Local Picard groups and theta series

Jan Hendrik Bruinier

December 14, 2001

Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 288,
D-69120 Heidelberg, Germany
E-mail: bruinier@mathi.uni-heidelberg.de

1 Introduction

The purpose of the present note is to present some of the results on local Picard groups and Borcherds products obtained in joint work with E. Freitag [BF].

Let (V,q) be a real quadratic space of signature $(2,l)$ with $l>2$ and $L\subset V$ an even lattice. To simplify the exposition we assume throughout that L *is unimodular*. For the general case we refer the reader to [BF].

Let $\mathrm{O}'(V)$ be the spinor kernel of the orthogonal group of V and K a maximal compact subgroup. Then the quotient $\mathcal{H}_l=\mathrm{O}'(V)/K$ is a Hermitean symmetric domain of complex dimension l. The orthogonal group $\mathrm{O}(L)$ of L is an arithmetic subgroup of $\mathrm{O}(V)$. The group $\Gamma=\mathrm{O}(L)\cap\mathrm{O}'(V)$ acts properly discontinuously on $\mathcal{H}_l$, and the quotient $\mathcal{H}_l/\Gamma$ is a non-compact normal complex space. Let X_L be the Baily-Borel compactification of $\mathcal{H}_l/\Gamma$. It is obtained by adjoining finitely many boundary curves to $\mathcal{H}_l/\Gamma$. According to the theory of Baily-Borel, X_L has a structure as a projective algebraic variety over $\mathbb{C}$.

The Picard group $\mathrm{Pic}(X_L)$ is an important invariant of X_L. It can be studied by looking at divisors on X_L. A particular class of divisors, called *Heegner divisors*, arises from embedded quotients of smaller dimension $l-1$ analogous to X_L.

Using Borcherds' lifting from nearly-holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ to meromorphic modular forms for Γ, one obtains explicit relations among these Heegner divisors, and thereby some information about their position in $\mathrm{Pic}(X_L)$ [Bo1, Bo2].

The local Picard group $\mathrm{Pic}(X_L,s)$ of X_L at a point $s\in X_L$ is defined as the direct limit

$$\mathrm{Pic}(X_\Gamma,s)=\varinjlim \mathrm{Pic}(U_{\mathrm{reg}}), \tag{1.1}$$

where U runs through all open neighborhoods of s, and U_{reg} denotes the regular locus of U. The group $\mathrm{Pic}(X_L,s)$ is zero, if s is a nonsingular point, and a torsion group, if s is an elliptic fixed point. It is much more complicated, if s is a boundary point of X_L. We will consider the case that s is a *generic* boundary point, which means in particular that s does not belong to the zero dimensional boundary components of X_L.

By means of certain local Borcherds products the positions of the images of Heegner divisors in $\mathrm{Pic}(X_L, s)$ can be precisely determined up to torsion. They can be described in terms of the Fourier coefficients of certain theta series attached to the 1 dimensional boundary component containing s.

In contrast to the general study of $\mathrm{Pic}(X_L, s)$, in our main result we exploit the unimodularity of L in a vital way. By means of a result of Waldspurger, it can be proved that a linear combination of Heegner divisors is torsion in the Picard group of X_L, if and only if it is torsion in $\mathrm{Pic}(X_L, s)$ for every one-dimensional boundary component B and a generic point $s \in B$. As a consequence we find a converse theorem for the Borcherds lifting [Bo1]: Any meromorphic modular form for the group Γ, whose divisor is a linear combination of Heegner divisors, must be a Borcherds product. This was also proved in greater generality in [Br1, Br2]. However, in these papers a completely different argument is used, which does not say anything about the local Picard groups of X_L.

2 Heegner divisors and modular forms

We stick to the notation introduced in the introduction.

Let m be a negative integer. If $\lambda \in L$ with $q(\lambda) = m$, then $(V \cap \lambda^\perp, q)$ is a real quadratic space of signature $(2, l-1)$. Let $\mathrm{O}'(V)_\lambda$ and K_λ be the stabilizers of λ in $\mathrm{O}'(V)$ and K, respectively. Then $H_\lambda = \mathrm{O}'(V)_\lambda / K_\lambda$ is a Hermitean symmetric space isomorphic to $\mathcal{H}_{l-1}$ and the natural embedding

$$\mathrm{O}'(V)_\lambda / K_\lambda \hookrightarrow \mathrm{O}'(V)/K$$

identifies H_λ with an analytic divisor of $\mathcal{H}_l$. The *Heegner divisor* of discriminant m is defined by

$$H(m) = \sum_{\substack{\lambda \in L \\ q(\lambda) = m}} H_\lambda.$$

It is a Γ-invariant divisor on $\mathcal{H}_l$, which is the inverse image of an algebraic divisor on X_L (also denoted by $H(m)$). These Heegner divisors can be viewed as higher dimensional generalizations of Heegner points on modular curves and Hirzebruch-Zagier divisors on Hilbert modular surfaces [Bo2].

