ALGEBRAIC FORMULAS FOR THE COEFFICIENTS OF HALF-INTEGRAL WEIGHT HARMONIC WEAK MAASS FORMS

JAN HENDRIK BRUINIER AND KEN ONO

ABSTRACT. We prove that the coefficients of certain weight $-1/2$ harmonic Maass forms are "traces" of singular moduli for weak Maass forms. To prove this theorem, we construct a theta lift from spaces of weight -2 harmonic weak Maass forms to spaces of weight $-1/2$ vector-valued harmonic weak Maass forms on $\mathrm{Mp}_2(\mathbb{Z})$, a result which is of independent interest. We then prove a general theorem which guarantees (with bounded denominator) when such Maass singular moduli are algebraic. As an example of these results, we derive a formula for the partition function $p(n)$ as a finite sum of algebraic numbers which lie in the usual discriminant $-24n+1$ ring class field.

1. Introduction and statement of results

A *partition* [4] of a positive integer n is any nonincreasing sequence of positive integers which sum to n. The partition function $p(n)$, which counts the number of partitions of n, defines the rapidly increasing sequence of integers:

$$1, 1, 2, 3, 5, \dots, p(100) = 190569292, \dots, p(1000) = 24061467864032622473692149727991, \dots.$$

In celebrated work [24], which gave birth to the "circle method", Hardy and Ramanujan quantified this rate of growth. They proved the asymptotic formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}.$$

Rademacher [35, 36] subsequently perfected this method to derive his famous "exact" formula

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \tag{1.1}$$

Here $I_{\frac{3}{2}}(\cdot)$ is a modified Bessel function of the first kind, and $A_k(n)$ is a Kloosterman sum.

Remark. Values of $p(n)$ can be obtained by rounding sufficiently accurate truncations of (1.1). Bounding the error between $p(n)$ and such truncations is a well known difficult problem. Recent work by Folsom and Masri [20] gives the best known nontrivial bounds on this problem.

We obtain a new formula for $p(n)$. Answering Questions 1 and 2 of [8], we express $p(n)$ as a *finite sum* of *algebraic numbers*. These numbers are *singular moduli* for a weak Maass form

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which we describe using Dedekind's eta-function $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ (note. $q := e^{2\pi i z}$ throughout) and the quasimodular Eisenstein series

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} dq^n. \tag{1.2}$$

To this end, we define the $\Gamma_0(6)$ weight -2 meromorphic modular form $F(z)$ by

$$F(z) := \frac{1}{2} \cdot \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2} = q^{-1} - 10 - 29q - \dots. \tag{1.3}$$

Using the convention that $z = x + iy$, with $x, y \in \mathbb{R}$, we define the *weak Maass form*

$$P(z) := -\left(\frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y}\right) F(z) = \left(1 - \frac{1}{2\pi y}\right) q^{-1} + \frac{5}{\pi y} + \left(29 + \frac{29}{2\pi y}\right) q + \dots. \tag{1.4}$$

This nonholomorphic form has weight 0, and is a weak Maass form (for more on weak Maass forms, see [10]). It has eigenvalue -2 with respect to the hyperbolic Laplacian

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

To describe our formula, we use discriminant $-24n + 1 = b^2 - 4ac$ positive definite integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ with the property that $6 \mid a$. The group $\Gamma_0(6)$ acts on such forms, and we let $\mathcal{Q}_n$ be any set of representatives of those equivalence classes with $a > 0$ and $b \equiv 1 \pmod{12}$. For each $Q(x, y)$, we let α_Q be the CM point in $\mathbb{H}$, the upper half of the complex plane, for which $Q(\alpha_Q, 1) = 0$. We then define the "trace"

$$\mathrm{Tr}(n) := \sum_{Q \in \mathcal{Q}_n} P(\alpha_Q). \tag{1.5}$$

The following theorem gives the finite algebraic formula for $p(n)$.

Theorem 1.1. *If n is a positive integer, then we have that*

$$p(n) = \frac{1}{24n - 1} \cdot \mathrm{Tr}(n).$$

The numbers $P(\alpha_Q)$, as Q varies over $\mathcal{Q}_n$, form a multiset of algebraic numbers which is the union of Galois orbits for the discriminant $-24n + 1$ ring class field. Moreover, for each $Q \in \mathcal{Q}_n$ we have that $(24n - 1)P(\alpha_Q)$ is an algebraic integer.

Remark. Theorem 1.1 gives an algorithm for computing $p(n)$, as well as the polynomial

$$H_n(x) = x^{h(-24n+1)} - (24n - 1)p(n)x^{h(-24n+1)-1} + \cdots := \prod_{Q \in \mathcal{Q}_n} (x - P(\alpha_Q)) \in \mathbb{Q}[x]. \tag{1.6}$$

One may simply compute sufficiently precise approximations of the singular moduli $P(\alpha_Q)$. Recently the authors and Sutherland [13] have devised a completely different algorithm for computing $H_n(x)$ and hence $p(n)$. They use the theory of elliptic curves with CM and the structure of *isogeny volcanos* to obtain a CRT-based algorithm for computing $H_n(x)$. This algorithm only requires computing singular moduli modulo a small number of primes p which are predetermined in terms of n.

Remark. Using the theory of Poincaré series and identities and formulas for Kloosterman-type sums, one can use Theorem 1.1 to give a new (and longer) proof of the exact formula (1.1).

Example. We give an amusing proof of the fact that $p(1) = 1$. In this case, we have that $24n - 1 = 23$, and we use the $\Gamma_0(6)$-representatives

$$\mathcal{Q}_1 = \{Q_1, Q_2, Q_3\} = \{6x^2 + xy + y^2, 12x^2 + 13xy + 4y^2, 18x^2 + 25xy + 9y^2\}.$$

The corresponding CM points are

$$\alpha_{Q_1} := -\frac{1}{12} + \frac{1}{12} \cdot \sqrt{-23}, \qquad \alpha_{Q_2} := -\frac{13}{24} + \frac{1}{24} \cdot \sqrt{-23}, \qquad \alpha_{Q_3} := -\frac{25}{36} + \frac{1}{36} \cdot \sqrt{-23}.$$

Using the explicit Fourier expansion of $P(z)$, we find that $P(\alpha_{Q_3}) = \overline{P(\alpha_{Q_2})}$, and that

$$P(\alpha_{Q_1}) \sim 13.965486281 \qquad \text{and} \qquad P(\alpha_{Q_2}) \sim 4.517256859 - 3.097890591i.$$

