SIGN CHANGES OF COEFFICIENTS OF HALF INTEGRAL WEIGHT MODULAR FORMS

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Abstract. For a half integral weight modular form f we study the signs of the Fourier coefficients $a(n)$. If f is a Hecke eigenform of level N with real Nebentypus character, and t is a fixed square-free positive integer with $a(t) \neq 0$, we show that for all but finitely many primes p the sequence $(a(tp^{2m}))_m$ has infinitely many signs changes. Moreover, we prove similar (partly conditional) results for arbitrary cusp forms f which are not necessarily Hecke eigenforms.

1. Introduction

Let f be a non-zero elliptic cusp form of positive real weight κ, with multiplier v and with real Fourier coefficients $a(n)$ for $n \in \mathbb{N}$. Then under quite general conditions, using the theory of L-functions, it was shown in [KKP] that the sequence $(a(n))_{n\in\mathbb{N}}$ has infinitely many sign changes, i.e., there are infinitely many n such that $a(n) > 0$ and there are infinitely many n such that $a(n) < 0$.

This is particularly interesting when κ is an integer and f is a Hecke eigenform of level N, and so the $a(n)$ are proportional to the Hecke eigenvalues. For recent work in this direction we refer to e.g. [KoSe], [IKS], [KSL].

In the present note we shall consider the case of half-integral weight $\kappa = k + 1/2$, $k \in \mathbb{N}$, and level N divisible by 4. Note that this case is distinguished through the celebrated works of Shimura [Sh] and Waldspurger [Wa] in the following way. First, for each square-free positive integer t, there exists a linear lifting from weight $k + 1/2$ to even integral weight $2k$ determined by the coefficients $a(tn^2)$ (where $n \in \mathbb{N}$), see [Ni], [Sh]. In particular, through these liftings, the theory of Hecke eigenvalues is the same as that in the integral weight case. Secondly, if f is a Hecke eigenform, then the *squares* $a(t)^2$ are essentially proportional to the central critical values of the Hecke L-function of F twisted with the quadratic character $\chi_{t,N} = (\frac{(-1)^k N^2 t}{\cdot})$, see [Wa]. Here F is a Hecke eigenform of weight $2k$ corresponding to f under the Shimura correspondence.

These facts motivate the following questions. First, it is natural to ask for sign changes of the sequence $(a(tn^2))_{n\in\mathbb{N}}$ where t is a fixed positive square-free integer. We start with a conditional result here, namely if the Dirichlet L-function associated to $\chi_{t,N}$ has no zeros in the interval $(0, 1)$ (Chowla's conjecture), then the sequence $(a(tn^2))_{n\in\mathbb{N}}$ –if not identically zero– changes sign infinitely often (Theorem 2.1). Note that by work of Conrey

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and Soundararajan [CS], Chowla's conjecture is true for a positive proportion of positive square-free integers t. If f is a Hecke eigenform, we can in fact prove an unconditional and much better result on the sign changes of the sequence $(a(tp^{2m}))_{m\in\mathbb{N}}$ where p is a prime not dividing N (see Theorem 2.2).

Secondly, one may ask for sign changes of the sequence $(a(t))_t$, where t runs through the square-free integers only. This question is more difficult to treat. Numerical calculations seem to suggest not only that there are infinitely many sign changes, but also that "half" of these coefficients are positive and "half" of them are negative.

It seems quite difficult to prove any general theorem here, and we can only prove a result that seems to point into the right direction (see Theorem 2.4): Under the (clearly necessary) assumption for $k = 1$ that f is contained in the orthogonal complement of the space of unary theta functions, there exist infinitely many positive square-free integers t and for each such t a natural number n_t, such that the sequence $(a(tn_t^2))_t$ has infinitely many sign changes. Note that we do not require f to be an eigenform. We in fact prove a slightly stronger result (Theorem 2.5).