With the theory of Borcherds products it is possible to construct explicit relations among Heegner divisors (cf. [Bo1] Theorem 13.3). Since the existence of Borcherds products is controlled by the space S_κ of elliptic cusp forms of weight $\kappa = 1 + l/2$ for the group $\mathrm{SL}_2(\mathbb{Z})$, the position of Heegner divisors in $\mathrm{Pic}(X_L)$ can be described in terms of such cusp forms (cf. [Bo2] Theorem 3.1). Let $\widetilde{\mathrm{Pic}}(X_L)$ be the quotient of $\mathrm{Pic}(X_L)$ modulo the subgroup generated by the canonical bundle.

Theorem 2.1 (Borcherds). *An integral linear combination $\sum_{m<0} c(m)H(m)$ of Heegner divisors is a torsion element of $\widetilde{\mathrm{Pic}}(X_L)$, if $\sum_{m<0} c(m)a(-m) = 0$ for every cusp form $f \in S_\kappa$ with Fourier coefficients $a(n)$. In other words, the generating series*

$$G(\tau) = \sum_{n>0} H(-n)e(n\tau)$$

is a cusp form of weight κ for $\mathrm{SL}_2(\mathbb{Z})$ with values in the (finite dimensional) subspace $\mathcal{P}$ of $\widetilde{\mathrm{Pic}}(X_L) \otimes_{\mathbb{Z}} \mathbb{C}$ generated by the Heegner divisors. Here τ denotes the variable in the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C};\ \Im(z) > 0\}$ and $e(\tau) = e^{2\pi i \tau}$ as usual.

This is a sufficient condition for a linear combination of Heegner divisors to being torsion in $\widetilde{\mathrm{Pic}}(X_L)$. Similar results, but without precise information on the level of G, were obtained earlier in a completely different way by Kudla-Millson.

3 Local Heegner divisors and theta series

Let B be a 1-dimensional boundary component of X_L and $s \in B$ a generic point. We want to determine the position of the image of $H(m)$ in the local Picard group $\mathrm{Pic}(X_L, s)$ at s.

The 1-dimensional boundary components of X_L are parametrized by Γ-classes of 2-dimensional isotropic subspaces of $L \otimes_{\mathbb{Z}} \mathbb{Q}$ (see [BF]). Let F be a 2-dimensional isotropic subspace of $L \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to B. It can be shown that the image of $H(m)$ in the local Picard group $\mathrm{Pic}(X_L, s)$ is given by the local Heegner divisor

$$H_F(m) = \sum_{\substack{\lambda \in L \cap F^\perp \\ q(\lambda) = m}} H_\lambda \subset \mathcal{H}_l.$$

This divisor is invariant under the stabilizer Γ_s of s and thereby defines an element of $\mathrm{Pic}(X_L, s)$.

Let $\tilde{F} \subset L \otimes_{\mathbb{Z}} \mathbb{Q}$ be a complementary isotropic subspace for F. Then $D = L \cap F^\perp \cap \tilde{F}^\perp$ is a negative definite even unimodular lattice of rank $l-2$. We consider theta series associated with D.

We write $(\cdot,\cdot)$ for the bilinear form corresponding to $q(\cdot)$ such that $q(x) = \frac{1}{2}(x,x)$. The homogeneous polynomial

$$Q(u,v) = (u,v)^2 - \frac{(u,u)(v,v)}{l-2}, \qquad u, v \in D \otimes_{\mathbb{Z}} \mathbb{R},$$

is harmonic in u and v. Hence standard results on theta series imply that

$$\Theta_D(\tau, v) = \sum_{\lambda \in D} Q(\lambda, v) e(-q(\lambda)\tau), \qquad \tau \in \mathbb{H},$$

belongs to S_κ for every $v \in D \otimes_{\mathbb{Z}} \mathbb{R}$. The minus sign in the exponential term comes from the fact that D is negative definite. We denote by S_κ^D the subspace of S_κ generated by the theta series $\Theta_D(\tau, v)$ with $v \in D$. Proposition 5.1 of [BF], specialized to our situation that L is unimodular, says:

Theorem 3.1. *An integral linear combination $\sum_{m<0} c(m)H_F(m)$ of local Heegner divisors is a torsion element of* $\operatorname{Pic}(X_L, s)$*, if and only if $\sum_{m<0} c(m)a(-m) = 0$ for every $f \in S_\kappa^D$ with Fourier coefficients $a(n)$.*

In the proof one uses local analogues of Borcherds products or more precisely of the generalized Borcherds products introduced in [Br1]. We briefly indicate how these local Borcherds products are defined, referring to [BF] for more details. We introduce adapted coordinates $Z = (z_1, z_2, \mathfrak{z})$ with respect to F for $Z \in \mathcal{H}_l$ as in [BF], where $\mathfrak{z} \in D \otimes_{\mathbb{Z}} \mathbb{C}$, $z_1, z_2 \in \mathbb{C}$, and $q(\Im(Z)) > 0$, $\Im(z_2) > 0$. If $e_1, e_3 \in L$ denote primitive isotropic vectors such that $L \cap F = \mathbb{Z}e_1 + \mathbb{Z}e_3$, then the local Heegner divisor $H_F(m)$ on $\mathcal{H}_l$ can be written as

$$H_F(m) = \sum_{\substack{\lambda \in D \\ q(\lambda)=m}} \sum_{\nu_1,\nu_3 \in \mathbb{Z}} H_{\lambda+\nu_1 e_1+\nu_3 e_3}.$$

