REGULARIZED THETA LIFTS FOR ORTHOGONAL GROUPS OVER TOTALLY REAL FIELDS

JAN HENDRIK BRUINIER

Abstract. We define a regularized theta lift from SL_2 to orthogonal groups over totally real fields. It takes harmonic 'Whittaker forms' to automorphic Green functions and weakly holomorphic Whittaker forms to meromorphic modular forms on orthogonal groups with zeros and poles supported on special divisors, generalizing Borcherds' work on automorphic products. To prove our results we use the spectral expansion of the lift and study its relationship with the cohomological theta lift of Kudla and Millson.

1. Introduction

The theory of dual reductive pairs and theta liftings provides an important tool for the construction of automorphic forms and for understanding the relationship between automorphic forms on different groups. A new aspect was added to the theory by the celebrated discovery of Harvey–Moore and Borcherds that divergent theta integrals can often be regularized [HM], [Bo1]. One can define a regularized theta lift of (vector valued) weakly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ to meromorphic modular forms on orthogonal groups associated to *rational* quadratic spaces of signature $(n, 2)$. These lifts have their zeros and poles on special divisors (also referred to as Heegner divisors or rational quadratic divisors). Their Fourier expansions are given by infinite products, so called Borcherds products. They have found various applications, for instance in the theory of generalized Kac-Moody algebras, in the study of moduli problems, and in the geometry and arithmetic of Shimura varieties, see e.g. [AF], [Bo2], [Bo3], [GN], [Ku3], [BBK], [BY1].

Since the work of Borcherds, there has been the question whether the regularized theta lift can be generalized to dual reductive pairs other than $(\mathrm{SL}_2, \mathrm{O}(V))$, where V is a quadratic space over $\mathbb{Q}$ (see [Bo1], Problem 16.4). For instance, one would like to define it for quadratic spaces over totally real number fields as well. There are two serious problems that arise. First, the special cycles on which the lift should have its singularities are not divisors in general, so one cannot expect that they are related to a single meromorphic function. Second, the straightforward generalization of weakly holomorphic elliptic modular forms, the "input" for the lift, would be meromorphic Hilbert modular forms whose poles are supported at the Baily–Borel boundary. However, by the Koecher principle, there are no non-trivial modular forms of this type.

Date: February 17, 2011.
2000 *Mathematics Subject Classification.* 11F55, 11G18, 14G35.
The author is partially supported by DFG grant BR-2163/2-1.

In the present paper, we propose a solution for the second problem. We introduce the notion of a weakly holomorphic Whittaker form and consider a regularized theta lift of such functions. It leads to meromorphic modular forms with singularities along special divisors, generalizing Borcherds' construction of automorphic products [Bo1]. More generally, we study harmonic Whittaker forms and their regularized theta lifts. They give rise to Green functions in the sense of Arakelov geometry (cf. [SABK], [BKK]).

We now describe the content of this paper in more detail. Let F be a totally real number field of degree d and discriminant D. We write $\hat{F}$ for the ring of finite adeles and $\mathcal{O}_F$ for the ring of integers of F. Let (V, Q) be a quadratic space over F of dimension $\ell = n + 2$ and assume that the signature of V at the archimedian places of F is equal to

$$((n, 2), (n + 2, 0), \dots, (n + 2, 0)).$$

We consider the algebraic group $H = \operatorname{Res}_{F/\mathbb{Q}} \operatorname{GSpin}(V)$ over $\mathbb{Q}$ given by Weil restriction of scalars. Our hypothesis on the signature guarantees that the symmetric space $\mathbb{D}$ associated to $H(\mathbb{R})$ carries an invariant hermitean structure and that there exist special *divisors* in the sense of [Ku1].

Let $L \subset V$ be an even $\mathcal{O}_F$-lattice. To simplify the exposition we assume throughout this introduction that L be unimodular. This implies in particular that n is even. The case of arbitrary even lattices of possibly odd rank is treated in the body of the paper. For a compact open subgroup $K \subset H(\hat{\mathbb{Q}})$ stabilizing L we consider the Shimura variety

$$X_K = H(\mathbb{Q})\backslash(\mathbb{D} \times H(\hat{\mathbb{Q}}))/K.$$

It is a quasi-projective variety of dimension n defined over F, see [Ku1]. It is projective if and only if V is anisotropic over F. By our assumption on the signature of V this is always the case if $d > 1$.

Associated to L there is a Siegel theta function $\Theta_S(\tau, z, h)$, where $\tau = u + iv \in \mathbb{H}^d$, $z \in \mathbb{D}$, and $h \in H(\hat{\mathbb{Q}})$, see Section 3. In the variable τ it transforms as a nonholomorphic Hilbert modular form of weight $(\frac{n-2}{2}, \frac{n+2}{2}, \dots, \frac{n+2}{2})$ for the group $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$. In the variable (z, h) it is $H(\mathbb{Q})$-invariant. For a Hilbert modular form f which is holomorphic in τ_1 and antiholomorphic in $\tau_2, \dots, \tau_d$ of weight $k =: (\frac{2-n}{2}, \frac{n+2}{2}, \dots, \frac{n+2}{2})$, we would like to consider the theta integral

$$\phi(z, h, f) = \frac{1}{\sqrt{D}} \int_{\Gamma\backslash\mathbb{H}^d} f(\tau)\Theta_S(\tau, z, h)(v_2 \cdots v_d)^{\ell/2} \, d\mu(\tau).$$

However, if $n \geq 2$ then there are no non-constant Hilbert modular forms of this type with moderate growth at the cusps. In view of Borcherds' work on regularized theta lifts for $F = \mathbb{Q}$ one could try to look at the integral for Hilbert modular forms with singularities at the cusps. But the Koecher principle implies that there are no non-constant forms of this type when $d > 1$. Now one could further relax the assumptions on f and allow singularities on some divisor in addition to the cusps. However, then the lift tends to behave poorly under the invariant differential operators for $H(\mathbb{R})$.

Instead, we use a different approach here. To motivate it, we briefly revisit the case where $F = \mathbb{Q}$. For simplicity we assume in this paragraph that $n > 2$. Any weakly holomorphic modular form of weight $k = \frac{2-n}{2}$ for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ can be constructed as a Poincaré series from a Whittaker form. For $s \in \mathbb{C}$ and $v \in \mathbb{R} \setminus \{0\}$ we let

$$\mathcal{M}_s(v) = |v|^{-k/2} M_{\mathrm{sgn}(v)k/2, s/2}(|v|) \cdot e^{-v/2},$$

where $M_{\nu,\mu}$ denotes the usual Whittaker function, and for any positive integer m we put

$$f_m(\tau, s) = \Gamma(s+1)^{-1} \mathcal{M}_s(-4\pi m v) e(-m\bar{\tau}),$$

where $e(u) := e^{2\pi i u}$. A *harmonic Whittaker form* f of weight k is a finite linear combination of the functions $f_m(\tau, 1-k)$ with $m > 0$ (see Remark 4.2 for a characterization by differential equations and growth conditions). Note that such a function is annihilated by the hyperbolic Laplacian in weight k and has exponential growth as $v \to \infty$. There is a differential operator ξ_k taking harmonic Whittaker forms of weight k to cusp forms of weight $2-k$. It is defined by

$$\xi_k(f) = 2i \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} v^k \overline{\frac{\partial}{\partial \bar{\tau}}(f)} \mid_{2-k} \gamma,$$

where $\Gamma_\infty = \{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right);\ b \in \mathbb{Z} \}$ denotes the subgroup of translations of Γ. We call f *weakly holomorphic* if $\xi_k(f) = 0$. If f is a harmonic Whittaker form of weight k, then the Poincaré series

$$\eta(f) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f \mid_k \gamma \tag{1.1}$$

converges and defines a harmonic weak Maass form of weight k for Γ in the sense of [BF]. The assignment $f \mapsto \eta(f)$ defines an isomorphism between the space of harmonic Whittaker forms and the space of harmonic weak Maass forms of weight k. Moreover, $\eta(f)$ is a weakly holomorphic modular form if and only if f is a weakly holomorphic Whittaker form (see Proposition 4.6). Many properties of harmonic weak Maass forms and their relationship to weakly holomorphic modular forms and cusp forms can also be rephrased using Whittaker forms, see Section 4 for details.

If f is a harmonic Whittaker form and $\eta(f)$ the corresponding weak Maass form, we can unfold the regularized theta integral

$$\begin{aligned} \Phi(z, h, \eta(f)) &= \int_{\Gamma \backslash \mathbb{H}} \eta(f)(\tau) \Theta_S(\tau, z, h)\, d\mu(\tau) \\ &= \int_{\Gamma_\infty \backslash \mathbb{H}} f(\tau) \Theta_S(\tau, z, h)\, d\mu(\tau). \end{aligned} \tag{1.2}$$

Consequently, regularized theta lifts of weakly holomorphic elliptic modular forms can also be viewed as regularized theta lifts of weakly holomorphic Whittaker forms.

We now come back to the more general setup above for an arbitrary totally real field F of degree d. The idea of the present paper is that Whittaker forms have a straightforward generalization to this situation (see Definitions 4.1 and 4.4). Since they are only invariant under translations there is no Koecher principle. If $d > 1$, then Poincaré series analogous to (1.1) diverge wildly. Nevertheless, in view of (1.2) we consider for a Whittaker form f of weight k the theta integral

$$\Phi(z,h,f) = \frac{1}{\sqrt{D}} \int_{\Gamma_\infty\backslash \mathbb{H}^d} f(\tau)\Theta_S(\tau,z,h)(v_2\cdots v_d)^{\ell/2}\, d\mu(\tau),$$

where $\Gamma_\infty = \{(\begin{smallmatrix}1&b\\0&1\end{smallmatrix});\ b\in\mathcal{O}_F\}$ denotes the subgroup of translations of $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$. Since Whittaker forms are exponentially increasing as $v_1\to\infty$, the integral has to be regularized.

In Section 5 we define the regularization and study its properties. It suffices to do this for the functions $f_m(\tau,1-k)$ for m totally positive, see (4.13). If $\Re(s)$ is sufficiently large, then the theta integral of $f_m(\tau,s)$ can be regularized by first integrating over u and afterwards over v, see Definition 5.1. The resulting function $\Phi_m(z,h,s)$ is an eigenfunction of the invariant Laplacian on $\mathbb{D}$ with a singularity along the special divisor $Z(m)$ of discriminant m. By means of ideas of Oda and Tsuzuki [OT] we compute the spectral expansion of $\Phi_m(z,h,s)$ and employ it to derive a meromorphic continuation to the whole s-plane and a functional equation in s (see Theorems 5.8 and 5.12). We define the regularized theta integral of $f_m(\tau,1-k)$ as the constant term in the Laurent expansion of $\Phi_m(z,h,s)$ at $s=1-k$. So for any harmonic Whittaker form

$$f = \sum_{m\gg 0} c(m) f_m(\tau,1-k) \tag{1.3}$$

of weight k we obtain a regularized theta lift $\Phi(z,h,f)$.

Theorem 1.1. *(See Theorem 5.14.) The regularized theta lift $\Phi(z,h,f)$ of f is a logarithmic Green function in the sense of Arakelov geometry for the divisor*

$$Z(f) = \sum_{m\gg 0} c(m) Z(m).$$

In Section 6 we investigate the relationship of the regularized theta lift and the Kudla–Millson lift (see e.g. [KM3]). Recall that Kudla and Millson constructed a theta function $\Theta_{KM}(\tau,z,h)$ which transforms in τ like a non-holomorphic Hilbert modular form of weight $\kappa = (\frac{n+2}{2},\dots,\frac{n+2}{2})$ and which takes values in the closed differential forms of type $(1,1)$ on X_K. For a Hilbert cusp form g of weight κ we may consider the theta integral

$$\Lambda(z,h,g) = \frac{1}{\sqrt{D}} \int_{\Gamma\backslash\mathbb{H}^d} \overline{g(\tau)}\Theta_{KM}(\tau,z,h) v^\kappa\, d\mu(\tau).$$

It gives rise to a map from Hilbert cusp forms to closed harmonic $(1,1)$-forms on X_K. Using the action of various differential operators on the Siegel and the Kudla–Millson theta kernels as in [BF], we prove (see Theorem 6.4):

Theorem 1.2. *Let f be a harmonic Whittaker form of weight k for Γ. Then*

$$dd^c\Phi(z,h,f) = \Lambda(z,h,\xi_k(f)) - B(f)\Omega.$$

Here $B(f)$ is a constant which is explicitly given by the Fourier coefficients of a certain Hilbert Eisenstein series of weight κ and Ω denotes the invariant Kähler form on $\mathbb{D}$.

If f is weakly holomorphic, then $\xi_k(f) = 0$. Hence the first term on the right hand side vanishes and $\Phi(z, h, f)$ is essentially a pluriharmonic function. This can be used to prove the main result of the present paper (Theorem 6.8).

Theorem 1.3. *Let f be a weakly holomorphic Whittaker form of weight k for Γ as in (1.3). Assume that the coefficients $c(m)$ are integral. Then there exists a meromorphic modular form $\Psi(z, h, f)$ for $H(\mathbb{Q})$ of level K with a multiplier system of finite order such that:*

(i) *The weight of Ψ is $-B(f)$.*
(ii) *The divisor of Ψ is equal to $Z(f)$.*
(iii) *The Petersson metric of Ψ is given by*

$$-\log \|\Psi(z, h, f)\|^2_{Pet} = \Phi(z, h, f).$$

Up to the statement about the Fourier expansion of Ψ, this result is completely analogous to Theorem 13.3 of [Bo1] on the regularized theta lift of weakly holomorphic modular forms. When $d = 1$, it is compatible via the map η with Borcherds' result (and gives a new proof of it). Notice that when $d > 1$, the variety X_K is compact and there are no Fourier expansions.

The third assertion of Theorem 1.3 provides a regularized integral representation for $\log \|\Psi\|^2_{Pet}$. In a follow-up paper [BY2] this is used to compute CM values and integrals of $\log \|\Psi\|^2_{Pet}$, and more generally of the Green functions $\Phi(z, h, f)$, extending results of [Ku3], [Scho], [BY1], [BK] to totally real fields. Such quantities can be interpreted as archimedian intersection pairings and therefore play an important role in arithmetic intersection theory.

Theorem 1.3 can be used (when $n > 2$) to show that the generating series

$$A(\tau) = -c_1(\mathcal{M}_1) + \sum_{m \gg 0} Z(m) q^m,$$

of the special divisors $Z(m)$ is a Hilbert modular form of weight κ with values in the first Chow group of X_K (see Theorem 7.1). Here $c_1(\mathcal{M}_1)$ denotes the Chern class of the line bundle of modular forms of weight 1. Our proof is a variant of the proof that Borcherds gave for $F = \mathbb{Q}$, see [Bo2]. This result was also proved in [YZZ] using the modularity result of Kudla–Millson [KM3] for the cohomology classes of special divisors.

In Section 8 we present some examples illustrating Theorem 1.3. In particular, we consider Shimura curves over totally real fields and Shimura varieties associated to orthogonal groups of even unimodular lattices over real quadratic fields.

I thank A. Deitmar for his help with Lemma 5.11 and G. Nebe for her help with Section 8.2. Moreover, I thank E. Freitag, J. Funke, K. Ono and T. Yang for many useful conversations and comments related to this paper.

2. Quadratic spaces and Shimura varieties

Throughout we use the setup of [Ku1], Section 1. Let F be a totally real number field of degree d over $\mathbb{Q}$. We write $\mathcal{O}_F$ for the ring of integers in F, and write $\partial = \partial_F$ for the different ideal of F. The discriminant of F is denoted by $D = \mathrm{N}(\partial_F) = \#\mathcal{O}_F/\partial_F$. Let

$\sigma_1, \dots, \sigma_d$ be the different embeddings of F into $\mathbb{R}$. We write $\mathbb{A}_F$ for the ring of adeles of F and $\hat{F}$ for the subring of finite adeles. Moreover, we put $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$.

Let (V, Q) be a non-degenerate quadratic space of dimension $\ell = n + 2$ over F. We put $V_{\sigma_i} = V \otimes_{F,\sigma_i} \mathbb{R}$ and identify $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_i V_{\sigma_i}$. We assume that V has signature

$$((n, 2), (n + 2, 0), \dots, (n + 2, 0)),$$

that is, V_{σ_1} has signature $(n, 2)$ and V_{σ_i} has signature $(n+2, 0)$ for $i = 2, \dots, d$. Sometimes we will also refer to the quantity

$$\operatorname{sig}(V) = (n - 2, n + 2, \dots, n + 2) \in \mathbb{Z}^d$$

as the signature. Let $\operatorname{GSpin}(V)$ be the 'general' Spin group of V, that is, the group of all invertible elements g in the even Clifford algebra of V such that $gVg^{-1} = V$. It is an algebraic group over F, and the vector representation gives rise to an exact sequence

$$1 \longrightarrow F^\times \longrightarrow \operatorname{GSpin}(V) \longrightarrow \operatorname{SO}(V) \longrightarrow 1.$$

We consider the algebraic group $H = \operatorname{Res}_{F/\mathbb{Q}} \operatorname{GSpin}(V)$ over $\mathbb{Q}$ given by Weil restriction of scalars. So $H(\mathbb{Q})$ can be identified with $\operatorname{GSpin}(V)(F)$.

We realize the hermitean symmetric space corresponding to H as the Grassmannian $\mathbb{D}$ of oriented negative definite 2-dimensional subspaces of V_{σ_1}. Note that $\mathbb{D}$ has two components corresponding to the two possible choices of the orientation. The complex structure on $\mathbb{D}$ is most easily realized as follows. We let $V_{\mathbb{C}} = V \otimes_{F,\sigma_1} \mathbb{C}$ and extend the bilinear form $\mathbb{C}$-bilinearly to $V_{\mathbb{C}}$. The open subset

$$\mathcal{K} = \{[Z] \in P(V_{\mathbb{C}});\ (Z, Z) = 0 \text{ and } (Z, \bar{Z}) < 0\} \tag{2.1}$$

of the zero quadric of the projective space $P(V_{\mathbb{C}})$ of $V_{\mathbb{C}}$ is isomorphic to $\mathbb{D}$ by mapping $[Z]$ to the subspace $\mathbb{R}\Re(Z) + \mathbb{R}\Im(Z) \subset V_{\sigma_1}$ with the appropriate orientation.

