{\alpha\mu} \right]$$

to the standard Gaussian $\varphi_0 \otimes 1 \in [\mathcal{S}(V(\mathbb{R})) \otimes \bigwedge^0(\mathfrak{p}^*)]^K$:

$$\varphi_{KM} = \mathcal{D}(\varphi_0 \otimes 1).$$

Here $A_{\alpha\mu}$ denotes the left multiplication by $\omega_{\alpha\mu}$. Note that this is $2^{q/2}$ times the corresponding quantity in [KM3]. It is easy to see that φ_{KM} is K-invariant, and by [KM1], Theorem 3.1 it is an eigenfunction of K' of weight $\frac{p+q}{2}$.

For a multi-index $\underline{\alpha} = (\alpha_1, \ldots, \alpha_q) \in \{1, \ldots, p\}^q$, we have

$$\varphi_{KM}(x) = \sum_{\underline{\alpha}} P_{\underline{\alpha}}(x) \, \varphi_0(x) \, \omega_{\alpha_1 p+1} \wedge \cdots \wedge \omega_{\alpha_q p+q},$$

where $P_{\underline{\alpha}}(x)$ is a (in general non-homogeneous) polynomial of degree q. For $\underline{\alpha} = (\alpha, \ldots, \alpha)$, $P_{\underline{\alpha}}(x)$ is given by

$$P_{\underline{\alpha}}(x) = (4\pi)^{-q/2} H_q\left(\sqrt{2\pi} x_\alpha \right),$$

where $H_q(t) = (-1)^q e^{t^2} \frac{d^q}{dt^q} e^{-t^2}$ is the q-th Hermite polynomial. For 'mixed' $\underline{\alpha}$, $P_{\underline{\alpha}}(x)$ is a product of Hermite functions in the x_α.

In particular, we have (see [KM1], Prop. 5.1)

(4.1) $$\varphi_{KM}(0) = e_q,$$

where for $q = 2l$ even, e_q is the Euler form of the symmetric space D (which is the Euler class (see e.g. [KN]) of the tautological vector bundle over D, i.e., the fiber over a point $z \in D$ is given by the negative q-plane z) and zero for q odd. Here e_q is normalized such that it is given in $\bigwedge^q(\mathfrak{p}^*)$ by

$$e_q = \left(-\frac{1}{4\pi} \right)^l \frac{1}{l!} \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma) \Omega_{p+\sigma(1),p+\sigma(2)} \ldots \Omega_{p+\sigma(2l-1),p+\sigma(2l)},$$

with

$$\Omega_{\mu\nu} = \sum_{\alpha=1}^{p} \omega_{\alpha\mu} \wedge \omega_{\alpha\nu}.$$

Note that for $q = 2$, $\Omega := -e_2$ is positive, i.e., it defines a Kähler form on the Hermitian domain D.

The space of K'-finite vectors in $\mathcal{S}(V(\mathbb{R}))$ is given by the so-called polynomial Fock space $S(V(\mathbb{R})) \subset \mathcal{S}(V(\mathbb{R}))$ which consists of those Schwartz functions on $V(\mathbb{R})$ of the form $p(x)\varphi_0(x)$, where $p(x)$ is a polynomial function on $V(\mathbb{R})$. Differentiating the action of $\operatorname{Mp}_2(\mathbb{R}) \times \operatorname{O}(V(\mathbb{R}))$ on $S(V(\mathbb{R}))$ we obtain the associated action of the Lie algebra $\mathfrak{sl}_2 \times \mathfrak{o}(V)$

which we also denote by ω. Then there is an intertwining map $\iota : S(V(\mathbb{R})) \longrightarrow \mathcal{P}(\mathbb{C}^{p+q})$ to the infinitesimal Fock model of the Weil representation acting on the space of complex polynomials $\mathcal{P}(\mathbb{C}^{p+q})$ in $p + q$ variables such that $\iota(\varphi_0) = 1$. We denote the variables in $\mathcal{P}(\mathbb{C}^{p+q})$ by z_α $(1 \leq \alpha \leq p)$ and z_μ $(p + 1 \leq \mu \leq p + q)$. Then the intertwining map ι satisfies

$$\iota\left(x_j - \frac{1}{2\pi}\frac{\partial}{\partial x_j}\right)\iota^{-1} = -\tfrac{i}{2\pi}z_j,$$

$$\iota\left(x_j + \frac{1}{2\pi}\frac{\partial}{\partial x_j}\right)\iota^{-1} = 4\pi i\frac{\partial}{\partial z_j}.$$

Note that this differs from [KM3], page 153, since there a slightly different realization of the Schrödinger model is used.

Clearly, the form φ_{KM} is in the polynomial Fock space. We see

$$(4.2) \qquad \varphi_{KM} = \left(\frac{-i\sqrt{2}}{4\pi}\right)^q \sum_{\alpha_1,\ldots,\alpha_q} z_{\alpha_1}\cdots z_{\alpha_q} \otimes \omega_{\alpha_1 p+1} \wedge \cdots \wedge \omega_{\alpha_q p+q},$$

when considered as an element in $[\mathcal{P}(\mathbb{C}^{p+q}) \otimes \bigwedge^q(\mathfrak{p}^*)]^K$.

In the Fock model, the Weil representation acts as follows: For $\mathfrak{o}(V(\mathbb{R}))$, we have

$$\omega(X_{\alpha\mu}) = 4\pi\frac{\partial^2}{\partial z_\alpha \partial z_\mu} - \frac{1}{4\pi}z_\alpha z_\mu.$$

The elements R and L in $\mathfrak{sl}_2(\mathbb{C})$ (which correspond to the raising and lowering operators R and L on $\mathbb{H}$, see section 3) are given by

$$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad \text{and} \qquad L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

and their action is

$$(4.3) \qquad \omega(R) = \frac{1}{8\pi}\sum_{\alpha=1}^{p} z_\alpha^2 - 2\pi\sum_{\mu=p+1}^{p+q} \frac{\partial^2}{\partial z_\mu^2},$$

$$(4.4) \qquad \omega(L) = -2\pi\sum_{\alpha=1}^{p} \frac{\partial^2}{\partial z_\alpha^2} + \frac{1}{8\pi}\sum_{\mu=p+1}^{p+q} z_\mu^2.$$

(The signs in [KM3] Theorem 7.1 are wrong with the exception of a)(iii); for details see [FM1]).

The differentiation d in the Lie algebra complex $[\mathcal{P}(\mathbb{C}^{p+q}) \otimes \bigwedge(\mathfrak{p}^*)]^K$ is given by

$$d = \sum_{\alpha,\mu} \omega(X_{\alpha\mu}) \otimes A_{\alpha\mu}.$$

From this it is easy to see that φ_{KM} is closed.

We also define a form

$$(4.5) \qquad \psi_{KM} = \psi \in [\mathcal{P}(\mathbb{C}^{p+q}) \otimes \bigwedge^{q-1}(\mathfrak{p}^*)]^K$$

(and therefore in $[S(V(\mathbb{R})) \otimes \mathcal{A}^{q-1}(D)]^G$) by

$$(4.6) \qquad \psi_{KM} = \frac{-1}{2(p+q-1)} h(\varphi_{KM}),$$

where h is the operator on $[\mathcal{P}(\mathbb{C}^{p+q}) \otimes \bigwedge^*(\mathfrak{p}^*)]^K$ given by

$$h = \sum_{\alpha,\mu} z_\mu \frac{\partial}{\partial z_\alpha} \otimes A^*_{\alpha\mu}.$$

Here $A^*_{\alpha\mu}$ denotes the (interior) multiplication with $X_{\alpha\mu}$, i.e., $A^*_{\alpha\mu}(\omega_{\alpha'\mu'}) = \delta_{\alpha\alpha'}\delta_{\mu\mu'}$.

Lemma 4.1.

$$\psi_{KM} = \left(\frac{-i\sqrt{2}}{4\pi}\right)^q \frac{-1}{2(q-1)!} \sum_{\alpha_1,\dots,\alpha_{q-1}} z_{\alpha_1} \cdots z_{\alpha_{q-1}} \det \begin{pmatrix} z_{p+1} & \cdots & z_{p+q} \\ \omega_{\alpha_1 p+1} & \cdots & \omega_{\alpha_1 p+q} \\ \vdots & & \vdots \\ \omega_{\alpha_{q-1} p+1} & \cdots & \omega_{\alpha_{q-1} p+q} \end{pmatrix},$$

where the determinant in the non-commutative algebra $\mathcal{P}(\mathbb{C}^{p+q}) \otimes \bigwedge^*(\mathfrak{p}^*)$ is defined inductively via expansion along the first row.