By means of these numerics we can prove that

$$H_1(x) := \prod_{m=1}^{3} (x - P(\alpha_{Q_m})) = x^3 - 23x^2 + \frac{3592}{23}x - 419,$$

and this confirms that $p(1) = \frac{1}{23}\mathrm{Tr}(1) = 1$. If $\beta := 161529092 + 18648492\sqrt{69}$, then we have

$$P(\alpha_{Q_1}) = \frac{\beta^{1/3}}{138} + \frac{2782}{3\beta^{1/3}} + \frac{23}{3},$$
$$P(\alpha_{Q_2}) = -\frac{\beta^{1/3}}{276} - \frac{1391}{3\beta^{1/3}} + \frac{23}{3} - \frac{\sqrt{-3}}{2} \cdot \left(\frac{\beta^{1/3}}{138} - \frac{2782}{3\beta^{1/3}}\right).$$

The claim in Theorem 1.1 that $p(n) = \mathrm{Tr}(n)/(24n - 1)$ is an example of a general theorem (see Theorem 3.6) on "traces" of CM values of certain weak Maass forms. This result pertains to weight 0 weak Maass forms which are images under the Maass raising operator of weight -2 *harmonic Maass forms*. We apply this to $F(z)$ which is a weight -2 *weakly holomorphic modular form*, a meromorphic modular form whose poles are supported at cusps. Theorem 3.6 is a new result which adds to the extensive literature (for example, see [7, 11, 12, 16, 17, 18, 25, 33, 34]) inspired by Zagier's seminal paper [39] on "traces" of singular moduli.

To obtain this result, we employ the theory of theta lifts as in earlier work by Funke and the first author [11, 21]. This paper can be thought of as a natural extension of these earlier works. For the sake of brevity, we shall assume that the reader is familiar with the general framework of the theory of theta lifts and the Weil representation. Readers may consult the references in these papers such as [10, 30] and the text [9].

Here we use the Kudla-Millson theta functions to construct a new theta lift (see Corollary 3.4), a result which is of independent interest. The lift maps spaces of weight -2 harmonic weak Maass forms to spaces of weight $-1/2$ vector valued harmonic weak Maass forms for $\mathrm{Mp}_2(\mathbb{Z})$. In Section 2 we recall properties of these theta functions, and in Section 3 we construct the lift, and we then employ an argument of Katok and Sarnak to prove Theorem 3.6. For the sake of brevity and the application to $p(n)$, we chose to focus on the case of weight -2 harmonic Maass forms. In Section 3 we indicate how Corollary 3.4 and Theorem 3.6 extend to general weights. In particular, we illustrate how to define theta lifts for general weights using the Kudla-Millson kernel and Maass differential operators. Alfes [3] has recently fully worked out the details of these generalizations.

To complete the proof of Theorem 1.1, we must show that the values $P(\alpha_Q)$ are *algebraic numbers*[1] with *bounded denominators*. To prove these claims, we require the classical theory of complex multiplication, as well as new results which bound denominators of suitable singular moduli. For example, we bound the denominators of the singular moduli (see Lemma 4.7) of suitable nonholomorphic modular functions which contain the nonholomorphic Eisenstein series $E_2^*(z)$ as a factor. Theorem 4.5 is our general result which bounds the denominators of algebraic Maass singular moduli such as $P(\alpha_Q)$. These results are contained in Section 4. In Section 5 we give further examples, and we then conclude with some natural questions.

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2. The Kudla-Millson theta functions

We assume that the reader is familiar with basic facts about weak Maass forms (for example, see [11]). Using the Kudla-Millson theta functions, we will construct a theta lift from spaces of weight -2 harmonic weak Maass forms on $\Gamma_0(N)$ to weight $-1/2$ vector-valued harmonic Maass forms on $\mathrm{Mp}_2(\mathbb{Z})$. This lift will be crucial to the proof of Theorem 3.6 which interprets coefficients of holomorphic parts of these weight $-1/2$ forms as "traces" of the CM values of weight 0 weak Maass forms with Laplacian eigenvalue -2.

We begin by recalling some important facts about these theta functions in the setting of the present paper (see [29], [11]). Let N be a positive integer. Let (V, Q) be the quadratic space over $\mathbb{Q}$ of signature $(1, 2)$ given by the trace zero 2×2 matrices

$$V := \left\{ X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Q}) \right\}, \tag{2.1}$$

with the quadratic form $Q(X) = N \det(X)$. The corresponding bilinear form is $(X, Y) = -N\,\mathrm{tr}(XY)$. We let $G = \mathrm{Spin}(V)$, viewed as an algebraic group over $\mathbb{Q}$, and write $\bar{G}$ for its image in $\mathrm{SO}(V)$. We realize the associated symmetric space $\mathbb{D}$ as the Grassmannian of lines in $V(\mathbb{R})$ on which the quadratic form Q is positive definite:

$$\mathbb{D} \simeq \{z \subset V(\mathbb{R});\ \dim z = 1 \text{ and } Q|_z > 0\}.$$

The group $\mathrm{SL}_2(\mathbb{Q})$ acts on V by conjugation

$$g.X := gXg^{-1}$$

for $X \in V$ and $g \in \mathrm{SL}_2(\mathbb{Q})$. This gives rise to the isomorphisms $G \simeq \mathrm{SL}_2$ and $\bar{G} \simeq \mathrm{PSL}_2$.

[1] The algebraicity of such CM values follows from results of Shimura [37]. Here we give a new proof of this algebraicity. Our proof has the advantage that it bounds denominators.

The hermitian symmetric space $\mathbb{D}$ can be identified with the complex upper half plane $\mathbb{H}$ as follows: We choose as a base point $z_0 \in \mathbb{D}$ the line spanned by

$$X_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.2}$$

Its stabilizer in $G(\mathbb{R})$ is equal to $K = \mathrm{SO}(2)$. For $z = x + iy \in \mathbb{H}$, we choose $g_z \in G(\mathbb{R})$ such that $g_z i = z$. If we associate to z the vector

$$X(z) := g_z.X_0 = \frac{1}{y}\begin{pmatrix} -x & z\bar{z} \\ -1 & x \end{pmatrix} \in V(\mathbb{R}), \tag{2.3}$$

then $Q(X(z)) = N$, and $g.X(z) = X(gz)$ for $g \in G(\mathbb{R})$. We obtain the isomorphism

$$\mathbb{H} \longrightarrow \mathbb{D}, \quad z \mapsto g_z z_0 = \mathbb{R}X(z). \tag{2.4}$$

The minimal majorant of $(\ ,\ )$ associated to $z \in \mathbb{D}$ is given by $(X,X)_z = (X, X(z))^2 - (X,X)$.