We choose a pair $a, b \in V_{\sigma_1}$ of isotropic vectors such that $(a, b) = 1$. The real quadratic space $V_0 := V_{\sigma_1} \cap a^\perp \cap b^\perp$ has signature $(n - 1, 1)$. The tube domain

$$\mathcal{H} = \{z \in V_0 \otimes_{\mathbb{R}} \mathbb{C};\ Q(\Im(z)) < 0\} \tag{2.2}$$

is isomorphic to $\mathcal{K}$ by mapping $z \in \mathcal{H}$ to the class in $P(V_{\mathbb{C}})$ of

$$w(z) = z + a - Q(z)b. \tag{2.3}$$

The domain $\mathcal{H}$ can be viewed as a generalized complex upper half plane. The linear action of $H(\mathbb{R})$ on $V_{\mathbb{C}}$ induces an action on $\mathcal{H}$ by fractional linear transformations. If $\gamma \in H(\mathbb{R})$, we have

$$\gamma w(z) = j(\gamma, z) w(\gamma z) \tag{2.4}$$

for an automorphy factor $j(\gamma, z) = (\gamma w(z), b)$.

The function

$$Z \mapsto -\frac{1}{2}(Z, \bar{Z}) = -(Y, Y) =: |Y|^2 \tag{2.5}$$

on $V_{\mathbb{C}}$ defines a hermitean metric on the tautological line bundle $\mathcal{L}$ over $\mathcal{K}$, where $Y = \Im(Z)$. Its first Chern form

$$\Omega = -dd^c \log |Y|^2 \tag{2.6}$$

is $H(\mathbb{R})$-invariant and positive. It corresponds to an invariant Kähler metric on $\mathbb{D} \cong \mathcal{K}$ and gives rise to an invariant volume form $d\mu(z) = \Omega^n$. Note that for $z \in \mathcal{H}$ we have

$$-\frac{1}{2}(w(z), \overline{w(z)}) = -(\Im(z), \Im(z)), \tag{2.7}$$

$$|\Im(\gamma z)|^2 = |j(\gamma, z)|^{-2} |\Im(z)|^2. \tag{2.8}$$

For $K \subset H(\hat{\mathbb{Q}})$ compact open we consider the Shimura variety

$$X_K = X_{K,V} := H(\mathbb{Q}) \backslash (\mathbb{D} \times H(\hat{\mathbb{Q}}))/K. \tag{2.9}$$

It is a quasi-projective variety of dimension n defined over F. It is projective if and only if V is anisotropic over F. By our assumption this is always the case when $d > 1$. Let $\mathbb{D}^+$ be one of the two components of $\mathbb{D}$. The connected component of the identity $H(\mathbb{R})^+$ of $H(\mathbb{R})$ acts on $\mathbb{D}^+$. We let $H(\mathbb{Q})^+ = H(\mathbb{Q}) \cap H(\mathbb{R})^+$ and write

$$H(\hat{\mathbb{Q}}) = \coprod_j H(\mathbb{Q})^+ h_j K, \tag{2.10}$$

as a finite disjoint union with $h_j \in H(\hat{\mathbb{Q}})$. Then we have

$$X_K = \coprod_j \Gamma_j \backslash \mathbb{D}^+, \tag{2.11}$$

where $\Gamma_j = H(\mathbb{Q})^+ \cap h_j K h_j^{-1}$. The different components of X_K all have the same finite volume.

2.1. **Modular forms.** We define modular forms for the group $H(\mathbb{Q})$ as follows. A function Ψ on $\mathcal{H} \times H(\hat{\mathbb{Q}})$ is called a meromorphic (holomorphic) modular form of weight $w \in \mathbb{Z}$ and level K if:

(i) For fixed $h \in H(\hat{\mathbb{Q}})$ the function $\Psi(z, h)$ is meromorphic (holomorphic) in $z \in \mathcal{H}$,
(ii) $\Psi(z, hk) = \Psi(z, h)$ for all $k \in K$,
(iii) $\Psi(\gamma z, \gamma h) = j(\gamma, z)^w \Psi(z, h)$ for all $\gamma \in H(\mathbb{Q})$,
(iv) Ψ is meromorphic (holomorphic) at the boundary.

The last condition is trivially fulfilled if V is anisotropic over F. By the Koecher principle, it is also automatically fulfilled if the Witt rank of V over F (i.e. the dimension of a maximal totally isotropic subspace over F) is smaller than n. Using the decomposition (2.10), one can define functions $\Psi_j(z) := \Psi(z, h_j)$ on $\mathbb{D}^+$. They are classical modular forms of weight w for the groups Γ_j. Consequently, the function $\Psi(z, h)$ corresponds to a tuple of classical modular forms (Ψ_j).

The transformation law (iii) can be relaxed by allowing characters or multiplier systems. By slight abuse of notation, a function $\sigma : H(\mathbb{Q}) \times H(\hat{\mathbb{Q}}) \to \mathbb{C}^\times$ is called a character for $H(\mathbb{Q})$, if $\sigma(\gamma, hk) = \sigma(\gamma, h)$ for all $k \in K$, and $\sigma(\gamma_1\gamma_2, h) = \sigma(\gamma_1, \gamma_2 h)\sigma(\gamma_2, h)$ for all $\gamma_1, \gamma_2 \in H(\mathbb{Q})$. Then the functions $\sigma_j(\gamma) := \sigma(\gamma, h_j)$ are homomorphisms $\Gamma_j \to \mathbb{C}^\times$. A function Ψ on $\mathcal{H} \times H(\hat{\mathbb{Q}})$ is called a modular form of weight $w \in \mathbb{Z}$ and level K with character σ, if it satisfies besides (i), (ii), (iv) that

(iii') $\Psi(\gamma z, \gamma h) = \sigma(\gamma, h) j(\gamma, z)^w \Psi(z, h)$ for all $\gamma \in H(\mathbb{Q})$.

More generally, one can define modular forms of rational weight $w \in \mathbb{Q}$ with a multiplier system $H(\mathbb{Q}) \times H(\hat{\mathbb{Q}}) \to \mathbb{C}^\times$, see e.g. [Br], Chapter 3.3.

Modular forms of weight w can be viewed as sections of the line bundle $\mathcal{M}_w$ of modular forms of weight w. The line bundle $\mathcal{M}_1$ of modular forms of weight 1 can be defined as the quotient

$$H(\mathbb{Q}) \backslash \mathcal{L} \times H(\hat{\mathbb{Q}})/K$$

of the tautological bundle $\mathcal{L} \times H(\hat{\mathbb{Q}})$. The w-th power of this bundle is the line bundle of modular forms of weight w. The hermitean metric (2.5) on $\mathcal{L}$ induces a metric on the bundle $\mathcal{M}_w$ called the Petersson metric. For a modular form Ψ of weight w it is given by

$$\|\Psi(z,h)\|_{Pet} = |\Psi(z,h)| \cdot |y|^w. \tag{2.12}$$

The first Chern form of the line bundle $\mathcal{M}_w$ with the Petersson metric is $w\Omega$.

2.2. **Lattices.** Let $L \subset V$ be an $\mathcal{O}_F$-lattice, that is, a finitely generated $\mathcal{O}_F$-submodule such that $L \otimes_{\mathcal{O}_F} F = V$. We assume that L is even, that is, $Q(L) \subset \partial^{-1}$. Then $Q_{\mathbb{Q}}(x) = \operatorname{tr}_{F/\mathbb{Q}} Q(x)$ defines an even $\mathbb{Z}$-valued quadratic form on L. Let L' be the $\mathbb{Z}$-dual lattice of L with respect to the quadratic form $Q_{\mathbb{Q}}$. The following is easily seen.

Proposition 2.1. *We have*

1. *L' is an $\mathcal{O}_F$-lattice.*
2. *$L \subset L'$ is a sublattice of finite index.*
3. *$(L, L') \subset \partial^{-1}$.*

The finite $\mathcal{O}_F$-module L'/L is called the discriminant group of L. We write $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. We have $L'/L \cong \hat{L}'/\hat{L}$. The lattice L is called unimodular if $L' = L$.

Recall that $H(\hat{\mathbb{Q}})$ acts on the set of lattices $M \subset V$ by $M \mapsto hM := (h\hat{M}) \cap V(F)$. This action induces an isomorphism $M'/M \to (hM)'/(hM)$, and hM lies in the same genus as M. Throughout we assume that the compact open subgroup $K \subset H(\hat{\mathbb{Q}})$ fixes the lattice $L \subset V$ and acts trivially on L'/L.

2.3. **Special divisors.** Here we define special divisors on X_K (see e.g. [Ku1], [Bo1], [Br]). They generalize Heegner divisors on modular curves. We follow the description in [Ku1].

Let $x \in V$ be a vector of totally positive norm. We write V_x for the orthogonal complement of x in V and H_x for the stabilizer of x in H. So $H_x \cong \operatorname{Res}_{F/\mathbb{Q}} \operatorname{GSpin}(V_x)$. The sub-Grassmannian

$$\mathbb{D}_x = \{z \in \mathbb{D};\ z \perp x\}$$

defines an analytic divisor on $\mathbb{D}$. For $h \in H(\hat{\mathbb{Q}})$ we consider the natural map

$$H_x(\mathbb{Q}) \backslash \mathbb{D}_x \times H_x(\hat{\mathbb{Q}})/(H_x(\hat{\mathbb{Q}}) \cap hKh^{-1}) \longrightarrow X_K, \quad (z, h_1) \mapsto (z, h_1 h).$$

Its image defines a divisor $Z(x,h)$ on X_K, which is rational over F. Let $m \in F$ be totally positive, and let $\varphi \in S(V(\hat{F}))^K$ be a K-invariant Schwartz function. If there is an $x_0 \in V(F)$ with $Q(x_0) = m$, we define the weighted cycle

$$Z(m,\varphi) = \sum_{h \in H_{x_0}(\hat{\mathbb{Q}})\backslash H(\hat{\mathbb{Q}})/K} \varphi(h^{-1}x_0) Z(x_0,h).$$

The sum is finite, and $Z(m,\varphi)$ is a divisor on X_K with complex coefficients. If there is no $x_0 \in V(F)$ with $Q(x_0) = m$, we put $Z(m,\varphi) = 0$. If $\mu \in L'/L$ is a coset, and $\chi_\mu = \operatorname{char}(\mu + \hat{L}) \in S(V(\hat{F}))^K$ is the characteristic function, we briefly write

$$Z(m,\mu) := Z(m,\chi_\mu).$$

3. Theta functions

3.1. **Special Schwartz functions.** We begin by recalling some properties of Schwartz functions on quadratic spaces over $\mathbb{R}$. Let (W, Q_W) be a real quadratic space of signature (p,q) and write $S(W)$ for the space of Schwartz functions on W. Let $\mathbb{D}_W$ be the symmetric domain associated to $\mathrm{SO}(W)$, realized as the Grassmannian of q-dimensional negative definite oriented subspaces of W. When $q > 0$, then $\mathbb{D}_W$ consists of two components corresponding to the two possible choices of the orientation, and when $q = 0$, then $\mathbb{D}_W$ is a point.

Let $\mathbb{H}$ be the upper complex half plane. Let $\mathrm{Mp}_2(\mathbb{R})$ be the two-fold metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$ realized by the two possible choices of a holomorphic square root of the automorphy factor $c\tau + d$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Recall that $\mathrm{Mp}_2(\mathbb{R})$ acts on $S(W)$ via the Weil representation ω_W associated to the standard additive character $u \mapsto e(u)$ of $\mathbb{R}$. Let $\widetilde{\mathrm{SO}}_2(\mathbb{R})$ be the inverse image of $\mathrm{SO}_2(\mathbb{R})$ in $\mathrm{Mp}_2(\mathbb{R})$, and let

$$k_\alpha = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}) \tag{3.1}$$

for $\alpha \in \mathbb{R}$. There is a character $\chi_{1/2} : \widetilde{\mathrm{SO}}_2(\mathbb{R}) \to \mathbb{C}^\times$ given by

$$\chi_{1/2}(k_\alpha, \phi) \mapsto \phi(i)^{-1} = \pm e^{i\alpha/2}. \tag{3.2}$$

We fix a base point $w_0 \in \mathbb{D}_W$, and let $l \in \frac{1}{2}\mathbb{Z}$. Let $\varphi \in S(W)$ be a Schwartz function which is invariant under the stabilizer in $\mathrm{SO}(W)$ of w_0, and which is an eigenfunction of weight l for $\widetilde{\mathrm{SO}}_2(\mathbb{R})$, that is, $\omega_W(\tilde{k})(\varphi) = \chi_{1/2}^{2l}(\tilde{k})\varphi$ for $\tilde{k} \in \widetilde{\mathrm{SO}}_2(\mathbb{R})$. Then we obtain a Schwartz function

$$\varphi(\tau, w, \lambda) = \phi(i)^{2l} \big(\omega_W(\tilde{g}_\tau, h_w)\varphi\big)(\lambda), \tag{3.3}$$

depending on $\tau \in \mathbb{H}$ and $w \in \mathbb{D}_W$. Here $\tilde{g}_\tau = (g_\tau, \phi) \in \mathrm{Mp}_2(\mathbb{R})$ with $g_\tau(i) = \tau$ and $h_w \in \mathrm{SO}(W)$ with $h_w w_0 = w$.

Every $w \in \mathbb{D}_W$ defines an orthogonal sum decomposition $W = w^\perp \oplus w$ into a positive definite subspace $w^\perp$ and a negative definite subspace w. If $\lambda \in W$ we write $\lambda_{w^\perp}$ and λ_w for the corresponding orthogonal projections. The Gaussian associated to w is the Schwartz function

$$\varphi_0^W(w,\lambda)=\exp\left(-2\pi Q_W(\lambda_{w^\perp})+2\pi Q_W(\lambda_w)\right).$$

It has weight $(p-q)/2$. Using the explicit formulas for the Weil representation, it is easily seen that the corresponding Schwartz function $\varphi_0^W(\tau,w,\lambda)$ as in (3.3) is given by

$$\varphi_0^W(\tau,w,\lambda)=\Im(\tau)^{q/2}e\big(Q(\lambda_{w^\perp})\tau+Q(\lambda_w)\bar\tau\big).$$

Kudla and Millson constructed a Schwartz form φ_{KM}^W on W taking values in $A^q(\mathbb{D}_W)$, the differential q-forms on $\mathbb{D}_W$ (see e.g. [KM3], and [BF] Section 4). We will require the following properties of this form. With respect to the natural action, we have the invariance

$$\varphi_{KM}^W(w,\lambda)\in[A^q(\mathbb{D}_W)\otimes S(W)]^{\mathrm{SO}(W)},$$

and $\varphi_{KM}^W(w,\lambda)$ is closed for all $\lambda\in W$. We have $\varphi_{KM}^W(w,\lambda)=P_{KM}(w,\lambda)\varphi_0^W(w,\lambda)$, where $P_{KM}(w,\lambda)$ is a polynomial on W taking values in $A^q(\mathbb{D}_W)$. Under the action of the Weil representation, the function $\varphi_{KM}^W(\lambda)$ has weight $(p+q)/2$. We also have the corresponding Schwartz function

$$\varphi_{KM}^W(\tau,w,\lambda)=P_{KM}(w,\sqrt{\Im(\tau)}\lambda)e\big(Q(\lambda_{w^\perp})\tau+Q(\lambda_w)\bar\tau\big). \tag{3.4}$$

When $q=0$, so that W is positive definite, φ_{KM}^W is simply the Gaussian φ_0^W.

In the present paper we only need the hermitean case $q=2$, for which a convenient construction of φ_{KM}^W is given in [Ku3], Section 4. We have that $\varphi_{KM}^W(0)=-\Omega$, where Ω is the negative of the invariant Kähler form Ω on $\mathbb{D}_W$ defined in (2.6). The following relationship of φ_{KM}^W and the Gaussian is obtained in [BF], Theorem 4.4.

Proposition 3.1. *Assume that $q=2$ so that $\mathbb{D}_W$ is hermitean. We have*

$$dd^c\varphi_0^W(\tau,w,\lambda)=-L\varphi_{KM}^W(\tau,w,\lambda),$$

where $L=-2i\Im(\tau)^2\frac{\partial}{\partial\bar\tau}$ denotes the Maass lowering operator, and d and $d^c=\frac{1}{4\pi i}(\partial-\bar\partial)$ are the usual differentials on $\mathbb{D}_W$.

3.2. **Discriminant forms and the Weil representation.** We now come back to our global quadratic space V over the totally real field F as in Section 2. Let $\mathbb{H}$ be the upper complex half plane. We use $\tau=(\tau_1,\dots,\tau_d)$ as a standard variable on $\mathbb{H}^d$ and put $u_i=\Re(\tau_i)$, $v_i=\Im(\tau_i)$. For a d-tuple $(w_1,\dots,w_d)$ of complex numbers, we put $\mathrm{tr}(w)=\sum_i w_i$ and $\mathrm{N}(w)=\prod_i w_i$.

We view $\mathbb{C}^d$ as a $\mathbb{R}^d$-module by putting $\lambda w=(\lambda_1w_1,\dots,\lambda_dw_d)$ for $\lambda=(\lambda_1,\dots,\lambda_d)\in\mathbb{R}^d$. For $x\in F$ we briefly write $x_i=\sigma_i(x)$ and identify x with its image $(x_1,\dots,x_d)\in\mathbb{R}^d$. The usual trace and norm of x coincide with the above definitions. Moreover, the inclusion $F\to\mathbb{R}^d$ defines an F-vector space structure on $\mathbb{C}^d$.

