

## VECTOR VALUED FORMAL FOURIER-JACOBI SERIES

JAN HENDRIK BRUINIER

(Communicated by Kathrin Bringmann)

ABSTRACT. H. Aoki showed that any symmetric formal Fourier-Jacobi series for the symplectic group  $\mathrm{Sp}_2(\mathbb{Z})$  is the Fourier-Jacobi expansion of a holomorphic Siegel modular form. We prove an analogous result for vector valued symmetric formal Fourier-Jacobi series, by combining Aoki's theorem with facts about vector valued modular forms. Recently, this result was also proved independently by M. Raum using a different approach. As an application, by means of work of W. Zhang, modularity results for special cycles of codimension 2 on Shimura varieties associated to orthogonal groups can be derived.

### 1. INTRODUCTION

Let  $\Gamma^{(g)}$  be the metaplectic extension of the integral symplectic group  $\mathrm{Sp}_g(\mathbb{Z})$  of genus  $g$ . The elements of  $\Gamma^{(g)}$  are pairs  $(M, \alpha)$ , where  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z})$ , and  $\alpha$  is a holomorphic function on the Siegel upper half plane  $\mathbb{H}_g$  such that  $\alpha(Z)^2 = \det(CZ + D)$ . The product of two elements  $(M_1, \alpha_1)$  and  $(M_2, \alpha_2)$  is defined by

$$(M_1, \alpha_1(Z)) \cdot (M_2, \alpha_2(Z)) = (M_1 M_2, \alpha_1(M_2 Z) \alpha_2(Z)).$$

Let  $k \in \frac{1}{2}\mathbb{Z}$ , and let

$$(1) \quad \rho : \Gamma^{(g)} \longrightarrow \mathrm{GL}(V_\rho)$$

be a unitary representation on a finite dimensional complex vector space  $V_\rho$ . Throughout we assume that  $\rho$  is trivial on some congruence subgroup of sufficiently large level. We denote by  $M_k^{(g)}(\rho)$  the vector space of holomorphic Siegel modular forms for the group  $\Gamma^{(g)}$  of weight  $k$  with representation  $\rho$ , that is, the space of holomorphic functions  $f : \mathbb{H}_g \rightarrow V_\rho$  on  $\mathbb{H}_g$  satisfying the transformation law

$$(2) \quad f(MZ) = \alpha(Z)^{2k} \rho(M, \alpha) f(Z)$$

for all  $(M, \alpha) \in \Gamma^{(g)}$  (and which are holomorphic at the cusp  $\infty$  if  $g = 1$ ); see [Fr]. For the trivial representation  $\rho_0$  on  $\mathbb{C}$ , we briefly write  $M_k^{(g)}$  instead of  $M_k^{(g)}(\rho_0)$ . Note that the transformation law for  $(1, -1) \in \Gamma^{(g)}$  implies that  $M_k^{(g)} = 0$  if  $k$  is not integral.

In the present note we are mainly interested in the case  $g = 2$ . If we write the variable  $Z \in \mathbb{H}_2$  as

$$Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$$

---

Received by the editors March 15, 2013 and, in revised form, May 16, 2013.

2010 *Mathematics Subject Classification*. Primary 11F46, 11F50.

The author was partially supported by DFG grants BR-2163/2-2 and FOR 1920.

with  $\tau, \tau' \in \mathbb{H}_1$  and  $z \in \mathbb{C}$ , then any  $f \in M_k^{(2)}(\rho)$  has a Fourier-Jacobi expansion of the form

$$(3) \quad f(Z) = \sum_{m \geq 0} \phi_m(\tau, z) q'^m,$$

where  $q' = e^{2\pi i \tau'}$ , and the coefficients  $\phi_m$  are Jacobi forms of weight  $k$ , index  $m$ , with representation  $\rho$ . To explain this more precisely, recall that the (metaplectic) Jacobi group  $\Gamma^J = \Gamma^{(1)} \ltimes \mathbb{Z}^2$  acts on the Jacobi half plane  $\mathbb{H}_1 \times \mathbb{C}$  in the usual way; see [EZ]. Moreover, there is an embedding  $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \rightarrow \mathrm{Sp}_2(\mathbb{Z})$ ,

