The Legacy of Srinivasa Ramanujan, RMS-Lecture Notes Series No. 20, 2013, pp. 139–151.

Recent Work on the Partition Function

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Abstract. This expository article describes recent work by the authors on the partition function $p(n)$. This includes a finite formula for $p(n)$ as a "trace" of algebraic singular moduli, and an overarching ℓ-adic structure which controls partition congruences modulo powers of primes $\ell \geq 5$.

Keywords. Partition function, harmonic Maass forms.

1. Introduction and statement of results

A *partition* [4] of a positive integer n is any nonincreasing sequence of positive integers which sum to n, and the *partition function* $p(n)$, which counts the number of partitions of n, defines the rapidly increasing provocative sequence:

$$\begin{aligned}1, 1, 2, 3, 5, \ldots, p(100) = 190569292, \ldots, p(1000)\\ = 24061467864032622473692149727991, \ldots.\end{aligned}$$

In famous work [34] which gave birth to the "circle method", Hardy and Ramanujan proved the asymptotic formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}.$$

Rademacher [49,50] subsequently perfected this method to derive the "exact" formula

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \qquad (1.1)$$

Here $I_{\frac{3}{2}}(\cdot)$ is a modified Bessel function of the first kind, and $A_k(n)$ is the Kloosterman-type sum

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} (-1)^{\{\frac{x}{6}\}} \cdot \exp\left(\frac{2\pi i x}{12k}\right),$$

where $\{\alpha\}$ denotes the integer nearest to α.

Remark. Individual values of $p(n)$ can be obtained by rounding sufficiently accurate truncations of (1.1). Recent work by the second author and Masri [26] gives the best known results on the problem of optimally bounding the error between $p(n)$ and such truncations.

In recent work [18], two of the authors answer questions raised in [14] by establishing a new formula for the partition function, one which expresses $p(n)$ as a *finite sum* of *algebraic numbers*. These numbers are *singular moduli* for a weak Maass form which we describe using Dedekind's eta-function $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ (note. $q := e^{2\pi i z}$ throughout) and the quasimodular Eisenstein series

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} d q^n. \tag{1.2}$$

To this end, we define the $\Gamma_0(6)$ weight -2 meromorphic modular form $F(z)$ by

$$F(z) := \frac{1}{2} \cdot \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2} = q^{-1} - 10 - 29q - \dots. \tag{1.3}$$

Using the convention that $z = x + iy$, with $x, y \in \mathbb{R}$, we define the *weak Maass form*

$$\begin{aligned} P(z) &:= -\left(\frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y}\right) F(z) \\ &= \left(1 - \frac{1}{2\pi y}\right) q^{-1} + \frac{5}{\pi y} + \left(29 + \frac{29}{2\pi y}\right) q + \dots. \end{aligned} \tag{1.4}$$

This nonholomorphic modular form has weight 0, and is a weak Maass form (for background on weak Maass forms, see [15]). It has eigenvalue -2 with respect to the hyperbolic Laplacian

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The formula uses discriminant $-24n + 1 = b^2 - 4ac$ positive definite integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ with the property that $6 \mid a$. The congruence subgroup $\Gamma_0(6)$ acts on such forms, and we let $\mathcal{Q}_n$ be any set of representatives of those equivalence classes with $a > 0$ and $b \equiv 1 \pmod{12}$. For each $Q(x, y)$, we let α_Q be the CM point in $\mathbb{H}$, the upper half of the complex plane, for which $Q(\alpha_Q, 1) = 0$. In terms of the "trace"

$$\mathrm{Tr}(n) := \sum_{Q \in \mathcal{Q}_n} P(\alpha_Q), \tag{1.5}$$

we have the following theorem.

Theorem 1.1 (Theorem 1.1 of [18]). *If n is a positive integer, then we have that*

$$p(n) = \frac{1}{24n - 1} \cdot \mathrm{Tr}(n).$$

The numbers $P(\alpha_Q)$, as Q varies over $\mathcal{Q}_n$, form a multiset of algebraic numbers which is the union of Galois orbits for the discriminant $-24n+1$ ring class field. Moreover, for each $Q \in \mathcal{Q}_n$ we have that $(24n-1)P(\alpha_Q)$ is an algebraic integer.