We are interested in certain vector valued modular forms for the Hilbert modular group associated to F. Let $G=\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SL}_2$. For $g\in\mathrm{SL}_2(F)\cong G(\mathbb{Q})$, we briefly write $g_i=\sigma_i(g)$. So the image of g in $G(\mathbb{R})\cong\mathrm{SL}_2(\mathbb{R})^d$ is given by $(g_1,\dots,g_d)$. The group $G(\mathbb{R})$ acts on $\mathbb{H}^d$ by fractional linear transformations. We denote by $\tilde G_{\mathbb{A}}$ the twofold metaplectic cover of

$G(\mathbb{A})$. Let $\tilde{G}_\mathbb{R}$ be the full inverse image in $\tilde{G}_\mathbb{A}$ of $G(\mathbb{R})$. We will frequently realize $\tilde{G}_\mathbb{R}$ as the group of pairs

$$(g, \phi(\tau)),$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{R})$ and $\phi(\tau)$ is a holomorphic function on $\mathbb{H}^d$ such that $\phi(\tau)^2 = \mathrm{N}(c\tau + d)$. The product of $(g_1, \phi_1(\tau))$, $(g_2, \phi_2(\tau))$ is given by

$$(g_1, \phi_1(\tau))\,(g_2, \phi_2(\tau)) = (g_1 g_2, \phi_1(g_2\tau)\phi_2(\tau)).$$

Let $\tilde{\Gamma}$ be the full inverse image in $\tilde{G}_\mathbb{R}$ of the Hilbert modular group

$$\Gamma = \mathrm{SL}_2(\mathcal{O}_F) \subset G(\mathbb{R}).$$

It follows from Vaserstein's theorem that $\tilde{\Gamma}$ is generated by the elements

$$\begin{aligned}
T_b &= \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1 \right), \qquad b \in \mathcal{O}_F, \\
S &= \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\mathrm{N}(\tau)} \right), \\
N &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -1 \right).
\end{aligned}$$

We also put $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i^d \right)$. Observe that we have the relations $(ST)^3 = S^2 = Z$. If d is odd then $Z^2 = N$, and if d is even then $Z^2 = N^2 = 1$. There are further relations corresponding to the elliptic fixed points of Γ.

Let $L \subset V$ be an $\mathcal{O}_F$-lattice. For $\mu \in L'/L$ we write $\chi_\mu = \mathrm{char}(\mu + \hat{L}) \in S(V(\hat{F}))$ for the characteristic function of the coset. Associated to the reductive dual pair $(\mathrm{SL}_2, \mathrm{O}(V))$ there is a Weil representation $\omega = \omega_\psi$ of $\tilde{G}_\mathbb{A}$ on the Schwartz space $S(V(\mathbb{A}_F))$, where ψ is the standard additive character of $F\backslash\mathbb{A}_F$ with $\psi_\infty(x) = e(\mathrm{tr}\, x)$ [We1]. The subspace

$$S_L = \bigoplus_{\mu \in L'/L} \mathbb{C}\chi_\mu \subset S(V(\hat{F}))$$

is preserved by the action of $\widetilde{\mathrm{SL}}_2(\hat{\mathcal{O}}_F)$, the full inverse image in $\tilde{G}_\mathbb{A}$ of $\mathrm{SL}_2(\hat{\mathcal{O}}_F) \subset G(\hat{\mathbb{Q}})$. The canonical splitting $G(F) \to \tilde{G}_\mathbb{A}$ defines a homomorphism

$$\tilde{\Gamma} \longrightarrow \widetilde{\mathrm{SL}}_2(\hat{\mathcal{O}}_F), \quad \gamma \mapsto \hat{\gamma},$$

where $\hat{\gamma}$ is the unique element such that $\gamma\hat{\gamma}$ is in the image of $G(F)$. This induces a representation ρ_L of $\tilde{\Gamma}$ on S_L by

$$\rho_L(\gamma)\varphi = \bar{\omega}(\hat{\gamma}), \quad \gamma \in \tilde{\Gamma}.$$

In terms of the above generators of $\tilde{\Gamma}$ the representation ρ_L is given by

$$\rho_L(T_b)(\chi_\mu) = e\left(\mathrm{tr}(Q(\mu)b)\right)\chi_\mu, \qquad b \in \mathcal{O}_F, \tag{3.5}$$

$$\rho_L(S)(\chi_\mu) = \frac{e(-\mathrm{tr}(\mathrm{sig}\, V)/8)}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e\left(-\mathrm{tr}(\mu, \nu)\right)\chi_\nu, \tag{3.6}$$

$$\rho_L(N)(\chi_\mu) = (-1)^\ell \chi_\mu. \tag{3.7}$$

Note that this is compatible with the conventions in [Bo1], [Ku3], [Br], where the case $F = \mathbb{Q}$ is considered. We also have the useful formula $\rho_L(Z)(\chi_\mu) = e(-\operatorname{tr}(\operatorname{sig} V)/4)\chi_{-\mu}$. For a totally positive unit $\varepsilon \in \mathcal{O}_F^\times$ the element $m(\varepsilon) = (\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{smallmatrix}\right), 1)$ acts by

$$\rho_L(m(\varepsilon))(\chi_\mu) = \chi_{\varepsilon^{-1}\mu}. \tag{3.8}$$

We denote the standard $\mathbb{C}$-bilinear pairing on S_L (the L^2 bilinear pairing) by

$$\langle a, b\rangle = \sum_{\mu\in L'/L} a_\mu b_\mu \tag{3.9}$$

for $a, b \in S_L$. The representation ρ_L is unitary, that is, we have $\langle \bar\rho_L a, \rho_L b\rangle = \langle a, b\rangle$. It is well known that ρ_L factors through a finite quotient of $\tilde\Gamma$.

3.3. **Siegel theta functions.** Next, we define Siegel theta functions for the lattice L. For the global quadratic space V over F, we obtain the Gaussian on $V(\mathbb{R})$ by piecing together the local data. For $\lambda = (\lambda_1, \dots, \lambda_d) \in V(\mathbb{R})$ and $z \in \mathbb{D}$ we define

$$\varphi_0(z, \lambda) = \varphi_0^{V_{\sigma_1}}(z, \lambda_1) \otimes \varphi_0^{V_{\sigma_2}}(\lambda_2) \otimes \cdots \otimes \varphi_0^{V_{\sigma_d}}(\lambda_d).$$

By our assumption on the signature of V, it has weight $(\frac{n-2}{2}, \frac{n+2}{2}, \dots, \frac{n+2}{2})$. We obtain the corresponding Schwartz function

$$\varphi_0(\tau, z, \lambda) = \varphi_0^{V_{\sigma_1}}(\tau_1, z, \lambda_1) \otimes \varphi_0^{V_{\sigma_2}}(\tau_2, \lambda_2) \otimes \cdots \otimes \varphi_0^{V_{\sigma_d}}(\tau_d, \lambda_d)$$

for $\tau \in \mathbb{H}^d$. It can be explicitly described as follows. We put

$$Q(\lambda) = (Q(\lambda_1), \dots, Q(\lambda_d)) \in \mathbb{R}^d.$$

If $z \in \mathbb{D}$ we let λ_{1z} (respectively $\lambda_{1z^\perp}$) be the orthogonal projection of λ_1 to z (respectively to $z^\perp$). Moreover, we put

$$\begin{aligned}\lambda_z &= (\lambda_{1z}, 0, \dots, 0),\\ \lambda_{z^\perp} &= (\lambda_{1z^\perp}, \lambda_2, \dots, \lambda_d).\end{aligned}$$

Hence

$$\lambda \mapsto \operatorname{tr} Q(\lambda_{z^\perp}) - \operatorname{tr} Q(\lambda_z)$$

defines a positive definite quadratic form on $V(\mathbb{R})$, the majorant associated to z. It is easily seen that

$$\varphi_0(\tau, z, \lambda) = v_1 e\big(\operatorname{tr} Q(\lambda_{z^\perp})\tau + \operatorname{tr} Q(\lambda_z)\bar\tau\big).$$

In particular, this Schwartz function is non-holomorphic in τ_1, but holomorphic in $\tau_2, \dots, \tau_d$.

We identify $V(F)$ with its image under the canonical embedding into $V(\mathbb{R})$. Let $\varphi_f \in S(V(\hat F))$ be a Schwartz-Bruhat function. For $\tau \in \mathbb{H}^d$, $z \in \mathbb{D}$, and $h \in H(\hat{\mathbb{Q}})$, we define the Siegel theta function associated to φ_f by

$$\begin{aligned}\theta_S(\tau, z, h; \varphi_f) &= \sum_{\lambda\in V(F)} \varphi_f(h^{-1}\lambda)\varphi_0(\tau, z, \lambda)\\ &= v_1 \sum_{\lambda\in V(F)} \varphi_f(h^{-1}\lambda) e\big(\operatorname{tr} Q(\lambda_{z^\perp})\tau + \operatorname{tr} Q(\lambda_z)\bar\tau\big).\end{aligned} \tag{3.10}$$

It satisfies the transformation formulas

(3.11) $\theta_S(\tau, \gamma z, \gamma h; \varphi_f) = \theta_S(\tau, z, h; \varphi_f), \quad \gamma \in H(\mathbb{Q}),$

(3.12) $\theta_S(\gamma\tau, z, h; \varphi_f) = (c_1\tau_1 + d_1)^{-2}\phi(\tau)^{\ell}\theta_S(\tau, z, h; \omega(\hat{\gamma}^{-1})\varphi_f), \quad \gamma = ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), \phi) \in \tilde{\Gamma}.$

We will also consider the S_L-valued theta function

$$\Theta_S(\tau, z, h) = \sum_{\mu \in L'/L} \theta_S(\tau, z, h; \chi_\mu)\chi_\mu. \tag{3.13}$$

It has the following transformation formula, which can be deduced from (3.12) or by applying the Poisson summation formula as in [Bo1].

Theorem 3.2. *For $\gamma = (g, \phi(\tau)) \in \tilde{\Gamma}$ with $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, we have*

$$\Theta_S(\gamma\tau, z, h) = (c_1\tau_1 + d_1)^{-2}\phi(\tau)^{\ell}\rho_L(\gamma)\Theta_S(\tau, z, h).$$

□

The following growth estimate will be used later. It is proved in the same way as the corresponding statement for holomorphic Hilbert modular forms.

Proposition 3.3. *The Siegel theta function satisfies uniformly in u that*

$$\theta_S(\tau, z, h; \chi_\mu) = O\left(v_1 N(v)^{-(n+2)/2}\right), \quad v_i \to 0.$$

3.4. **Kudla–Millson theta functions.** For $\lambda = (\lambda_1, \dots, \lambda_d) \in V(\mathbb{R})$ and $z \in \mathbb{D}$ the Kudla–Millson Schwartz form on $V(\mathbb{R})$ is defined by

$$\varphi_{KM}(z, \lambda) = \varphi_{KM}^{V_{\sigma_1}}(z, \lambda_1) \otimes \varphi_{KM}^{V_{\sigma_2}}(\lambda_2) \otimes \cdots \otimes \varphi_{KM}^{V_{\sigma_d}}(\lambda_d).$$

It has parallel weight $\ell/2 = (n+2)/2$. We also have the corresponding Schwartz form $\varphi_{KM}(\tau, z, \lambda)$ for $\tau \in \mathbb{H}^d$. It is non-holomorphic in τ_1, but holomorphic in $\tau_2, \dots, \tau_d$.

Let $\varphi_f \in S(V(\hat{F}))$ be a Schwartz-Bruhat function. For $\tau \in \mathbb{H}^d$, $z \in \mathbb{D}$, and $h \in H(\hat{\mathbb{Q}})$, the Kudla–Millson theta function associated to φ_f is given by

$$\theta_{KM}(\tau, z, h; \varphi_f) = \sum_{\lambda \in V(F)} \varphi_f(h^{-1}\lambda)\varphi_{KM}(\tau, z, \lambda), \tag{3.14}$$

see [KM1], [KM2], [KM3] for details. Its geometric significance lies in the fact that the Fourier coefficients with totally positive index m are Poincaré dual forms for the cycles $Z(m, \varphi_f)$ (however, we do not need that here).

The characteristic functions of the cosets of L can be used to define the S_L-valued valued Kudla–Millson theta function

$$\Theta_{KM}(\tau, z, h) = \sum_{\mu \in L'/L} \theta_{KM}(\tau, z, h; \chi_\mu)\chi_\mu. \tag{3.15}$$

In the variable (z, h) it defines a closed 2-form on X_K. In τ it satisfies the transformation formula

$$\Theta_{KM}(\gamma\tau, z, h) = \phi(\tau)^{\ell}\rho_L(\gamma)\Theta_{KM}(\tau, z, h),$$

for $\gamma = (g, \phi(\tau)) \in \tilde{\Gamma}$. Hence it is a non-holomorphic vector valued Hilbert modular form for $\tilde{\Gamma}$ of parallel weight $\ell/2$ with representation ρ_L. Since it has moderate growth at the cusps, it can be integrated against cusp forms.

Proposition 3.4. *The Kudla–Millson theta function satisfies uniformly in u that*

$$\theta_{KM}(\tau, z, h; \chi_\mu) = O\left(N(v)^{-(n+2)/2}\right), \quad v_i \to 0.$$

4. Whittaker forms

Following Harvey, Moore, and Borcherds (see [HM] and [Bo1]) we would like to construct automorphic forms on X_K with singularities along special cycles by integrating weakly holomorphic modular forms or weak Maass forms against the Siegel theta function. Unfortunately, because of the Koecher principle, there are no such automorphic forms with singularities at the cusps when $d > 1$.

Here we overcome this problem by viewing weak Maass forms as *formal* Poincaré series ignoring the issue of convergence. Then the theta integral can be formally unfolded leading to an integral over $\tilde{\Gamma}_\infty \backslash \mathbb{H}^d$, where

$$\tilde{\Gamma}_\infty := \{T_b \cdot (1, \pm 1); \quad b \in \mathcal{O}_F\} \tag{4.1}$$

is the subgroup of translations of $\tilde{\Gamma}$. It turns out that such integrals still make sense when they are suitably regularized. It is natural to consider them for translation invariant functions which are eigenfunctions for the Laplacians. This leads to the definition of Whittaker forms below. Moreover, we define weakly holomorphic Whittaker forms, which serve as substitutes of weakly holomorphic modular forms. We derive some properties of these functions which will be important later.

We begin by fixing some notation. Let $k = (k_1, \dots, k_d) \in (\frac{1}{2}\mathbb{Z})^d$ be a weight. Throughout we assume that $k \equiv (\frac{\ell}{2}, \dots, \frac{\ell}{2}) \pmod{\mathbb{Z}^d}$. We define a Petersson slash operator in weight k for the representation ρ_L on functions $f : \mathbb{H}^d \to S_L$ by

$$(f\mid_{k,\rho_L}(g, \phi))(\tau) = (c_1\tau_1 + d_1)^{-k_1+\ell/2} \cdots (c_d\tau_d + d_d)^{-k_d+\ell/2} \phi(\tau)^{-\ell} \rho_L(g, \phi)^{-1} f(g\tau),$$

where $(g, \phi) \in \tilde{G}_\mathbb{R}$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The Petersson slash operator for the dual representation $\bar{\rho}_L$ is defined analogously. We write S_{k,ρ_L} for the space of vector valued Hilbert cusp forms of weight k for $\tilde{\Gamma}$ with representation ρ_L.

We have the usual hyperbolic Laplace operators in weight k acting on smooth functions on $\mathbb{H}^d$. They are given by

$$\Delta_k^{(j)} = -v_j^2 \left(\frac{\partial^2}{\partial u_j^2} + \frac{\partial^2}{\partial v_j^2}\right) + ik_j v_j \left(\frac{\partial}{\partial u_j} + i\frac{\partial}{\partial v_j}\right) \tag{4.2}$$

for $j = 1, \dots, d$. Moreover, we have the Maass lowering and raising operators

$$R_k^{(j)} = 2i\frac{\partial}{\partial \tau_j} + k_j v_j^{-1},$$

$$L_k^{(j)} = -2iv_j^2 \frac{\partial}{\partial \bar{\tau}_j}.$$

The raising operator $R_k^{(j)}$ raises the weight of an automorphic form in the j-th component by 2 while $L_k^{(j)}$ lowers it by 2. The Laplacian $\Delta_k^{(j)}$ satisfies the identity

$$-\Delta_k^{(j)} = L_{k+2}^{(j)} R_k^{(j)} + k_j = R_{k-2}^{(j)} L_k^{(j)}.$$

We consider functions which are invariant under the group of translations $\tilde{\Gamma}_\infty$. For $s \in \mathbb{C}$ we let $A_{k,\bar{\rho}_L}(s)$ be the space of smooth functions $f : \mathbb{H}^d \to S_L$ satisfying:

(1) $f(T_b\tau) = \bar{\rho}_L(T_b)f(\tau), \qquad b \in \mathcal{O}_F$,
(2) $\Delta_k^{(1)} f = \frac{1}{4}(k_1 - 1 + s)(k_1 - 1 - s)f$,
(3) f is antiholomorphic in τ_j for $j = 2, \dots, d$.

To describe the Fourier expansion of such a function we recall some properties of Whittaker functions, see [AbSt] Chap. 13 pp. 189 or [Er1] Vol. I Chap. 6 p. 264. Kummer's confluent hypergeometric function is defined by

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \tag{4.3}$$

where $(a)_n = \Gamma(a + n)/\Gamma(a)$ and $(a)_0 = 1$. The Whittaker functions are defined by

$$M_{\nu,\mu}(z) = e^{-z/2} z^{1/2+\mu} M(1/2 + \mu - \nu, 1 + 2\mu, z), \tag{4.4}$$

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - \nu)} M_{\nu,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - \nu)} M_{\nu,-\mu}(z). \tag{4.5}$$

They are linearly independent solutions of the Whittaker differential equation. The M-Whittaker function has the asymptotic behavior

$$M_{\nu,\,\mu}(z) = z^{\mu+1/2}(1 + O(z)), \qquad z \to 0, \tag{4.6}$$

$$M_{\nu,\,\mu}(z) = \frac{\Gamma(1 + 2\mu)}{\Gamma(\mu - \nu + 1/2)} e^{z/2} z^{-\nu}(1 + O(z^{-1})), \qquad z \to \infty, \tag{4.7}$$

while $W_{\nu,\,\mu}(z)$ is exponentially decreasing for real $z \to \infty$ and behaves like a constant times $z^{-\mu+1/2}$ as $z \to 0$.