$$(4) \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \mapsto \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \lambda\mu \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is compatible with the group actions on  $\mathbb{H}_1 \times \mathbb{C}$  and on  $\mathbb{H}_2$ . It lifts to an embedding  $j : \Gamma^J \rightarrow \Gamma^{(2)}$ , taking the cocycle  $\alpha$  to itself. The restriction of  $\rho$  defines a representation of the metaplectic Jacobi group. A holomorphic function  $\phi : \mathbb{H}_1 \times \mathbb{C} \rightarrow V_\rho$  is called a Jacobi form for  $\Gamma^J$  of weight  $k$  and index  $m$  with representation  $\rho$ , if the function

$$(5) \quad \tilde{\phi}(Z) = \phi(\tau, z) q'^m$$

on  $\mathbb{H}_2$  satisfies the transformation law (2) for all  $(M, \alpha) \in j(\Gamma^J)$ , and  $\phi$  is holomorphic at  $\infty$ . Here the latter condition means that the Fourier expansion of  $\phi$  has the form

$$(6) \quad \phi(\tau, z) = \sum_{\substack{n \in \mathbb{Q} \\ n \geq 0}} \sum_{\substack{r \in \mathbb{Q} \\ r^2 \leq 4mn}} c(\phi; n, r) q^n \zeta^r$$

with  $q = e^{2\pi i \tau}$ ,  $\zeta = e^{2\pi i z}$ , and  $c(\phi; n, r) \in V_\rho$ . Because of our assumption that  $\rho$  be trivial on some congruence subgroup, the coefficients  $c(\phi; n, r)$  are supported on rational numbers with bounded denominators. We write  $J_{k,m}(\rho)$  for the vector space of Jacobi forms for  $\Gamma^J$  of weight  $k$ , index  $m$ , with representation  $\rho$ .

The element

$$(7) \quad \delta = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, i \right) \in \Gamma^{(2)}$$

acts on  $\mathbb{H}_2$  by  $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \mapsto \begin{pmatrix} \tau' & z \\ z & \tau \end{pmatrix}$ . The transformation law (2) implies that the Fourier-Jacobi coefficients (3) of any  $f \in M_k^{(2)}(\rho)$  satisfy the symmetry relation

$$(8) \quad c(\phi_m; n, r) = i^{2k} \rho(\delta) c(\phi_n; m, r).$$

This motivates the following definition.

**Definition 1.1.** A formal Fourier-Jacobi series for the group  $\Gamma^{(2)}$  of weight  $k$  with representation  $\rho$  is a formal series

$$(9) \quad f = \sum_{m \geq 0} \phi_m(\tau, z) q'^m$$

whose coefficients  $\phi_m(\tau, z)$  belong to  $J_{k,m}(\rho)$ . We denote the vector space of such formal Fourier-Jacobi series by  $N_k^{(2)}(\rho)$ . We call  $f \in N_k^{(2)}(\rho)$  *symmetric* if its coefficients satisfy (8) for all triples  $m, n, r$ .

For the trivial representation  $\rho_0$  on  $\mathbb{C}$ , we briefly write  $N_k^{(2)}$  instead of  $N_k^{(2)}(\rho_0)$ . The above considerations show that the Fourier-Jacobi expansion of any  $f \in M_k^{(2)}(\rho)$  is a symmetric Fourier-Jacobi series. In the present note we consider the following converse.