Theorem 1.1 shows the partition numbers are natural invariants associated with the "class polynomials" defined by

$$H_n(x) = x^{h(-24n+1)} - (24n-1)p(n)x^{h(-24n+1)-1} + \cdots := \prod_{Q \in \mathcal{Q}_n} (x - P(\alpha_Q)) \in \mathbb{Q}[x]. \tag{1.6}$$

In a forthcoming paper [19], Sutherland and the first and fourth authors will describe an efficient algorithm for computing these class polynomials by combining the Chinese Remainder Theorem with the theory of isogeny volcanoes for elliptic curves with complex multiplication.

Remark. Using Maass-Poincaré series and identities and formulas for Kloosterman-type sums, one can use Theorem 1.1 to give a new proof of the exact formula (1.1). Work along these lines has been done by Dewar and Murty [20].

Example. We give an amusing proof that $p(1) = 1$. We have that $24n - 1 = 23$, and we use the $\Gamma_0(6)$-representatives

$$\mathcal{Q}_1 = \{Q_1, Q_2, Q_3\} = \{6x^2 + xy + y^2, 12x^2 + 13xy + 4y^2, 18x^2 + 25xy + 9y^2\}.$$

The corresponding CM points are

$$\alpha_{Q_1} := -\frac{1}{12} + \frac{1}{12} \cdot \sqrt{-23}, \quad \alpha_{Q_2} := -\frac{13}{24} + \frac{1}{24} \cdot \sqrt{-23},$$
$$\alpha_{Q_3} := -\frac{25}{36} + \frac{1}{36} \cdot \sqrt{-23}.$$

Using the explicit Fourier expansion of $P(z)$, we find that $P(\alpha_{Q_3}) = \overline{P(\alpha_{Q_2})}$, and we have that

$$P(\alpha_{Q_1}) = \frac{\beta^{1/3}}{138} + \frac{2782}{3\beta^{1/3}} + \frac{23}{3},$$
$$P(\alpha_{Q_2}) = -\frac{\beta^{1/3}}{276} - \frac{1391}{3\beta^{1/3}} + \frac{23}{3} - \frac{\sqrt{-3}}{2} \cdot \left(\frac{\beta^{1/3}}{138} - \frac{2782}{3\beta^{1/3}}\right),$$

where $\beta := 161529092 + 18648492\sqrt{69}$. Therefore, we have that

$$H_1(x) := \prod_{m=1}^{3} (x - P(\alpha_{Q_m})) = x^3 - 23x^2 + \frac{3592}{23}x - 419,$$

and this confirms that $p(1) = \frac{1}{23}\mathrm{Tr}(1) = 1$.

Example. Here are the first four partition class polynomials.

n	$(24n-1)p(n)$	$H_n(x)$
1	23	$x^3 - 23x^2 + \frac{3592}{23}x - 419$
2	94	$x^5 - 94x^4 + \frac{169659}{47}x^3 - 65838x^2 + \frac{1092873176}{47^2}x + \frac{1454023}{47}$
3	213	$x^7 - 213x^6 + \frac{1312544}{71}x^5 - 723721x^4 + \frac{44648582886}{71^2}x^3$ $+\frac{9188934683}{71}x^2 + \frac{166629520876208}{71^3}x + \frac{2791651635293}{71^2}$
4	475	$x^8 - 475x^7 + \frac{9032603}{95}x^6 - 9455070x^5 + \frac{3949512899743}{95^2}x^4$ $-\frac{97215753021}{19}x^3 + \frac{9776785708507683}{95^3}x^2$ $-\frac{53144327916296}{19^2}x - \frac{134884469547631}{5^4 \cdot 19}.$

Theorem 1.1 is an example of a general theorem on "traces" of CM values of certain weight 0 weak Maass forms which are images under the Maass raising operator of weight -2 *harmonic Maass forms*. Theorem 1.1 is obtained by applying this general theorem to $F(z)$. This general theorem is a new result which adds to the extensive literature (for example, see [13,16,17,21–23,37,43,44]) inspired by Zagier's seminal paper [57] on "traces" of singular moduli.

The congruence properties of $p(n)$ have served as a testing ground for fundamental constructions in the theory of modular forms. Indeed, some of the deepest results on partition congruences have been obtained by making use of modular equations, Hecke operators, Shimura's correspondence, and the Deligne-Serre theory of ℓ-adic Galois representations.