For convenience we put for $s \in \mathbb{C}$ and $v_1 \in \mathbb{R}$:

$$\mathcal{M}_s(v_1) = |v_1|^{-k_1/2} M_{\operatorname{sgn}(v_1)k_1/2,\, s/2}(|v_1|) \cdot e^{-v_1/2}, \tag{4.8}$$

$$\mathcal{W}_s(v_1) = |v_1|^{-k_1/2} W_{\operatorname{sgn}(v_1)k_1/2,\, s/2}(|v_1|) \cdot e^{-v_1/2}. \tag{4.9}$$

The functions $\mathcal{M}_s(v_1)$ and $\mathcal{W}_s(v_1)$ are holomorphic in s. Later we will be interested in their special value at

$$s_0 = 1 - k_1. \tag{4.10}$$

We have

$$\mathcal{M}_{s_0}(v_1) = (-\operatorname{sgn}(v_1))^{k_1-1} \cdot e^{-v_1}\big(\Gamma(2 - k_1) - (1 - k_1)\Gamma(1 - k_1, -v_1)\big), \tag{4.11}$$

$$\mathcal{W}_{s_0}(v_1) = \begin{cases} e^{-v_1}, & v_1 > 0, \\ e^{-v_1} \cdot \Gamma(1 - k_1, -v_1), & v_1 < 0, \end{cases} \tag{4.12}$$

where $\Gamma(a,z)$ denotes the incomplete Gamma function. Any $f\in A_{k,\bar\rho_L}(s)$ has a Fourier expansion of the form

$$f(\tau)=\sum_{\substack{\mu\in L'/L\\ Q(\mu)\in\partial_F^{-1}}}\big(a(0,\mu,s)v_1^{(1-k_1-s)/2}+b(0,\mu,s)v_1^{(1-k_1+s)/2}\big)\chi_\mu$$
$$+\sum_{\mu\in L'/L}\sum_{\substack{m\in\partial_F^{-1}-Q(\mu)\\ m\neq 0}}\big(a(m,\mu,s)\mathcal{W}_s(4\pi m_1v_1)+b(m,\mu,s)\mathcal{M}_s(4\pi m_1v_1)\big)e(\operatorname{tr}(m\bar\tau))\chi_\mu.$$

Special elements of $A_{k,\bar\rho_L}(s)$ are the functions

$$f_{m,\mu}(\tau,s):=C(m,k,s)\mathcal{M}_s(-4\pi m_1v_1)e(-\operatorname{tr}(m\bar\tau))\chi_\mu \tag{4.13}$$

for $\mu\in L'/L$, $m\in\partial^{-1}+Q(\mu)$, and $m\gg 0$. Here $C(m,k,s)$ denotes the normalizing factor

$$C(m,k,s):=\frac{(4\pi m_2)^{k_2-1}\cdots(4\pi m_d)^{k_d-1}}{\Gamma(s+1)\Gamma(k_2-1)\cdots\Gamma(k_d-1)}, \tag{4.14}$$

which turns out to be convenient later (for instance in Proposition 4.5).

Definition 4.1. A *Whittaker form* of weight k and parameter s (for $\tilde\Gamma$ and $\bar\rho_L$) is a finite linear combination of the functions $f_{m,\mu}(\tau,s)$ for $\mu\in L'/L$, $m\in\partial_F^{-1}+Q(\mu)$, and $m\gg0$. A *harmonic Whittaker form* is a Whittaker form with parameter s_0. We denote the $\mathbb{C}$-vector space of such harmonic Whittaker forms by $H_{k,\bar\rho_L}$.

So a harmonic Whittaker form is a finite linear combination of the functions

$$f_{m,\mu}(\tau):=f_{m,\mu}(\tau,s_0)$$

for $\mu\in L'/L$, $m\in\partial_F^{-1}+Q(\mu)$, and $m\gg0$. Explicitly we have

$$f_{m,\mu}(\tau)=C(m,k,s_0)\Gamma(2-k_1)\left(1-\frac{\Gamma(1-k_1,4\pi m_1v_1)}{\Gamma(1-k_1)}\right)e^{4\pi m_1v_1}e(\operatorname{tr}(-m\bar\tau))\chi_\mu. \tag{4.15}$$

We define the *dual weight* for k by $\kappa=(2-k_1,k_2,\dots,k_d)$. We consider the differential operator δ_k on functions $f:\mathbb{H}^d\to S_L$ given by

$$\delta_k(f)=v_1^{k_1-2}\overline{L_k^{(1)}f(\tau)}. \tag{4.16}$$

If $f\in H_{k,\bar\rho_L}$, then $\delta_k(f)$ is a holomorphic function satisfying $f(T_b\tau)=\rho_L(T_b)f(\tau)$ for all $b\in\mathcal{O}_F$. In particular, we have

$$\delta_k(f_{m,\mu})(\tau)=\frac{(4\pi m_1)^{\kappa_1-1}\cdots(4\pi m_d)^{\kappa_d-1}}{\Gamma(\kappa_1-1)\cdots\Gamma(\kappa_d-1)}e(\operatorname{tr}(m\tau))\chi_\mu. \tag{4.17}$$

We have $\Delta_k^{(1)}f=0$ for a harmonic Whittaker form f. The functions $f_{m,\mu}(\tau,-s_0)$ have eigenvalue zero for this Laplacian as well, but they are *not* harmonic Whittaker forms in the sense of our definition. The point of the definition is to prescribe a particular growth as $v_j\to\infty$ and $v_j\to0$ such that there is a smooth relationship with Hilbert cusp forms (see below) and such that the theta lift of a harmonic Whittaker form has singularities along special divisors (see Section 5). More precisely, we have the following characterization:

Remark 4.2. Assume that $k_1 < 1$ (which will be the case in all later applications). Then $H_{k,\bar\rho_L}$ is the space of functions in $A_{k,\bar\rho_L}(s_0)$ satisfying:

(4) only finitely many Fourier coefficients are non-zero,
(5) there is a totally positive $\varepsilon \in F$ such that $\delta_k(f)(\tau) = O(e^{-\operatorname{tr}(\varepsilon v)})$ for $v_j \to \infty$,
(6) we have $f(\tau) = O(v_1^{s_0})$, for $v_1 \to 0$.

For the rest of this section we assume that $\kappa_j \geq 3/2$ for $j = 1, \dots, d$. We define an operator $\xi_k : H_{k,\bar\rho_L} \to S_{\kappa,\rho_L}$ by

$$\xi_k(f) = \sum_{\gamma \in \tilde\Gamma_\infty \backslash \tilde\Gamma} \delta_k(f) \mid_{\kappa,\rho_L} \gamma. \tag{4.18}$$

When $\kappa_j > 2$ for $j = 1, \dots, d$, the Poincaré series on the right hand side converges normally by a standard estimate (see e.g. [Ga], Chapter 1.13) and defines a holomorphic cusp form. When $\kappa_j \geq 3/2$ we define the Poincaré series using "Hecke summation" as the value at $s' = 0$ of the holomorphic continuation in s' of

$$\sum_{\gamma \in \tilde\Gamma_\infty \backslash \tilde\Gamma} \delta_k(f)\, \mathrm{N}(v)^{s'} \mid_{\kappa,\rho_L} \gamma. \tag{4.19}$$

Proposition 4.3. *Assume that $\kappa_j \geq 3/2$ for $j = 1, \dots, d$. The map $\xi_k : H_{k,\bar\rho_L} \to S_{\kappa,\rho_L}$ is surjective.*

Proof. The assertion follows from (4.17) and the fact that S_{κ,ρ_L} is generated by Poincaré series. □

Definition 4.4. A Whittaker form f is called *weakly holomorphic* if it is harmonic and satisfies $\xi_k(f) = 0$. We denote by $M^!_{k,\bar\rho_L}$ the subspace of weakly holomorphic Whittaker forms in $H_{k,\bar\rho_L}$.

Note that a weakly holomorphic Whittaker form is *not* holomorphic as a function on $\mathbb{H}^d$. It is rather holomorphic in a weak distribution sense. In view of Proposition 4.3, we have the exact sequence

$$0 \longrightarrow M^!_{k,\bar\rho_L} \longrightarrow H_{k,\bar\rho_L} \xrightarrow{\xi_k} S_{\kappa,\rho_L} \longrightarrow 0\,. \tag{4.20}$$

Recall that the Petersson scalar product of $f, g \in S_{\kappa,\rho_L}$ is given by

$$(f,g)_{Pet} = \frac{1}{\sqrt{D}} \int_{\tilde\Gamma \backslash \mathbb{H}^d} \langle f, \bar g\rangle v^\kappa \, d\mu(\tau), \tag{4.21}$$

where $d\mu(\tau) = \frac{du_1\, dv_1}{v_1^2} \dots \frac{du_d\, dv_d}{v_d^2}$ is the invariant measure on $\mathbb{H}^d$, and v^κ is understood in multi-index notation. We define a bilinear pairing between the spaces S_{κ,ρ_L} and $H_{k,\bar\rho_L}$ by putting

$$\{g, f\} = \big(g,\, \xi_k(f)\big)_{Pet} \tag{4.22}$$

for $g \in S_{\kappa,\rho_L}$ and $f \in H_{k,\bar\rho_L}$. The pairing vanishes when f is weakly holomorphic. Because of Proposition 4.3 the induced pairing between S_{κ,ρ_L} and $H_{k,\bar\rho_L}/M^!_{k,\bar\rho_L}$ is non-degenerate. So an $f \in H_{k,\bar\rho_L}$ is weakly holomorphic, if and only if $\{g, f\} = 0$ for all $g \in S_{\kappa,\rho_L}$.

Proposition 4.5. *Let $g \in S_{\kappa,\rho_L}$ with Fourier expansion $g = \sum_{n,\nu} b(n,\nu)e(\mathrm{tr}(n\tau))\chi_\nu$, and let*

$$f = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu) f_{m,\mu}(\tau) \in H_{k,\bar\rho_L}.$$

Then the pairing of g and f is equal to

$$\{g, f\} = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu) b(m,\mu). \tag{4.23}$$

Proof. This follows from (4.17) and (4.18) using the formula for the Petersson scalar product of Poincaré series with cusp forms, see e.g. [Luo, Section 2]. □

4.1. **Whittaker forms and weak Maass forms.** We end this section by explaining the relationship between Whittaker forms and weak Maass forms as defined in [BF]. Assume that $d = 1$ so that $F = \mathbb{Q}$. Then there is no Koecher principle and there are nontrivial weak Maass forms. For simplicity we also assume that $k = k_1 < 0$.

A smooth function $f : \mathbb{H} \to S_L$ is called a *harmonic weak Maass form* (of weight k with respect to $\tilde\Gamma$ and $\bar\rho_L$) if it satisfies:

(i) $f \mid_{k,\bar\rho_L} \gamma = f$ for all $\gamma \in \tilde\Gamma$,
(ii) $\Delta_k f = 0$, where Δ_k is the weight k Laplacian,
(iii) there is a S_L-valued Fourier polynomial

$$P_f(\tau) = \sum_{\mu \in L'/L} \sum_{m \geq 0} c^+(m,\mu) q^{-m} \chi_\mu$$

such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$.

The Fourier polynomial P_f is called the *principal part* of f. Here we denote the vector space of these harmonic weak Maass forms by $\mathcal{H}_{k,\bar\rho_L}$. Any weakly holomorphic modular form is a harmonic weak Maass form. We denote the subspace of weakly holomorphic modular forms by $\mathcal{M}^!_{k,\bar\rho_L}$.

Proposition 4.6. *If $f \in H_{k,\bar\rho_L}$ is a harmonic Whittaker form, then*

$$\eta(f) = \sum_{\gamma \in \tilde\Gamma_\infty \backslash \tilde\Gamma} f \mid_{k,\bar\rho_L} \gamma$$

converges and defines an element of $\mathcal{H}_{k,\bar\rho_L}$. The map $\eta : H_{k,\bar\rho_L} \to \mathcal{H}_{k,\bar\rho_L}$ defined by $f \mapsto \eta(f)$ is an isomorphism. Its inverse is given by mapping a harmonic weak Maass form g with principal part $P_g = \sum_\mu \sum_{m \geq 0} c^+(m,\mu) q^{-m} \chi_\mu$ to the harmonic Whittaker form

$$f = \sum_\mu \sum_{m > 0} c^+(m,\mu) f_{m,\mu}(\tau).$$

The restriction of η induces an isomorphism $M^!_{k,\bar\rho_L} \to \mathcal{M}^!_{k,\bar\rho_L}$.

The proof of the proposition follows from well known properties of non-holomorphic Poincaré series, see e.g. [He] or [Br], Chapter 1. Moreover, the operator ξ_k on harmonic Whittaker forms is compatible with the corresponding operator on harmonic weak Maass forms of [BF].

5. The theta lift

Here we define the regularized theta lift of Whittaker forms with parameter s. The images of the lift are automorphic Green functions for special divisors. We first do this when $\Re(s)$ is sufficiently large. For the general case we use meromorphic continuation in s, which can be obtained by means of spectral theory for X_K.

We use the setup of the previous sections. Throughout we assume that the compact open subgroup $K \subset H(\hat{\mathbb{Q}})$ fixes the lattice $L \subset V$ and acts trivially on L'/L. Let $\Delta_{\mathbb{D}}$ be the Laplace operator induced by the the Casimir element of the Lie algebra of $\mathrm{SO}(V_{\sigma_1})$ (or by the invariant metric on $\mathbb{D}$). We normalize it as in [Br] (4.1).

Let f be a Whittaker form of weight

$$k = \left(\frac{2-n}{2}, \frac{2+n}{2}, \dots, \frac{2+n}{2}\right) \tag{5.1}$$

with parameter s for $\tilde{\Gamma}$ and $\bar{\rho}_L$ (see Definition 4.1). We assume that $s > s_0 := 1 - k_1 = n/2$. Recall from Theorem 3.2 that the Siegel theta function $\Theta_S(\tau, z, h)$ associated to the lattice L has weight $\left(\frac{n-2}{2}, \frac{2+n}{2}, \dots, \frac{2+n}{2}\right)$ and representation ρ_L. Consequently, the pairing

$$\langle f(\tau), \Theta_S(\tau, z, h)\rangle (v_2 \cdots v_d)^{\ell/2}$$

is a scalar valued $\tilde{\Gamma}_\infty$-invariant function in τ. We want to consider the theta integral

$$\Phi(z, h, f) = \frac{1}{\sqrt{D}} \int_{\tilde{\Gamma}_\infty \backslash \mathbb{H}^d} \langle f(\tau), \Theta_S(\tau, z, h)\rangle (v_2 \cdots v_d)^{\ell/2}\, d\mu(\tau).$$

Because of the exponential growth of f, the integral does not converge. Similarly as in [Bo1] and [Br], we regularize it by prescribing the order of integration, namely we first integrate over u and then over v. In the notation the regularization is indicated by the superscript "reg" at the integral.

Definition 5.1. The regularized theta lift of f is defined as

$$\begin{aligned}\Phi(z, h, f) &= \frac{1}{\sqrt{D}} \int^{reg}_{\tilde{\Gamma}_\infty \backslash \mathbb{H}^d} \langle f(\tau), \Theta_S(\tau, z, h)\rangle (v_2 \cdots v_d)^{\ell/2}\, d\mu(\tau)\\ &= \frac{1}{\sqrt{D}} \int_{v \in (\mathbb{R}_{>0})^d} \left(\int_{u \in \mathcal{O}_F \backslash \mathbb{R}^d} \langle f(\tau), \Theta_S(\tau, z, h)\rangle\, du \right) (v_2 \cdots v_d)^{\ell/2} \frac{dv}{\mathrm{N}(v)^2}.\end{aligned}$$

Theorem 5.2. *Let f be a Whittaker form of parameter s as above. If $\Re(s) > s_0 + 2$, then the regularized theta integral converges for (z, h) outside a subset of X_K of measure zero. It defines an integrable function on X_K.*

To prove this Theorem, it suffices to consider for any $\mu \in L'/L$ and any totally positive $m \in \partial_F^{-1} + Q(\mu)$ the theta integral

$$\Phi_{m,\mu}(z, h, s) := \Phi(z, h, f_{m,\mu}(\cdot, s)) \tag{5.2}$$

of the special Whittaker forms $f_{m,\mu}(\tau, s)$, see (4.13). We call $\Phi_{m,\mu}(z, h, s)$ the *automorphic Green function* of the divisor $Z(m, \mu)$. Later we will show that its regularized value at s_0

gives rise to a subharmonic Arakelov Green function for $Z(m,\mu)$ in the sense of [SABK]. Theorem 5.2 will follow from Theorem 5.3 below.

Let $F(a,b,c;z)$ denote the Gauss hypergeometric function

$$F(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{5.3}$$

see e.g. [AbSt] Chap. 15 or [Er1] Vol. I Chap. 2. The circle of convergence of the series (5.3) is the unit circle $|z|=1$. For $\lambda\in V(\mathbb{R})$, $z\in\mathbb{D}$, and $s\in\mathbb{C}$ we put

$$\phi(\lambda,z,s):=\frac{\Gamma(\frac{s}{2}+\frac{n}{4})}{\Gamma(s+1)}\left(\frac{Q(\lambda_1)}{Q(\lambda_{1z^\perp})}\right)^{\frac{s}{2}+\frac{n}{4}}F\left(\frac{s}{2}+\frac{n}{4},\,\frac{s}{2}-\frac{n}{4}+1,\,s+1;\,\frac{Q(\lambda_1)}{Q(\lambda_{1z^\perp})}\right). \tag{5.4}$$

This function has the invariance property $\phi(\lambda,z,s)=\phi(h\lambda,hz,s)$ for $h\in H(\mathbb{R})$. Notice that it is closely related to the secondary spherical function $\phi_s^{(2)}(h)$ on $\mathrm{SO}(n,2)$ considered in [OT]. More precisely, for a totally positive $\lambda\in V$, we fix a base point $z_0\in\mathbb{D}$ such that $z_0\perp\lambda$. As in the proof of Theorem 4.7 of [BK] it is easily verified that

$$\phi(\lambda,hz_0,s)=\frac{-2}{\Gamma(\frac{s}{2}-\frac{s_0}{2}+1)}\phi_s^{(2)}(h) \tag{5.5}$$

for $h\in\mathrm{SO}(V_{\sigma_1})\cong\mathrm{SO}(n,2)$. It follows from [OT], Proposition 2.4.2, that $\phi(\lambda,z,s)$ satisfies the differential equation

$$\Delta_{\mathbb{D}}\phi(\lambda,z,s)=\frac{1}{8}(s^2-s_0^2)\phi(\lambda,z,s). \tag{5.6}$$

Theorem 5.3. *Let $\mu\in L'/L$, $m\in\partial_F^{-1}+Q(\mu)$, and $m\gg 0$. If $\Re(s)>s_0+2$, the regularized theta integral of $f_{m,\mu}(\tau,s)$ converges and is equal to*

$$\Phi_{m,\mu}(z,h,s)=\sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}}\phi(\lambda,z,s). \tag{5.7}$$

The sum converges for $\Re(s)>s_0$ and (z,h) outside a subset of X_K of measure 0. It defines an integrable function on X_K.