Let $L \subset V$ be an even lattice and write L' for the dual lattice. Let Γ be a congruence subgroup of $\mathrm{Spin}(L)$, which takes L to itself and acts trivially on the discriminant group L'/L. We write $M = \Gamma\backslash\mathbb{D}$ for the associated modular curve.

Heegner points in M are given as follows. If $X \in V(\mathbb{Q})$ with $Q(X) > 0$, we put

$$\mathbb{D}_X = \mathrm{span}(X) \in \mathbb{D}. \tag{2.5}$$

The stabilizer $\Gamma_X \subset \Gamma$ of X is finite. We denote by $Z(X)$ the image of $\mathbb{D}_X$ in M, counted with multiplicity $\frac{1}{|\Gamma_X|}$. For $m \in \mathbb{Q}_{>0}$ and $h \in L'/L$, the group Γ acts on $L_{m,h} = \{X \in L + h;\ Q(X) = m\}$ with finitely many orbits. We define the *Heegner divisor* of discriminant (m,h) on M by

$$Z(m,h) = \sum_{X \in \Gamma\backslash L_{m,h}} Z(X). \tag{2.6}$$

2.1. **The Kudla-Millson function.** Kudla and Millson defined [29] a Schwartz function φ_{KM} on $V(\mathbb{R})$ valued in $\Omega^{1,1}(\mathbb{D})$, the differential forms on $\mathbb{D}$ of Hodge type $(1,1)$, by

$$\varphi_{KM}(X,z) = \left((X, X(z))^2 - \frac{1}{2\pi} \right) e^{-\pi (X,X)_z}\, \Omega, \tag{2.7}$$

where $\Omega = \frac{dx\wedge dy}{y^2} = \frac{i}{2}\frac{dz\wedge d\bar{z}}{y^2}$. We have $\varphi_{KM}(g.X, gz) = \varphi_{KM}(X,z)$ for $g \in G(\mathbb{R})$. We define

$$\varphi^0_{KM}(X,z) = e^{\pi(X,X)}\varphi_{KM}(X,z) = \left((X, X(z))^2 - \frac{1}{2\pi} \right) e^{-2\pi R(X,z)}\, \Omega, \tag{2.8}$$

where, following [28], we set

$$R(X,z) := \frac{1}{2}(X,X)_z - \frac{1}{2}(X,X) = \frac{1}{2N}(X,X(z))^2 - (X,X). \tag{2.9}$$

The quantity $R(X,z)$ is always non-negative. It equals 0 if and only if $z = \mathbb{D}_X$, that is, if X lies in the line generated by $X(z)$. Hence, for $X \neq 0$, this does not occur if $Q(X) \le 0$. Recall that for $Q(X) > 0$, the 2-form $\varphi^0_{KM}(X,z)$ is a Poincaré dual form for the Heegner point $\mathbb{D}_X$, while it is exact for $Q(X) < 0$.

2.2. **The Weil representation.** We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic two-fold cover of $\mathrm{SL}_2(\mathbb{R})$. The elements of this group are pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi(\tau)^2 = c\tau + d$. The multiplication is defined by

$$(M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

We denote the integral metaplectic group, the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map, by $\tilde{\Gamma} = \mathrm{Mp}_2(\mathbb{Z})$. It is well known that $\tilde{\Gamma}$ is generated by $T := (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1)$, and $S := (\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau})$. We let $\tilde{\Gamma}_\infty := \langle T \rangle \subset \tilde{\Gamma}$.

We denote the standard basis elements of the group ring $\mathbb{C}[L'/L]$ by $\mathfrak{e}_h$ for $h \in L'/L$. Recall (for example, see [5], [9]) that the Weil representation ρ_L associated with the discriminant form L'/L is the unitary representation of $\tilde{\Gamma}$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)(\mathfrak{e}_h) := e(h^2/2)\mathfrak{e}_h, \tag{2.10}$$

$$\rho_L(S)(\mathfrak{e}_h) := \frac{e(-\operatorname{sig}(L'/L)/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-(h, h'))\mathfrak{e}_{h'}. \tag{2.11}$$

Here $\operatorname{sig}(L'/L)$ denotes the signature of the discriminant form L'/L modulo 8.

For $k \in \frac{1}{2}\mathbb{Z}$, we let H_{k,ρ_L} be the space of $\mathbb{C}[L'/L]$-valued harmonic Maass forms of weight k for the group $\tilde{\Gamma}$ and the representation ρ_L. We write $M^!_{k,\rho_L}$ for the subspace of weakly holomorphic forms (see [11] for definitions). We note that H_{k,ρ_L} is denoted $H^+_{k,L}$ in [10].

2.3. **Theta series.** For $\tau = u + iv \in \mathbb{H}$ with $u, v \in \mathbb{R}$, we put $g'_\tau = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}$. We denote by ω the Weil representation of $\mathrm{Mp}_2(\mathbb{R})$ on the space of Schwartz functions $S(V(\mathbb{R}))$. For $h \in L'/L$ and $\varphi \in S(V(\mathbb{R}))$ of weight k with respect to the action of $\widetilde{\mathrm{SO}}(2, \mathbb{R}) \subset \mathrm{Mp}_2(\mathbb{R})$, we define the $\mathbb{C}[L'/L]$-valued theta function

$$\Theta_L(\tau, \varphi) = \sum_{X \in L'} v^{-k/2}(\omega(g'_\tau)\varphi)(X)\mathfrak{e}_X. \tag{2.12}$$

It is well known that $\Theta_L(\tau, \varphi)$ is a (in general non-holomorphic) modular form of weight k for $\tilde{\Gamma}$ with representation ρ_L. In particular, for the Kudla-Millson Schwartz function, we obtain, in the variable τ, that

$$\Theta_L(\tau, z, \varphi_{KM}) := \Theta_L(\tau, z, \varphi_{KM}(\cdot, z))$$

is a non-holomorphic modular form of weight $3/2$ for $\tilde{\Gamma}$ with representation ρ_L. In z it is a Γ-invariant $(1,1)$-form on $\mathbb{D}$.