Proof. We give a brief sketch. We first note (easily checked by induction)

$$(4.7) \qquad \varphi_{KM} = \left(\frac{-i\sqrt{2}}{4\pi}\right)^q \frac{1}{q!} \sum_{\alpha_1,\dots,\alpha_q} z_{\alpha_1} \cdots z_{\alpha_q} \det(\Omega),$$

where

$$\Omega = \begin{pmatrix} \omega_{\alpha_1 p+1} & \cdots & \omega_{\alpha_1 p+q} \\ \vdots & & \vdots \\ \omega_{\alpha_q p+1} & \cdots & \omega_{\alpha_q p+q} \end{pmatrix}.$$

The action of the operator $A^*_{\beta\mu}$ on $\det(\Omega)$ can be computed by expanding $\det(\Omega)$ along the $(p-\mu)$-th column. If we write $\Omega^{\nu\mu}$ for the $(q-1) \times (q-1)$ matrix obtained from Ω by canceling the ν-th row and the $(p-\mu)$-th column, we have

$$A^*_{\beta\mu} \det(\Omega) = -\sum_{\nu=1}^q (-1)^{\mu-p} A^*_{\beta\mu} \omega_{\alpha_\nu \mu} \det(\Omega^{\nu\mu}) = -\sum_{\nu=1}^q (-1)^{\mu-p} \delta_{\beta\alpha_\nu} \det(\Omega^{\nu\mu}).$$

A little calculation then shows

$$\psi_{KM} = \frac{-(-i\sqrt{2}/4\pi)^q}{2(p+q-1)(q-1)!} \sum_{\alpha_1,\dots,\alpha_q} \frac{\partial}{\partial z_{\alpha_q}} z_{\alpha_1} \cdots z_{\alpha_q} \det \begin{pmatrix} z_{p+1} & \cdots & z_{p+q} \\ \omega_{\alpha_1 p+1} & \cdots & \omega_{\alpha_1 p+q} \\ \vdots & & \vdots \\ \omega_{\alpha_{q-1} p+1} & \cdots & \omega_{\alpha_{q-1} p+q} \end{pmatrix}.$$

One now applies the product rule to obtain the assertion. $\square$

Theorem 4.2 (Kudla-Millson [KM3])**.**

(i) *We have the identity*

$$\omega(L)\varphi_{KM} = d\psi_{KM}.$$

(ii) ψ_{KM} *is an eigenfunction of K' with weight $\frac{p+q}{2} - 2$.*

Example 4.3. For signature $(p, 1)$, we have in the Fock model (Schrödinger model) of the Weil representation

$$\varphi_{KM} = -i\frac{\sqrt{2}}{4\pi}\sum_{\alpha=1}^{p} z_\alpha \otimes \omega_{\alpha p+1} = (\iota \otimes 1)\left(\sqrt{2}\sum_{\alpha=1}^{p} x_\alpha \varphi_0 \otimes \omega_{\alpha p+1}\right)$$

and

$$\psi_{KM} = i\frac{\sqrt{2}}{8\pi}z_{p+1} \otimes 1 = (\iota \otimes 1)\left(\frac{-1}{\sqrt{2}}x_{p+1}\varphi_0 \otimes 1\right).$$

We now specialize to the case of signature $(p, 2)$, i.e., $q = 2$, when we have an underlying complex structure on D. Then by the conventions of section 2, the operators ∂ and $\bar{\partial}$ on D are given by

$$\partial = \frac{1}{2}\sum_\alpha \omega(X_{\alpha p+1} - iX_{\alpha p+2}) \otimes (\omega_{\alpha p+1} + i\omega_{\alpha p+2}),$$

$$\bar{\partial} = \frac{1}{2}\sum_\alpha \omega(X_{\alpha p+1} + iX_{\alpha p+2}) \otimes (\omega_{\alpha p+1} - i\omega_{\alpha p+2}).$$

We set $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$, so that $dd^c = -\frac{1}{2\pi i}\partial\bar{\partial}$. Note that in this case φ_{KM} is actually a $(1, 1)$-form.

Theorem 4.4. *In the case of signature $(p, 2)$, we have*

$$\omega(L)\varphi_{KM} = -dd^c\varphi_0.$$

We prove the theorem by computing the two sides separately. For the left hand side, by (4.4) and (4.2) we easily see

Lemma 4.5. *For signature $(p, 2)$,*

$$\omega(L)\varphi_{KM} = \frac{1}{2\pi}\sum_{\alpha=1}^{p}\omega_{\alpha p+1} \wedge \omega_{\alpha p+2} - \frac{1}{64\pi^3}(z_{p+1}^2 + z_{p+2}^2)\sum_{\alpha_1,\alpha_2} z_{\alpha_1}z_{\alpha_2}\,\omega_{\alpha_1 p+1} \wedge \omega_{\alpha_2 p+2}.$$

For the right hand side we get the same:

Lemma 4.6. *For signature $(p, 2)$,*

$$-dd^c\varphi_0 = \frac{1}{2\pi}\sum_{\alpha=1}^{p}\omega_{\alpha p+1} \wedge \omega_{\alpha p+2} - \frac{1}{64\pi^3}(z_{p+1}^2 + z_{p+2}^2)\sum_{\alpha_1,\alpha_2} z_{\alpha_1}z_{\alpha_2}\,\omega_{\alpha_1 p+1} \wedge \omega_{\alpha_2 p+2}.$$

Proof. We have

$$\frac{1}{2}\omega(X_{\alpha p+1} \pm iX_{\alpha p+2}) = 2\pi\frac{\partial}{\partial z_\alpha}\left(\frac{\partial}{\partial z_{p+1}} \pm i\frac{\partial}{\partial z_{p+2}}\right) - \frac{1}{8\pi}z_\alpha(z_{p+1} \pm iz_{p+2}).$$

We first see

$$\bar{\partial}(\varphi_0) = -\frac{1}{8\pi}\sum_\alpha z_\alpha(z_{p+1} + iz_{p+2})(\omega_{\alpha p+1} - i\omega_{\alpha p+2})$$

and then

$$\partial\bar{\partial}(\varphi_0) = -\frac{1}{2}\sum_\alpha (\omega_{\alpha p+1} + i\omega_{\alpha p+2}) \wedge (\omega_{\alpha p+1} + i\omega_{\alpha p+2})$$

$$-\frac{1}{64\pi^2}\sum_{\alpha_1,\alpha_2} z_{\alpha_1}z_{\alpha_2}(z_{p+1} - iz_{p+2})(z_{p+1} + iz_{p+2})$$

$$\times (\omega_{\alpha_1 p+1} + i\omega_{\alpha_1 p+2}) \wedge (\omega_{\alpha_2 p+1} - i\omega_{\alpha_2 p+2})$$

$$= +i\sum_\alpha \omega_{\alpha p+1} \wedge \omega_{\alpha p+2}$$

$$-\frac{i}{32\pi^2}\sum_{\alpha_1,\alpha_2} z_{\alpha_1}z_{\alpha_2}(z_{p+1}^2 + z_{p+2}^2)\,\omega_{\alpha_1 p+1} \wedge \omega_{\alpha_2 p+2}.$$

The lemma follows. $\qquad\square$

Remark 4.7. Theorem 4.2 states that there is $(q-1)$-form ψ such that $\omega(L)\varphi_{KM} = d\psi_{KM}$ (for any signature). So Theorem 4.4 suggests (and it is easily checked) that in the Hermitian case one has

$$(4.8) \qquad\qquad\qquad \psi = -d^c\varphi_0.$$

From this perspective we can consider Theorem 4.4 as the Weil representation-theoretic analogue of the dd^c-Lemma in complex geometry.

5. The theta lifts

5.1. The Kudla-Millson lift.

We consider the Schwartz function φ_{KM} from the previous section. Since φ_{KM} is closed and an eigenfunction of K' of weight $\kappa = (p+q)/2$, we get

$$\Theta(\tau, z, \varphi_{KM}) \in A_{\kappa,L} \otimes \mathcal{Z}^q(X),$$

i.e., $\Theta(\tau, z, \varphi_{KM})$ is a non-holomorphic modular form of weight κ associated to the representation ϱ_L with values in $\mathcal{Z}^q(X)$, the closed q-forms on X. (In fact, it extends to a closed q-form on the Borel-Serre compactification $\bar{X}$ of X, see [FM2]). We then can consider the assignment

$$\eta \mapsto \Lambda_{KM}(\tau, \eta) := \int_X \eta \wedge \Theta(\tau, z, \varphi_{KM}),$$

which is defined for rapidly decreasing $(p-1)q$-forms on X. This map factors through $H_c^{(p-1)q}(X)$, the de Rham cohomology with compact support of X.