As an immediate application, in the integral weight as well as in the half-integral weight case, we may consider representation numbers of quadratic forms. Let Q be a positive definite integral quadratic form, and for a positive integer n let $r_Q(n)$ be the number of integral representations of n by Q. Then the associated theta series is the sum of a modular form lying in the space of Eisenstein series and a corresponding cusp form. An infinity of sign changes of the coefficients of the latter (the "error term" for $r_Q(n)$) means that $r_Q(n)$ for infinitely many n is larger (respectively less) than the corresponding Eisenstein coefficient (the "main term" for $r_Q(n)$).

Exact statements of our results are given in Section 2, while Section 3 contains their proofs. These are based on the existence of the Shimura lifts, the theory of L-functions, and on results on quadratic twists proved in [Br]. In Section 4 some numerical examples are given.

2. Notation and statement of results

We denote by $\mathbb{N}$ the set of positive integers. The set of square-free positive integers is denoted by $\mathbb{D}$. Throughout we write $q = e^{2\pi i z}$ for z in the upper complex half plane $\mathbb{H}$.

Let k be a positive integer. Let N be a positive integer divisible by 4, and let χ be a Dirichlet character modulo N. We write χ^* for the Dirichlet character modulo N given by $\chi^*(a) = \left(\frac{-4}{a}\right)^k \chi(a)$. Moreover, we write $S_{k+1/2}(N,\chi)$ for the space of cusp forms of weight $k + 1/2$ for the group $\Gamma_0(N)$ with character χ in the sense of Shimura [Sh].

If m and r are positive integers and ψ is an odd primitive Dirichlet character modulo r, then the unary theta function

$$\theta_{\psi,m}(z) = \sum_{n\in\mathbb{Z}} \psi(n) n q^{mn^2}$$

belongs to $S_{3/2}\left(N, \left(\frac{-4m}{\cdot}\right)\psi\right)$ for all N divisible by $4r^2m$, cf. [Sh]. Let $S^*_{3/2}(N,\chi)$ be the orthogonal complement with respect to the Petersson scalar product of the subspace of

$S_{3/2}(N,\chi)$ spanned by such theta series. For $k \geq 2$ we simply put $S^*_{k+1/2}(N,\chi) = S_{k+1/2}(N,\chi)$. It is well known that the Shimura lift maps $S^*_{k+1/2}(N,\chi)$ to the space $S_{2k}(N/2,\chi^2)$ of cusp forms of integral weight $2k$ for $\Gamma_0(N/2)$ with character χ^2.

Throughout this section, let $f = \sum_{n=1}^\infty a(n)q^n \in S^*_{k+1/2}(N,\chi)$ be a non-zero cup form with Fourier coefficients $a(n) \in \mathbb{R}$. In our first result we consider the coefficients $a(tn^2)$ for fixed $t \in \mathbb{D}$ and varying n.

Theorem 2.1. *Let $t \in \mathbb{D}$ such that $a(t) \neq 0$, and write $\chi_{t,N}$ for the quadratic character $\chi_{t,N} = (\frac{(-1)^k N^2 t}{\cdot})$. Assume that the Dirichlet L-function $L(s,\chi_{t,N})$ has no zeros in the interval $(0,1)$. Then the sequence $(a(tn^2))_{n\in\mathbb{N}}$ has infinitely many sign changes.*

For Hecke eigenforms, we prove the following unconditional result.

Theorem 2.2. *Suppose that the character χ of f is real, and suppose that f is an eigenform of all Hecke operators $T(p^2)$ with corresponding eigenvalues λ_p for p coprime to N. Let $t \in \mathbb{D}$ such that $a(t) \neq 0$. Then for all but finitely many primes p coprime to N the sequence $(a(tp^{2m}))_{m\in\mathbb{N}}$ has infinitely many sign changes.*

Remark 2.3. Let K_f be the number field generated by the Hecke eigenvalues λ_p of f. The number of exceptional primes in Theorem 2.2 is bounded by r where 2^r is the highest power of 2 dividing the degree of K_f over $\mathbb{Q}$.