Notice that the first sum over λ is finite. The local Borcherds product associated with $H_F(m)$ is defined by

$$\Psi_{F,m}(Z) = \prod_{\substack{\lambda \in D \\ q(\lambda)=m}} \prod_{n \in \mathbb{Z}} \left[1 - e\big(\sigma_n(nz_2 + (\lambda, \mathfrak{z}))\big)\right]$$

for $Z = (z_1, z_2, \mathfrak{z}) \in \mathcal{H}_l$. Here

$$\sigma_n = \begin{cases} +1, & \text{if } n \geq 0, \\ -1, & \text{if } n < 0, \end{cases}$$

is a sign, which is needed in order that the product converges. It is easily checked that the divisor of $\Psi_{F,m}(Z)$ is precisely $H_F(m)$. Consequently the image of $H_F(m)$ in $\operatorname{Pic}(X_L, s)$ is determined by the automorphy factor

$$J_{F,m}(g, Z) = \Psi_{F,m}(gZ) \big/ \Psi_{F,m}(Z) \qquad (g \in \Gamma_s).$$

It can be computed explicitly in terms of the above polynomials $Q(u, v)$. Thereby the Chern class of $H_F(m)$ in $H^2(\Gamma_s, \mathbb{Z})$ can be described. Combining this with results on the local cohomology due to Ballweg [Ba], the theorem is derived.

The fact that the condition in Theorem 3.1 is both, necessary and sufficient, may be used to infer:

Corollary 3.2. *The condition in Theorem 2.1 is also necessary. In other words, the natural map $\mathcal{P}^* \to S_\kappa$ given by $\ell \mapsto \sum_{n>0} \ell(H(-n))e(n\tau)$ is surjective. Here $\mathcal{P}^*$ denotes the dual vector space of $\mathcal{P}$.*

Proof. Let $H = \sum_{m<0} c(m)H(m)$ be an integral linear combination of global Heegner divisors which is a torsion element of $\widetilde{\operatorname{Pic}}(X_L)$. This means that H is the divisor of a meromorphic modular form for Γ with some multiplier system of finite order. Then the local Heegner divisors $\sum_{m<0} c(m) H_F(m)$ are clearly torsion elements of $\mathrm{Pic}(X_L, s)$ for all 1-dimensional boundary components B and a generic point $s \in B$. In view of Theorem 3.1 this implies that

$$\sum_{m<0} c(m)a(-m) = 0 \tag{3.1}$$

for every $f = \sum_{n>0} a(n)e(n\tau) \in S_\kappa^D$ and every negative definite even unimodular sublattice $D \subset L$ of rank $l-2$.

According to a theorem of Waldspurger, the sum $\sum_D S_\kappa^D$, where D runs through all negative definite even unimodular lattices of rank $l-2$, is equal to S_κ (see [Wal] and also [EZ] Theorem 7.4). But since every negative definite even unimodular lattice of rank $l-2$ can be realized as a sublattice of L, we find that (3.1) actually holds for all cusp forms $f \in S_\kappa$ with Fourier coefficients $a(n)$. □

Corollary 3.3. *Any meromorphic modular form for the group* Γ*, whose divisor is a linear combination of Heegner divisors, must be a Borcherds product in the sense of [Bo1] Theorem 13.3.*

Proof. This is an immediate consequence of Corollary 3.2 and [Bo2] Therem 3.1. □

References

[Ba] *L. Ballweg*, Die lokalen Cohomologiegruppen der Baily-Borel-Kompaktifizierung in generischen Randpunkten, Dissertation, University of Heidelberg (1992).

[Bo1] *R. E. Borcherds*, Automorphic forms with singularities on Grassmannians, Invent. Math. **132** (1998), 491–562.

[Bo2] *R. E. Borcherds*, The Gross-Kohnen-Zagier theorem in higher dimensions, Duke Math. J. **97** (1999), 219–233.

[Br1] *J. H. Bruinier*, Borcherds products on $\mathrm{O}(2,l)$ and Chern classes of Heegner divisors, preprint (May 2000), to appear in the Springer Lecture Notes in Mathematics series.

[Br2] *J. H. Bruinier*, Borcherds products and Chern classes of Hirzebruch-Zagier divisors, Invent. math. **138** (1999), 51–83.

[BF] *J. H. Bruinier and E. Freitag*, Local Borcherds Products, Ann. Inst. Fourier, Grenoble **51** (2001), 1–27.

[EZ] *M. Eichler and D. Zagier*, The Theory of Jacobi Forms, Progress in Math. **55** (1985), Birkhäuser.

[Wal] *J.-L. Waldspurger*, Engendrement par des séries thêta de certains espaces de formes modulaires, Invent. math. **50** (1979), 135–168.