Here we describe recent work [25] by some of the authors on questions inspired by Ramanujan's celebrated congruences, which assert for $m \in \mathbb{N}$ and primes $\ell \in \{5, 7, 11\}$ that

$$\begin{aligned} p(5^m n + \delta_5(m)) &\equiv 0 \pmod{5^m},\\ p(7^m n + \delta_7(m)) &\equiv 0 \pmod{7^{\lfloor m/2 \rfloor + 1}},\\ p(11^m n + \delta_{11}(m)) &\equiv 0 \pmod{11^m},\end{aligned}$$

where $0 < \delta_\ell(m) < \ell^m$ satisfies the congruence $24\delta_\ell(m) \equiv 1 \pmod{\ell^m}$. To prove these congruences, Atkin, Ramanujan, and Watson [6,51,52,54] relied heavily on the properties of the generating functions

$$P_\ell(b; z) := \sum_{n=0}^{\infty} p\left(\frac{\ell^b n + 1}{24}\right) q^{\frac{n}{24}} \tag{1.7}$$

(note that $p(0) = 1$, and $p(\alpha) = 0$ if $\alpha < 0$ or $\alpha \notin \mathbb{Z}$).

Expanding on these works, some of the authors studied the general ℓ-adic properties of the $P_\ell(b;z)$, as $b \to +\infty$, for fixed primes $\ell \geq 5$. Despite the absence of modular equations, which played a central role for the primes $\ell \in \{5,7,11\}$, it turns out that these generating functions are strictly constrained ℓ-adically. They are "self-similar", with resolution that improves as one "zooms in" appropriately. Throughout, if $\ell \geq 5$ is prime and $m \geq 1$, then we let

$$b_\ell(m) := 2\left(\left\lfloor \frac{\ell-1}{12} \right\rfloor + 2\right) m - 3. \tag{1.8}$$

To illustrate the general theorem (see Theorem 1.3), we first explain the situation for the primes $5 \leq \ell \leq 31$.

Theorem 1.2 (Theorem 1.1 of [25]). *Suppose that $5 \leq \ell \leq 31$ is prime, and that $m \geq 1$. If $b_1 \equiv b_2 \pmod 2$ are integers for which $b_2 > b_1 \geq b_\ell(m)$, then there is an integer $A_\ell(b_1, b_2, m)$ such that for every non-negative integer n we have*

$$p\left(\frac{\ell^{b_2} n + 1}{24}\right) \equiv A_\ell(b_1, b_2, m) \cdot p\left(\frac{\ell^{b_1} n + 1}{24}\right) \pmod{\ell^m}.$$

If $\ell \in \{5,7,11\}$, then $A_\ell(b_1, b_2, m) = 0$.

Remark. Boylan and Webb [12] have recently lengthened the range on b in Theorem 1.2. Their work and numerics suggest that one can generically take $b_\ell(m) := 2m - 1$.

Example. For $m = 1$ and $\ell = 13$, Theorem 1.2 applies for every pair of positive integers $b_1 < b_2$ with the same parity. We let $b_1 := 1$ and $b_2 := 3$. It turns out that $A_{13}(1,3,1) = 6$, and so

$$p(13^3 n + 1007) \equiv 6p(13n + 6) \pmod{13}.$$

We "zoom" in and consider $m = 2$. It turns out that $b_1 := 2$ and $b_2 := 4$ satisfy the conclusion of Theorem 1.2 with $A_{13}(2,4,2) = 45$, which in turn implies that

$$p(13^4 n + 27371) \equiv 45p(13^2 n + 162) \pmod{13^2}.$$

Theorem 1.2 follows from a more general theorem. If $\ell \geq 5$ is prime and $m \geq 1$, then let $k_\ell(m) := \ell^{m-1}(\ell - 1)$. We define a certain alternating sequence of operators applied to $S_{k_\ell(m)} \cap \mathbb{Z}[[q]]$, the space of weight $k_\ell(m)$ cusp forms on $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients. We define $\Omega_\ell(m)$ to be the $\mathbb{Z}/\ell^m\mathbb{Z}$-module of the reductions modulo ℓ^m of those forms which arise as images after applying at least the first $b_\ell(m)$ operators. We bound the dimension of $\Omega_\ell(m)$ independently of m, and we relate the partition generating functions to the forms in this space.