Next, we consider the coefficients $a(tn^2)$ for varying $t \in \mathbb{D}$.

Theorem 2.4. *For every $t \in \mathbb{D}$ there is an $n_t \in \mathbb{N}$ such that the sequence $(a(tn_t^2))_{t\in\mathbb{D}}$ has infinitely many sign changes.*

In the special case when f is a Hecke eigenform, Theorem 2.4 is an easy consequence of Theorem 2.2. However, we do not assume this in Theorem 2.4. Notice that the statement is obviously wrong for the theta functions $\theta_{\psi,m}$. We shall also prove the following slightly stronger statement.

Theorem 2.5. *Let $p_1,\dots,p_r$ be distinct primes not dividing N and let $\varepsilon_1,\dots,\varepsilon_r \in \{\pm 1\}$. Write $\mathbb{D}'$ for the set of $t \in \mathbb{D}$ satisfying $(\frac{t}{p_j}) = \varepsilon_j$ for $j = 1,\dots,r$. Then for every $t \in \mathbb{D}'$ there is an $n_t \in \mathbb{N}$ such that the sequence $(a(tn_t^2))_{t\in\mathbb{D}'}$ has infinitely many sign changes.*

3. Proofs

Here we prove the results of the previous section.

Proof of Theorem 2.1. Put

$$A(n) := \sum_{d|n} \chi_{t,N}(d) d^{k-1} a(\frac{n^2}{d^2}t). \tag{3.1}$$

According to [Sh, Ni], the series

$$F(z) := \sum_{n\geq 1} A(n) q^n$$

is in $S_{2k}(N/2,\chi^2)$ and is non-zero due to our assumption $a(t)\neq 0$. Note that (3.1) is equivalent to the Dirichlet series identity

$$\sum_{n\geq 1} a(tn^2)n^{-s} = \frac{1}{L(s-k+1,\chi_{t,N})}\cdot L(F,s) \tag{3.2}$$

in the range of absolute convergence, where $L(F,s)$ is the Hecke L-function attached to F.

Now suppose that $a(tn^2)\geq 0$ for all but finitely many n. Then by a classical theorem of Landau, either the Dirichlet series on the left hand side of (3.2) has a singularity at the real point of its line of convergence or must converge everywhere.

By our hypothesis, $L(s,\chi_{t,N})$ has no real zeros for $\Re(s)>0$. Hence the series on the left hand side of (3.2) converges for $\Re(s)>k-1$. In particular, we have

$$a(tn^2)\ll_\epsilon n^{k-1+\epsilon} \qquad (\epsilon>0).$$

From (3.1) we therefore deduce that

$$A(n)\ll_\epsilon \sum_{d|n} d^{k-1}\left(\frac{n}{d}\right)^{k-1+\epsilon} \ll_\epsilon n^{k-1+2\epsilon} \qquad (\epsilon>0).$$

Consequently, the Rankin-Selberg Dirichlet series

$$R_F(s)=\sum_{n\geq 1} A(n)^2 n^{-s}$$

must be convergent for $\Re(s)>2k-1$. However, it is well-known that the latter has a pole at $s=2k$ with residue $c_k\|F\|^2$, where $c_k>0$ is a constant depending only on k, and $\|F\|^2$ is the square of the Petersson norm of F. Since $F\neq 0$, we obtain a contradiction. This proves the claim. □

Proof of Theorem 2.2 and Remark 2.3. We use the same notation as in the proof of Theorem 2.1. Since f is an eigenfunction of $T(p^2)$, the function F is an eigenfunction under the usual Hecke operator $T(p)$ with eigenvalue λ_p. Since $\chi^2=1$, the eigenvalue λ_p is real. One has

$$\sum_{m\geq 1} a(tp^{2m})p^{-ms} = a(t)\frac{1-\chi_{t,N}(p)p^{k-1-s}}{1-\lambda_p p^{-s}+p^{2k-1-2s}} \tag{3.3}$$

for $\Re(s)$ sufficiently large, which is the local variant of (3.2).