**Theorem 1.2.** *Every symmetric  $f \in N_k^{(2)}(\rho)$  is the Fourier-Jacobi expansion of some Siegel modular form in  $M_k^{(2)}(\rho)$ . In particular,  $f$  converges absolutely.*

Such a result was first proved by Aoki in [Ao] for the *trivial* representation  $\rho$ . His proof relies on classical facts on Taylor expansions of Jacobi forms [EZ] and comparisons of dimension formulas. Variants for paramodular groups of small level and the trivial representation are proved in [IPY, Theorem 1.2]. In a recent preprint [Ra], Raum proves Theorem 1.2 (for possibly non-trivial representations as in (1)), by employing the Hirzebruch-Riemann-Roch theorem and the Lefschetz fixed point formula, combined with asymptotic estimates for the dimensions of symmetric formal Fourier-Jacobi series.

In the present note we show that the full statement of Theorem 1.2 can be derived from Aoki’s original result for the scalar case by employing some facts about vector valued Siegel modular forms and their Fourier-Jacobi expansions. We hope that this approach might also help to attack the analogous problem in other situations (e.g. in higher genus) by reducing the vector valued to the scalar case.

An important application of Theorem 1.2 is an analogue of the Gross-Kohnen-Zagier theorem [GKZ] for codimension 2 special cycles on Shimura varieties associated with orthogonal groups of signature  $(n, 2)$ . It states that the generating series of special cycles of codimension 2 is a vector valued Siegel modular form for the group  $\Gamma^{(2)}$  of weight  $1 + n/2$  with values in the second Chow group. This result was conjectured (in greater generality) by Kudla in [Ku3], motivated by his joint work with Millson on geometric generating series; see e.g. [KM], [Ku1]. Employing [Bo], it was proved by Zhang [Zh] that the generating series is a symmetric formal Fourier-Jacobi series. Theorem 1.2 implies that it actually converges and therefore defines a Siegel modular form. We explain this application in Section 3.

2. VECTOR VALUED MODULAR FORMS AND THE PROOF OF THEOREM 1.2

Let  $(\rho, V_\rho)$  and  $(\sigma, V_\sigma)$  be finite dimensional representations of  $\Gamma^{(2)}$ . Denote the canonical bilinear pairing  $\text{Hom}(V_\rho, V_\sigma) \times V_\rho \rightarrow V_\sigma$  by  $\langle \lambda, v \rangle = \lambda(v)$  for  $\lambda \in \text{Hom}(V_\rho, V_\sigma)$  and  $v \in V_\rho$ . The following two lemmas are easily seen.

**Lemma 2.1.** *Let  $f = \sum_{m \geq 0} \phi_m q^m \in N_k^{(2)}(\rho)$  and  $g = \sum_{m \geq 0} \psi_m q^m \in N_{k_1}^{(2)}(\sigma)$ . Then*

$$f \otimes g = \sum_{M \geq 0} \sum_{\substack{m, m_1 \geq 0 \\ m + m_1 = M}} \phi_m \otimes \psi_{m_1} q^M$$

*belongs to  $N_{k+k_1}^{(2)}(\rho \otimes \sigma)$ . If  $f$  and  $g$  are symmetric, then  $f \otimes g$  is symmetric as well.*

**Lemma 2.2.** *Let  $f = \sum_{m \geq 0} \phi_m q^m \in N_k^{(2)}(\rho)$ , and let  $g = \sum_{m \geq 0} \psi_m q^m$  be a formal Fourier-Jacobi series of weight  $k_1$  and representation  $\text{Hom}(\overline{V}_\rho, V_\sigma)$ . Then*

$$\langle g, f \rangle = \sum_{M \geq 0} \sum_{\substack{m, m_1 \geq 0 \\ m+m_1=M}} \langle \psi_{m_1}, \phi_m \rangle q^M$$

*belongs to  $N_{k+k_1}^{(2)}(\sigma)$ . If  $f$  and  $g$  are symmetric, then  $\langle f, g \rangle$  is symmetric as well.*