Theorem 1.3 (Theorem 1.2 of [25]). *If $\ell \geq 5$ is prime and $m \geq 1$, then $\Omega_\ell(m)$ is a $\mathbb{Z}/\ell^m\mathbb{Z}$-module with rank $\leq \left\lfloor \frac{\ell-1}{12} \right\rfloor$. Moreover, if $b \geq b_\ell(m)$, then we have that*

$$P_\ell(b;z) \equiv \begin{cases} \frac{F_\ell(b;z)}{\eta(z)} \pmod{\ell^m} & \text{if } b \text{ is even}, \\ \frac{F_\ell(b;z)}{\eta(\ell z)} \pmod{\ell^m} & \text{if } b \text{ is odd}, \end{cases}$$

where $F_\ell(b;z) \in \Omega_\ell(m)$.

Four remarks.

(1) As the proof will show, each form $F_\ell(b;z) \in \Omega_\ell(m)$ is congruent modulo ℓ to a cusp form in $S_{\ell-1} \cap \mathbb{Z}[[q]]$. Since these spaces are trivial for $\ell \in \{5, 7, 11\}$, then for all $b \geq b_\ell(m)$ we have

$$p(\ell^b n + \delta_\ell(b)) \equiv 0 \pmod{\ell^m},$$

giving a weak form of the Ramanujan congruences modulo powers of 5, 7, and 11.

(2) As mentioned earlier, Boylan and Webb [12] have improved the bound for $b_\ell(m)$.

(3) Theorem 1.3 shows that the partition numbers are self-similar ℓ-adically with resolutions that improve as one zooms in properly using the iterative process which defines the $P_\ell(b;z)$. Indeed, the $P_\ell(b;z) \pmod{\ell^m}$, for $b \geq b_\ell(m)$, form periodic orbits. This is "fractal"-type behavior where the simple iterative process possesses self-similar structure with increasing resolution.

(4) In October 2010 Mazur [42] asked the fourth author questions about the modules $\Omega_\ell(m)$. Calegari has answered some of these questions by fitting Theorem 1.3 into the theory of *overconvergent* half-integral weight p-adic modular forms as developed in the recent works of Ramsey. The Appendix of [25] by Ramsey includes a detailed discussion of this result.

Theorem 1.3 is inspired by work of Atkin and O'Brien [7,8,10] from the 1960s, which suggested the existence of a richer theory of partition congruences than was known at the time. Although Ramanujan's congruences had already been the subject of many works (for example, see [4,6,7,10,36,45,46,54,55] to name a few),[1] mathematicians had difficulty finding further congruences until the pioneering work by Atkin and O'Brien [7,10] which surprisingly produced congruences modulo the primes $13 \leq \ell \leq 31$. For example, Atkin proved that

$$p(1977147619n + 815655) \equiv 0 \pmod{19}.$$

In the late 1990s, the fourth author revisited their work using ℓ-adic Galois representations and the theory of half-integral weight modular forms [47], and he proved that there are such congruences modulo every prime $\ell \geq 5$. Ahlgren and the fourth author [1,2] subsequently extended this to include all moduli coprime to 6. Other works by the fourth author and Lovejoy, Garvan, Weaver, and Yang [29,40,55,56] gave more results along these lines, further removing much of the mystery behind the wild congruences of Atkin and O'Brien.

Despite these advances, one problem in Atkin's program on "congruence Hecke operators" remained open. In [7] he writes:

"*The theory of Hecke operators for modular forms of negative dimension [i.e. positive weight] shows that under suitable conditions their Fourier coefficients*

[1] Ramanujan's congruences have continued to inspire research. Indeed, the subject of ranks and cranks represents a different thread in number theory which has grown out of the problem of trying to better understand partition congruences (for example, see [5,11,13,24,28,30,35,41] to name a few).

possess multiplicative properties... I have overwhelming numerical evidence, and some theoretical support, for the view that a similar theory exists for forms of positive dimension [i.e. negative weight] and functions... ; the multiplicative properties being now congruential and not identical."

Remark. Guerzhoy [32,33] has confirmed this speculation for level 1 modular functions using the theory of integer weight p-adic modular forms as developed by Hida, and refined by Wan.

For negative half-integral weights, Atkin offered $p(n)$ as evidence of his speculation. He suspected that the $P_\ell(b; 24z) \pmod{\ell^m}$, where the $b, m \to +\infty$, converge to Hecke eigenforms for $\ell = 13$ and 17. Since the $P_\ell(b; 24z)$, as $m \to +\infty$, lie in spaces whose dimensions grow exponentially in m, Atkin believed in the existence of a theory of "congruence Hecke operators", one which depends on ℓ but is independent of m.