Theorem 5.1 (Kudla-Millson, see [KM3]). *For η closed and rapidly decreasing, the function $\Lambda_{KM}(\tau, \eta)$ is holomorphic on $\mathbb{H}$ and at the cusp (even though $\Theta(\tau, z, \varphi_{KM})$ is not). We therefore have a map*

$$\Lambda_{KM} : H_c^{(p-1)q}(X) \longrightarrow M_{\kappa,L}.$$

The Fourier expansion is given by

$$\Lambda_{KM}(\tau, \eta)_h = \delta_{h0}\left(\int_X \eta \wedge e_q\right) + \sum_{n>0}\left(\int_{Z(h,n)} \eta\right) e(n\tau),$$

i.e., the n-th Fourier coefficient of $\Theta(\tau, z, \varphi_{KM})_h$ *is a Poincaré dual form of the cycle* $Z(h, n)$. *Here (for q even),* e_q *is the Euler form of the symmetric space* D, *see (4.1). For q odd,* $\Lambda_{KM}(\tau, \eta)$ *is a cusp form.*

Remark 5.2. For $p = 1$ and X non-compact, we have $H_c^0(X) = 0$, so that Theorem 5.1 would be empty. However, in that case, the q-form $\Theta(\tau, z, \varphi_{KM})$ is actually rapidly decreasing on X, see [FM2], and therefore defines a class in $H_c^q(X)$. This corresponds to the fact that the cycles $Z(h, n)$ are a collection of points in this situation. Hence Λ_{KM} is defined on $H^0(X) \simeq \mathbb{R}$ and for $q \geq 3$, the lift $\Lambda_{KM}(\tau, \eta)$ is a holomorphic modular form ([FM2]), while for $q = 2$, it is *non*-holomorphic, see [Fu].

5.2. **The singular theta lift.**

5.2.1. *The Hermitian case.* Here we assume that V has signature $(p, 2)$ such that the corresponding symmetric domain is Hermitian. We put $k = 1 - p/2$ and $\kappa = 1 + p/2$. In this section we briefly recall some facts on the Borcherds lift from weakly holomorphic elliptic modular forms of weight k to automorphic forms on the orthogonal group $\mathrm{O}(V(\mathbb{R}))$ [Bo1] and its generalization to weak Maass forms [Br1].

We consider the Siegel theta function $\Theta(\tau, z, \varphi_0^{p,2}) \in A_{-k,L} \otimes C^\infty(X)$. The (additive) Borcherds lift of a weak Maass form $f \in H_{k,L^-}^+$ is defined by the theta integral

$$(5.1) \qquad \Phi(z, f) = \Phi(z, f, \varphi_0) = \int_{\Gamma' \backslash \mathbb{H}}^{reg} \langle f(\tau), \overline{\Theta(\tau, z, \varphi_0^{p,2})} \rangle \, d\mu.$$

The integral is typically divergent and needs to be regularized in the following way (indicated by the superscript "reg"): If F is a Γ'-invariant function on $\mathbb{H}$, we consider for an additional complex parameter $s \in \mathbb{C}$ the function

$$(5.2) \qquad \lim_{t \to \infty} \int_{\mathcal{F}_t} F(\tau) v^{-s} \, d\mu,$$

where $\mathcal{F}_t$ denotes the truncated fundamental domain (3.11) for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Formally, at $s = 0$, the quantity (5.2) equals the usual integral of F over $\Gamma' \backslash \mathbb{H}$. However, even if this diverges, (5.2) sometimes converges for $\Re(s) \gg 0$ and has a meromorphic continuation to a neighborhood of $s = 0$. Then we define

$$(5.3) \qquad \int_{\Gamma' \backslash \mathbb{H}}^{reg} F(\tau) \, d\mu$$

to be the constant term in the Laurent expansion of (5.2) at $s = 0$. So one feature of the regularization consists in prescribing the order of integration, the other in introducing s and looking at the Laurent expansion. The above regularization of the theta integral for weakly holomorphic modular forms was discovered by Harvey and Moore [HM] and vastly generalized by Borcherds [Bo1]. In connection with weak Maass forms it was investigated in [Br1].

Remark 5.3. Note that in [Bo1] and [Br1] the Borcherds lift is actually defined for signature $(2, p)$. Identifying the symmetric spaces $D_{p,2}$ and $D_{2,p}$ and switching to the space V^- one has

$$\Phi(z, f) = \big(f(\tau), \Theta(\tau, z, \varphi_0^{2,p}, L^-)\big)_{k,L^-}^{reg},$$

where the integral defining the Peterson scalar product is regularized as above. The right hand side is the definition used in [Bo1] and [Br1]. The definition given here is more convenient for our purposes.

Proposition 5.4. *Denote the Fourier expansion of $f \in H^+_{k,L^-}$ as in (3.2). The regularized integral for $\Phi(z,f)$ converges to a Γ-invariant C^∞ function on D with a logarithmic singularity along the divisor*

$$-2 \sum_h \sum_{n \in \mathbb{Q}_{<0}} a^+(h,n) Z(h,-n).$$

Proof. See [Bo1] section 6 and [Br1] section 2.2. $\qquad\square$

We list some further properties of $\Phi(z,f)$: Outside its singularities, the function $\Phi(z,f)$ is almost an eigenfunction of the invariant Laplacian Δ on D induced by the Casimir element of the universal enveloping algebra of the Lie algebra of $O(V(\mathbb{R}))$. More precisely, if we normalize Δ as in [Br1] chapter 4.1, we have $\Delta\Phi(z,f) = \frac{p}{4}a^+(0,0)$. Consequently, since Δ is an elliptic differential operator on D, the function $\Phi(z,f)$ is actually real analytic outside its singularities.

It is proved in [Br1] chapter 3.2 that $\Phi(z,f)$ can be split into a sum

$$(5.4) \qquad\qquad \Phi(z,f) = -2\log|\Psi(z,f)| + \xi(z,f),$$

where $\xi(z,f)$ is real analytic on the whole domain D and $\Psi(z,f)$ is a meromorphic function on D whose divisor equals

$$(5.5) \qquad\qquad Z(f) := \sum_h \sum_{n \in \mathbb{Q}_{<0}} a^+(h,n) Z(h,-n).$$

If $f \in M^!_{k,L^-}$ and its Fourier coefficients $a^+(h,n)$ with negative index are integral, then $\xi(z,f)$ reduces to a "simple" function and $\Psi(z,f)$ is a meromorphic modular form for the group Γ of weight $a^+(0,0)$. This is the (multiplicative) Borcherds lift from $M^!_{k,L^-}$ to meromorphic modular forms for Γ. If f is not holomorphic, then $\Psi(z,f)$ is far from being modular under Γ. (Caution: It is in general not true that $\xi(z,f)$ is equal to the regularized theta integral of f^- as one might think.)

From these considerations it can be deduced that the $(1,1)$-form

$$(5.6) \qquad\qquad \Lambda_B(f) := dd^c\Phi(z,f) = dd^c\xi(z,f)$$

is closed, harmonic, Γ-invariant, and of moderate growth. It represents the Chern class of the divisor (5.5) in the second cohomology $H^2(X)$. In particular, we have

$$(5.7) \qquad\qquad \int_{Z(f)} \eta = \int_X \eta \wedge \Lambda_B(f)$$

for any rapidly decreasing $2(p-1)$-form η. If $f \in M^!_{k,L^-}$, then $\Lambda_B(f) = a^+(0,0)\Omega$, i.e., a multiple of the Kähler form $\Omega = -e_2$ on D and actually vanishes for $f \in M^{!!}_{k,L^-}$ and $k < 0$.

We summarize part of the above discussion in the following theorem.

Theorem 5.5. *(See* [Br1] *chapter 5.) The assignment $f \mapsto \Lambda_B(f)$ defines a linear mapping $H^+_{k,L^-} \to \mathcal{Z}^{1,1}_{mg}(X)$ to the space $\mathcal{Z}^{1,1}_{mg}(X)$ of closed $(1,1)$-forms of moderate growth on X. The induced mapping $H^+_{k,L^-} \to \mathcal{Z}^{1,1}_{mg}(X)/\mathbb{C}\Omega$ factors through $H^+_{k,L^-}/M^!_{k,L^-}$.*