We will also be interested in the standard Siegel theta function

$$\Theta_L(\tau, z, \varphi_S) := \Theta_L(\tau, z, \varphi_S(\cdot, z)),$$

where $\varphi_S(X, z) = e^{-\pi(X,X)_z}$ is the Gaussian on $V(\mathbb{R})$ associated to the majorant $(\cdot, \cdot)_z$. In τ, it is a non-holomorphic modular form of weight $-1/2$ for $\tilde{\Gamma}$ with representation ρ_L, while it is a Γ-invariant function in z. Explicitly we have

$$\Theta_L(\tau, z, \varphi_S) := v \sum_{X \in L'} e^{-2\pi v R(X,z)} e(Q(X)\tau)\mathfrak{e}_X.$$

2.4. **Differential operators.** Let $k \in \frac{1}{2}\mathbb{Z}$. Recall that the hyperbolic Laplace operator of weight k on functions in the variable τ on $\mathbb{H}$ is given by

$$\Delta_k = \Delta_{k,\tau} = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{2.13}$$

The Maass raising and lowering operators on non-holomorphic modular forms of weight k are defined as the differential operators

$$R_k = R_{k,\tau} = 2i\frac{\partial}{\partial \tau} + kv^{-1}, \tag{2.14}$$

$$L_k = L_{k,\tau} = -2iv^2\frac{\partial}{\partial \bar{\tau}}. \tag{2.15}$$

The raising operator R_k raises the weight of an automorphic form by 2, while L_k lowers it by 2. The Laplacian Δ_k can be expressed in terms of R_k and L_k by

$$-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k. \tag{2.16}$$

We let ∂, $\bar{\partial}$ and d be the usual differentials on $\mathbb{D}$. We set $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$, so that $dd^c = -\frac{1}{2\pi i}\partial\bar{\partial}$. According to [10, Theorem 4.4], the Kudla-Millson theta function and the Siegel theta function are related by the identity

$$L_{3/2,\tau}\Theta_L(\tau, z, \varphi_{KM}) = -dd^c\Theta_L(\tau, z, \varphi_S) = \frac{1}{4\pi}\Delta_{0,z}\Theta_L(\tau, z, \varphi_S) \cdot \Omega. \tag{2.17}$$

Moreover, by [9, Prop. 4.5], it follows that the Laplace operators on the Kudla-Millson theta kernel are related by

$$\Delta_{3/2,\tau}\Theta_L(\tau, z, \varphi_{KM}) = \frac{1}{4}\Delta_{0,z}\Theta_L(\tau, z, \varphi_{KM}). \tag{2.18}$$

2.5. **A lattice related to $\Gamma_0(N)$.** For the rest of this section, we let L be the even lattice

$$L := \left\{ \begin{pmatrix} b & a/N \\ c & -b \end{pmatrix} : \quad a, b, c \in \mathbb{Z} \right\}. \tag{2.19}$$

The dual lattice is given by

$$L' := \left\{ \begin{pmatrix} b/2N & a/N \\ c & -b/2N \end{pmatrix} : \quad a, b, c \in \mathbb{Z} \right\}. \tag{2.20}$$

We identify L'/L with $\mathbb{Z}/2N\mathbb{Z}$, and the quadratic form on L'/L is identified with the quadratic form $x \mapsto -x^2$ on $\mathbb{Z}/2N\mathbb{Z}$. The level of L is $4N$. The group $\Gamma_0(N)$ is contained in $\mathrm{Spin}(L)$ and acts trivially on L'/L. We denote by ℓ, ℓ' the primitive isotropic vectors

$$\ell = \begin{pmatrix} 0 & 1/N \\ 0 & 0 \end{pmatrix}, \qquad \ell' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

in L, and write K for the one-dimensional lattice $\mathbb{Z}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \subset L$. We have $L = K + \mathbb{Z}\ell + \mathbb{Z}\ell'$ and $L'/L \cong K'/K$. For $\lambda \in V(\mathbb{R})$ and $z = x + iy \in \mathbb{H}$ we let $\lambda_z \in V(\mathbb{R})$ be the orthogonal projection of λ to $\mathbb{R}X(z)$. It is easily checked that

$$\ell_z = \frac{1}{2Ny}X(z) \quad \text{and} \quad \ell_z^2 = \frac{1}{2Ny^2}.$$

Following [5] we define a theta function for the smaller lattice K as follows. For $\alpha, \beta \in K \otimes \mathbb{R}$ and $h \in K'/K$ we put

$$\xi_h(\tau, \alpha, \beta) = \sqrt{v} \sum_{\lambda \in K+h} e\big(Q(\lambda+\beta)\bar{\tau} - (\lambda + \beta/2, \alpha)\big),$$
$$\Xi_K(\tau, \alpha, \beta) = \sum_{h \in K'/K} \xi_h(\tau, \alpha, \beta)\mathfrak{e}_h.$$

According to [5, Theorem 4.1], the function $\Xi_K(\tau, \alpha, \beta)$ transforms like a non-holomorphic modular form of weight $-1/2$ for $\tilde{\Gamma}$ with representation ρ_K. For $z \in \mathbb{H}$ we put $\mu(z) = \left(\begin{smallmatrix} -x & 0 \\ 0 & x \end{smallmatrix}\right) \in K \otimes \mathbb{R}$. Theorem 5.2 of [5] allows us to rewrite Θ_L as a Poincaré series:

Proposition 2.1. *We have that*

$$\Theta_L(\tau, z, \varphi_S) = \frac{1}{\sqrt{2\ell_z^2}} \cdot \Xi_K(\tau, 0, 0) + \frac{1}{2\sqrt{2\ell_z^2}} \sum_{n=1}^{\infty} \sum_{\gamma \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} \left[\exp\left(-\frac{\pi n^2}{2v\ell_z^2}\right) \Xi(\tau, n\mu(z), 0)\right] |_{-1/2, \rho_K}\, \gamma.$$

2.6. **Poincaré series.** We now recall some facts on Poincaré series with exponential growth at the cusps. Let $k \in \frac{1}{2}\mathbb{Z}$. Let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ be the usual Whittaker functions (see p. 190 of [1]). For convenience, we put for $s \in \mathbb{C}$ and $y \in \mathbb{R}_{>0}$:

$$\mathcal{M}_{s,k}(y) = y^{-k/2} M_{-k/2,\, s-1/2}(y). \tag{2.21}$$

For $s = k/2$, we have the identity

$$\mathcal{M}_{k/2,k}(y) = y^{-k/2} M_{-k/2,\, k/2-1/2}(y) = e^{y/2}. \tag{2.22}$$

Let Γ_∞ be the subgroup of $\Gamma = \Gamma_0(N)$ generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. If k is integral, and m is a positive integer, we define the Poincaré series

$$F_m(z, s, k) = \frac{1}{2\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} [\mathcal{M}_{s,k}(4\pi m y) e(-mx)] \,|_k\, \gamma, \tag{2.23}$$

where $z = x + iy \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\Re(s) > 1$ (for example, see [9]). This Poincaré series converges for $\Re(s) > 1$, and it is an eigenfunction of Δ_k with eigenvalue $s(1-s) + (k^2 - 2k)/4$. Its specialization at $s_0 = 1 - k/2$ is a harmonic Maass form [9, Proposition 1.10]. Its principal part at the cusp ∞ is given by $q^{-m} + C$ for some constant $C \in \mathbb{C}$, and the principal parts at the other cusps are constant.