To be precise, Atkin considered the weight $-\frac{1}{2}$ Hecke operator with Nebentypus $\chi_{12}(\bullet) = \left(\frac{12}{\bullet}\right)$. Recall that if λ is an integer and c is prime, then the Hecke operator $T(c^2)$ on the space of forms of weight $\lambda + \frac{1}{2}$ with Nebentypus χ is defined by

$$\left(\sum_n a(n)q^n\right) \mid T(c^2)$$
$$:= \sum_n \left(a(c^2n) + c^{\lambda-1}\left(\frac{(-1)^\lambda n}{c}\right)\chi(c)\, a(n) + c^{2\lambda-1}\chi(c^2)\, a(n/c^2)\right) q^n, \tag{1.9}$$

where $a(n/c^2) = 0$ if $c^2 \nmid n$. Atkin and O'Brien found instances in which these series, as b varies, behave like Hecke eigenforms modulo increasing powers of 13 and 17. For 13 (see Theorem 5 of [10]) they prove this observation modulo 13 and 13^2, and for 17 Atkin claims (see Section 6.3 of [7]) to have a proof modulo 17, 17^2, and 17^3.

Some of the authors have confirmed [25] Atkin's speculation for the primes $\ell \leq 31$.

Theorem 1.4 (Theorem 1.3 of [25]). *If $5 \leq \ell \leq 31$ and $m \geq 1$, then for $b \geq b_\ell(m)$ we have that $P_\ell(b; 24z) \pmod{\ell^m}$ is an eigenform of all of the weight $k_\ell(m) - \frac{1}{2}$ Hecke operators on $\Gamma_0(576)$.*

As an immediate corollary, we have the following congruences for $p(n)$.

Corollary 1.5 (Corollary 1.4 of [25]). *Suppose that $5 \leq \ell \leq 31$ and that $m \geq 1$. If $b \geq b_\ell(m)$, then for every prime $c \geq 5$ there is an integer $\lambda_\ell(m, c)$ such that for all n coprime to c we have*

$$p\left(\frac{\ell^b nc^3 + 1}{24}\right) \equiv \lambda_\ell(m, c)\, p\left(\frac{\ell^b nc + 1}{24}\right) \pmod{\ell^m}.$$

Remark. Atkin [7] found such congruences modulo 13^2, 17^3, 19^2, 23^6, 29, and 31.

The results described here are obtained in the two papers [18] and [25]. Due to the technical nature of the proof of Theorem 1.1, it is not possible to provide a succinct summary in this expository paper. A proper treatment requires a discussion of theta lifts, Kudla-Milsson theta functions, the Weil representation, Maass-Poincareé series, Maass operators, work of Katok and Sarnak, and the theory of complex multiplication. Instead, in Section 2 we shall offer a brief discussion of the strategy and method of proof. In Section 3 we offer a slightly more detailed discussion of the ideas which must be assembled to prove the theorems on the ℓ-adic properties of $p(n)$, and we sketch the proof of Theorem 1.2.

2. Discussion of finite formula for $p(n)$

Here we provide a brief overview of the proof of the general theorem which implies Theorem 1.1, the finite algebraic formula for $p(n)$. To obtain this general result, we employ the theory of theta lifts as in earlier work by Funke and the first author [15,27]. The idea is to use the Kudla-Millson theta functions, combined with the action of the Maass lowering and raising operators, to construct a new theta lift which works for arbitrary level N, a result which is already of independent interest. The new lift maps spaces of weight -2 harmonic weak Maass forms to spaces of weight $-1/2$ vector valued harmonic weak Maass forms. Then we employ an argument of Katok and Sarnak [38] to obtain formulas for the coefficients of the weight $-1/2$ forms as formal "traces" of the values of a weight 0 nonholomorphic modular function which is obtained by applying a suitable differential operator.

Remark. Alfes [3] has generalized this new theta lift to obtain lifts from harmonic Maass forms of weight $-2k$ to harmonic Maass forms of weight $1/2-k$, when $k \geq 0$ is odd, and to weight $3/2+k$ when $k \geq 0$ is even.