5.2.2. *The real case.* We now consider the case of general signature (p,q). If $x \in V(\mathbb{R})$, we put $|x| = |(x,x)|^{1/2}$. Moreover, as before we let $\kappa = (p+q)/2$ and $k = 2 - \kappa$. The Schwartz form $\psi = \psi_{KM}$ (see section 4) has weight $-k = (q+p)/2 - 2$ by Theorem 4.2 (ii), hence

$$\Theta(\tau, z, \psi) \in A_{-k,L} \otimes \mathcal{A}^{q-1}(X).$$

Note that the components of $\Theta(\tau, z, \psi)$ are given by

$$(\Theta(\tau, z, \psi))_h = v^{1-q/2} \sum_{\lambda \in L+h} P_\psi(\sqrt{v}\lambda, z) \varphi^{p,q}_0(\lambda, \tau, z),$$

with a (in general non-homogeneous) polynomial $P_\psi(x, z) \in [\mathcal{P}(V(\mathbb{R})) \otimes \mathcal{A}^{q-1}(D)]^G$ of degree q. By Lemma 4.1, P_ψ can be written as

$$(5.8) \qquad P_\psi(x, z) = \sum_{\underline{\alpha}} Q_{\underline{\alpha}}(x, z) R_{\underline{\alpha}}(x, z),$$

where the sum extends over all multi-indices $\underline{\alpha} = (\alpha_1, \ldots, \alpha_{q-1})$. Here $Q_{\underline{\alpha}}$ is polynomial in $[\mathcal{P}(V(\mathbb{R})) \otimes \mathcal{A}^0(D)]^G$ of degree $q-1$ (in fact, a product of Hermite functions as for φ_{KM}), which only depends on $x_{z^\perp}$; while $R_{\underline{\alpha}}$ is a homogeneous linear polynomial in $[\mathcal{P}(V(\mathbb{R})) \otimes \mathcal{A}^{q-1}(D)]^G$, which only depends on x_z. In particular, we have $P_\psi(0, z) = 0$. We normalize the polynomials such that at the base point z_0, we have

$$R_{\underline{\alpha}}(x) = R_{\underline{\alpha}}(x, z_0) = \frac{-1}{\sqrt{2}(q-1)!} \det \begin{pmatrix} x_{p+1} & \cdots & x_{p+q} \\ \omega_{\alpha_1 p+1} & \cdots & \omega_{\alpha_1 p+q} \\ \vdots & & \vdots \\ \omega_{\alpha_{q-1} p+1} & \cdots & \omega_{\alpha_{q-1} p+q} \end{pmatrix},$$

and the leading term of $Q_{\underline{\alpha}}(x) = Q_{\underline{\alpha}}(x, z_0)$ is given by $2^{(q-1)/2} \prod_{i=1}^{q-1} x_{\alpha_i}$.

Let $f \in H^+_{k,L^-}$ be a weak Maass form. We then define a singular theta lift of Borcherds' type for signature (p,q) by

$$(5.9) \qquad \Phi(z, f, \psi) := \left(\Theta(\tau, z, \psi), v^k \overline{f}\right)^{reg}_{-k,L} = \int_{\Gamma' \backslash \mathbb{H}}^{reg} \langle \Theta(\tau, z, \psi), \overline{f} \rangle \, d\mu.$$

We define the cycle $Z(f)$ associated to f by (5.5), as in the Hermitian case. Note that the Fourier coefficients satisfy $a^+(-h, n) = (-1)^q a^+(h, n)$, corresponding to $Z(-h, n) = (-1)^q Z(h, n)$. In particular, we have $Z(F_{h,m}) = 2Z(h, -m)$, where $F_{h,m}$ is the Hejhal Poincaré series, see Remark 3.10.

Proposition 5.6. *The regularized integral for $\Phi(z, f, \psi)$ converges to a Γ-invariant C^∞ function on D with singularities along $Z(f)$. More precisely, there exist G-invariant scalar-valued polynomials $\widetilde{Q}_{\underline{\alpha}}(x, z) \in [\mathcal{P}(V(\mathbb{R})) \otimes \mathcal{A}^0(D)]^G$ of degree $(q-1)$ such that in a small neighborhood of $w \in D$ the singularity is of type*

$$\sum_h \sum_{n \in \mathbb{Q}_{<0}} a^+(h, n) \sum_{\lambda \in w^\perp \cap L_{h,-n}} \sum_{\underline{\alpha}} \widetilde{Q}_{\underline{\alpha}} \left(\lambda_{z^\perp}/|\lambda_z|, z\right) R_{\underline{\alpha}} \left(\lambda_z/|\lambda_z|, z\right).$$

The leading term $\widetilde{Q}_{\underline{\alpha},q-1}$ of $\widetilde{Q}_{\underline{\alpha}}$ at the base point z_0 is given by

$$\widetilde{Q}_{\underline{\alpha},q-1}(x) = \frac{\Gamma\left(q/2\right)}{\sqrt{2}\pi^{q/2}} \prod_{i=1}^{q-1} x_{\alpha_i}.$$

Note that all sums in the above formula are finite. The polynomials $\widetilde{Q}_{\underline{\alpha}}$ and $R_{\underline{\alpha}}$ only depend on the signature (p,q) and not on f.

Example 5.7. In the hyperbolic case $(q = 1)$, we have $\widetilde{Q} = 1/\sqrt{2}$ and $R(x,z) = \frac{1}{\sqrt{2}}(x, gv_{p+q})$ with $z = gz_0$ for some $g \in G$, and the singularity near $w \in D$ is given by

$$\frac{1}{2} \sum_h \sum_{n \in \mathbb{Q}_{<0}} a^+(h,n) \sum_{\lambda \in w^\perp \cap L_{h,-n}} \sum_{\underline{\alpha}} \text{sgn}\left(\lambda, gv_{p+q}\right).$$

In particular, $\Phi(z,f,\psi)$ is locally bounded.

Proof. The argument follows in large parts [Bo1], section 6. Since the function f^- is exponentially decreasing as $v \to \infty$, and $\Theta(\tau,z,\psi)$ has moderate growth, the integral

$$\int_{\mathcal{F}} \langle f^-(\tau), \overline{\Theta(\tau,z,\psi)} \rangle \, d\mu$$

over the standard fundamental domain $\mathcal{F} = \{\tau \in \mathbb{H}; \; |\tau| \geq 1, \; |u| \leq 1/2\}$ for $\text{SL}_2(\mathbb{Z})$ converges absolutely and defines a real analytic function on D. Moreover, the integral of $\langle f^+(\tau), \overline{\Theta(\tau,z,\psi)} \rangle$ over the compact subset $\mathcal{F}_1 \subset \mathbb{H}$ converges absolutely and defines a real analytic function on D. Hence we are left with considering the function

$$h(z,s) = \int_1^\infty \int_{-1/2}^{1/2} \langle f^+(\tau), \overline{\Theta(\tau,z,\psi)} \rangle v^{-2-s} \, du \, dv.$$

If we insert the Fourier expansions, the integration over u picks out the constant term in the Fourier expansion of $\langle f^+(\tau), \overline{\Theta(\tau,z,\psi)} \rangle$. Thus

$$(5.10) \qquad h(z,s) = \sum_{\lambda \in L^\#} a^+(\lambda, -q(\lambda)) \int_1^\infty P_\psi(\sqrt{v}\lambda, z) \exp(4\pi v q(\lambda_z)) v^{-1-s} dv.$$

(Note that $q(\lambda_z) \leq 0$ for all λ in the above sum.) By the growth of f, it has only finitely many non-zero Fourier coefficients $a^+(h,n)$ with negative index n. Hence, if $U \subset D$ is a relatively compact neighborhood of w and $\varepsilon > 0$, then by reduction theory the set

$$S_f(U,\varepsilon) = \left\{\lambda \in L^\#; \quad a^+(\lambda, -q(\lambda)) \neq 0 \text{ and } |q(\lambda_z)| < \varepsilon \text{ for some } z \in U\right\}$$

is finite. We split the sum over $\lambda \in L^\#$ in (5.10) into the sum over $\lambda \in S_f(U,\varepsilon)$ and the sum over $\lambda \in L^\# - S_f(U,\varepsilon)$. The latter sum is up to a constant factor majorized by

$$\sum_{\lambda \in L^\# - S_f(U,\varepsilon)} |a^+(\lambda, -q(\lambda))| \exp(2\pi q(\lambda_z)),$$

locally uniformly in $s \in \mathbb{C}$ and $z \in U$. According to Lemma 3.4 there exists a constant $C > 0$ such that this sum is majorized by

$$\sum_{\substack{\lambda \in L^\# \\ a^+(\lambda, -q(\lambda)) \neq 0}} \exp\left(C\sqrt{|q(\lambda)|} + 2\pi q(\lambda_z)\right)$$

$$= \sum_{\substack{\lambda \in L^\# \\ a^+(\lambda, -q(\lambda)) \neq 0}} \exp\left(C\sqrt{|q(\lambda)|} + \pi q(\lambda)\right) \exp\left(-\frac{\pi}{2}(\lambda, \lambda)_z\right) \ll \sum_{\lambda \in L^\#} \exp\left(-\frac{\pi}{2}(\lambda, \lambda)_z\right),$$

which clearly converges. We may conclude that the sum over $\lambda \in L^\# - S_f(U, \varepsilon)$ in (5.10) (and all its derivatives) converges locally uniformly absolutely for $s \in \mathbb{C}$ and $z \in U$. In particular, it defines a C^∞ function for $z \in U$ at $s = 0$.

Hence the singularity of $\Phi(z, f, \psi)$ for $z \in U$ is dictated by the finite sum

$$h_1(z, s) = \sum_{\lambda \in S_f(U, \varepsilon)} a^+(\lambda, -q(\lambda)) \int_1^\infty P_\psi(\sqrt{v}\lambda, z) \exp(4\pi v q(\lambda_z)) v^{-1-s} dv.$$

Note that so far the argument works for any polynomial $P(x, z)$.