For the trivial representation, the direct sum over  $k \in \mathbb{Z}_{\geq 0}$  of all spaces  $N_k^{(2)}$  is a graded algebra with the multiplication of Lemma 2.1. Any  $f \in N_k^{(2)}$  has a multiplicative inverse in the field  $\mathcal{M}(\mathbb{H}_1 \times \mathbb{C})((q'))$  of formal Laurent series with coefficients in the field  $\mathcal{M}(\mathbb{H}_1 \times \mathbb{C})$  of meromorphic functions on  $\mathbb{H}_1 \times \mathbb{C}$ . It is easily seen that the  $m$ -th coefficient of the inverse is a meromorphic Jacobi form of weight  $-k$  and index  $m$ .

**Definition 2.3.** Let  $f$  be a meromorphic modular form for  $\Gamma^{(2)}$  of weight  $k$  with representation  $\rho$ . Write  $f = g/h$  with  $g \in M_{k+k_1}^{(2)}(\rho)$  and  $h \in M_{k_1}^{(2)}$ . We define the formal Fourier-Jacobi expansion of  $f$  as the product of the Fourier-Jacobi expansion of  $g$  with the inverse of the Fourier-Jacobi expansion of  $h$  in  $\mathcal{M}(\mathbb{H}_1 \times \mathbb{C})((q'))$ . Its  $m$ -th coefficient is a meromorphic Jacobi form of weight  $k$  for  $\Gamma^{(2)}$  with representation  $\rho$ .

Our assumption that  $\rho$  is trivial on some congruence subgroup implies that it is always possible to find  $g$  and  $h$  as required. The definition is independent of their choice. The map taking a meromorphic modular form to its formal Fourier-Jacobi expansion is injective. Note that the question where the formal Fourier-Jacobi expansion actually converges is subtle. We do not address it here. Moreover, note that our approach to (formal) Fourier-Jacobi expansions of meromorphic Siegel modular forms is different from the one pursued in [DMZ] by means of Fourier integrals along suitably chosen contours.

The center  $C = \{(\pm 1, \pm 1)\} \subset \Gamma^{(2)}$  acts trivially on  $\mathbb{H}_2$ . We consider the  $\Gamma^{(2)}$ -invariant subspace

$$V_\rho(k) = \{v \in V_\rho : \rho(-1, 1)(v) = v = (-1)^{2k} \rho(1, -1)(v)\} \subset V_\rho.$$

The transformation law (2) for  $j((-1, i), 0)$  and the symmetry relation (8) imply that any  $f \in N_k^{(2)}(\rho)$  actually takes values in  $V_\rho(k)$ .

**Proposition 2.4.** *There exists a positive  $k_0 \in k + \mathbb{Z}$  with the following property: For every  $a \in \mathbb{H}_2$  which is not a fixed point of the action of  $\Gamma^{(2)}/C$ , there are modular forms  $g_1, \dots, g_d \in M_{k_0}^{(2)}(\rho)$  whose values  $g_1(a), \dots, g_d(a)$  generate the space  $V_\rho(k)$ .*

*Proof.* Let  $X$  be the Satake compactification of  $\Gamma^{(2)} \backslash \mathbb{H}_2$ . For  $r \in \frac{1}{2}\mathbb{Z}$  we let  $\mathcal{M}_r$  be the sheaf of scalar valued modular forms of weight  $r$  on  $X$ , and let  $\mathcal{M}_r(\rho)$  be the sheaf of modular forms of weight  $r$  with representation  $\rho$  on  $X$ . These sheaves are coherent  $\mathcal{O}_X$ -modules and  $\mathcal{M}_{r+k}(\rho) = \mathcal{M}_r \otimes \mathcal{M}_k(\rho)$ . By the work of Satake and Baily-Borel, there exists a positive integer  $r_0$  such that  $\mathcal{M}_r$  is a very ample line bundle for all positive integers  $r$  which are divisible by  $r_0$ . Hence, according to a theorem of Serre (see e.g. Theorem 5.17 in [Ha, Chapter II]), the sheaf  $\mathcal{M}_{r+k}(\rho)$  is generated by global sections for all integers  $r$  which are divisible by  $r_0$  and sufficiently large. We fix such an  $r$  and put  $k_0 = r + k$ .