The next proposition describes the images of these series under the Maass raising operator.

Proposition 2.2. *We have that*

$$\frac{1}{4\pi m} R_k F_m(z, s, k) = (s + k/2) F_m(z, s, k+2).$$

Proof. Since R_k commutes with the slash operator, it suffices to show that

$$\frac{1}{4\pi m} R_k \mathcal{M}_{s,k}(4\pi m y) e(-mx) = (s + k/2)\mathcal{M}_{s,k+2}(4\pi m y) e(-mx).$$

This identity follows from (13.4.10) and (13.1.32) in [1]. □

We also define $\mathbb{C}[L'/L]$-valued analogues of these series. Let $h \in L'/L$, and let $m \in \mathbb{Z} - Q(h)$ be positive. For $k \in \mathbb{Z} - \frac{1}{2}$ we define

$$\mathcal{F}_{m,h}(\tau, s, k) = \frac{1}{2\Gamma(2s)} \sum_{\gamma \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} [\mathcal{M}_{s,k}(4\pi m y) e(-mx)\mathfrak{e}_h] \mid_{k,\rho_L} \gamma. \tag{2.24}$$

The series converges for $\Re(s) > 1$ and defines a weak Maass form of weight k for $\tilde{\Gamma}$ with representation ρ_L. The special value at $s = 1 - k/2$ is harmonic. If $k \in \mathbb{Z} - \frac{1}{2}$, it has the principal part $q^{-m}\mathfrak{e}_h + q^{-m}\mathfrak{e}_{-h} + C$ for some constant $C \in \mathbb{C}[L'/L]$.

3. The theta lift and "traces" of CM values of weak Maass forms

Here we construct the theta lift which we then use to prove that the coefficients of certain weight $-1/2$ harmonic weak Maass forms are "traces" of CM values of weak Maass forms.

3.1. **A theta lift.** Let L be the lattice (2.19). For $k \in \frac{1}{2}\mathbb{Z}$, we let $H_k(N)$ denote the space of harmonic Maass forms of weight k for $\Gamma := \Gamma_0(N)$. We let $H_k^\infty(N)$ denote the subspace of $H_k(N)$ consisting of those harmonic Maass forms whose principal parts at all cusps other than ∞ are constant. We write $M_k^!(N)$ for the subspace of weakly holomorphic forms in $H_k(N)$, and we put $M_k^{!,\infty}(N) = M_k^!(N) \cap H_k^\infty(N)$.

For a weak Maass form f of weight -2 for Γ we define

$$\Lambda(\tau, f) = L_{3/2,\tau} \int_M (R_{-2,z} f(z)) \Theta_L(\tau, z, \varphi_{KM}). \tag{3.1}$$

According to [11, Proposition 4.1], the Kudla-Millson theta kernel has exponential decay as $O(e^{-Cy^2})$ for $y \to \infty$ at all cusps of Γ with some constant $C > 0$. Therefore the theta integral converges absolutely. It defines a $\mathbb{C}[L'/L]$-valued function on $\mathbb{H}$ that transforms like a non-holomorphic modular form of weight $-1/2$ for $\tilde{\Gamma}$. We denote by $\Lambda_h(\tau, f)$ the components of the lift $\Lambda(\tau, f)$ with respect to the standard basis $(\mathfrak{e}_h)_h$ of $\mathbb{C}[L'/L]$.

The group $\mathrm{O}(L'/L)$ can be identified with the group generated by the Atkin-Lehner involutions. It acts on weak Maass forms for Γ by the Petersson slash operator. It also acts on $\mathbb{C}[L'/L]$-valued modular forms with respect to the Weil representation ρ_L through the natural action on $\mathbb{C}[L'/L]$. The following proposition, which is easily checked, shows that the theta lift is equivariant with respect to the action of $\mathrm{O}(L'/L)$.

Proposition 3.1. *For $\gamma \in \mathrm{O}(L'/L)$ and $h \in L'/L$, we have*

$$\Lambda_{\gamma h}(\tau, f) = \Lambda_h(\tau, f \mid_{-2} \gamma^{-1}).$$

Proposition 3.2. *If f is an eigenform of the Laplacian $\Delta_{-2,z}$ with eigenvalue λ, then $\Lambda(\tau, f)$ is an eigenform of $\Delta_{-1/2,\tau}$ with eigenvalue $\lambda/4$.*

Proof. The result follows from (2.18) and the fact that

$$R_k \Delta_k = (\Delta_{k+2} - k) R_k, \qquad \Delta_{k-2} L_k = L_k(\Delta_k + 2 - k). \tag{3.2}$$

We may use symmetry of the Laplacian on the functions in the integral because of the very rapid decay of the Kudla-Millson theta kernel [11, Proposition 4.1]. $\square$

We now compute the lift of the Poincaré series.