Since $P_\psi(0, z) = 0$, the term for $\lambda = 0$ does not contribute (and this implies that $h_1(z, s)$ will be actually holomorphic at $s = 0$). We are left to consider (for $\lambda \neq 0$)

$$
\begin{aligned}
\tilde{\psi}(\lambda, z, s) &:= \int_1^\infty P_\psi(\sqrt{v}\lambda, z) \exp(4\pi v q(\lambda_z)) v^{-1-s} dv \\
&= \sum_{\underline{\alpha}} R_{\underline{\alpha}}(\lambda_z, z) \int_1^\infty Q_{\underline{\alpha}}(\sqrt{v}\lambda_{z^\perp}, z) \exp(4\pi v q(\lambda_z)) v^{1/2-s} \frac{dv}{v} \\
&= \sum_{\underline{\alpha}} R_{\underline{\alpha}}(\lambda_z, z) \sum_{\ell=0}^{q-1} Q_{\underline{\alpha}, \ell}(\lambda_{z^\perp}, z) \int_1^\infty \exp(4\pi v q(\lambda_z)) v^{\ell/2+1/2-s} \frac{dv}{v} \\
&= \sum_{\underline{\alpha}} R_{\underline{\alpha}}(\lambda_z, z) \sum_{\ell=0}^{q-1} Q_{\underline{\alpha}, \ell}(\lambda_{z^\perp}, z) |4\pi q(\lambda_z)|^{s-(\ell+1)/2} \Gamma\left(\frac{\ell+1}{2} - s, |4\pi q(\lambda_z)|\right).
\end{aligned}
$$

(5.11)

Here $Q_{\underline{\alpha}, \ell}(\lambda_{z^\perp}, z)$ denotes the homogeneous component of degree ℓ of $Q_{\underline{\alpha}}(\lambda_{z^\perp}, z)$ and $\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt$ is the incomplete Gamma function as in [AbSt] (6.5.3). This shows that $h_1(z, s)$ has a meromorphic continuation in s to the whole complex plane and is holomorphic at $s = 0$. The claim now follows from the recurrence relations of the incomplete Gamma function and from $\Gamma(q/2, |4\pi q(\lambda_z)|) = \Gamma(q/2) + O(|\lambda_z|)$ as $\lambda_z \to 0$. $\square$

Analogously to the Hermitian situation we define the map Λ_ψ by

$$\Lambda_\psi(z, f) = -d\Phi(z, f, \psi). \tag{5.12}$$

This is the analogue to the map Λ_B considered in the Hermitian case. The corresponding geometric properties will easily follow from the relationship to the Kudla-Millson lift which we establish in the next section.

Remark 5.8. In the Hermitian case of signature $(p, 2)$ we now have defined two singular theta lifts, the Borcherds lift $\Phi(z, f) = \Phi(z, f, \varphi_0)$ and $\Phi(z, f, \psi) = \Phi(z, f, -d^c\varphi_0)$. We will see later that we actually have (outside $Z(f)$)

$$d^c\Phi(z, f) = -\Phi(z, f, \psi);$$

i.e., one can interchange the differential operator d^c on the orthogonal domain D with the regularized $\mathrm{SL}_2(\mathbb{R})$-integral.

Remark 5.9. Analogously to [Br1], chapter 4, one can show that $\Phi(z, f, \psi)$ is harmonic, i.e., annihilated by the Laplacian for the symmetric space D. In particular, $\Phi(z, f, \psi)$ is real analytic.

6. Main results

Recall (with $\kappa = (p + q)/2$ and $k = 2 - \kappa$) the map $\xi_k(f) = R_{-k}(v^k\bar{f}) \in S_{\kappa,L}$ for $f \in H^+_{k,L^-}$. Since $\Theta(\tau, z, \varphi_{KM})$ is moderately increasing in τ, we can therefore study the scalar product

$$(\Theta(\tau, z, \varphi_{KM}), \xi_k(f))_{\kappa,L} \in \mathcal{Z}^q(X).$$

Theorem 6.1. *Let V be of signature (p, q) and let $f \in H^+_{k,L^-}$. Then, outside $Z(f)$, the set of singularities of the lift $\Phi(z, f, \psi)$, we have*

$$(6.1) \qquad (\Theta(z, \varphi_{KM}), \xi_k(f))_{\kappa,L} - a^+(0, 0)e_q = -d\Phi(z, f, \psi) = \Lambda_\psi(z, f).$$

For the Borcherds lift in signature $(p, 2)$, we have, outside $Z(f)$,

$$(6.2) \qquad (\Theta(z, \varphi_{KM}), \xi_k(f))_{\kappa,L} + a^+(0, 0)\Omega = dd^c\Phi(z, f) = \Lambda_B(z, f).$$

Here $a^+(0, 0)$ is the constant coefficient of the Fourier expansion of f, e_q is the Euler form on D for q even (and zero for q odd), and $\Omega = -e_2$ is the Kähler form for the Hermitian domain $D_{p,2}$.

Theorem 6.2.

   (i) $\Lambda_\psi(f)$ *extends to a smooth closed q-form of moderate growth on X.*
   (ii) *If $f \in M^!_{k,L^-}$ is weakly holomorphic, then*

$$\Lambda_\psi(f) = -a^+(0, 0)e_q.$$

   (iii) *In particular, Λ_ψ induces a map*

$$\Lambda_\psi : H^+_{k,L^-}/M^!_{k,L^-} \longrightarrow H^q(X)/\mathbb{C}e_q.$$

Proof. This follows immediately from Theorem 6.1: The left hand side of (6.1) is smooth and has moderate growth, which gives (i). The assertion (ii) follows from $\xi_k(f) = 0$ for $f \in M^!_{k,L^-}$. $\qquad\square$

Observe that in view of (6.2) one obtains new proofs of the analogous properties of the Borcherds lift for $O(p, 2)$ listed in section 6.

We denote by $(\cdot, \cdot)_X$ the natural pairing between $\mathcal{Z}^{qp-\ell}_{rd}(X)$, the closed rapidly decreasing $(qp-\ell)$-forms, and $\mathcal{Z}^\ell_{mg}(X)$, the closed ℓ-forms of moderate growth. Recall that by Poincaré duality this induces a non-degenerate pairing between $H^{qp-\ell}_c(X)$ and $H^\ell(X)$, see [BT].

On the other hand, recall the pairings $\{\,,\,\}$ and $\{\,,\,\}'$ between $M_{\kappa,L}$ and H_{k,L^-}^+ defined in (3.9) and (3.15).

Theorem 6.3. *The Kudla-Millson lift $\Lambda_{KM} : \mathcal{Z}_{rd}^{q(p-1)}(X) \to M_{\kappa,L}$ and the lift $\Lambda_\psi : H_{k,L^-}^+ \to \mathcal{Z}_{mg}^q(X)$ are adjoint in the following sense: For any $\eta \in \mathcal{Z}_{rd}^{q(p-1)}(X)$ and any $f \in H_{k,L^-}^+$ we have*

$$\left(\eta, \Lambda_\psi(f)\right)_X = \{\Lambda_{KM}(\eta), f\}'.$$

Moreover, this duality factors through the non-degenerate pairings on the cohomology level and the one of $M_{\kappa,L}$ with $H_{k,L^-}^+/M_{k,L^-}^{!!}$.

Proof. Using (6.1) we see that

$$
\begin{aligned}
\left(\eta, \Lambda_\psi(f)\right)_X + a^+(0,0)\left(\eta, e_q\right)_X &= \left(\eta, \left(\Theta_{KM}(\tau, Z), \xi_k(f)\right)_{\kappa,L}\right)_X \\
&= \left(\left(\eta, \Theta_{KM}(\tau, Z)\right)_X, \xi_k(f)\right)_{\kappa,L} \\
&= \{\Lambda_{KM}(\eta), f\}.
\end{aligned}
$$

Here, in the second equality, we have exchanged the order of integration. This is valid because the latter integrals converge absolutely. Now the assertion follows from the Fourier expansion of the Kudla-Millson lift (see Theorem 5.1) and the definition of the pairing $\{\,,\,\}'$. $\square$

Remark 6.4. In signature $(p, 2)$, the same statement of Theorem 6.3 holds with Λ_ψ replaced by the lift Λ_B.

Corollary 6.5. *For $\eta \in \mathcal{Z}_{rd}^{q(p-1)}(X)$ and $f \in H_{k,L^-}^+$, we have*

$$\left(\eta, \Lambda_\psi(f)\right)_X = \int_{Z(f)} \eta.$$

For q even, this also holds with $\eta = e_q^{p-1}$. In particular, $\Lambda_\psi(f)$ is a harmonic representative of the Poincaré dual form of the cycle $Z(f)$.