If  $a$  is not an elliptic fixed point, then the stalk  $\mathcal{M}_{k_0,a}(\rho)$  at  $a$  is equal to  $V_\rho(k) \otimes_{\mathbb{C}} \mathcal{O}_{X,\rho}$ . By the assumption on  $k_0$  there exist global sections of  $\mathcal{M}_{k_0}(\rho)$  which generate the stalk  $\mathcal{M}_{k_0,a}(\rho)$ . Their values at  $a$  must generate  $V_\rho(k)$ .  $\square$

E. Freitag has pointed out that this proposition could also be proved using Poincaré series as in Theorem 4.4 of [Fr, Chapter I].

*Proof of Theorem 1.2.* Let  $f = \sum_m \phi_m q^m \in N_k^{(2)}(\rho)$ . By replacing  $\rho$  by its restriction to  $V_\rho(k)$ , we may assume without loss of generality that  $V_\rho = V_\rho(k)$ .

Put  $d = \dim(V_\rho)$ , and let  $k_0 \in k + \mathbb{Z}$  be a positive number which has the property of Proposition 2.4 for the dual representation  $\rho^\vee$ . Let  $a \in \mathbb{H}_2$  be a point which is not a fixed point of the action of  $\Gamma^{(2)}/C$ . We choose  $g_1, \dots, g_d \in M_{k_0}^{(2)}(\rho^\vee)$  such that the vectors  $g_1(a), \dots, g_d(a)$  form a basis of  $V_\rho^\vee$ . The  $d$ -tuple

$$g = {}^t(g_1, \dots, g_d)$$

defines a vector valued Siegel modular form with values in  $\text{Hom}(V_\rho, \mathbb{C}^d)$  transforming with the representation induced by  $\rho$  and the trivial representation  $\rho_0^d$  on  $\mathbb{C}^d$ . The pairing

$$h = \langle g, f \rangle$$

as in Lemma 2.2 defines an element on  $N_{k+k_0}^{(2)}(\rho_0^d)$ , that is, a  $d$ -tuple of scalar formal Fourier-Jacobi series of weight  $k + k_0$ . Since  $f$  and  $g$  are symmetric,  $h$  is also symmetric by Lemma 2.2. According to [Ao] (Theorem 1.2 for the trivial representation),  $h$  is the Fourier-Jacobi expansion of some Siegel modular form in  $M_{k+k_0}^{(2)}(\rho_0^d)$ , which we also denote by  $h$ .

For  $Z \in \mathbb{H}_2$  the value  $g(Z) \in \text{Hom}(V_\rho, \mathbb{C}^d)$  is an invertible linear map if and only if  $\det(g(Z)) \neq 0$ . Since  $g_1(a), \dots, g_d(a)$  are linearly independent,  $g(Z)$  is invertible in a neighborhood of  $a$ . Consequently, the assignment  $Z \mapsto g(Z)^{-1}$  defines a meromorphic modular form  $g^{-1}$  for  $\Gamma^{(2)}$  of weight  $-k_0$  with values in  $\text{Hom}(\mathbb{C}^d, V_\rho)$ .

The natural pairing

$$\langle g^{-1}, h \rangle = \langle g^{-1}, \langle g, f \rangle \rangle$$

is a meromorphic Siegel modular form for  $\Gamma^{(2)}$  of weight  $k$  with representation  $\rho$ , whose formal Fourier-Jacobi expansion in the sense of Definition 2.3 has to agree with  $f$ . It is holomorphic in a neighborhood of  $a$ .