Proof. We first compute the theta integral $\Phi_{m,\mu}(z,h,s)$ formally. Afterwards we show the convergence of the infinite series representation in the statement of the theorem. The interchange of integration and summation in the computation of the theta integral is then justified a posteriori by the theorem of monotone convergence.

Inserting the definitions and carrying out the integration over u, we obtain

$$\begin{aligned}\Phi_{m,\mu}(z,h,s)&=C(m,k,s)\\ &\quad\times\int_{(\mathbb{R}_{>0})^d}\sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}}\mathcal{M}_s(-4\pi m_1v_1)e^{-4\pi Q(\lambda_{1z^\perp})v_1}v_1^{-1}(v_2\cdots v_d)^{\ell/2-2}\,dv.\end{aligned}$$

Here we have also used that $\mathrm{vol}(\mathcal{O}_F\backslash\mathbb{R}^d)=\sqrt{D}$. In view of Proposition 3.3 and (4.6), the integral converges for $\Re(s)>s_0+2$ and $(z,h)\in X_K\backslash Z(m,\mu)$.

We put in the definition of $\mathcal{M}_s$ and interchange integration and summation. We find

$$\Phi_{m,\mu}(z,h,s) = C(m,k,s) \sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}} \int_0^\infty \frac{M_{-k_1/2,\,s/2}(4\pi m_1 v_1)}{(4\pi m_1 v_1)^{k_1/2}} e^{-2\pi m_1 v_1 + 4\pi Q(\lambda_{1z})v_1} \frac{dv_1}{v_1}$$
$$\times \int_0^\infty e^{-4\pi m_2 v_2} v_2^{k_2-1} \frac{dv_2}{v_2} \cdots \int_0^\infty e^{-4\pi m_d v_d} v_d^{k_d-1} \frac{dv_d}{v_d}.$$

Inserting the value of $C(m,k,s)$ and carrying out the integration over $v_2,\dots,v_d$, we obtain

$$\Phi_{m,\mu}(z,h,s) = \frac{1}{\Gamma(s+1)(4\pi m_1)^{k_1/2}}$$
$$\times \sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}} \int_0^\infty M_{-k_1/2,\,s/2}(4\pi m_1 v_1) e^{-2\pi m_1 v_1 + 4\pi Q(\lambda_{1z})v_1} v_1^{-k_1/2} \frac{dv_1}{v_1}.$$

The latter integral is a Laplace transform. It is equal to

$$\frac{(4\pi m_1)^{s/2+1/2}\Gamma(\frac{s}{2}+\frac{n}{4})}{(4\pi Q(\lambda_{1z^\perp}))^{s/2+n/4}} F\left(\frac{s}{2}+\frac{n}{4},\, \frac{s}{2}-\frac{n}{4}+1,\, s+1;\, \frac{m_1}{Q(\lambda_{1z^\perp})}\right),$$

see e.g. [Er2] p. 215 (11). Consequently,

$$\Phi_{m,\mu}(z,h,s) = \frac{\Gamma(\frac{s}{2}+\frac{n}{4})}{\Gamma(s+1)}$$
$$\times \sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}} \left(\frac{m_1}{Q(\lambda_{1z^\perp})}\right)^{\frac{s}{2}+\frac{n}{4}} F\left(\frac{s}{2}+\frac{n}{4},\, \frac{s}{2}-\frac{n}{4}+1,\, s+1;\, \frac{m_1}{Q(\lambda_{1z^\perp})}\right).$$

We now prove that the sum converges for (z,h) outside a subset of measure zero to an integrable function on X_K. By reduction theory, the arithmetic group $\Gamma_K = H(\mathbb{Q})\cap K$ acts with finitely many orbits on the set of $\lambda\in\mu+L$ with $Q(\lambda)=m$. We consider for a fixed $\lambda\in\mu+L$ with $Q(\lambda)=m$ the sum

$$S(z) := \sum_{\gamma\in\Gamma_{K,\lambda}\backslash\Gamma_K} \phi(\gamma\lambda,z,s),$$

where $\Gamma_{K,\lambda} = H_\lambda(\mathbb{Q})\cap K$ and H_λ denotes the stabilizer of λ in H. It suffices to show that $S(z)$ converges outside a subset of measure zero and that $\int_{\Gamma_K\backslash\mathbb{D}} S(z)\,d\mu(z) < \infty$. According to Fubini's theorem, we have

$$\int_{\Gamma_K\backslash\mathbb{D}} S(z)\,d\mu(z) = \int_{\Gamma_{K,\lambda}\backslash\mathbb{D}} \phi(\lambda,z,s)\,d\mu(z),$$

and the desired convergence statement follows if we show that the integral on the right hand side is finite. Fixing a base point $z_0\in\mathbb{D}$ with $z_0\perp\lambda$, we may realize $\mathbb{D}$ as the coset space of $H(\mathbb{R})$ modulo the maximal compact subgroup given by the stabilizer of z_0. Hence it suffices to show that

$$\int_{\Gamma_{K,\lambda}\backslash H(\mathbb{R})} \phi(\lambda, hz_0, s)\, dh < \infty,$$

where dh denotes a Haar measure on $H(\mathbb{R})$. The latter integral is equal to

$$\begin{aligned}\int_{\Gamma_{K,\lambda}\backslash H(\mathbb{R})} \phi(h^{-1}\lambda, z_0, s)\, dh &= \int_{h\in\Gamma_{K,\lambda}\backslash H_\lambda(\mathbb{R})} \int_{h'\in H_\lambda(\mathbb{R})\backslash H(\mathbb{R})} \phi(h'^{-1}h^{-1}\lambda, z_0, s)\, dh'\, dh\\ &= \operatorname{vol}(\Gamma_{K,\lambda}\backslash H_\lambda(\mathbb{R})) \int_{H_\lambda(\mathbb{R})\backslash H(\mathbb{R})} \phi(h'^{-1}\lambda, z_0, s)\, dh'.\end{aligned}$$

The convergence of the latter integral for $\Re(s) > s_0$ is proved in [OT], Proposition 3.1.1 (see also [BK], Section 4.2, for a comparison of the different setups). □

Next we consider the singularities of $\Phi_{m,\mu}(z, h, s)$.

Lemma 5.4. *Let $m \in F$ be totally positive. i) For all $\lambda \in L'$ with $Q(\lambda) = m$ we have $0 < m_1 \leq Q(\lambda_{1z^\perp})$, and*

$$\frac{m_1}{Q(\lambda_{1z^\perp})} = \frac{2m_1}{m_1 + (Q(\lambda_{1z^\perp}) - Q(\lambda_{1z}))}.$$

ii) For any $\varepsilon > 0$ and any compact subset $C \subset \mathbb{D}$, there are only finitely many $\lambda \in L'$ with $Q(\lambda) = m$ and $\varepsilon < m_1/Q(\lambda_{1z^\perp})$ for some $z \in C$.

Proof. This follows by a straightforward computation and the fact that $Q(\lambda_{1z^\perp}) - Q(\lambda_{1z})$ is a positive definite quadratic form on V_{σ_1}. □

Theorem 5.5. *Let $\mu \in L'/L$, $m \in \partial_F^{-1} + Q(\mu)$, and $m \gg 0$. The series* (5.7) *and all its partial derivatives converge normally for $\Re(s) > s_0$ and $(z, h) \in X_K \setminus Z(m, \mu)$. For any point $(z_0, h_0) \in \mathbb{D} \times H(\hat{\mathbb{Q}})$ there is a neighborhood U such that the function*

$$\Phi_{m,\mu}(z, h, s) - \sum_{\substack{\lambda\in h_0(\mu+L)\\ Q(\lambda)=m\\ \lambda_1\perp z_0}} \phi(\lambda, z, s) \tag{5.8}$$

is C^∞ on U. Here the latter sum is finite.

Proof. Replacing the lattice L by h_0L, we may assume without loss of generality that $h_0 = 1$. The condition $\lambda_1 \perp z_0$ means that λ_1 is contained in the n-dimensional positive definite subspace $z_0^\perp$ of V_{σ_1}. Since $V_{\sigma_2}, \dots, V_{\sigma_d}$ are positive definite, there are only finitely many λ in the coset $\mu + L$ satisfying the condition under the sum in (5.8).

Let $U' \subset \mathbb{D}$ be a compact neighborhood of z_0. Let S_1 be the set of all $\lambda \in \mu + L$ with $Q(\lambda) = m$ and $m_1/Q(\lambda_{1z^\perp}) < 1/2$ for all $z \in U'$. Let S_2 be the set of all $\lambda \in \mu + L$ with $Q(\lambda) = m$ and $m_1/Q(\lambda_{1z^\perp}) \geq 1/2$ for some $z \in U'$. Then we have for $h \in K \subset H(\hat{\mathbb{Q}})$ that

$$\Phi_{m,\mu}(z, h, s) = \sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}} \phi(\lambda, z, s) = \sum_{\lambda\in S_1} \phi(\lambda, z, s) + \sum_{\lambda\in S_2} \phi(\lambda, z, s).$$

According to Lemma 5.4, the sum over S_2 is finite. Moreover,

$$\sum_{\lambda\in S_2}\phi(\lambda,z,s)-\sum_{\substack{\lambda\in\mu+L\\ Q(\lambda)=m\\ \lambda_1\perp z_0}}\phi(\lambda,z,s)$$

is a smooth function in a small neighborhood U of $(z_0,1)$.

Hence it suffices to show that the sum over S_1 converges normally for $z\in U'$. Using the power series expansion of the Gauss hypergeometric function, we see that

$$\phi(\lambda,z,s)\ll Q(\lambda_{1z^\perp})^{-s/2-n/4}$$

for all $\lambda\in S_1$ and all $z\in U'$. The same bound (with different implied constants) holds for all iterated partial derivatives of $\phi(\lambda,z,s)$. Consequently, it suffices to show that

$$\sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}} Q(\lambda_{1z^\perp})^{-s/2-n/4}$$

converges normally on U. This can be proved by comparing with an integral as in the proof of Theorem 5.3. □

Since the Green function $\Phi_{m,\mu}(z,h,s)$ belongs to $L^1(X_K)$, we may view it as a current $[\Phi_{m,\mu,s}]$, that is, as a functional on smooth top degree differential forms on X_K with compact support. For a bounded C^∞-function α on X_K, we write

$$[\Phi_{m,\mu,s}](\alpha)=\int_{X_K}\Phi_{m,\mu}(z,h,s)\alpha(z)\,d\mu(z).$$

Applying the Laplace operator, we obtain another current $\Delta_{\mathbb{D}}[\Phi_{m,\mu,s}]$, given by

$$(\Delta_{\mathbb{D}}[\Phi_{m,\mu,s}])(\alpha)=\int_{X_K}\Phi_{m,\mu}(z,h,s)(\Delta_{\mathbb{D}}\alpha)(z)\,d\mu(z).$$

Moreover, for any divisor Y on X_K, there is a Dirac current δ_Y, given by

$$\delta_Y(\alpha)=\int_Y\alpha(z)\Omega^{n-1}.$$

We define the degree of the divisor Y by $\deg(Y)=\delta_Y(1)$, provided the integral converges.

Remark 5.6. The degree of the first Chern class in $\mathrm{CH}^1(X_K)$ of the line bundle of modular forms of weight w is given by $\deg c_1(\mathcal{M}_w)=w\operatorname{vol}(X_K)$.

Proof. The Petersson metric defines a hermitean metric on the line bundle of modular forms of weight w. Its first Chern form is $w\Omega$. The remark follows from the Poincaré–Lelong lemma. □

Theorem 5.7. *Let $\Re(s)>s_0$. The Green current $[\Phi_{m,\mu,s}]$ satisfies the following differential equation*

$$\Delta_{\mathbb{D}}[\Phi_{m,\mu,s}]=\frac{1}{8}\left(s^2-s_0^2\right)[\Phi_{m,\mu,s}]-\frac{n}{4\Gamma(\frac{s}{2}-\frac{s_0}{2}+1)}\delta_{Z(m,\mu)}.$$

Proof. In view of (5.5), the result follows from [OT], Theorem 3.2.1 (3) and Corollary 3.2.1. The comparison of the different normalizations of the Laplacian and the invariant measures is similar as in the proof of Theorem 4.7 of [BK]. □

5.1. **The spectral expansion.** For $d = 1$ (when $F = \mathbb{Q}$) a meromorphic continuation of the Green function $\Phi_{m,\mu}(z, h, s)$ is obtained in [Br], employing Poincaré series built out of the functions $f_{m,\mu}(\tau, s)$ similarly as in Proposition 4.6. Since the corresponding Poincaré series do not converge when $d > 1$, we cannot argue this way. Instead, we use the approach of [OT] §6 and [MW] §4 to prove meromorphic continuation in the sense of distributions by means of spectral theory. Moreover, we refine the argument to obtain a meromorphic continuation as a smooth function on $X_K \setminus Z(m, \mu)$.

We begin by computing the spectral expansion of $\Phi_{m,\mu}(z, h, s)$. Throughout this subsection we assume that X_K is compact such that the Laplace operator has a discrete spectrum. This is always the case when $d > 1$.

Recall that the Laplace operator $-\Delta_{\mathbb{D}}$ gives rise to a densely defined self-adjoint operator on $L^2(X_K)$ which is positive. Let $\Lambda \subset \mathbb{R}_{\geq 0}$ be the set of eigenvalues of $-\Delta_{\mathbb{D}}$. It is a countable set with no accumulation points. So we may write $\Lambda = \{\lambda_k;\ k \in \mathbb{Z}_{\geq 0}\}$ with

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots,$$

where every $\lambda \in \mathbb{R}$ occurs with multiplicity $d(\lambda)$ given by the dimension of the corresponding eigenspace. Let $\{\varphi_k\} \subset \mathbb{C}^\infty(X_K)$ be an orthonormal system of eigenfunctions such that $-\Delta_{\mathbb{D}}\varphi_k = \lambda_k \varphi_k$. For any k we choose $\alpha_k \in \mathbb{C}$ such that

$$\lambda_k = -\frac{1}{8}\left(\alpha_k^2 - s_0^2\right).$$

Then $\alpha_k \in [-s_0, s_0] \cup i\mathbb{R}$. For the eigenvalue 0, the multiplicity $d(0)$ is equal to the number of connected components of X_K. We choose the suitably normalized characteristic functions of the components of X_K as an orthonormal basis of the eigenspace.

Any function $\varphi \in L^2(X_K)$ has a spectral decomposition

$$\varphi = \sum_{k=0}^{\infty} (\varphi, \varphi_k)\varphi_k, \tag{5.9}$$

where $(\varphi, \psi) = \int_{X_K} \varphi(z)\overline{\psi(z)}\, d\mu(z)$ denotes the scalar product on $L^2(X_K)$. The series converges in the L^2-norm. We now compute this expansion for $\Phi_{m,\mu}(z, h, s)$.

Theorem 5.8. *Assume that X_K is compact. The Green function $\Phi_{m,\mu}(z, h, s)$ has the spectral expansion*

$$\Phi_{m,\mu}(z, h, s) = \frac{2n}{\Gamma(\frac{s}{2} - \frac{s_0}{2} + 1)} \sum_{k=0}^{\infty} \frac{\delta_{Z(m,\mu)}(\overline{\varphi_k})}{s^2 - \alpha_k^2} \cdot \varphi_k(z, h).$$

Proof. This is an immediate consequence of Theorem 5.7 and (5.9). □

Corollary 5.9. *As a distribution on X_K, the Green function $\Phi_{m,\mu}(z,h,s)$ has a meromorphic continuation in s to the whole complex plane. At $s = s_0$ it has a simple pole with residue*

$$A(m,\mu) := 2\frac{\deg(Z(m,\mu))}{\operatorname{vol}(X_K)}. \tag{5.10}$$

Proof. Using the Bessel inequality, one sees that the spectral expansion of $\Phi_{m,\mu}(z,h,s)$ given in Theorem 5.8 converges locally uniformly for $s \in \mathbb{C}$ with $s \neq \pm\alpha_k$ to an element of $L^2(X_K)$. This proves the meromorphic continuation.

The singularity at $s = s_0$ comes from the terms in the spectral expansion with eigenvalue $\lambda_k = 0$. The sum over the remaining terms is holomorphic at s_0. Hence, the singularity is given by

$$\frac{2n}{\Gamma(\frac{s}{2} - \frac{s_0}{2} + 1)} \sum_{\substack{k\geq 0\\ \lambda_k=0}} \frac{\delta_{Z(m,\mu)}(\overline{\varphi_k})}{s^2 - s_0^2} \cdot \varphi_k. \tag{5.11}$$

The eigenfunctions φ_k contributing to this sum are the characteristic functions of the components of X_K multiplied by the normalizing factor $(d(0)/\operatorname{vol}(X_K))^{1/2}$. It is easily seen that for such eigenfunctions $\delta_{Z(m,\mu)}(\overline{\varphi_k}) = (d(0)\operatorname{vol}(X_K))^{-1/2}\delta_{Z(m,\mu)}(1)$. Hence the sum (5.11) is equal to

$$\frac{2n(s^2 - s_0^2)^{-1}}{\Gamma(\frac{s}{2} - \frac{s_0}{2} + 1)} \cdot \frac{\deg(Z(m,\mu))}{\operatorname{vol}(X_K)}.$$

This function has a simple pole at $s = s_0$ with the claimed residue. □

Remark 5.10. The spectral expansion also implies that $\Gamma(\frac{s}{2} - \frac{s_0}{2} + 1)\Phi_{m,\mu}(z,h,s)$ is invariant under the substitution $s \mapsto -s$.

We now refine the above argument, to obtain a meromorphic continuation in s of $\Phi_{m,\mu}(z,h,s)$ as a continuous *function* on $X_K \setminus Z(m,\mu)$. Then the distribution differential equation of Theorem 5.7 and the elliptic regularity theorem imply that $\Phi_{m,\mu}(z,h,s)$ is actually real analytic on $X_K \setminus Z(m,\mu)$. The following lemma is known, see e.g. [Shu], Proposition 10.2. We include it here for completeness.