The denominator of the right-hand side of (3.3) factors as

$$1-\lambda_p p^{-s}+p^{2k-1-2s}=(1-\alpha_p p^{-s})(1-\beta_p p^{-s})$$

where $\alpha_p+\beta_p=\lambda_p$ and $\alpha_p\beta_p=p^{2k-1}$. Explicitly one has

$$\alpha_p,\beta_p=\frac{\lambda_p\pm\sqrt{\lambda_p^2-4p^{2k-1}}}{2}. \tag{3.4}$$

Now assume that $a(tp^{2m})\geq 0$ for almost all m. Then by Landau's theorem the Dirichlet series on the left hand side of (3.3) either converges everywhere or has a singularity at the real point of its abscissa of convergence. The first alternative clearly is impossible, since the right-hand side of (3.3) has a pole for $p^s = \alpha_p$ or $p^s = \beta_p$.

Thus the second alternative must hold, and in particular α_p or β_p must be real. By Deligne's theorem, we have

$$\lambda_p^2 \leq 4p^{2k-1},$$

hence in combination with (3.4) we find that

$$\lambda_p = \pm 2p^{k-1/2}.$$

In particular we conclude that $\sqrt{p}$ is contained in the number field K_f generated by the Hecke eigenvalues of f.

Since for different primes $p_1, \ldots, p_r$ the degree of the field extension

$$\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_r})/\mathbb{Q}$$

is 2^r, we deduce our assertion. □

Throughout the rest of this section, let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+1/2}(N, \chi)$ be an arbitrary non-zero cup form with Fourier coefficients $a(n) \in \mathbb{R}$. For the proof of Theorem 2.4 we need the following three propositions.

Proposition 3.1. *There exist infinitely many $n \in \mathbb{N}$ such that $a(n)$ is negative.*

Proof. This result is proved in [KKP] in much greater generality. For the convenience of the reader we sketch the argument in the present special case.

Assume that there exist only finitely many $n \in \mathbb{N}$ such that $a(n) < 0$. Then the Hecke L-function of f (which is entire, cf. [Sh]) converges for all $s \in \mathbb{C}$ by Landau's theorem. Consequently,

$$a(n) \ll_C n^C$$

for all $C \in \mathbb{R}$.

This implies that the Rankin L-function of f also converges for all $s \in \mathbb{C}$. Arguing as at the end of the proof of Theorem 2.1 we find that f vanishes identically, contradicting the assumption $f \neq 0$. □

Proposition 3.2. *Let p be a prime not dividing N, and let $\varepsilon \in \{\pm 1\}$. Assume that $a(n) \geq 0$ for all positive integers n with $\left(\frac{n}{p}\right) = \varepsilon$. Then f is an eigenform of the Hecke operator $T(p^2)$ with eigenvalue $-\varepsilon\chi^*(p)(p^k + p^{k-1})$.*

Proof. We consider the cusp form

$$\tilde{f} := \sum_{\substack{n \geq 1 \\ \left(\frac{n}{p}\right) = \varepsilon}} a(n)q^n.$$

According to [Br], Section 2 (iii) and (iv), it belongs to the space $S_{k+1/2}(Np^2, \chi)$. By assumption, $\tilde{f}$ has non-negative real coefficients. Using Proposition 3.1 we find that $\tilde{f} = 0$. Consequently, $a(n) = 0$ for all positive integers n with $\left(\frac{n}{p}\right) = \varepsilon$. Now the assertion follows from [Br], Lemma 1. □