Varying the point  $a$  and the corresponding modular forms  $g_\nu$ , we find that  $f$  is holomorphic on the complement of the set of elliptic fixed points. By a standard argument, we get a holomorphic continuation of  $f$  over the fixed point manifolds<sup>1</sup> of codimension  $> 1$  of elements of  $\text{Sp}_2(\mathbb{Z})$ . To get a continuation over the fixed point manifolds of codimension 1, we note that by [St, §7.3] the group  $\text{Sp}_2(\mathbb{Z})$  has up to conjugation only the two elements  $\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}$  and  $\begin{pmatrix} H & J \\ 0 & H \end{pmatrix}$  whose fixed point manifolds have codimension 1. Here  $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The corresponding fixed point manifolds are given by the divisors  $D_1 = \{z = 0\}$  and  $D_2 = \{z = 1/2\}$  on  $\mathbb{H}_2$ .

Now there exists a holomorphic scalar valued Siegel modular form  $u$  with divisor supported on the  $\text{Sp}_2(\mathbb{Z})$ -translates of  $D_1 + D_2$  and a holomorphic  $V_\rho$ -valued modular form  $v$  such that  $u \cdot f = v$ . All Fourier-Jacobi coefficients of  $u$  must vanish along the divisor  $\{z = 0\}$  on  $\mathbb{H}_1 \times \mathbb{C}$  with order at least  $\text{ord}_{D_1}(u)$ . Since the Fourier-Jacobi

<sup>1</sup>By [St, Theorem 6], for general  $g \geq 2$ , the minimal codimension of fixed point manifolds of the action of  $\text{Sp}_g(\mathbb{Z})$  on  $\mathbb{H}_g$  is  $g - 1$ .

coefficients of  $f$  are holomorphic, we find that all Fourier-Jacobi coefficients of  $v$  have to vanish along  $\{z = 0\}$  with order at least  $\text{ord}_{D_1}(u)$ . But this implies that  $f$  is holomorphic along  $D_1$ . Analogously it is holomorphic along  $D_2$ .

Consequently, the formal Fourier-Jacobi series  $f$  is the Fourier-Jacobi expansion of an element of  $M_k^{(2)}(\rho)$ .  $\square$

*Remark 2.5.* For simplicity we stated Theorem 1.2 and its proof for vector valued Fourier-Jacobi series with scalar  $K$ -type  $\det^k$  at  $\infty$ . The above proof can be generalized to vector valued  $K$ -type  $\det^k \otimes \text{Sym}^j$  at  $\infty$  in a straightforward way.

### 3. GENERATING SERIES OF SPECIAL CYCLES

Here we describe the application of Theorem 1.2 to Kudla's modularity conjecture; see [Ku3, Section 3, Problem 1].

Let  $(V, Q)$  be a quadratic space over  $\mathbb{Q}$  of signature  $(n, 2)$ . The hermitian symmetric space corresponding to the orthogonal group of  $V$  can be realized as a connected component  $\mathcal{D}^+$  of the complex manifold

$$\mathcal{D} = \{[z] \in P(V(\mathbb{C})) : (z, z) = 0 \text{ and } (z, \bar{z}) < 0\}.$$

Here  $P(V(\mathbb{C}))$  denotes the projective space of  $V(\mathbb{C}) = V \otimes_{\mathbb{Q}} \mathbb{C}$ . Let  $L \subset V$  be an even lattice, and write  $L'$  for its dual. Let  $O(L)$  be the orthogonal group of  $L$ , and let  $\Gamma_L \subset O(L)$  be a subgroup of finite index which acts trivially on  $L'/L$  and which takes  $\mathcal{D}^+$  to itself. By the theory of Baily-Borel the quotient  $X(\Gamma_L) = \Gamma_L \backslash \mathcal{D}^+$  has a structure as a quasi-projective algebraic variety. The tautological line bundle  $\mathcal{L}$  over  $\mathcal{D}$  descends to a line bundle over  $X(\Gamma_L)$ , the line bundle of modular forms of weight 1.