Theorem 1.1, the finite algebraic formula for $p(n)$, corresponds to the theta lift in the special case when $N = 6$. The function $\eta(z)^{-1}$, which is essentially the generating function for $p(n)$, can be viewed as a component of a weight $-1/2$ vector-valued modular form which transforms suitably with respect to the Weil representation. More precisely, we have the vector-valued modular form

$$G(z) := \sum_{r \in \mathbb{Z}/12\mathbb{Z}} \chi_{12}(r)\eta(z)^{-1}\mathfrak{e}_r,$$

where $\chi_{12}(\bullet) = \left(\frac{12}{\bullet}\right)$ and $\{\mathfrak{e}_1, \mathfrak{e}_2, \dots \mathfrak{e}_{12}\}$ is the corresponding standard basis for the Weil representation in this case. The transformation law for the eta-function under $z \mapsto z+1$ and $z \mapsto -1/z$ easily confirms that $G(z)$ transforms according to the Weil representation. Moreover, the principal part of G is given by $q^{-1/24}(\mathfrak{e}_1 - \mathfrak{e}_5 - \mathfrak{e}_7 + \mathfrak{e}_{11})$.

On the other hand, $G(z)$ can be obtained as a theta lift. Let $F(z)$ be the function defined in (1.3). It is invariant under the Fricke involution W_6, and under the Atkin-Lehner involution W_3 it is taken to its negative. By making use of the theory of Poincaré series, which corresponds nicely with principal parts of vector-valued forms, we then find that the function $P(z)$ defined by (1.4) is the nonholomorphic weight 0 modular function whose traces give the coefficients of $1/\eta(z)$, namely the partition numbers.

To complete the proof of Theorem 1.1, we must show that the values $P(\alpha_Q)$ are *algebraic numbers*. In [18] some of the authors were able to obtain the algebraicity of these values with *bounded denominators*. The proof of this claim required the classical theory of complex multiplication, as well as new results which bound denominators of suitable singular moduli. In particular, it was proved that $6(24n-1)P(\alpha_Q)$ is always an algebraic integer. This result was refined by Larson and Rolen [39] who used some work of Masser to prove indeed that each $(24n-1)P(\alpha_Q)$ is an algebraic integer.

3. ℓ-adic properties of the partition function

Here we briefly describe the main ideas which are involved in the proofs of the results on partition congruences. We begin by recalling the crucial generating functions.

3.1 *Partition generating functions*

For every prime $\ell \geq 5$ we define a sequence of q-series that naturally contain the generating functions $P_\ell(b; z)$ as factors. Throughout, suppose that $\ell \geq 5$ is prime, and let

$$\Phi_\ell(z) := \frac{\eta(\ell^2 z)}{\eta(z)}. \tag{3.1}$$

We recall Atkin's $U(\ell)$-operator

$$\left(\sum a(n)q^n\right) \mid U(\ell) := \sum a(n\ell)q^n, \tag{3.2}$$

and we define $D(\ell)$ by

$$f(z) \mid D(\ell) := (\Phi_\ell(z) \cdot f(z)) \mid U(\ell). \tag{3.3}$$

This paper depends on a special sequence of modular functions. We begin by letting

$$L_\ell(0; z) := 1. \tag{3.4}$$

If $b \geq 1$, we then let

$$L_\ell(b; z) := \begin{cases} L_\ell(b-1; z) \mid U(\ell) & \text{if } b \text{ is even,} \\ L_\ell(b-1; z) \mid D(\ell) & \text{if } b \text{ is odd.} \end{cases} \tag{3.5}$$

We have the following elementary lemma.

Lemma 3.1 (Lemma 2.1 of [25]). *If b is a nonnegative integer, then*

$$L_\ell(b; z) = \begin{cases} \eta(z) \cdot P_\ell(b; z) & \text{if } b \text{ is even,} \\ \eta(\ell z) \cdot P_\ell(b; z) & \text{if } b \text{ is odd.} \end{cases}$$

As usual, let $M_k(\Gamma_0(N))$ denote the space of weight k holomorphic modular forms on $\Gamma_0(N)$. We let $M_k^!(\Gamma_0(N))$ denote the space of weight k *weakly holomorphic modular forms* on $\Gamma_0(N)$, those forms whose poles (if any) are supported at the cusps of $\Gamma_0(N)$. When $N = 1$ we use the notation M_k and $M_k^!$.

We have the following crucial lemma about the q-series $L_\ell(b; z)$ which can be used to force these series to live in the finite dimensional $\mathbb{Z}/\ell^m\mathbb{Z}$ modules.