Proof. By Theorem 5.1 and Proposition 3.5, we have

$$
\begin{aligned}
\{\Lambda_{KM}(\eta), f\} &= a^+(0,0)\int_X \eta \wedge e_q + \sum_{h \in L^\#/L} \sum_{n<0} a^+(h,n) \int_{Z(h,n)} \eta \\
&= a^+(0,0)\left(\eta, e_q\right)_X + \int_{Z(f)} \eta.
\end{aligned}
$$

The claim now follows from Theorem 6.3. The above calculation also holds for $\eta = e_q^{p-1}$, see [KM4]; hence the corollary is valid in this case as well. $\Lambda_\psi(f)$ is harmonic by Remark 5.9 or alternatively by [KM4], Theorem 4.1. $\square$

Proof of Theorem 6.1. We would like to use the adjointness of R_{-k} and $-L_{2-k}$ to compute

$$(\Theta(z, \varphi_{KM}), \xi_k(f))_{2-k,L} = (\Theta(z, \varphi_{KM}), R_{-k}(v^k \bar{f}))_{2-k,L}.$$

However, due to the rapid growth of f, the scalar product $(L_{2-k}\Theta(z, \varphi_{KM}), v^k \bar{f})_{-k,L}$ does not converge. We have to proceed more carefully.

Let $\mathcal{F}_t$ be the truncated fundamental domain (see (3.11)) for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. For $g \in A_{2-k,L}$ and $h \in A_{-k,L^-}$, we then have (see [Br1], Lemma 4.2, correcting a sign error)

$$(6.3) \quad \int_{\mathcal{F}_t} \langle g, R_{-k}h \rangle v^{-k}\, du\, dv = -\int_{\mathcal{F}_t} \langle L_{2-k}g, h \rangle v^{2-k}\, du\, dv + \int_{-1/2}^{1/2} \langle g(u+it), h(u+it) \rangle v^{-k}\, du.$$

Note that we do not require any regularity for g or h at the cusp. We apply (6.3) for $g = \Theta(\tau, z, \varphi_{KM})$ and $h = v^k \bar{f}$ and obtain

Lemma 6.6.

$$(6.4) \qquad (\Theta(z, \varphi_{KM}), R_{-k}(v^k \bar{f}))_{2-k,L} = \lim_{t \to \infty} \int_{\mathcal{F}_t} \langle \Theta(\tau, z, \varphi_{KM}), R_{-k} v^k \bar{f} \rangle v^{-k}\, du\, dv$$

$$(6.5) \qquad\qquad = \lim_{t \to \infty} \left(-\int_{\mathcal{F}_t} \langle L_{2-k}\Theta(\tau, z, \varphi_{KM}), \bar{f} \rangle \frac{du\, dv}{v^2} \right.$$

$$(6.6) \qquad\qquad\qquad \left. + \int_{-1/2}^{1/2} \langle \Theta(u+it, z, \varphi_{KM}), \overline{f(u+it)} \rangle\, du \right).$$

Next, we would like to compute the limit in (6.5) and (6.6) separately. We will see that this is possible for $z \notin Z(f)$, while for $z \in Z(f)$ the limits of both terms do not exist (in (6.5) and (6.6), the singularities cancel each other out). We first consider the term (6.6). The integration picks out the 0-th Fourier coefficient of $\langle \Theta(u+it, z, \varphi_{KM}), \overline{f(u+it)} \rangle$.

Lemma 6.7. *Outside $Z(f)$, we have*

$$\lim_{t \to \infty} \int_{-1/2}^{1/2} \langle \Theta(u+it, z, \varphi_{KM}), \overline{f(u+it)} \rangle\, du = a^+(0,0)e_q,$$

while the limit does not exist for $z \in Z(f)$. In particular, the limit defines a smooth differential form on $D - Z(f)$, which extends smoothly to D.

Proof. Since f^- is rapidly decreasing in v, we see

$$\lim_{t \to \infty} \int_{-1/2}^{1/2} \langle \Theta(u+it, z, \varphi_{KM}), \overline{f^-(u+it)} \rangle\, du = 0.$$

We write $\varphi_{KM}(x, z) = P_{KM}(x, z)\varphi_0(x, z)$, where P_{KM} is a polynomial in $[\mathcal{P}(V(\mathbb{R})) \otimes \mathcal{A}^q(D)]^G$ only depending on $x_{z^\perp}$. For f^+, one then has

$$\int_{-1/2}^{1/2} \langle \Theta(u+it, z, \varphi_{KM}), \overline{f^+(u+it)} \rangle\, du$$

$$= \varphi_{KM}(0)a^+(0,0) + \sum_{\lambda \in L^\# - 0} P_{KM}(\sqrt{t}\lambda, z)a^+(\lambda, -q(\lambda))e^{4\pi q(\lambda_z)t}.$$

For $z \in Z(f)$, hence $\lambda_z = 0$, the sum clearly diverges as $t \to \infty$, otherwise, the sum is rapidly decreasing in t by arguments similar to the proof of Proposition 5.6. The lemma now follows from $\varphi_{KM}(0) = e_q$, see (4.1). $\qquad\square$

Lemma 6.8. *The quantity in (6.5) is given by*

$$\lim_{t\to\infty}\int_{\mathcal{F}_t}\langle L_{2-k}\Theta(\tau,z,\varphi_{KM}),\bar{f}\rangle\,d\mu = \int_{\Gamma'\backslash\mathbb{H}}^{reg}\langle L_{2-k}\Theta(\tau,z,\varphi_{KM}),\bar{f}\rangle\,d\mu$$
$$= \int_{\Gamma'\backslash\mathbb{H}}^{reg}\langle\Theta(\tau,z,d\psi),\bar{f}\rangle\,d\mu.$$

This defines a smooth form on $D - Z(f)$ which extends smoothly to all of D.

Proof. The second equality follows from the fundamental fact

$$L_{2-k}\Theta(\tau,z,\varphi_{KM}) = \Theta(\tau,z,d\psi),$$

which is Theorem 4.2. From the Lemmas 6.6 and 6.7 it follows that the limit exists, defining a smooth form outside $Z(f)$. As (6.4) defines a smooth form on D, the limit extends to all of D. As the limit exists, it is per definitionem the regularized integral. (This corresponds to $L\varphi_{KM}(0) = d\psi(0) = 0$.) □

Combining Lemmas 6.6, 6.7, 6.8 we obtain outside $Z(f)$:

$$(6.7)\qquad (\Theta(z,\varphi_{KM}),\xi_k(f))_{2-k,L} = -\int_{\Gamma'\backslash\mathbb{H}}^{reg}\langle\Theta(\tau,z,d\psi),\bar{f}\rangle\,d\mu\ +\ a^+(0,0)e_q.$$

The identity (6.1) now follows from

Lemma 6.9.

$$\int_{\Gamma'\backslash\mathbb{H}}^{reg}\langle\Theta(\tau,z,d\psi),\bar{f}\rangle\,d\mu = d\int_{\Gamma'\backslash\mathbb{H}}^{reg}\langle\Theta(\tau,z,\psi),\bar{f}\rangle\,d\mu = d\Phi(z,f,\psi).$$

Proof. We recall that because of $P_\psi(0,z) = 0$, we have

$$\int_{\Gamma'\backslash\mathbb{H}}^{reg}\langle\Theta(\tau,z,\psi),\bar{f}\rangle\,d\mu = \lim_{t\to\infty}\int_{\mathcal{F}_t}\langle\Theta(\tau,z,\psi),\bar{f}\rangle\,d\mu.$$

Since $\mathcal{F}_t$ is compact, we only have to justify the interchange of the differentiation d with the limit $t \to \infty$. Arguing as in the proof of Proposition 5.6 (using the notation from there) we find that it suffices to show

$$(6.8)\qquad dh(z,0) = \sum_{\lambda\in L^\#}a^+(\lambda,-q(\lambda))\int_1^\infty dP_\psi(\sqrt{v}\lambda,z)\exp(4\pi vq(\lambda_z))\frac{dv}{v}.$$

If $U \subset D$ is any relatively compact open subset and $\varepsilon > 0$, we split the sum over $\lambda \in L^\#$ into the finite sum over $\lambda \in S_f(U,\varepsilon)$ and the sum over $\lambda \in L^\# - S_f(U,\varepsilon)$. An estimate for the latter sum as in Proposition 5.6 shows that it converges uniformly absolutely on U, justifying the interchange of differentiation and the limit. Thus it suffices to show that

$$d\int_1^\infty P_\psi(\sqrt{v}\lambda,z)\exp(4\pi vq(\lambda_z))\frac{dv}{v} = \int_1^\infty dP_\psi(\sqrt{v}\lambda,z)\exp(4\pi vq(\lambda_z))\frac{dv}{v}$$

for $\lambda \in S_f(U,\varepsilon)$ and $z \in U - Z(f)$. This follows from the exponential decay of the integrands as $v \to \infty$. □