Special cycles on  $X(\Gamma_L)$  can be defined as follows; see [Ku1], [Ku3]. For  $1 \leq r \leq n$ , let  $S_{L,r}$  be the complex vector space of functions  $\varphi : (L'/L)^r \rightarrow \mathbb{C}$ . Recall that there is a Weil representation

$$\omega_{L,r} : \Gamma^{(r)} \longrightarrow \text{GL}(S_{L,r}).$$

If  $n$  is even, it is a subrepresentation of the restriction to  $\text{Sp}_r(\mathbb{Z})$  of the usual Weil representation of  $\text{Sp}_r(\hat{\mathbb{Q}})$  on the space of Schwartz-Bruhat functions on  $V(\hat{\mathbb{Q}})^r$ . If  $n$  is odd, in addition, the cocycles have to be matched; see e.g. [Ku2] for the case  $r = 1$ , the case of general  $r$  is analogous.

For an  $r$ -tuple  $\lambda = (\lambda_1, \dots, \lambda_r) \in V^r$  we let  $Q(\lambda) = \frac{1}{2}((\lambda_i, \lambda_j))_{i,j} \in \mathbb{Q}^{r \times r}$  be the corresponding matrix of inner products. If  $Q(\lambda)$  is positive semidefinite of rank  $r(\lambda) \in \{0, \dots, r\}$ , then

$$Z(\lambda) = \{[z] \in \mathcal{D}^+ : (z, \lambda_1) = \dots = (z, \lambda_r) = 0\}$$

is a submanifold of codimension  $r(\lambda)$ . If  $Q(\lambda)$  is not positive semidefinite, then  $\mathcal{D}^+(\lambda) = \emptyset$ . For every positive semidefinite symmetric matrix  $T \in \mathbb{Q}^{r \times r}$  of rank  $r(T)$ , and for  $\varphi \in S_{L,r}$ , we define a cycle

$$Z(T, \varphi) = \sum_{\substack{\lambda \in L'^r \\ Q(\lambda) = T}} \varphi(\lambda) \cdot Z(\lambda)$$

of codimension  $r(T)$  with complex coefficients. By reduction theory, it descends to an algebraic cycle on the quotient  $X(\Gamma_L)$ , which we also denote by  $Z(T, \varphi)$ . If the denominators of the entries of  $T$  do not divide the level of  $L$ , then  $Z(T) = \emptyset$ . By taking the intersection pairing  $Z(T, \varphi) \cdot (\mathcal{L}^\vee)^{r-r(T)}$  in the sense of [Fu, Chapter 2.5]

with a power of the dual bundle of  $\mathcal{L}$ , we obtain a cycle class of codimension  $r$ , that is, an element of  $\mathrm{CH}^r(X(\Gamma_L))_{\mathbb{C}}$ . We write  $Z(T)$  for the element

$$\varphi \mapsto Z(T, \varphi) \cdot (\mathcal{L}^\vee)^{r-r(T)}$$

of  $\mathrm{Hom}(S_{L,r}, \mathrm{CH}^r(X(\Gamma_L))_{\mathbb{C}})$ .

**Conjecture 3.1** (Kudla). *The formal generating series*

$$A_r(Z) = \sum_{\substack{T \in \mathbb{Q}^{r \times r} \\ T \geq 0}} Z(T) \cdot q^T,$$

valued in  $S_{L,r}^\vee \otimes_{\mathbb{C}} \mathrm{CH}^r(X(\Gamma_L))_{\mathbb{C}}$ , is a Siegel modular form in  $M_{1+n/2}^{(r)}(\omega_{L,r}^\vee)$  with values in  $\mathrm{CH}^r(X(\Gamma_L))_{\mathbb{C}}$ . Here we have put  $q^T = e^{2\pi i \mathrm{tr}(TZ)}$  for  $Z \in \mathbb{H}_r$ .

For  $r = 1$  this conjecture was proved by Borcherds in [Bo].