Proposition 3.3. *Let p be a prime not dividing N, and let $\varepsilon \in \{\pm 1\}$. Assume that f is contained in $S^*_{k+1/2}(N,\chi)$. Then there exist positive integers n, n' such that*

$$\left(\frac{n}{p}\right) = \varepsilon \quad \text{and } a(n) < 0,$$
$$\left(\frac{n'}{p}\right) = \varepsilon \quad \text{and } a(n') > 0.$$

Proof. Suppose that $a(n) \geq 0$ for all $n \in \mathbb{N}$ with $(\frac{n}{p}) = \varepsilon$. Then, by Proposition 3.2, f is an eigenform of $T(p^2)$ with eigenvalue $\lambda_p = -\varepsilon\chi^*(p)(p^k + p^{k-1})$. Using the Shimura lift, we see that λ_p is also an eigenvalue of the integral weight Hecke operator $T(p)$ on $S_{2k}(N/2,\chi^2)$. But it is easy to see that any eigenvalue λ of this Hecke operator satisfies the bound $|\lambda| < p^k + p^{k-1}$, see e.g. [Ko2] (or use the stronger Deligne bound). We obtain a contradiction.

Finally, replacing f by $-f$ we deduce the existence of n' with the claimed properties. □

Proof of Theorem 2.4. Suppose that there exist finitely many square-free $t_1, \dots, t_h \in \mathbb{N}$ such that $a(tn^2) \leq 0$ for all square-free integers t different from t_ν, $\nu = 1, \dots, h$, and all $n \in \mathbb{N}$. Choose a prime p coprime to N such that

$$\left(\frac{t_\nu}{p}\right) = 1, \quad \text{for all } \nu = 1, \dots, h.$$

Then $a(n) \leq 0$ for all $n \in \mathbb{N}$ with $(\frac{n}{p}) = -1$. But this contradicts Proposition 3.3. Hence there exist infinitely many square-free $t \in \mathbb{N}$ for which there is an $n_t \in \mathbb{N}$ such that $a(tn_t^2) > 0$.

Finally, replacing f by $-f$ we deduce the existence of infinitely many square-free $t \in \mathbb{N}$ for which there is an $n_t \in \mathbb{N}$ such that $a(tn_t^2) < 0$. □

Proof of Theorem 2.5. The assertion follows combining Theorem 2.4 and Proposition 3.3. □

4. Examples

Let $f \in S^*_{k+1/2}(N,\chi_0)$ be a cusp form with trivial character χ_0, square-free level, and real coefficients $a(n)$. We suppose that f is contained in the plus space, that is, $a(n) = 0$ when $(-1)^k n \equiv 2, 3 \pmod 4$, see [KZ], [Ko1]. For a positive number X, we define the quantity

$$R^+_{tot}(f,X) = \frac{\#\{n \leq X;\quad a(n) > 0\}}{\#\{n \leq X;\quad a(n) \neq 0\}}.$$

Moreover, in view of Waldspurger's theorem, it is natural to consider the coefficients $a(d)$ especially for *fundamental* discriminants d. Therefore we put

$$R^+_{fund}(f,X) = \frac{\#\{d \leq X;\quad d \text{ fundamental discriminant and } a(d) > 0\}}{\#\{d \leq X;\quad d \text{ fundamental discriminant and } a(d) \neq 0\}}.$$

The numerical experiments below suggest that

$$\lim_{X\to\infty} R^+_{tot}(f,X) = 1/2, \qquad \lim_{X\to\infty} R^+_{fund}(f,X) = 1/2.$$

In our first example, we consider the Delta function $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}$. A cusp form of weight 13/2 of level 4 in the plus space corresponding to Δ under the Shimura lift is given by

$$\delta(z) = \frac{1}{8\pi i}\big(2E_4(4z)\theta'(z) - E_4'(4z)\theta(z)\big) \in S^+_{13/2}(4,\chi_0),$$

see [KZ]. Here $E_4(z) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n$ is the classical Eisenstein series of weight 4 and $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$. The Fourier expansion of δ starts as follows:

$$\delta(z) = q - 56q^4 + 120q^5 - 240q^8 + 9q^9 + 1440q^{12} - 1320q^{13} - 704q^{16} - 240q^{17} + \dots.$$

Computational data for δ is listed in Table 1.