Theorem 3.3. *If m is a positive integer, then we have*
$$\Lambda\big(\tau, F_m(z,s,-2)\big) = \frac{2^{2-s}\sqrt{\pi}Ns(1-s)}{\Gamma(\frac{s}{2}-\frac{1}{2})}\sum_{n|m} n\cdot \mathcal{F}_{\frac{m^2}{4Nn^2},\frac{m}{n}}(\tau,\tfrac{s}{2}+\tfrac{1}{4},-\tfrac{1}{2}).$$

Proof. By definition we have
$$\Lambda\big(\tau, F_m(z,s,-2)\big) = L_{3/2,\tau}\int_M (R_{-2,z}F_m(z,s,-2))\Theta_L(\tau,z,\varphi_{KM}).$$
Employing Propsition 2.2 and (2.17), we see that this is equal to
$$m(s-1)\int_M F_m(z,s,0)\Delta_{0,z}\Theta_L(\tau,z,\varphi_S)\Omega.$$
Using definition (2.23), we find, by the usual unfolding argument, that
$$\Lambda\big(\tau, F_m(z,s,-2)\big) = \frac{m(s-1)}{\Gamma(2s)}\int_{\Gamma_\infty\backslash\mathbb{H}} \mathcal{M}_{s,0}(4\pi my)e(-mx)\Delta_{0,z}\Theta_L(\tau,z,\varphi_S)\Omega.$$
By Proposition 2.1, we may replace $\Delta_{0,z}\Theta_L(\tau,z,\varphi_S)$ by $\Delta_{0,z}\tilde{\Theta}_L(\tau,z,\varphi_S)$, where
$$\tilde{\Theta}_L(\tau,z,\varphi_S) = \frac{1}{2\sqrt{2\ell_z^2}}\sum_{n=1}^\infty\sum_{\gamma\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}}\left[\exp\left(-\frac{\pi n^2}{2v\ell_z^2}\right)\Xi(\tau,n\mu(z),0)\right]|_{-1/2,\rho_K}\,\gamma.$$
Recall that $\ell_z^2 = \frac{1}{2Ny^2}$. The function $\tilde{\Theta}_L(\tau,z,\varphi_S)$ and its partial derivatives have square exponential decay as $y\to\infty$. Therefore, for $\Re(s)$ large, we may move the Laplace operator to the Poincaré series and obtain
$$\begin{aligned}\Lambda\big(\tau, F_m(z,s,-2)\big) &= \frac{m(s-1)}{\Gamma(2s)}\int_{\Gamma_\infty\backslash\mathbb{H}}\big(\Delta_{0,z}\mathcal{M}_{s,0}(4\pi my)e(-mx)\big)\tilde{\Theta}_L(\tau,z,\varphi_S)\Omega\\ &= -\frac{ms(s-1)^2}{\Gamma(2s)}\int_{\Gamma_\infty\backslash\mathbb{H}}\mathcal{M}_{s,0}(4\pi my)e(-mx)\tilde{\Theta}_L(\tau,z,\varphi_S)\Omega\\ &= -\frac{ms(s-1)^2}{\Gamma(2s)}\sum_{n=1}^\infty\sum_{\gamma\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}} I(\tau,s,m,n)\,|_{-1/2,\rho_K}\,\gamma,\end{aligned}\tag{3.3}$$
where
$$I(\tau,s,m,n) = \int_{y=0}^\infty\int_{x=0}^1 \mathcal{M}_{s,0}(4\pi my)e(-mx)\frac{1}{2\sqrt{2\ell_z^2}}\exp\left(-\frac{\pi n^2}{2v\ell_z^2}\right)\Xi(\tau,n\mu(z),0)\,\frac{dx\,dy}{y^2}.$$
If we use the fact that $K' = \mathbb{Z}\left(\begin{smallmatrix}1/2N & 0\\ 0 & -1/2N\end{smallmatrix}\right)$, and identify $K'/K\cong\mathbb{Z}/2N\mathbb{Z}$, then we have
$$\Xi(\tau,n\mu(z),0) = \sqrt{v}\sum_{b\in\mathbb{Z}} e\left(-\frac{b^2}{4N}\bar{\tau}-nbx\right)\mathfrak{e}_b.$$
Inserting this in the formula for $I(\tau,s,m,n)$, and by integrating over x, we see that $I(\tau,s,m,n)$ vanishes when $n\nmid m$. If $n\mid m$, then only the summand for $b=-m/n$ occurs, and so
$$I(\tau,s,m,n) = \frac{\sqrt{Nv}}{2}\int_0^\infty \mathcal{M}_{s,0}(4\pi my)\exp\left(-\frac{\pi Nn^2y^2}{v}\right)\frac{dy}{y}e\left(-\frac{m^2}{4Nn^2}\bar{\tau}\right)\mathfrak{e}_{-m/n}.$$

To compute this last integral, we note that

$$\mathcal{M}_{s,0}(4\pi my) = M_{0,s-1/2}(4\pi my) = 2^{2s-1}\Gamma(s+1/2)\sqrt{4\pi my}\cdot I_{s-1/2}(2\pi my)$$

(for example, see (13.6.3) in [1]). Substituting $t = y^2$ in the integral, we obtain

$$\begin{aligned}
&\int_0^\infty \mathcal{M}_{s,0}(4\pi my)\exp\left(-\frac{\pi N n^2 y^2}{v}\right)\frac{dy}{y}\\
&= 2^{2s-1}\Gamma(s+1/2)\int_0^\infty \sqrt{4\pi my} I_{s-1/2}(2\pi my)\exp\left(-\frac{\pi N n^2 y^2}{v}\right)\frac{dy}{y}\\
&= 2^{2s-1}\Gamma(s+1/2)\sqrt{\pi m}\int_0^\infty I_{s-1/2}(2\pi m\sqrt{t})\exp\left(-\frac{\pi N n^2 t}{v}\right)t^{-3/4}\,dt.
\end{aligned}$$

The latter integral is a Laplace transform which is computed in [19] (see (20) on p.197). Inserting the evaluation, we obtain

$$\begin{aligned}
&\int_0^\infty \mathcal{M}_{s,0}(4\pi my)\exp\left(-\frac{\pi N n^2 y^2}{v}\right)\frac{dy}{y}\\
&= 2^{2s-1}\Gamma(s/2)\left(\frac{Nn^2}{\pi m^2 v}\right)^{1/4} M_{1/4,s/2-1/4}\left(\frac{\pi m^2 v}{Nn^2}\right)\exp\left(\frac{\pi m^2 v}{2Nn^2}\right)\\
&= 2^{2s-1}\Gamma(s/2)\left(\frac{Nn^2}{\pi m^2 v}\right)^{1/2} \mathcal{M}_{s/2+1/4,-1/2}\left(\frac{\pi m^2 v}{Nn^2}\right)\exp\left(\frac{\pi m^2 v}{2Nn^2}\right).
\end{aligned}$$

Consequently, we have in the case $n \mid m$ that

$$I(\tau, s, m, n) = \frac{2^{2s-2}Nn}{\sqrt{\pi m}}\Gamma(s/2)\,\mathcal{M}_{s/2+1/4,-1/2}\left(\frac{\pi m^2 v}{Nn^2}\right)e\left(-\frac{m^2}{4Nn^2}u\right)\mathfrak{e}_{-m/n}.$$