Lemma 5.11. *Assume the notation of the beginning of this subsection.*

1. *If $t > n$, then the series $\sum_{k\geq 0}(\lambda_k + 1)^{-t}$ converges.*
2. *For any integer $N > n$, there is a constant $C > 0$ such that for all $k \geq 0$ we have*
$$\|\varphi_k\|_\infty \leq C(\lambda_k + 1)^N.$$
3. *If $\psi \in C^\infty(X_K)$, then for any $N \in \mathbb{Z}_{\geq 0}$ and for all $k \in \mathbb{Z}_{\geq 0}$ we have*
$$|(\psi,\varphi_k)| \leq (\lambda_k + 1)^{-N}\|(-\Delta_{\mathbb{D}} + 1)^N\psi\|_2.$$
4. *If $\psi \in C^\infty(X_K)$, then the spectral expansion* (5.9) *converges uniformly towards ψ.*

Proof. The first assertion is a consequence of Weyl's law which states that

$$\#\{k;\ \lambda_k \leq x\} \sim cx^{\delta/2}, \quad x \to \infty,$$

where $c > 0$ is a constant and δ is the dimension (over $\mathbb{R}$) of the compact real Riemann manifold X_K.

The second assertion follows from the fact that for any integer $N > \delta/2$ the pseudo differential operator $(-\Delta_{\mathbb{D}} + 1)^{-N}$ has a continuous kernel function in $C(X_K \times X_K)$.

The third statement is an easy consequence of the self-adjointness of $-\Delta_{\mathbb{D}}$ and the Cauchy-Schwartz inequality. Finally, the last statement follows from (1), (2) and (3). □

Theorem 5.12. *For $(z,h) \in X_K \setminus Z(m,\mu)$, the Green function $\Phi_{m,\mu}(z,h,s)$ has a meromorphic continuation in s to the whole complex plane. For fixed s outside the set of poles, the resulting function in (z,h) is real analytic on $X_K \setminus Z(m,\mu)$.*

Proof. Let $\sigma : [0,1] \to \mathbb{R}$ be a monotonous C^∞-function such that $\sigma(t) = 1$ for $t \leq 1/2$ and $\sigma(t) = 0$ for $t \geq 3/4$. Besides the Poincaré series $\Phi_{m,\mu}(z,h,s)$ (see Theorem 5.3), we consider the Poincaré series

$$\tilde{\Phi}_{m,\mu}(z,h,s) = \sum_{\substack{\lambda \in h(\mu+L) \\ Q(\lambda)=m}} \sigma\left(\frac{m_1}{Q(\lambda_{1z^\perp})}\right) \phi(\lambda,z,s),$$

$$F(z,h,s) = \sum_{\substack{\lambda \in h(\mu+L) \\ Q(\lambda)=m}} \left[\Delta_{\mathbb{D}}\left(\sigma\left(\frac{m_1}{Q(\lambda_{1z^\perp})}\right)\phi(\lambda,z,s)\right) - \sigma\left(\frac{m_1}{Q(\lambda_{1z^\perp})}\right)\Delta_{\mathbb{D}}\left(\phi(\lambda,z,s)\right)\right].$$

The difference of $\Phi_{m,\mu}(z,h,s)$ and $\tilde{\Phi}_{m,\mu}(z,h,s)$ is the Poincaré series

$$\sum_{\substack{\lambda \in h(\mu+L) \\ Q(\lambda)=m}} \left(1 - \sigma\left(\frac{m_1}{Q(\lambda_{1z^\perp})}\right)\right)\phi(\lambda,z,s).$$

As in the proof of Theorem 5.5 we see that it is locally finite, that is, for (z,h) in any compact subset of $\mathbb{D} \times H(\hat{\mathbb{Q}})$, only finitely many terms are non-zero. Hence it defines a holomorphic function for all $s \in \mathbb{C}$, which is smooth for $(z,h) \in X_K \setminus Z(m,\mu)$. We now show that $\tilde{\Phi}_{m,\mu}(z,h,s)$ has a meromorphic continuation in s which is continuous for $(z,h) \in X_K$. This implies the desired continuation of $\Phi_{m,\mu}(z,h,s)$.

As in the proof of Theorem 5.5 we see that $\tilde{\Phi}_{m,\mu}(z,h,s)$ converges normally for $\Re(s) > s_0$ and defines a smooth function on X_K. The Poincaré series $F(z,h,s)$ is locally finite and defines a smooth function, which is holomorphic in s on the whole complex plane. Moreover, the differential equation (5.6) implies that

$$\Delta_{\mathbb{D}}\tilde{\Phi}_{m,\mu}(z,h,s) = \frac{1}{8}\left(s^2 - s_0^2\right)\tilde{\Phi}_{m,\mu}(z,h,s) + F(z,h,s) \tag{5.12}$$

for $\Re(s) > s_0$. Hence the coefficients of the spectral expansion of $\tilde{\Phi}_{m,\mu}(z,h,s)$ are given by

$$(\tilde{\Phi}_{m,\mu}(\cdot,s),\varphi_k) = \frac{8}{\alpha_k^2 - s^2}(F(\cdot,s),\varphi_k),$$

and we have

$$\tilde{\Phi}_{m,\mu}(z,h,s) = \sum_{k=0}^{\infty} \frac{8}{\alpha_k^2 - s^2}(F(\cdot,s),\varphi_k)\varphi_k(z,h).$$

Lemma 5.11 implies that the series converges locally uniformly for $s \in \mathbb{C}$ and $(z,h) \in X_K$. Consequently, it defines a meromorphic continuation in s which is continuous in (z,h).

Now the distribution differential equation of Theorem 5.7 and the elliptic regularity theorem imply that for fixed s outside the set of poles, $\Phi_{m,\mu}(z,h,s)$ is actually real analytic for $(z,h) \in X_K \setminus Z(m,\mu)$. □

5.2. **Regularized Green functions.** Here we define the regularized theta lift of a *harmonic* Whittaker form, that is, a Whittaker form with parameter s_0. We determine the singularities of the lift. In this subsection we do not have to assume that X_K is compact. So we come back to the general setup of Section 2.

Definition 5.13. Let $f \in H_{k,\bar\rho_L}$ and write

$$f = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu) f_{m,\mu}(\tau). \tag{5.13}$$

We define the regularized theta lift $\Phi(z,h,f)$ of f to be the constant term in the Laurent expansion at $s = s_0$ of

$$\Phi(z,h,s,f) := \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu) \Phi_{m,\mu}(z,h,s).$$

If $\mu \in L'/L$ and $m \in \partial_F^{-1} + Q(\mu)$ is totally positive, we briefly write $\Phi_{m,\mu}(z,h)$ for the regularized theta lift of the harmonic Whittaker form $f_{m,\mu}(\tau)$, that is, for the constant term in the Laurent expansion of $\Phi_{m,\mu}(z,h,s)$ at $s = s_0$.

For a harmonic Whittaker form $f \in H_{k,\bar\rho_L}$ as in (5.13) we define a divisor $Z(f) \in \operatorname{Div}(X_K)_{\mathbb{C}}$ by

$$Z(f) = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu) Z(m,\mu). \tag{5.14}$$

Moreover, by means of the quantities $A(m,\mu)$ of Corollary 5.9 we define

$$A(f) = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu) A(m,\mu). \tag{5.15}$$

In view of Corollary 5.9 we have

$$\Phi(z,h,f) = \lim_{s \to s_0} \left(\Phi(z,h,s,f) - \frac{A(f)}{s - s_0} \right). \tag{5.16}$$

If Y is an irreducible Cartier divisor on a normal complex space X, we say that a real analytic function F on $X \setminus Y$ has a logarithmic singularity along Y, if for any $x \in Y$ there is a neighborhood $U \subset X$ and a local equation G for Y such that $F - \log|G|$ can be continued to a real analytic function on U. We extend this definition $\mathbb{C}$-linearly to $\operatorname{Div}(X_K)_{\mathbb{C}}$.

Theorem 5.14. *Assume that $n > 0$. For $f \in H_{k,\bar\rho_L}$ the function $\Phi(z,h,f)$ is real analytic on $X_K \setminus Z(f)$. It has a logarithmic singularity along the divisor $-2Z(f)$.*

Proof. It suffices to show that for $\mu \in L'/L$ and totally positive $m \in \partial_F^{-1} + Q(\mu)$, the function $\Phi_{m,\mu}(z,h)$ is real analytic on $X_K \setminus Z(m,\mu)$ and has a logarithmic singularity along the divisor $-2Z(m,\mu)$. To this end we show that for any point $(z_0,h_0) \in \mathbb{D} \times H(\hat{\mathbb{Q}})$ the function

$$\Phi_{m,\mu}(z,h) + \sum_{\substack{\lambda \in h_0(\mu+L) \\ Q(\lambda)=m \\ \lambda_1 \perp z_0}} \log|Q(\lambda_{1z})| \tag{5.17}$$

is real analytic in a neighborhood of (z_0,h_0).

Since the residue of $\Phi_{m,\mu}(z,h,s)$ at $s = s_0$ does not depend on (z,h), the proof of Theorem 5.12 also shows that the function

$$\Phi_{m,\mu}(z,h) - \sum_{\substack{\lambda \in h_0(\mu+L) \\ Q(\lambda)=m \\ \lambda_1 \perp z_0}} \phi(\lambda, z, s_0)$$

is real analytic in a neighborhood of (z_0,h_0). Hence it suffices to show that

$$\phi(\lambda,z,s_0) + \log|Q(\lambda_{1z})|$$

extends to a real analytic function on $\mathbb{D}$. This follows from Lemma 5.15 below. □

Lemma 5.15. *If $n > 0$, the function*

$$\frac{2}{n} w^{n/2} F(n/2, 1, n/2+1, w) + \log(1-w)$$

extends to a real analytic function near w=1.

Proof. We use the integral representation

$$\frac{2}{n} w^{n/2} F(n/2, 1, n/2+1, w) = \int_0^1 (tw)^{n/2}(1-tw)^{-1}\frac{dt}{t} = \int_0^w t^{n/2}(1-t)^{-1}\frac{dt}{t},$$

see for instance [AbSt] (15.3.1). Comparing this with $\log(1-w) = -\int_0^w \frac{dt}{1-t}$, we see that

$$\frac{2}{n} w^{n/2} F(n/2, 1, n/2+1, w) + \log(1-w) = \int_0^w \frac{t^{n/2-1}-1}{1-t}\, dt.$$

If $n=1$ this is equal to $\int_0^w \frac{dt}{t+\sqrt{t}}$, if $n=2$ it vanishes identically, and if $n \geq 3$ it is equal to

$$-\sum_{k=0}^{n-3} \int_0^w \frac{t^{k/2}}{1+\sqrt{t}}\, dt.$$

In all cases, the resulting function is real analytic near $w=1$. □

Corollary 5.16. *The differential form $dd^c\Phi(f)$ extends to a smooth form on X_K, which is a (harmonic) Poincaré dual form for $Z(f)$. The current $[\Phi(f)]$ induced by $\Phi(z,h,f)$ satisfies the dd^c-equation*

$$dd^c[\Phi(f)] + \delta_{Z(f)} = [dd^c\Phi(f)].$$

Proof. The corollary follows from Theorem 5.14 by means of the usual Poincaré-Lelong argument, see e.g. [SABK], Chapter II.1.4, Theorem 2. □

Remark 5.17. The current $[\Phi(f)]$ induced by $\Phi(z,h,f)$ satisfies the differential equation

$$\Delta_{\mathbb{D}}[\Phi(f)] + \frac{n}{4}\delta_{Z(f)} = \frac{n}{8}[A(f)].$$

Proof. This is a direct consequence of Theorem 5.7 and Corollary 5.9. □

6. The theta lift and meromorphic modular forms

We continue to use the notation of the previous section. Here we investigate the relationship of the regularized theta lift and the Kudla–Millson lift (see [KM1], [KM2], [KM3]). We use the approach of [BF]. As an application we construct explicit meromorphic modular forms on X_K whose divisors are supported on Heegner divisors. They are analogous to the automorphic products constructed by Borcherds [Bo1]. However, notice that there are no Fourier expansions and therefore no product expansions when X_K is compact. Consequently, Borcherds' argument to prove important properties of the lift (such as e.g. meromorphicity) cannot be employed when $d > 1$.

6.1. **The relationship with a regularized Kudla–Millson lift.** Recall that κ is the dual weight for k given by

$$\kappa = (2-k_1, k_2, \dots, k_d) = \left(\frac{n+2}{2}, \dots, \frac{n+2}{2}\right).$$

Proposition 6.1. *Let $\mu \in L'/L$ and $m \in \partial_F^{-1} + Q(\mu)$ be totally positive. For $\Re(s) > s_0 + 2$ we have the identity*

$$dd^c \Phi_{m,\mu}(z,h,s) = \frac{1}{\sqrt{D}} \int_{\tilde{\Gamma}_\infty \backslash \mathbb{H}^d}^{reg} \left\langle \overline{\delta_k(f_{m,\mu}(\tau,s))}, \Theta_{KM}(\tau,z,h) \right\rangle v^\kappa \, d\mu(\tau).$$

Here δ_k is the differential operator defined in (4.2)*.*

Proof. According to Proposition 3.1 we have

$$dd^c \Theta_S(\tau,z,h) = -L_\kappa^{(1)} \Theta_{KM}(\tau,z,h).$$

Moreover, writing $\eta = (v_2 \cdots v_d)^{\ell/2} d\tau_1 d\mu(\tau_2) \cdots d\mu(\tau_d)$, we have the identity of differential forms on $\mathbb{H}^d$:

$$-\left(L_\kappa^{(1)} \Theta_{KM}(\tau,z,h)\right)(v_2 \cdots v_d)^{\ell/2} \, d\mu(\tau) = \bar{\partial}(\Theta_{KM}(\tau,z,h)\eta).$$

In view of (4.6), when $\Re(s)$ is sufficiently large, we may interchange the regularized theta integral in the definition of $\Phi_{m,\mu}(z,h,s)$ with the operator dd^c. By means of the above identities we obtain

$$\begin{aligned} dd^c\Phi_{m,\mu}(z,h,s) &= \frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}^{reg} \langle f_{m,\mu}(\tau,s), dd^c\Theta_S(\tau,z,h)\rangle (v_2\cdots v_d)^{\ell/2}\,d\mu(\tau)\\ &= \frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}^{reg} \langle f_{m,\mu}(\tau,s), -L_\kappa^{(1)}\Theta_{KM}(\tau,z,h)\rangle (v_2\cdots v_d)^{\ell/2}\,d\mu(\tau)\\ &= \frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}^{reg} \langle f_{m,\mu}(\tau,s), \bar\partial\Theta_{KM}(\tau,z,h)\eta\rangle. \end{aligned}$$

Using the product rule, we find

$$\begin{aligned} dd^c\Phi_{m,\mu}(z,h,s) &= \frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}^{reg} d\langle f_{m,\mu}(\tau,s), \Theta_{KM}(\tau,z,h)\eta\rangle\\ &\quad - \frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}^{reg} \langle \bar\partial(f_{m,\mu}(\tau,s)), \Theta_{KM}(\tau,z,h)\eta\rangle. \end{aligned} \tag{6.1}$$

For the second summand on the right hand side we notice that

$$\begin{aligned} \bar\partial(f_{m,\mu}(\tau,s))\eta &= -(L_k^{(1)} f_{m,\mu}(\tau,s))(v_2\cdots v_d)^{\ell/2}\,d\mu(\tau)\\ &= -\overline{\delta_k(f_{m,\mu}(\tau,s))}(v_1\cdots v_d)^{\ell/2}\,d\mu(\tau). \end{aligned} \tag{6.2}$$

Hence this term gives the right hand side of the formula stated in the proposition.

Consequently, it suffices to prove that the first summand on the right hand side of (6.1) vanishes for $\Re(s) > s_0+2$. For $T>0$ we let $R_T\subset\mathbb{R}^d_{>0}$ be the rectangle

$$R_T = [1/T,T]\times\cdots\times[1/T,T].$$

Using the invariance of the integrand under translations, we find by Stokes' theorem that

$$\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}^{reg} d\langle f_{m,\mu}(\tau,s), \Theta_{KM}(\tau,z,h)\eta\rangle = \lim_{T\to\infty}\int_{\partial R_T}\int_{\mathcal{O}_F\backslash\mathbb{R}^d} \langle f_{m,\mu}(\tau,s), \Theta_{KM}(\tau,z,h)\eta\rangle.$$

Inserting (3.4), (3.14), and (4.13), and carrying out the integration over u, we see that this is equal to

$$\begin{aligned} &\sqrt{D}C(m,k,s)\lim_{T\to\infty}\int_{\partial R_T}\mathcal{M}_s(-4\pi m_1 v_1)\\ &\times \sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}} P_{KM}(z,\sqrt{v_1}\lambda_1)e^{-4\pi Q(\lambda_{1z^\perp})v_1}(v_2\cdots v_d)^{\ell/2-2}\,dv_2\cdots dv_d. \end{aligned}$$

Only the parts of the boundary where $v_1=1/T$ or $v_1=T$ give a non-zero contribution. Carrying out the integration over $v_2,\dots,v_d$, we see that the $v_1=1/T$ contribution is equal to a constant times

$$\lim_{T\to\infty} T^{k_1/2}M_{-k_1/2,\,s/2}(4\pi m_1/T)\sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}} P_{KM}(z,T^{-1/2}\lambda_1)e^{-2\pi(Q(\lambda_{1z^\perp})-Q(\lambda_{1z}))/T}.$$

The later sum is up to a constant factor equal to the m-th Fourier coefficient of

$$\Theta_{KM}(u + i(1/T, 1, \dots, 1), z, h).$$

Hence it converges and satisfies the growth estimate of Proposition 3.4 as $1/T \to 0$. So the asymptotic behavior of the M-Whittaker function (4.6) implies that the limit vanishes for $\Re(s) > s_0 + 2$. On the other hand, the $v_1 = T$ contribution is easily seen to vanish for $(z, h) \in X_K \backslash Z(m, \mu)$. This proves the proposition. $\square$

6.2. **Eisenstein series and theta integrals.** The right hand side of the formula of Proposition 6.1 converges for $\Re(s) > s_0 + 2$. It has a meromorphic continuation to the whole complex plane, since the left hand side has. It is actually holomorphic at $s = s_0$. We now modify the integral representation on the right hand side in order to obtain an expression which converges near s_0. This is done by subtracting the "Eisenstein contribution" of $\Theta_{KM}(\tau, z, h)$. The remaining "cuspidal contribution" satisfies a better growth estimate as $v_i \to 0$ and therefore leads to a larger domain of convergence.