**Theorem 3.2.** *For  $r = 2$  the conjecture is true.*

*Proof.* Zhang showed in [Zh] that  $A_2(Z)$  is a formal Fourier-Jacobi series in  $N_{1+n/2}^{(2)}(\omega_{L,2}^\vee)$  with values in  $\mathrm{CH}^2(X(\Gamma_L))_{\mathbb{C}}$ . Theorem 1.2 therefore implies the assertion.  $\square$

#### ACKNOWLEDGEMENTS

The author thanks Martin Raum and the referee for very helpful comments on this paper.

#### REFERENCES

- [Ao] Hiroki Aoki, *Estimating Siegel modular forms of genus 2 using Jacobi forms*, J. Math. Kyoto Univ. **40** (2000), no. 3, 581–588. MR1794522 (2001i:11055)
- [Bo] Richard E. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. **97** (1999), no. 2, 219–233, DOI 10.1215/S0012-7094-99-09710-7. MR1682249 (2000f:11052)
- [DMZ] Atish Dabholkar and Suresh Nampuri, *Quantum black holes*, Strings and fundamental physics, Lecture Notes in Phys., vol. 851, Springer, Heidelberg, 2012, pp. 165–232, DOI 10.1007/978-3-642-25947-0\_5. MR2920326
- [EZ] Martin Eichler and Don Zagier, *The theory of Jacobi forms*, Progress in Mathematics, vol. 55, Birkhäuser Boston Inc., Boston, MA, 1985. MR781735 (86j:11043)
- [Fr] E. Freitag, *Siegelsche Modulfunktionen* (German), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 254, Springer-Verlag, Berlin, 1983. MR871067 (88b:11027)
- [Fu] William Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR1644323 (99d:14003)
- [GKZ] B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L-series. II*, Math. Ann. **278** (1987), no. 1-4, 497–562, DOI 10.1007/BF01458081. MR909238 (89i:11069)
- [Ha] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)
- [IPY] T. Ibukiyama, C. Proor, and D. Yuen, *Jacobi forms that characterize paramodular forms*, preprint (2012), arXiv:1209.3438 [math.NT].
- [Ku1] Stephen S. Kudla, *Algebraic cycles on Shimura varieties of orthogonal type*, Duke Math. J. **86** (1997), no. 1, 39–78, DOI 10.1215/S0012-7094-97-08602-6. MR1427845 (98e:11058)
- [Ku2] Stephen S. Kudla, *Integrals of Borcherds forms*, Compositio Math. **137** (2003), no. 3, 293–349, DOI 10.1023/A:1024127100993. MR1988501 (2005c:11052)

- [Ku3] Stephen S. Kudla, *Special cycles and derivatives of Eisenstein series*, Heegner points and Rankin  $L$ -series, Math. Sci. Res. Inst. Publ., vol. 49, Cambridge Univ. Press, Cambridge, 2004, pp. 243–270, DOI 10.1017/CBO9780511756375.009. MR2083214 (2005g:11108)
- [KM] Stephen S. Kudla and John J. Millson, *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, Inst. Hautes Études Sci. Publ. Math. **71** (1990), 121–172. MR1079646 (92e:11035)
- [Ra] M. Raum, *Formal Fourier Jacobi expansions and special cycles of codimension 2*, preprint (2013), arXiv:1302.0880 [math.NT].
- [St] B. Steinle, *Fixpunktmannigfaltigkeiten symplektischer Matrizen*, Acta Arith. **20** (1972), 63–106.
- [Zh] Wei Zhang, *Modularity of generating functions of special cycles on Shimura varieties*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)—Columbia University. MR2717745

FACHBEREICH MATHEMATIK, TECHNISCHE UNIVERSITÄT DARMSTADT, SCHLOSSGARTENSTRASSE  
7, D-64289 DARMSTADT, GERMANY

*E-mail address:* `bruinier@mathematik.tu-darmstadt.de`