Substituting this in (3.3), we find

$$\Lambda\big(\tau, F_m(z,s,-2)\big) = \frac{2^{2-s}\sqrt{\pi}Ns(1-s)}{\Gamma(\frac{s}{2}-\frac{1}{2})}\sum_{n\mid m} n\cdot\mathcal{F}_{\frac{m^2}{4Nn^2},-\frac{m}{n}}(\tau, \tfrac{s}{2}+\tfrac{1}{4}, -\tfrac{1}{2}).$$

Since $\mathcal{F}_{m,h}(\tau,s,-1/2) = \mathcal{F}_{m,-h}(\tau,s,-1/2)$, this concludes the proof of the theorem. □

Corollary 3.4. *Assume that N is squarefree. If $f \in H_{-2}(N)$ is a harmonic Maass form of weight -2 for $\Gamma_0(N)$, then $\Lambda(\tau, f)$ belongs to $H_{-1/2,\rho_L}$. In particular, we have*

$$\Lambda\big(\tau, F_m(z,2,-2)\big) = -2N\sum_{n\mid m} n\cdot\mathcal{F}_{\frac{m^2}{4Nn^2},\frac{m}{n}}(\tau, \tfrac{5}{4}, -\tfrac{1}{2}).$$

Proof. The formula for the image of the Poincaré series $F_m(z,2,-2)$ is a direct consequence of Theorem 3.3. These Poincaré series for $m \in \mathbb{Z}_{>0}$ span the subspace $H^\infty_{-2}(N) \subset H_{-2}(N)$ of harmonic Maass forms whose principal parts at all cusps other than ∞ are constant. Consequently, the image of $H^\infty_{-2}(N)$ is contained in $H_{-1/2,\rho_L}$.

Since N is squarefree, the group $\mathrm{O}(L'/L)$ of Atkin-Lehner involutions acts transitively on the cusps of $\Gamma_0(N)$. Hence, we have

$$H_{-2}(N) = \sum_{\gamma\in\mathrm{O}(L'/L)} \gamma H^\infty_{-2}(N).$$

Using Proposition 3.1, we see that the whole space $H_{-2}(N)$ is mapped to $H_{-1/2,\rho_L}$. □

Theorem 3.5. *Assume that N is squarefree. The theta lift Λ maps weakly holomorphic modular forms to weakly holomorphic modular forms.*

Proof. First, let $F \in M^{!,\infty}_{-2}(N)$ and denote the Fourier expansion of F at the cusp ∞ by

$$F(z) = \sum_{m\in\mathbb{Z}} a_F(m)e(mz).$$

We may write F as a linear combination of Poincaré series as

$$F(z) = \sum_{m>0} a_F(-m)F_m(z, 2, -2).$$

According to Corollary 3.4, we find that the principal part of $\Lambda(F)$ is equal to

$$-2N \sum_{m>0} a_F(-m) \sum_{n|m} n \cdot e\left(-\frac{m^2}{4Nn^2}z\right)\left(\mathfrak{e}_{m/n} + \mathfrak{e}_{-m/n}\right).$$

We now use the pairing $\{\cdot,\cdot\}$ of $H_{-1/2,\rho_L}$ with the space of cusp forms $S_{5/2,\bar\rho_L}$ (see [10], Proposition 3.5) to prove that $\Lambda(F)$ is weakly holomorphic. We need to show that $\{\Lambda(F), g\} = 0$ for every cusp form $g \in S_{5/2,\bar\rho_L}$. If we denote the coefficients of g by $b(M, h)$, we have

$$\begin{aligned}\{\Lambda(F), g\} &= -4N \sum_{m>0} a_F(-m) \sum_{n|m} n \cdot b\left(\frac{m^2}{4Nn^2}, \frac{m}{n}\right)\\ &= -4N\{F, \mathcal{S}_1(g)\}.\end{aligned}$$

Here $\mathcal{S}_1(g) \in S_4(N)$ denotes the (first) Shimura lift of g as in [38]. Since F is weakly holomorphic, the latter quantity vanishes.

Since N is squarefree, one obtains the result for the full space $M^!_{-2}(N)$ using the action of $\mathrm{O}(L'/L)$ as in the proof of Corollary 3.4. □

Remark. The statements of Corollary 3.4 and Theorem 3.5 also hold for general (not necessarily squarefree) level N. In this case one can use the action of $G(\mathbb{Q})$ to move an arbitrary cusp to ∞. A straightforward generalization of Theorem 3.3 to arbitrary even lattices in V implies the more general statement. For brevity we chose to not include the technical details in this paper.

Remark. We have that $\Lambda : H_{-2}(N) \to H_{-1/2,\rho_L}$ is injective on the subspace of forms which are invariant under the Fricke involution W_N. It vanishes identically on the subspace of forms which have eigenvalue -1 under W_N.

Theorem 3.6. *Let $f \in H_{-2}(N)$ and put $\partial f := \frac{1}{4\pi}R_{-2,z}f$. For $m \in \mathbb{Q}_{>0}$ and $h \in L'/L$ the (m, h)-th Fourier coefficient of the holomorphic part of $\Lambda(\tau, f)$ is equal to*

$$\mathrm{tr}_f(m, h) = -\frac{1}{2m} \sum_{z\in Z(m,h)} \partial f(z).$$

Proof. Inserting the definition of the theta lifting and using (2.17), we have

$$\begin{aligned}\Lambda\big(\tau, f\big) &= 4\pi L_{3/2,\tau} \int_M \partial f(z) \Theta_L(\tau, z, \varphi_{KM}) \\ &= \int_M \partial f(z) \Delta_{0,z} \Theta_L(\tau, z, \varphi_S) \Omega.\end{aligned}$$

For $X \in V(\mathbb{R})$ and $z \in \mathbb{D}$ we define $\varphi_S^0(X, z) = e^{2\pi Q(X)} \varphi_S(X, z)$. Then the Fourier expansion of the Siegel theta function in the variable τ is given by

$$\Theta_L(\tau, z, \varphi_S) = \sum_{X \in L'} \varphi_S^0(\sqrt{v}X, z) q^{Q(X)} \mathfrak{e}_X. \tag{3.4}$$