We briefly summarize some facts on the Siegel–Weil formula, see e.g. [KR], [Ku3] for more details. Let χ_V denote the quadratic character of $\mathbb{A}_F^\times / F^\times$ associated to V given by

$$\chi_V(x) = (x, (-1)^{\ell(\ell-1)/2} \det(V))_F.$$

Here $\det(V)$ denotes the Gram determinant of V and $(\cdot, \cdot)_F$ is the Hilbert symbol of F. Let $P \subset G$ be the parabolic subgroup of upper triangular matrices. For $s \in \mathbb{C}$ and a standard section $\Phi(s)$ of the principal series representation $I(s, \chi_V)$ induced by $\chi_V |\cdot|^s$, we have the Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in P(F) \backslash G(F)} \Phi(\gamma g).$$

It converges for $\Re(s) > 1$ and has a meromorphic continuation to the whole complex plane.

Recall that if $v = \sigma_j$ is an infinite prime, then the corresponding local induced representation $I(s, \chi_{V,\sigma_j})$ is generated by the sections

$$\Phi_{\mathbb{R}}^{l_j}(k_\alpha, \phi) = \chi_{1/2}(k_\alpha, \phi)^{2l_j} = \pm e^{i l_j \alpha}$$

for $l_j \in \frac{1}{2}\mathbb{Z}$ satisfying $l_j \equiv \ell/2 \pmod{\mathbb{Z}}$. Here $(k_\alpha, \phi) \in \widetilde{\mathrm{SO}}_2(\mathbb{R})$ is given by (3.1) and $\chi_{1/2}$ is the character defined in (3.2). If $l = (l_1, \dots, l_d)$ is a d-tuple of such half-integers we put $\Phi_\infty^l = \prod_j \Phi_{\mathbb{R}}^{l_j}$. If $\Phi_f(s)$ is a standard section of the non-archimedian induced representation, we obtain an Eisenstein series of weight l on $\mathbb{H}^d$ by putting

$$E(\tau, s, l; \Phi_f) = v^{-l/2} E(\tilde{g}_\tau, s, \Phi_f \otimes \Phi_\infty^l),$$

where $\tilde{g}_\tau \in \tilde{G}_{\mathbb{R}}$ with the property that $\tilde{g}_\tau(i, \dots, i) = \tau$.

The Weil representation gives rise to a $\tilde{G}_{\mathbb{A}}$-intertwining map

$$\lambda : S(V(\mathbb{A}_F)) \longrightarrow I(s_0, \chi_V), \quad \lambda(\varphi)(g) = (\omega(g)\varphi)(0), \tag{6.3}$$

where $s_0 = \ell/2 - 1 = n/2$. We also write $\lambda(\varphi)$ for the unique standard section of $I(s, \chi_V)$ whose value at s_0 is equal to $\lambda(\varphi)$. The map λ factors into $\lambda = \lambda_\infty \otimes \lambda_f$, where λ_∞ and λ_f

are the analogous intertwining maps at the finite and the infinite places, respectively. We obtain a vector valued Eisenstein series for $\tilde{\Gamma}$ of weight l with representation ρ_L by putting

$$E_L(\tau, s, l) = \sum_{\mu \in L'/L} E(\tau, s, l; \lambda_f(\chi_\mu))\chi_\mu. \tag{6.4}$$

Note that if the class number of F is one, we have that

$$E_L(\tau, s, l) = \sum_{\gamma \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} \left(v_1^{(s+1-l_1)/2} \cdots v_d^{(s+1-l_d)/2} \chi_0 \right) |_{l,\rho_L} \gamma.$$

In general, it is a finite sum over such Eisenstein series. We will be interested in the special value $E_L(\tau, \kappa) := E_L(\tau, s_0, \kappa)$ at s_0.

For the rest of this section we assume that V is anisotropic over F or that its Witt rank is smaller than n. Note that this condition is automatically fulfilled when $d > 1$ or $n > 2$. Employing the Siegel–Weil formula (see e.g. [KR] and [We2]), it can be shown that the average value of the Kudla–Millson theta function on X_K is the given by an Eisenstein series of weight κ. More precisely, we have

$$E_L(\tau, \kappa) = -\frac{1}{\operatorname{vol}(X_K)} \int_{X_K} \Theta_{KM}(\tau, z, h)\Omega^{n-1}. \tag{6.5}$$

Kudla proved this for $F = \mathbb{Q}$ in [Ku3, Corollary 4.16], and the argument for general F is analogous. Moreover, as in [Ku3, Remark 2.8], it can be proved that $E_L(\tau, \kappa)$ is holomorphic in τ and therefore defines an element of M_{κ,ρ_L}.

At the cusp ∞, the Eisenstein series has a Fourier expansion of the form

$$E_L(\tau, \kappa) = \chi_0 + \sum_{\mu \in L'/L} \sum_{m > 0} B(m, \mu) e(\operatorname{tr}(m\tau))\chi_\mu. \tag{6.6}$$

The coefficients $B(m, \mu)$ can be computed explicitly using the argument of [KY], [Scho], or [BK]. However, we will not need that.

The differential form $-E_L(\tau, \kappa)\Omega$ can be viewed as the average of $\Theta_{KM}(\tau, z, h)$. We define the cuspidal part of the Kudla–Millson theta function by

$$\tilde{\Theta}_{KM}(\tau, z, h) = \Theta_{KM}(\tau, z, h) + E_L(\tau, \kappa)\Omega. \tag{6.7}$$

It is rapidly decreasing at all cusps of $\tilde{\Gamma}$.

Proposition 6.2. *Assume the above hypothesis on V.*

(i) *The function $\tilde{\Theta}_{KM}(\tau, z, h)v^{\kappa/2}$ is bounded on $\mathbb{H}^d$.*
(ii) *For $v_i \to 0$ we have uniformly in u that $\tilde{\Theta}_{KM}(\tau, z, h) = O(v^{-\kappa/2})$.*
(iii) *For $m \in \partial_F^{-1}$ the m-th Fourier coefficient of $\tilde{\Theta}_{KM}(\tau, z, h)$ is bounded by $O(v_1^{-\kappa_1/2})$ as $v_i \to 0$.*

Proof. Since $\tilde{\Theta}_{KM}(\tau, z, h)$ is rapidly decreasing, (i) and (ii) follow by the usual argument. It remains to prove (iii). The behavior of the Fourier coefficients as $v_1 \to 0$ is a direct consequence of (ii). Moreover, since $\tilde{\Theta}_{KM}(\tau, z, h)$ is holomorphic in $\tau_2, \ldots, \tau_d$ its Fourier coefficients are bounded as $v_i \to 0$ for $i = 2, \ldots, d$. □

Proposition 6.3. *We have*

$$dd^c\Phi_{m,\mu}(z,h,s) = \frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}^{reg} \left\langle \overline{\delta_k(f_{m,\mu}(\tau,s))}, \tilde\Theta_{KM}(\tau,z,h)\right\rangle v^\kappa\, d\mu(\tau) - \frac{B(m,\mu)\Omega}{\Gamma(\frac{s}{2}-\frac{s_0}{2}+1)}.$$

Here the regularized integral converges locally uniformly for $\Re(s)>1$.

Proof. According to Proposition 6.1 we have

$$\begin{aligned}(6.8)\qquad dd^c\Phi_{m,\mu}(z,h,s) = {}&\frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}^{reg} \left\langle \overline{\delta_k(f_{m,\mu}(\tau,s))}, \tilde\Theta_{KM}(\tau,z,h)\right\rangle v^\kappa\, d\mu(\tau)\\ &- \frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}^{reg} \left\langle \overline{\delta_k(f_{m,\mu}(\tau,s))}, E_L(\tau,\kappa)\Omega\right\rangle v^\kappa\, d\mu(\tau).\end{aligned}$$

We have to compute the latter integral. A direct computation shows that

$$\begin{aligned}(6.9)\qquad \overline{\delta_k(f_{m,\mu}(\tau,s))} = {}& C(m,k,s)(s+s_0)(4\pi m_1)^{-k_1/2}\\ &\times v_1^{k_1/2-1}M_{1-k_1/2,s/2}(4\pi m_1v_1)e^{-2\pi m_2v_2}\cdots e^{-2\pi m_dv_d}e(-\operatorname{tr}(mu))\chi_\mu.\end{aligned}$$

Inserting this and carrying out the integrations over u and $v_2,\dots,v_d$, we see that the second integral on the right hand side of (6.8) is equal to

$$B(m,\mu)\Omega\cdot\frac{s+s_0}{\Gamma(s+1)}\int_0^\infty (4\pi m_1v_1)^{-k_1/2}M_{1-k_1/2,s/2}(4\pi m_1v_1)e^{-2\pi m_1v_1}\frac{dv_1}{v_1}.$$

This is a Laplace transform, which can be computed by means of [Er2] p. 215 (11). We obtain for the second integral on the right hand side of (6.8)

$$\frac{B(m,\mu)\Omega}{\Gamma(\frac{s}{2}-\frac{s_0}{2}+1)}.$$

This proves the formula of the proposition.

We now prove the convergence statement for the integral. According to (6.9) and (4.6) we have

$$\overline{\delta_k(f_{m,\mu}(\tau,s))} = O(v_1^{\Re(s)/2+(k_1-1)/2}),\quad v_i\to 0.$$

By means of Proposition 6.2 (iii), we see that

$$\int_{\mathcal{O}_F\backslash\mathbb{R}^d} \left\langle \overline{\delta_k(f_{m,\mu}(\tau,s))}, \tilde\Theta_{KM}(\tau,z,h)\right\rangle v^\kappa\, du = O\left(v_1^{\Re(s)/2+1/2}(v_2\cdots v_d)^{n/2+1}\right),$$

as $v_i\to 0$. On the other hand, in view of (4.7), this quantity is bounded as $v_i\to\infty$. Consequently,

$$\int_{v\in\mathbb{R}^d_{>0}}\left(\int_{\mathcal{O}_F\backslash\mathbb{R}^d} \left\langle \overline{\delta_k(f_{m,\mu}(\tau,s))}, \tilde\Theta_{KM}(\tau,z,h)\right\rangle v^\kappa\, du\right)\frac{dv}{\mathrm{N}(v)^2}$$

converges when $\Re(s)>1$. □

6.3. **Regularized Green functions and the Kudla–Millson lift of cusp forms.** For a cusp form $g \in S_{\kappa,\rho_L}$ we define the Kudla–Millson lift by

$$\Lambda(z,h,g) = \big(\Theta_{KM}(\tau,z,h), g(\tau)\big)_{Pet}. \tag{6.10}$$

The theta integral converges and defines a closed harmonic 2-form on X_K. The following theorem is a generalization of [BF] Theorem 6.1 to our situation.

Theorem 6.4. *Let $f \in H_{k,\bar\rho_L}$. We write*

$$f = \sum_{\mu\in L'/L}\sum_{m\gg 0} c(m,\mu) f_{m,\mu}(\tau),$$

and define

$$B(f) = \sum_{\mu\in L'/L}\sum_{m\gg 0} c(m,\mu) B(m,\mu). \tag{6.11}$$

Then we have the identity

$$dd^c\Phi(z,h,f) = \Lambda(z,h,\xi_k(f)) - B(f)\Omega.$$

Proof. We first assume that $n>2$. Then it follows from Proposition 6.3 that the regularized Green function $\Phi_{m,\mu}(z,h)$ satisfies the identity

$$dd^c\Phi_{m,\mu}(z,h) = \frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}\left\langle\overline{\delta_k(f_{m,\mu}(\tau))}, \tilde\Theta_{KM}(\tau,z,h)\right\rangle v^\kappa\, d\mu(\tau) - B(m,\mu)\Omega. \tag{6.12}$$

Notice that the regularized theta integral in Proposition 6.3 converges near $s=s_0$ when $n>2$. Moreover, because of (4.17) and Proposition 6.2, the theta integral in (6.12) is well defined without any regularization! By the unfolding argument we see that it is equal to

$$\int_{\tilde\Gamma\backslash\mathbb{H}^d}\left\langle\overline{\xi_k(f_{m,\mu}(\tau))}, \tilde\Theta_{KM}(\tau,z,h)\right\rangle v^\kappa\, d\mu(\tau).$$

Since the integral of the cusp form $\xi_k(f_{m,\mu})$ against the Eisenstein series $E_L(\tau,\kappa)$ vanishes, we obtain the assertion.

If $n\geq 1$, then one can show by means of Proposition 6.3 that $dd^c\Phi_{m,\mu}(z,h)$ is equal to the value at $s'=0$ of the holomorphic continuation in s' of

$$\frac{1}{\sqrt{D}}\int_{\tilde\Gamma_\infty\backslash\mathbb{H}^d}\left\langle\overline{\delta_k(f_{m,\mu}(\tau))}, \tilde\Theta_{KM}(\tau,z,h)\right\rangle v^\kappa\, \mathrm{N}(v)^{s'/2}\, d\mu(\tau) - B(m,\mu)\Omega.$$

Again the assertion follows by unfolding the theta integral. □

Remark 6.5. Let $\mu\in L'/L$ and $m\in\partial_F^{-1}+Q(\mu)$ be totally positive. We have

$$B(m,\mu) = -\frac{\deg(Z(m,\mu))}{\mathrm{vol}(X_K)} = -\frac{1}{2}A(m,\mu).$$

Proof. Integrating the identity of (6.12) (respectively its analogue for $n\geq 1$) against Ω^{n-1} we obtain by means of (6.5) that

$$\int_{X_K}(dd^c\Phi_{m,\mu})\Omega^{n-1} = -B(m,\mu)\int_{X_K}\Omega^n.$$

On the other hand, Corollary 5.16 implies that the left hand side is equal to $\delta_{Z(m,\mu)}(\Omega^{n-1})$. Consequently, $\deg(Z(m,\mu)) = -B(m,\mu)\operatorname{vol}(X_K)$. (Alternatively, this can be proved as in [Ku3] Theorem 4.20, using (6.5) and the Thom form property of φ_{KM}.) □

6.4. **Meromorphic modular forms and special divisors.** We now use Theorem 6.4 to derive an analogue of Borcherds' result on automorphic products (see [Bo1], Theorem 13.3).

Lemma 6.6. *Let $U \subset \mathbb{C}^n$ be a convex domain. Let C be an analytic divisor on U, and let $\psi : U \setminus C \to \mathbb{R}$ be a C^2-function with a logarithmic singularity along C. If ψ is pluriharmonic (i.e. $\partial\bar\partial\psi = 0$), then there exists a meromorphic function Ψ on U such that $\psi = \log|\Psi|$.*

Proof. By the assumption on U we have $H^1(U, \mathcal{O}_U) = H^2(U, \mathbb{Z}) = 0$ and the multiplicative Cousin problem is universally solvable. Hence there is a meromorphic function G on U such that $C = \operatorname{div}(G)$. The assumption on ψ implies that

$$\psi - \log|G|$$

extends to a pluriharmonic real analytic function on U. Since U is simply connected, there exists a holomorphic function $H : U \to \mathbb{C}$ such that

$$\Re(H) = \psi - \log|G|,$$

see e.g. [GR] Chapter IX, Section C. Rewriting this as

$$\psi = \log|e^H \cdot G|,$$

we see that we can take $\Psi = e^H \cdot G$. □

The function Ψ in the lemma has divisor C. By the maximum modulus principle, it is uniquely determined up to a constant of modulus 1.

Lemma 6.7. *(cp.* [Bo1] *Lemma 13.1.) Let $r \in \mathbb{Q}$. Suppose that Ψ is a meromorphic function on $\mathcal{H} \times H(\hat{\mathbb{Q}})/K$ for which $|\Psi(z,h)| \cdot |y|^r$ is invariant under $H(\mathbb{Q})$. Then there exists a unitary multiplier system $\chi : H(\mathbb{Q}) \times H(\hat{\mathbb{Q}}) \to \mathbb{C}^\times$ of weight r such that Ψ is a meromorphic modular form of weight r, level K, and multiplier system χ.*

Proof. The hypothesis implies that for every $\gamma \in H(\mathbb{Q})$ and $h \in H(\hat{\mathbb{Q}})$, the function

$$\frac{\Psi(\gamma z, \gamma h)}{\Psi(z,h)} j(\gamma, z)^{-r}$$

is holomorphic on $\mathbb{D}$ and has constant modulus 1. By the maximum modulus principle it has to be constant, say equal to $\chi(\gamma, h)$. The right-invariance of Ψ under K implies that χ is right-invariant under K. Moreover, it is easily checked that χ satisfies the cocycle condition of a multiplier system. □

Theorem 6.8. *Let*

$$f = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu) f_{m,\mu}(\tau) \in M^!_{k,\bar\rho_L}$$

be a weakly holomorphic Whittaker form with coefficients $c(m,\mu) \in \mathbb{Z}$. Then there exists a function $\Psi(z,h,f)$ on $\mathcal{H} \times H(\hat{\mathbb{Q}})$ with the following properties:

(i) Ψ *is a meromorphic modular form for* $H(\mathbb{Q})$ *of weight* $-B(f)$ *and level* K *with a unitary multiplier system of finite order.*
(ii) *The divisor of* Ψ *is equal to* $Z(f)$*.*
(iii) *The Petersson metric of* Ψ *is given by*

$$-\log \|\Psi(z,h,f)\|^2_{Pet} = \Phi(z,h,f).$$

Proof. We use Theorem 6.4. The assumption that f is weakly holomorphic means that $\xi_k(f) = 0$. Consequently, we have

$$\begin{aligned} dd^c\Phi(z,h,f) &= -B(f)\Omega \\ &= B(f) dd^c \log|y|^2. \end{aligned}$$

Hence the function $\Phi(z,h,f) - B(f)\log|y|^2$ is pluriharmonic on $\mathcal{H}$ and has a logarithmic singularity along $-2Z(f)$. According to Lemma 6.6 there exists a meromorphic function $\Psi(z,h,f)$ on $\mathcal{H} \times H(\hat{\mathbb{Q}})/K$ such that

$$\Phi(z,h,f) - B(f)\log|y|^2 = -2\log|\Psi(z,h,f)|.$$

So $\Phi(z,h,f) = -\log\|\Psi(z,h,f)\|^2_{Pet}$, where the Petersson metric is in weight $-B(f)$. By construction, the divisor of Ψ is equal to $Z(f)$.

The invariance properties of $\Phi(z,h,f)$ and Lemma 6.7 imply that there is a unitary multiplier system χ such that Ψ is a meromorphic modular form of weight $-B(f)$, level K, and multiplier system χ.