Lemma 3.2 (Lemma 2.2 of [25]). *If b is a nonnegative integer, then $L_\ell(b; z)$ is in $M_0^!(\Gamma_0(\ell)) \cap \mathbb{Z}[[q]]$. In particular, if $b \geq 1$, then $L_\ell(b; z)$ vanishes at $i\infty$, and its only pole is at the cusp at 0.*

3.2 *The space $\Omega_\ell(m)$*

The basic theory of modular forms mod ℓ as developed by Serre [53] is well suited for defining a distinguished space of modular forms modulo ℓ^m, a space we denote by $\Omega_\ell(m)$. It will turn out that $\Omega_\ell(m)$ contains large ranges of the $L_\ell(b; z) \pmod{\ell^m}$.

To define these spaces, we consider the alternating sequence of operators

$$\{D(\ell), U(\ell), D(\ell), U(\ell), D(\ell), U(\ell), \ldots\}.$$

For a cusp form $G(z)$, to ease notation, we let $G_\ell(0; z) := G(z)$, and for $b \geq 1$ we then let

$$G_\ell(b; z) := \begin{cases} G_\ell(b-1; z) \mid U(\ell) & \text{if } b \text{ is even,} \\ G_\ell(b-1; z) \mid D(\ell) & \text{if } b \text{ is odd.} \end{cases} \tag{3.6}$$

We say that a cusp form $G(z) \in S_{k_\ell(m)} \cap \mathbb{Z}[[q]]$ is *good for* (ℓ, m) if for each $b \geq b_\ell(m)$ we have that $G_\ell(b; z)$ is the reduction modulo ℓ^m of a cusp form in $S_{k_\ell(m)} \cap \mathbb{Z}[[q]]$. It will turn out that each $L_\ell(b; z)$, for $b \geq b_\ell(m)$, is the reduction modulo ℓ^m of a cusp form in $S_{k_\ell(m)} \cap \mathbb{Z}[[q]]$.

We define the space $\Omega_\ell(m)$ to be the $\mathbb{Z}/\ell^m\mathbb{Z}$-module generated by the set

$$\{G_\ell(b; z) \pmod{\ell^m} : \text{where } b \geq b_\ell(m) \text{ and } G(z) \text{ is good for } (\ell, m)\}. \tag{3.7}$$

Using the theory of modular forms modulo ℓ and Serre's contraction property for filtrations, one can prove the following theorem.

Theorem 3.3 (Theorem 3.4 of [25]). *If $\ell \geq 5$ and $m \geq 1$, then $\Omega_\ell(m)$ is a $\mathbb{Z}/\ell^m\mathbb{Z}$-module with rank $\leq \left\lfloor \frac{\ell-1}{12} \right\rfloor$.*

Using Lemma 3.2 and the theory of modular forms modulo ℓ, some of the authors have proved the following crucial fact about the forms $L_\ell(b; z)$.

Theorem 3.4 (Theorem 4.3 of [25]). *If $\ell \geq 5$ is prime, $m \geq 1$, and $b \geq b_\ell(m)$, then $L_\ell(b; z)$ is in $\Omega_\ell(m)$.*

Proof of Theorem 1.2. The theorem follows trivially from the Ramanujan congruences when $\ell \in \{5, 7, 11\}$. More generally, we consider the two subspaces,

$\Omega_\ell^{\text{odd}}(m)$ and $\Omega_\ell^{\text{even}}(m)$, of $\Omega_\ell(m)$ generated by $L_\ell(b; z)$ for odd b and even b, respectively. We observe that applying $D(\ell)$ to a form gives q-expansions satisfying

$$F \mid D(\ell) = \sum_{n > \frac{\ell^2-1}{24\ell}} a(n)q^n.$$

Combining this observation with Theorem 1.3 and the fact that the full space $\Omega_\ell(m)$ is generated by alternately applying $D(\ell)$ and $U(\ell)$, we have that the ranks of $\Omega_\ell^{\text{odd}}(m)$ and $\Omega_\ell^{\text{even}}(m)$ are $\leq \left\lfloor \frac{\ell-1}{12} \right\rfloor - \left\lfloor \frac{\ell^2-1}{24\ell} \right\rfloor$. If $13 \leq \ell \leq 31$, then direct calculation, when $m = 1$, shows that each of these subspaces has dimension 1. The theorem now follows immediately.

Acknowledgements

The first author thanks the DFG for its support. The authors thank the American Institute of Mathematics for their generous support. The second and fourth authors thank the NSF for its support, and the fourth author is also grateful for the support of the Candler Fund.

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