For (6.2), we note that in this case, we have $\psi = -d^c\varphi_0$, see (4.8). Therefore, we just need to show

$$(6.9) \qquad \int_{\Gamma'\backslash\mathbb{H}}^{reg} d^c\langle\Theta(\tau,z,\varphi_0),\bar{f}\rangle\,d\mu = d^c\int_{\Gamma'\backslash\mathbb{H}}^{reg}\langle\Theta(\tau,z,\varphi_0),\bar{f}\rangle\,d\mu = d^c\Phi(z,f,\varphi_0).$$

This goes through as above with one extra point: Since $\varphi_0(0) = 1 \neq 0$, the constant term of $\langle\Theta(\tau,z,\varphi_0),\bar{f}\rangle$ involves also the term $a^+(0,0)v$. One sees that the regularized integral is actually given by

$$(6.10) \qquad \int_{\Gamma'\backslash\mathbb{H}}^{reg}\langle\Theta(\tau,z,\varphi_0),\bar{f}\rangle\,d\mu = a^+(0,0)C + \lim_{t\to\infty}\int_{\mathcal{F}_t}\left(\langle\Theta(\tau,z,\varphi_0),\bar{f}\rangle - a^+(0,0)v\right)\,d\mu,$$

where C is a constant not depending on z (namely the constant term in the Laurent expansion at $s = 0$ of $\lim_{t\to\infty}\int_{\mathcal{F}_t} v^{1-s}d\mu$). Applying d^c to the right hand side, one can now interchange differentiation and the limit as above. One obtains (6.9), which implies (6.2).

This completes the proof of Theorem 6.1. □

7. The singular theta lift as a current

In this section, we consider the current which is induced by the singular theta lift $\Phi(f,\psi)$ defined in (5.9).

Consider a top degree form $\phi \in A^{pq}(\Gamma\backslash D)$. We then have

$$\int_{\Gamma\backslash D}\phi(z) = \left(\int_{\Gamma\backslash G}\phi(g)dg\right)(1_{\mathfrak{p}}),$$

where ϕ on the right hand side is considered as an element in $[C^\infty(\Gamma\backslash G)\otimes\bigwedge^{pq}\mathfrak{p}^*]^K$. (We will frequently use this identification without further comment.) Here $1_{\mathfrak{p}}$ is a properly oriented basis vector for $\bigwedge^{pq}\mathfrak{p} \simeq \mathbb{R}$ of length one with respect to the Killing form. Moreover, $dg = dz\,dk$, where dz is the measure on D coming form the Killing form and $\mathrm{vol}(K,dk) = 1$.

We now pick appropriate coordinates for D. We set $H = G_{v_1}$, the stabilizer in G of the first basis vector v_1 of V. Then H is the fixed point set of an involution τ on G. On the Lie algebra level we obtain a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ into ± 1 eigenspaces of τ. Then $L = H \cap K$ is a maximal compact subgroup of H. We write $\Gamma_H = \Gamma_{v_1}$. There is a diffeomorphism (see [KM2], section 4, [FJ], section 2)

$$\Psi : H \times_L (\mathfrak{p}\cap\mathfrak{q}) \longrightarrow D = G/K,$$
$$(h,Y) \mapsto h\exp(Y)K.$$

Let $a_t = \exp(tX_{1p+q})$ for $t \in \mathbb{R}$, and let $A = \{a_t;\ t \in \mathbb{R}\} \simeq \mathbb{R}$ be the one-parameter subgroup of G associated to the maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$ generated by X_{1p+q}. We write $A_\varepsilon = \{a_t;\ t \geq \varepsilon\}$. We have a decomposition $G = HAK$ and, with a positive constant C depending on the normalizations of the invariant measures, the integral formula (see [FJ], section 2)

$$\int_G\phi(g)dg = C\int_K\int_{A_0}\int_H\phi(ha_tk)|\sinh(t)|^{q-1}\cosh(t)^{p-1}dh\,dt\,dk.$$

(If $q = 1$, one has to replace A_0 by A, corresponding to the fact that in this case D_{v_1} splits D into two components).

Proposition 7.1. *The $(q-1)$-form $\Phi(f,\psi)$ is locally integrable, i.e.,*

$$\int_X \eta \wedge \Phi(f,\psi) < \infty$$

for $\eta \in A_c^{(p-1)q+1}(X)$.

Proof. For $q = 1$, the statement is clear since in that case $\Phi(f,\psi)$ is locally bounded, see Proposition 5.6 and Example 5.7. For $q \neq 1$, we need to show

$$\int_{\Gamma\backslash G} \eta(g) \wedge \Phi(f,\psi)(g)\, dg < \infty.$$

We can replace $\Phi(f,\psi)(g)$ by the part which gives rise to its singularities

$$\sum_h \sum_{n\in\mathbb{Q}_{<0}} a^+(h,n) \sum_{\lambda\in L_{h,-n}} \tilde{\psi}(g^{-1}\lambda)$$

with $\tilde{\psi}(\lambda) := \tilde{\psi}(\lambda, z_0, 0)$ given by setting $s = 0$ in (5.11). Now

$$\sum_{\lambda\in L_{h,-n}} \tilde{\psi}(g^{-1}\lambda) = \sum_{\lambda\in\Gamma\backslash L_{h,-n}} \sum_{\gamma\in\Gamma_\lambda\backslash\Gamma} \tilde{\psi}(\gamma^{-1}g^{-1}\lambda),$$

and therefore by unfolding

$$\int_{\Gamma\backslash G} \sum_h \sum_{n\in\mathbb{Q}_{<0}} a^+(h,n) \sum_{\lambda\in L_{h,-n}} \eta(g) \wedge \tilde{\psi}(g^{-1}\lambda)\, dg$$

$$= \sum_h \sum_{n\in\mathbb{Q}_{<0}} a^+(h,n) \sum_{\lambda\in\Gamma\backslash L_{h,-n}} \int_{\Gamma_\lambda\backslash G} \eta(g) \wedge \tilde{\psi}(g^{-1}\lambda)\, dg.$$

This is a finite sum (since $a^+(h,n) = 0$ for $n \ll 0$), so that we only have to show the existence of the integral for a given λ of positive length. We can assume $\lambda = \sqrt{m}v_1$ for some $m > 0$. We have

$$(7.1) \quad \int_{\Gamma_H\backslash G} \eta(g) \wedge \tilde{\psi}(g^{-1}\sqrt{m}v_1)\, dg$$

$$= C' \int_{\Gamma_H\backslash H} \int_0^\infty \eta(ha_t) \wedge \tilde{\psi}(a_t^{-1}h^{-1}\sqrt{m}v_1) \sinh(t)^{q-1} \cosh(t)^{p-1}\, dt\, dh$$

with a positive constant C'. But now $h^{-1}v_1 = v_1$ and

$$a_t^{-1}\sqrt{m}v_1 = \cosh(t)\sqrt{m}v_1 - \sinh(t)\sqrt{m}v_{p+q}.$$

Hence

$$(7.2\mathrm{a}) \qquad\qquad (a_t^{-1}\sqrt{m}v_1)_{z_0} = -\sinh(t)\sqrt{m}v_{p+q},$$

$$(7.2\mathrm{b}) \qquad\qquad (a_t^{-1}\sqrt{m}v_1)_{z_0^\perp} = \cosh(t)\sqrt{m}v_1,$$

so that, see (5.11),

$$(7.3) \quad \tilde{\psi}(a_t^{-1}\sqrt{m}v_1) = \frac{-1}{\sqrt{2\pi}} \sum_{\underline{\alpha}} R_{\underline{\alpha}}(v_{p+q}) \sum_{\ell=0}^{q-1} Q_{\underline{\alpha},\ell}\left(\sqrt{m}\cosh(t)v_1\right)$$
$$\times \left(2\pi m\sinh^2(t)\right)^{-\ell/2} \Gamma\left(\tfrac{\ell+1}{2}, 2\pi m\sinh^2(t)\right).$$

Therefore the integrand in (7.1) is bounded as $t \to 0$. On the other hand, $\tilde{\psi}(a_t^{-1}h^{-1}v_1)$ is exponentially decreasing in e^t (uniformly in h). Since η has compact support and $\Gamma_H\backslash H$ has finite volume, we conclude that (7.1) converges. Hence the above unfolding is valid and the proposition is proved. □