When $n > 2$, the Lie group $H(\mathbb{R})$ has no almost simple factor of real rank 1. Hence, according to [Mar] (Proposition 6.19 on p. 333), the multiplier system χ has finite order. When $n \leq 2$ we will prove that χ has finite order in the next section by means of the embedding trick, see Corollary 7.4. □

Remark 6.9. When $d = 1$, then Theorem 6.8 is compatible (up to a constant) via Proposition 4.6 with the Borcherds lift of weakly holomorphic modular forms [Bo1, Theorem 13.3]. This follows from [Br], Proposition 2.11.

7. Modularity of special divisors

We first assume that $n > 2$. We use Theorem 6.8 to prove that the generating series of special divisors is a Hilbert modular form of weight κ with values in the Chow group. This result is proved in [YZZ] using the modularity result for the cohomology classes of Kudla–Millson [KM1, KM2, KM3]. Our proof is a variant of the proof that Borcherds gave for $F = \mathbb{Q}$, see [Bo2].

Next we consider the cases of small dimension $n = 1, 2$. Following [YZZ] and [Bo1], the modularity result can be extended to this case by means of the embedding trick. As an application we prove the finiteness of the multiplier system of the meromorphic modular form $\Psi(z,h,f)$ of Theorem 6.8 for $n = 1, 2$.

7.1. **The case $n > 2$.** For a Schwartz function $\varphi \in S_L \subset S(V(\hat{F}))$ we consider the special divisors $Z(m, \varphi)$. According to Remark 5.6, the divisors

$$Z^0(m, \varphi) := Z(m, \varphi) - \frac{\deg Z(m, \varphi)}{\operatorname{vol}(X_K)} c_1(\mathcal{M}_1)$$

have degree 0. We define the generating functions

$$A(\tau, \varphi) = A_V(\tau, \varphi) := -c_1(\mathcal{M}_1) + \sum_{m \gg 0} Z(m, \varphi) q^m,$$
$$A^0(\tau, \varphi) = A_V^0(\tau, \varphi) := \sum_{m \gg 0} Z^0(m, \varphi) q^m,$$

which we view as formal power series with coefficients in $\operatorname{CH}^1(X_K)$. Here $q^m := e(\operatorname{tr}(m\tau))$ and $c_1(\mathcal{M}_1)$ is the Chern class in $\operatorname{CH}^1(X_K)$ of $\mathcal{M}_1$ given by the divisor of a rational section. Moreover, we define the ρ_L-valued generating functions

$$A(\tau) = \sum_{\mu \in L'/L} A(\tau, \chi_\mu) \chi_\mu,$$
$$A^0(\tau) = \sum_{\mu \in L'/L} A^0(\tau, \chi_\mu) \chi_\mu.$$

When the Eisenstein series $E_L(\tau, \kappa)$ is holomorphic, then Remark 6.5 implies that

$$A^0(\tau) = A(\tau) + c_1(\mathcal{M}_1) E_L(\tau, \kappa).$$

Theorem 7.1. *The generating series $A^0(\tau)$ belongs to $S_{\kappa, \rho_L} \otimes \operatorname{CH}^1(X_K)$.*

For the proof of the theorem we need the following linear algebra lemma.

Lemma 7.2. *Let X, Y be vector spaces over a field E, and let $\beta : X \times Y \to E$ be a non-degenerate bilinear form. Let $X_1 \subset X$ be a subspace and put*

$$X_1^\perp = \{y \in Y;\ \beta(x, y) = 0 \text{ for all } x \in X_1\},$$
$$X_1^{\perp\perp} = \{x \in X;\ \beta(x, y) = 0 \text{ for all } y \in X_1^\perp\}.$$

If X_1 is finite dimensional, then $X_1^{\perp\perp} = X_1$. □

Proof of Theorem 7.1. We write P_L for the vector space of ρ_L-valued formal power series with vanishing constant term, that is, the vector space of formal power series of the form

$$g = \sum_{\mu \in L'/L} \sum_{\substack{m \in \partial^{-1} + Q(\mu) \\ m \gg 0}} b(m, \mu) q^m \chi_\mu.$$

We may view S_{κ, ρ_L} as a subspace of P_L by taking the Fourier expansion of a cusp form. We extend the pairing $\{\cdot, \cdot\}$ between $H_{k, \bar{\rho}_L}$ and S_{κ, ρ_L} defined in (4.22) to a non degenerate pairing between H_{k, ρ_L} and P_L using the formula (4.23).

We apply Lemma 7.2 with $X = P_L$, $X_1 = S_{\kappa,\rho_L}$, $Y = H_{k,\bar\rho_L}$, and the pairing $\{\cdot,\cdot\}$. We have that $M^!_{k,\bar\rho_L} = S^\perp_{\kappa,\rho_L}$. The generating series $A^0(\tau)$ is an element of $P_L \otimes \mathrm{CH}^1(X_K)$. According to the lemma it belongs to $S_{\kappa,\rho_L} \otimes \mathrm{CH}^1(X_K)$ if and only if

$$\{A^0, f\} = 0 \in \mathrm{CH}^1(X_K)_{\mathbb{C}} \tag{7.1}$$

for all $f \in M^!_{k,\bar\rho_L}$. Adapting the argument of [McG], it can be proved that S_{κ,ρ_L} has a basis of cusp forms with coefficients in $\mathbb{Z}$. Hence it suffices to verify (7.1) for those

$$f = \sum_\mu \sum_{m \gg 0} c(m,\mu) f_{m,\mu}$$

which are integral linear combinations of the $f_{m,\mu}$. So for such f we have to show that

$$\sum_\mu \sum_{m\gg 0} c(m,\mu) Z(m,\mu) = \sum_\mu \sum_{m \gg 0} c(m,\mu) \frac{\deg Z(m,\varphi)}{\mathrm{vol}(X_K)} c_1(\mathcal{M}_1) \in \mathrm{CH}^1(X_K)_{\mathbb{Q}}.$$

In view of Remark 6.5 this is equivalent to

$$Z(f) = -B(f) c_1(\mathcal{M}_1) \in \mathrm{CH}^1(X_K)_{\mathbb{Q}}.$$

But this relation is exactly produced by Theorem 6.8. □

7.2. **The embedding trick.** Let $V_1 \subset V$ be a quadratic subspace defined over F, and assume that V_1 has signature

$$((n_1, 2), (n_1+2, 0), \dots, (n_1+2, 0))$$

with $1 \leq n_1 \leq n$. Let $V_2 = V_1^\perp$. Then V_2 is totally positive definite and $V = V_1 \oplus V_2$. We view $H_1 := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSpin}(V_1)$ as a subgroup of H acting trivially on V_2 and put $K_1 := H_1(\hat{\mathbb{Q}}) \cap K$. We let $\mathbb{D}_{V_1}$ be the sub-Grassmannian of $\mathbb{D}$ given by the oriented negative definite 2-dimensional subspaces z of $V_{1,\sigma_1} \subset V_{\sigma_1}$. We obtain an embedding of Shimura varieties

$$\iota : X_{K_1,V_1} := H_1(\mathbb{Q}) \backslash (\mathbb{D}_{V_1} \times H_1(\hat{\mathbb{Q}}))/K_1 \longrightarrow X_{K,V}.$$

It induces a pull-back homomorphism of the Chow groups

$$\iota^* : \mathrm{CH}^1(X_{K,V}) \longrightarrow \mathrm{CH}^1(X_{K_1,V_1}).$$

The pull back of the generating series $A_V(\tau,\varphi)$ is computed in [YZZ], Proposition 3.1. Let $\varphi_i \in S(V_i(\hat F))$ and assume that X_{K_1,V_1} is compact. Then

$$\iota^*(A_V(\tau, \varphi_1 \otimes \varphi_2)) = A_{V_1}(\tau,\varphi_1) \cdot \theta_{S,V_2}(\tau;\varphi_2), \tag{7.2}$$

where

$$\theta_{S,V_2}(\tau;\varphi_2) = \sum_{\lambda \in V_2(F)} \varphi_2(\lambda) e(\mathrm{tr}\, Q(\lambda)\tau)$$

is the usual theta series of the positive definite quadratic space V_2. By embedding quadratic spaces over F of dimension $n = 1$ or 2 into larger spaces, employing the pull back-formula (7.2), and varying φ_2, one obtains (see [YZZ], proof of Theorem 1.3):

Proposition 7.3. *Theorem 7.1 also holds for $n = 1, 2$.*

Corollary 7.4. *Let $f \in M^!_{k,\bar\rho_L}$ be a weakly holomorphic Whittaker form as in Theorem 6.8 and let $\Psi(z,h,f)$ be the corresponding meromorphic modular form for $H(\mathbb{Q})$. The multiplier system of Ψ has also finite order when $n = 1, 2$.*

Proof. Since f is weakly holomorphic and $A^0 \in S_{\kappa,\rho_L} \otimes \mathrm{CH}^1(X_K)$ by Proposition 7.3, we have $\{A^0, f\} = 0 \in \mathrm{CH}^1(X_K)_\mathbb{Q}$. This is equivalent to

$$Z(f) = -B(f)c_1(\mathcal{M}_1) \in \mathrm{CH}^1(X_K)_\mathbb{Q}.$$

So there is a meromorphic modular form $\tilde\Psi$ of weight $-B(f)$ and level K with a multiplier system of *finite* order such that $\mathrm{div}(\tilde\Psi) = Z(f)$. Consequently, $\Psi/\tilde\Psi$ is a holomorphic modular form of weight 0 with no zeros and a multiplier system of possibly infinite order. But it is easily seen that such a modular form must be constant. This proves the corollary. □

8. Examples

Here we give some examples illustrating Theorem 6.8 and Theorem 7.1.

8.1. **Shimura curves.** Let B/F be a quaternion algebra which is split at σ_1 and ramified at all the other real places of F. Let $\delta \in \partial_F^{-1}$ be an element such that $\sigma_i(\delta) > 0$ for $i = 2, \dots, d$. We write $\mathrm{N}(x)$ for the reduced norm of $x \in B$. Then $(B, \delta\,\mathrm{N})$ is a quadratic space over F of signature $((2,2),(4,0),\dots,(4,0))$. The group $\mathrm{GSpin}(B)$ can be computed using similar arguments as in [KuRa] §0. One finds that

$$\mathrm{GSpin}(B) \cong \{(g_1,g_2) \in B^\times \times B^\times;\ \mathrm{N}(g_1) = \mathrm{N}(g_2)\}.$$

Under this identification the natural action of $\mathrm{GSpin}(B)$ on B is identified with $(g_1,g_2).b = g_1 b g_2^{-1}$ for $b \in B$. We may view an $\mathcal{O}_F$-order $\mathcal{O} \subset B$ as an $\mathcal{O}_F$-lattice in the quadratic space B. The Shimura variety X_K can be viewed as the product of two Shimura curves or as product of a Shimura curve with itself depending on the choice of the compact open subgroup K.

The subspace $B^0 \subset B$ of trace zero elements of B defines a quadratic subspace. When $\sigma_1(\delta)$ is also positive, it has signature $((1,2),(3,0),\dots,(3,0))$. We may identify

$$\mathrm{GSpin}(B^0) \cong B^\times,$$

the action on B^0 being given by conjugation. So Theorem 6.8 gives rise to automorphic forms on Shimura curves over F with known divisor supported on special divisors.

It is interesting to consider the examples of Shimura curves investigated in [El] in that way. Here we briefly discuss the Shimura curve X associated to the triangle group $G_{2,3,7}$, see [El] Section 5.3. It is a genus zero curve with a number of striking properties. For instance, the minimal quotient area of a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is $1/42$, and it is only attained by the triangle group $G_{2,3,7}$. Let F be the totally real cubic field $\mathbb{Q}(\cos(2\pi/7)) = \mathbb{Q}[x]/(x^3 + x^2 - 2x - 1)$. It has discriminant 49 and class number 1. The inverse different ∂_F^{-1} has a generator δ such that $\sigma_i(\delta) > 0$ for $i = 2, 3$. Let B be the quaternion algebra over F which is ramified at exactly the real places σ_2, σ_3. We view $(B, \delta\,\mathrm{N})$ as a quadratic space over F as above. Let L be a maximal $\mathcal{O}_F$-order of B. Then at all finite places $\mathfrak{p}$ of F the lattice $L_\mathfrak{p}$ is isomorphic to the 2×2 matrices with entries in $\mathcal{O}_{F,\mathfrak{p}}$. This implies that L is even unimodular and the corresponding Weil representation ρ_L is trivial. The underlying lattice $(L, Q_{\mathbb{Q}})$ over $\mathbb{Z}$ is isometric to $E_8 \oplus H \oplus H$, where H denotes a hyperbolic plane over $\mathbb{Z}$. We let $K \subset H(\hat{\mathbb{Q}})$ be the stabilizer of the lattice $\hat{L}$. Then the corresponding Shimura variety X_K is isomorphic to $X \times X$, where X is the quotient of $\mathbb{H}$ by the group of units of norm 1 of L.

The Jacquet–Langlands correspondence provides an isomorphism between the space of Hilbert cusp forms S_κ of parallel weight $\kappa = (2, 2, 2)$ for $\mathrm{SL}_2(\mathcal{O}_F)$ and the space of holomorphic differential 1-forms on X. Since X has genus 0, we see that $S_\kappa = \{0\}$. Consequently, any harmonic Whittaker form f of weight $k = (0, 2, 2)$ is weakly holomorphic. Its regularized theta lift gives rise to a meromorphic modular form of weight $-B(f)$ on X_K. Its divisor $Z(f)$ is a linear combination of Hecke correspondences. The generating series $A(\tau)$ of special divisors is equal to

$$c_1(\mathcal{M}_1)E_L(\tau, \kappa).$$

For any totally positive $m \in \partial_F^{-1}$ there exists a holomorphic modular form Ψ_m on X_K with divisor $Z(m, 0)$ and weight $B(m, 0)$, given by Theorem 6.8. These modular forms are analogues of the form $j(z_1) - j(z_2)$ and its multiplicative Hecke translates on $Y(1) \times Y(1)$, where j denotes the classical j-function and $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. In view of the work of Gross and Zagier on singular moduli, it would be interesting to compute the CM values of the functions Ψ_m.

Similar unimodular lattices can be constructed over any totally real field of odd degree for which the different is a principal ideal in the narrow sense. But clearly the space of cusp forms for $\mathrm{SL}_2(\mathcal{O}_F)$ of parallel weight 2 will be non-trivial in general.

8.2. **Even unimodular lattices over real quadratic fields.** It would be interesting to have some existence or classification results for even unimodular $\mathcal{O}_F$-lattices (in the sense of Section 2.2), since the Weil representation ρ_L is trivial in this case. Adapting the arguments of [Scha] and [Ch], one can obtain some results in this direction. (Notice that the setup in these references is slightly different from ours. For instance, they consider $\mathcal{O}_F$-valued quadratic forms and define the dual lattice as the $\mathcal{O}_F$-dual.)

Here we briefly consider the case where F is a real quadratic field of discriminant D.

Lemma 8.1. *There exists an even unimodular $\mathcal{O}_F$-lattice of signature $((n, 2), (n+2, 0))$ if and only if n is divisible by 4 and $\mathcal{O}_F$ contains a totally positive unit ε such that $-\varepsilon$ is a square modulo $4\mathcal{O}_F$.*

Proof. A lattice (L, Q) is even unimodular of signature $((n, 2), (n + 2, 0))$ in the sense of Section 2.2, if and only if $(L, \sqrt{D}Q)$ is a unimodular lattice of signature $((n, 2), (0, n + 2))$ in the sense of [Ch]. Hence the lemma follows from Theorem 3 in [Ch]. □

Remark 8.2. i) If $F = \mathbb{Q}(\sqrt{a})$ with $a > 0$ squarefree and $a \equiv 3 \pmod 4$ then the condition of the lemma is always fulfilled when n is divisible by 4. In this case we can take $\varepsilon = 1$.

ii) If the fundamental unit of F has norm -1, then the condition of the lemma is never fulfilled, since every totally positive unit is a square and therefore -1 would have to be a square modulo $4\mathcal{O}_F$. This is not the case as an elementary computation shows.

iii) If $\varepsilon \in \mathcal{O}_F$ is a totally positive unit such that $-\varepsilon = \alpha^2 - 4\beta$ with $\alpha, \beta \in \mathcal{O}_F$, then the lattice L_1 given by the Gram matrix

$$\frac{-1}{\sqrt{D}} \begin{pmatrix} 2 & \alpha \\ \alpha & 2\beta \end{pmatrix}$$

is even unimodular of signature $((0,2),(2,0))$.

It also follows from Theorem 3 in [Ch] that for any real quadratic field F there exists an even unimodular $\mathcal{O}_F$-lattice L_0 of signature $((4,0),(4,0))$. The corresponding lattice over $\mathbb{Z}$ is isometric to E_8. One can construct such a lattice explicitly by modifying the construction of [Scha] Section 3. For instance, for $F = \mathbb{Q}(\sqrt{3})$, one can take the lattice with the Gram matrix

$$\begin{pmatrix} -\sqrt{3}+2 & \frac{\sqrt{3}-1}{2} & \frac{-\sqrt{3}+3}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}-1}{2} & 3 & \frac{\sqrt{3}}{2} & \frac{-\sqrt{3}-3}{6} \\ \frac{-\sqrt{3}+3}{6} & \frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}+1}{2} \\ \frac{\sqrt{3}}{6} & \frac{-\sqrt{3}-3}{6} & \frac{\sqrt{3}+1}{2} & 2+\sqrt{3} \end{pmatrix}.$$

If n is divisible by 4 and F satisfies the condition of the lemma, then $L = L_0^{\oplus n/4} \oplus L_1$ is even unimodular of signature $((n,2),(n+2,0))$.

Let now $F = \mathbb{Q}(\sqrt{3})$ and let L be the lattice $L_0 \oplus L_1$ as above. Let $K \subset H(\hat{\mathbb{Q}})$ be the stabilizer of $\hat{L}$. One can show that the space of Hilbert cusp forms S_κ of parallel weight $\kappa = (3,3)$ for $\mathrm{SL}_2(\mathcal{O}_F)$ vanishes. Consequently, any harmonic Whittaker form f of weight $k = (-1,3)$ is weakly holomorphic. For any totally positive $m \in \partial_F^{-1}$ there exists a holomorphic modular form Ψ_m on X_K with divisor $Z(m,0)$ and weight $B(m,0)$, given by Theorem 6.8. The generating series $A(\tau)$ of special divisors is equal to $c_1(\mathcal{M}_1)E_L(\tau,\kappa)$.

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Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, D–64289 Darmstadt, Germany

E-mail address: bruinier@mathematik.tu-darmstadt.de