TABLE 1. The proportion of positive coefficients of δ

X	10	10^2	10^3	10^4	10^5	10^6
$R^+_{tot}(\delta,X)$	0.600	0.520	0.518	0.504600	0.499600	0.499822
$R^+_{fund}(\delta,X)$	0.667	0.548	0.515	0.501643	0.500016	0.499836

In our second example, we consider the cusp form $G = \eta(z)^2\eta(11z)^2$ of weight 2 and level 11 corresponding to the elliptic curve $X_0(11)$. Here $\eta = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function. A cusp form of weight 3/2 and level 44 in the plus space corresponding to G under the Shimura lift is

$$g(z) = \big(\theta(11z)\eta(2z)\eta(22z)\big)|U_4 \in S^+_{3/2}(44,\chi_0),$$

see [Du] §2. Here U_4 denotes the usual Hecke operator of index 4. The Fourier expansion of g starts as follows:

$$g(z) = q^3 - q^4 - q^{11} - q^{12} + q^{15} + 2q^{16} + q^{20} - q^{23} - q^{27} - q^{31} + q^{44} + q^{55} + \dots.$$

Computational data for g is listed in Table 2.

TABLE 2. The proportion of positive coefficients of g

X	10	10^2	10^3	10^4	10^5	10^6
$R^+_{tot}(g,X)$	0.500	0.500	0.500	0.496042	0.501022	0.499544
$R^+_{fund}(g,X)$	1.000	0.500	0.503	0.491968	0.500861	0.499589

References

[Br] *J. H. Bruinier*, On a theorem of Vigneras, Abh. Math. Sem. Univ. Hamburg **68** (1998), 163–168.

[CS] *J. B. Conrey and K. Soundararajan*, Real zeros of quadratic Dirichlet L-functions, Invent. Math. **150** (2002), 1–44.

[Du] *N. Dummigan*, Congruences of modular forms and Selmer groups, Math. Res. Lett. **8** (2001), 479–494.

[IKS] *H. Iwaniec, W. Kohnen, and J. Sengupta*, The first negative Hecke eigenvalue, Intern. J. Number Theory, to appear.

[KKP] *M. Knopp, W. Kohnen, and W. Pribitkin*, On the signs of Fourier coefficients of cusp forms, Ramanujan J. **7** (2003), 269–277.

[Ko1] *W. Kohnen*, Fourier coefficients of modular forms of half integral weight, Math. Ann. **271** (1985), 237–268.

[Ko2] *W. Kohnen*, A simple remark on eigenvalues of Hecke operators on Siegel modular forms, Abh. Math. Sem. Univ. Hamburg **57** (1986), 33–36.

[KoSe] *W. Kohnen and J. Sengupta*, On the first sign change of Hecke eigenvalues of newforms, Math. Z. **254** (2006), 173–184.

[KSL] *W. Kohnen, I. Shparlinski, and Y. K. Lau*, On the number of sign changes of Hecke eigenvalues of newforms, J. Australian Math. Soc., to appear.

[KZ] *W. Kohnen and D. Zagier*, Values of L-series of modular forms at the center of the critical strip, Invent. Math. **64** (1981), 175–198.

[Ni] *S. Niwa*, Modular forms of half-integral weight and the integral of certain theta-functions, Nagoya Math. J. **56** (1975), 147–161.

[Sh] *G. Shimura*, On modular forms of half integral weight, Ann. of Math. (2) **97** (1973), 440–481.

[Wa] *J.-L. Waldspurger*, Sur les coefficients de Fourier des formes modulaires de poids demi-entier, J. Math. Pures et Appl. **60** (1981), 375–484.

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