Recall that a locally integrable ℓ-form ω on X defines a current $[\omega]$ on X, i.e., a linear functional on $\mathcal{A}_c^{pq-\ell}(X)$, via

$$[\omega](\eta) := \int_X \eta \wedge \omega$$

for $\eta \in \mathcal{A}_c^{pq-\ell}(X)$. For a current T, we define its exterior derivative by

$$(dT)(\eta) = (-1)^{\deg(\eta)+1} T(d\eta).$$

Theorem 7.2. *Let $\delta_{Z(f)}$ be the delta-current for the special cycle $Z(f)$, and for a locally integrable differential form ω, we denote the associated current by $[\omega]$. Then*

$$(7.4) \qquad\qquad d[\Phi(f,\psi)] + \delta_{Z(f)} = [\Lambda_\psi(f)].$$

Proof. Let $\eta \in \mathcal{A}_c^{(p-1)q}(X)$. Then

$$d[\Phi(f,\psi)](\eta) = (-1)^{(p-1)q+1} \int_X d\eta \wedge \Phi(f,\psi)$$
$$= (-1)^{(p-1)q+1} \int_X d\left(\eta \wedge \Phi(f,\psi)\right) + \int_X \eta \wedge d\Phi(f,\psi).$$

To obtain the theorem we therefore only need to show

$$(-1)^{(p-1)q+1} \int_X d\left(\eta \wedge \Phi(f,\psi)\right) = \int_{Z(f)} \eta.$$

By Stokes' theorem, we can modify the differential form $\Phi(f,\psi)$ by a smooth form without changing the integral. Hence we can replace it by its singular part as in the previous proposition. By unfolding we see

$$\int_X d\left(\eta \wedge \Phi(f,\psi)\right) = \sum_h \sum_{n\in\mathbb{Q}_{<0}} a^+(h,n) \sum_{\lambda\in\Gamma\backslash L_{h,-n}} \int_{\Gamma_\lambda\backslash D} d\left(\eta \wedge \tilde{\psi}(\lambda,z,0)\right).$$

Again, this is a finite sum, so it suffices to consider the integral on the right hand side for $\lambda = \sqrt{m}v_1$.

For $\varepsilon > 0$,

$$U_\varepsilon := D - \Psi\left(H \times \{tX_{1p+q}; \; t \geq \varepsilon\}\right)$$

defines an open neighborhood of the cycle D_{v_1}. (For $q = 1$, replace the condition $t \geq \varepsilon$ by $|t| \geq \varepsilon$.) Then by Stokes' theorem we obtain

$$(7.5) \qquad \int_{\Gamma_H \backslash D} d \left(\eta \wedge \tilde{\psi}(\sqrt{m}v_1, z, 0) \right) = \lim_{\varepsilon \to 0} \int_{\partial(\Gamma_H \backslash (D - U_\varepsilon))} \eta \wedge \tilde{\psi}(\sqrt{m}v_1, z, 0).$$

By the analogue for (7.1) (which follows from the considerations in [FJ], section 2), we see that (7.5) is equal to

$$(7.6) \qquad C \lim_{\varepsilon \to 0} \int_{\Gamma_H \backslash H} \eta(ha_\varepsilon) \wedge \tilde{\psi}(a_{-\varepsilon} \sqrt{m}v_1) \sinh(\varepsilon)^{q-1} \cosh(\varepsilon)^{p-1} \, dh \left(1_{\mathfrak{p}/\mathbb{R}X_{1_{p+q}}} \right).$$

for some universal constant $C \neq 0$.

We consider (7.2) and (7.3) (with $t = \varepsilon$). For (7.6) only the terms with $\ell = q - 1$ can contribute. But

$$Q_{\underline{\alpha},q-1}(\sqrt{m}\cosh(\varepsilon)v_1, z_0) = \begin{cases} \left(\sqrt{2m} \cosh(\varepsilon) \right)^{q-1}, & \underline{\alpha} = (1, 1, \ldots, 1), \\ 0, & \text{otherwise.} \end{cases}$$

We obtain

$$C \lim_{\varepsilon \to 0} \int_{\Gamma_H \backslash H} \eta(ha_\varepsilon) \wedge \tilde{\psi}(a_{-\varepsilon} \sqrt{m}v_1) \sinh(\varepsilon)^{q-1} \cosh(\varepsilon)^{p-1} \, dh = C' \int_{\Gamma_H \backslash H} \eta(h) \, dh$$

with a constant C and $\tilde{\psi}(\lambda) = \tilde{\psi}(\lambda, z_0, 0)$ as before. Therefore, the theorem holds with $\delta_{Z(f)}$ replaced by $C'' \delta_{Z(f)}$ with a certain constant C'' independent of η. We conclude that the constant is equal to 1 by noting that this is the case for η closed: In (7.4), $d[\Phi(f, \psi)](\eta) = 0$, while $[\Lambda_\psi(f)](\eta) = (\eta, \Lambda_\psi(f))_X = \int_{Z(f)} \eta$ by Corollary 6.5. $\qquad \square$

The previous theorem can be reformulated saying that the pair $(Z(f), \Phi(z, f, \psi))$ defines a class in the group of differential characters introduced by Cheeger and Simons, see [C, CS]. Note that we can consider the group of differential characters as the analogue for real manifolds of the arithmetic Chow group, see [GS].

Theorem 7.3. *In the Hermitian case of $q = 2$ the Borcherds lift $\Phi(z, f)$ defines a Green's current for the divisor $Z(f)$, i.e.,*

$$dd^c[\Phi(z, f)] + \delta_{Z(f)} = [\Lambda_B(z, f)].$$

Proof. Using $-d^c\Phi(z, f) = \Phi(z, f, \psi)$, one easily obtains

$$(dd^c\eta)\Phi(z, f) = \eta \wedge dd^c\Phi(z, f) + d\left((d^c\eta)\Phi(z, f) + \eta \wedge \Phi(z, f, \psi) \right).$$

The theorem now follows from the previous and the logarithmic growth of $\Phi(z, f)$ along $Z(f)$. $\qquad \square$

This result also follows from the splitting (5.4) of $\Phi(z, f)$, obtained in [Br1] via the Fourier expansion, and the Poincaré-Lelong formula.

Observe that in the generic case the space X will be non-compact. Thus, to study intersection theory, one has to consider a smooth compactification $\widetilde{X}$ of X. In particular, in the arithmetic intersection theory of special divisors at the Archimedian place, one has to study $\Phi(z, f)$ as a current on $\widetilde{X}$. This requires a detailed analysis of the growth of $\Phi(z, f)$ at the boundary, which turns out to be of log- and log-log-type (for the case of

Hilbert modular surfaces see [BBK]). One has to work with the extension of arithmetic intersection theory provided in [BKK].

Oda and Tsuzuki [OT] constructed Green's functions for the special divisors by means of Poincaré series involving the secondary spherical function. By Theorem 4.7 of [BK], $\Phi(z, f)$ can be written as a linear combination of the Green's functions of [OT] in an explicit way.

We also compare $\Phi(z, f)$ with the Green's functions constructed by Kudla [Ku1, Ku2]. He introduces for $x \in V(\mathbb{R}), x \neq 0$ and $z \in D$, the function

$$\xi^0(x, z) = -\operatorname{Ei}(2\pi(x_z, x_z)),$$

which for $(x, x) > 0$ turns out to be a Green's function for the cycle D_x on the Hermitian domain D and is smooth otherwise. Here, for $w \in \mathbb{C}$, $\operatorname{Ei}(w) = \int_{-\infty}^{w} \frac{e^t}{t} dt$ is the exponential integral. While ξ^0 is *not* a Schwartz function on $V(\mathbb{R})$, we can still define $\xi(x, \tau, z) = \xi^0(\sqrt{v}x, z)e^{-\pi\tau(x,x)}$ for $\tau \in \mathbb{H}$, and one then has, see [Ku1, Ku2],

$$dd^c\xi(x, \tau, z) = \varphi_{KM}(x, \tau, z)$$

for $x \neq 0$. On the other hand, we can apply the lowering operator L_κ on $\mathbb{H}$ to ξ, and one easily checks

$$L_\kappa\xi(x, \tau, z) = -\varphi_0(x, \tau, z).$$

This provides a different proof of Theorem 4.4, i.e., $\phi := L_\kappa\varphi_{KM} = -dd^c\varphi_0$, and in summary, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\xi(x, \tau, z) & \xmapsto{\quad L_\kappa \quad} & -\varphi_0(x, \tau, z) \; . \\
\Big\uparrow{\scriptstyle dd^c} & & \Big\uparrow{\scriptstyle dd^c} \\
\varphi_{KM}(x, \tau, z) & \xmapsto{\quad L_\kappa \quad} & \phi(x, \tau, z)
\end{array}
$$

One can then use the ideas of the proof of Theorem 6.1 together with (6.10) to account for the problem that ξ is not defined for $x = 0$, to express $\Phi(z, f)$ in terms of ξ.

Finally, we would like to mention that one can use the ideas of this paper to give a somewhat different proof of the results in [Ku1] and [BK] relating the geometric degrees (and the Archimedean contribution to the arithmetic degrees) of the special cycles to the coefficients of certain Eisenstein series and its derivatives (see also [Ku2, KRY]).

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