For $m \in \mathbb{Q}_{>0}$ and $h \in L'/L$, we put $L_{m,h} = \{X \in L + h;\ Q(X) = m\}$. The group Γ acts on $L_{m,h}$ with finitely many orbits. We write $C(m, h)$ for the (m, h)-th Fourier coefficient of the holomorphic part of $\Lambda(\tau, f)$. Using (3.4), we see that

$$C(m, h) = \int_M \partial f(z) \Delta_{0,z} \sum_{X \in L_{m,h}} \varphi_S^0(\sqrt{v}X, z) \Omega.$$

According to [11, Proposition 3.2], for $Q(X) > 0$ the function $\varphi_S^0(X, z)$ has square exponential decay as $y \to \infty$. This implies that we may move the Laplacian in the integral to the function ∂f. Since $\Delta_0 \partial f = -2\partial f$, we see that

$$C(m, h) = -2 \int_M \partial f(z) \sum_{X \in L_{m,h}} \varphi_S^0(\sqrt{v}X, z) \Omega.$$

Using the usual unfolding argument, we obtain

$$C(m, h) = -2 \sum_{X \in \Gamma \backslash L_{m,h}} \frac{1}{|\bar{\Gamma}_X|} \int_{\mathbb{D}} \partial f(z) \varphi_S^0(\sqrt{v}X, z) \Omega. \tag{3.5}$$

It is convenient to rewrite the integral over $\mathbb{D}$ as an integral over $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$. If we normalize the Haar measure such that the maximal compact subgroup $K = \mathrm{SO}(2)$ has volume 1, we have

$$I(X) := \int_{\mathbb{D}} \partial f(z) \varphi_S^0(\sqrt{v}X, z)\, \Omega = \int_{G(\mathbb{R})} \partial f(gi) \varphi_S^0(\sqrt{v}X, gi)\, dg.$$

Following Katok and Sarnak [26, pp.208], we choose $g_1 \in G(\mathbb{R})$ such that $g_1^{-1}.X = \sqrt{m/N} X_0$. Then $g_1(i)$ is the Heegner point in $\mathbb{H}$ corresponding to $\mathbb{D}_X$. Because of the invariance of the Haar measure we have

$$I(X) = \int_{G(\mathbb{R})} \partial f(g_1 gi) \varphi_S^0 \left(\sqrt{\frac{mv}{N}} X_0, gi \right) dg.$$

Putting $f_1(g) = \partial f(g_1 gi)$ and using the Cartan decomposition $G(\mathbb{R}) = KA^+K$, we obtain

$$I(X) = 4\pi \int_1^\infty \varphi_S^0 \left(\sqrt{\frac{mv}{N}} \alpha(a)^{-1} X_0, i \right) \int_K \int_K f_1(k_1 \alpha(a) k_2)\, dk_1\, dk_2\, \frac{a^2 - a^{-2}}{2} \frac{da}{a}.$$

Here we have set $\alpha(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in A$ for $a \in \mathbb{R}_{>0}$. The function

$$\tilde{f}_1(g) = \int_K \int_K f_1(k_1 g k_2)\, dk_1\, dk_2$$

has the same eigenvalue as $\partial f(gi)$ under the Casimir operator, and it is K-binvariant. Using the uniqueness of spherical functions (see e.g. [31], Chapters 5.4, 7.2, and 10.3), we find that

$$\tilde{f}_1(g) = f_1(1) \cdot \omega_\lambda(g) = \partial f(\mathbb{D}_X) \cdot \omega_\lambda(g),$$

where $\omega_\lambda(g)$ is the standard spherical function with eigenvalue $\lambda = -2$. Consequently,

$$I(X) = \partial f(\mathbb{D}_X) \cdot Y_\lambda\left(\sqrt{\frac{mv}{N}}\right),$$

where

$$Y_\lambda(t) = 4\pi \int_1^\infty \varphi_S^0\left(t\alpha(a)^{-1} X_0,\, i\right) \omega_\lambda(\alpha(a))\, \frac{a^2 - a^{-2}}{2}\, \frac{da}{a}.$$

By means of the identities $\omega_{-2}(\alpha(a)) = \frac{a^2+a^{-2}}{2}$ and

$$\varphi_S^0\left(t\alpha(a)^{-1} X_0,\, i\right) = v e^{-\pi N t^2 (a^2 - a^{-2})^2},$$

we obtain

$$Y_\lambda(t) = 2\pi v \int_0^\infty e^{-4\pi N t^2 \sinh(r)^2} \cosh(r) \sinh(r)\, dr = \frac{v}{4Nt^2},$$

and therefore $Y_\lambda(\sqrt{mv/N}) = \frac{1}{4m}$. Inserting this into (3.5), we obtain the assertion. $\square$

Remark. In an earlier version of this paper, the authors speculated on the generalization of the results in this paper. Alfes [3] has recently rigorously obtained this generalization. We briefly describe it here. We can define similar theta liftings for other weights. For $k \in \mathbb{Z}_{\geq 0}$ odd we define a theta lifting of weak Maass forms of weight $-2k$ to weak Maass forms of weight $1/2 - k$ by

$$\Lambda(\tau, f, -2k) = (L_\tau)^{\frac{k+1}{2}} \int_M (R_z^k f)(z) \Theta_L(\tau, z, \varphi_{KM}). \tag{3.6}$$

In view of Proposition 3.2 and identity (3.2), the authors speculated that this lifting takes $H_{-2k}(N)$ to $H_{1/2-k,\rho_L}$ and maps weakly holomorphic forms to weakly holomorphic forms. Analogously, for $k \in \mathbb{Z}_{\geq 0}$ even we define a theta lifting of weak Maass forms of weight $-2k$ to weak Maass forms of weight $3/2 + k$ by

$$\Lambda(\tau, f, -2k) = (R_\tau)^{\frac{k}{2}} \int_M (R_z^k f)(z) \Theta_L(\tau, z, \varphi_{KM}). \tag{3.7}$$

This lifting takes $H_{-2k}(N)$ to $H_{3/2+k,\rho_L}$. These maps give interpretations in terms of theta lifts of the results discussed in [39, §9]. Moreover, they yield generalizations to congruence subgroups, arbitrary weights, and to harmonic Maass forms at the same time. The lifting considered in [11] is (3.7) in the case $k = 0$. The lifting considered in the present paper is (3.6) for $k = 1$.

3.2. **The case of the partition function.** Here we derive the formula for the partition function stated in Theorem 1.1 from Theorem 3.6 and Corollary 3.4. We consider the theta lift of Section 3.1 in the special case when $N = 6$. We identify the discriminant form L'/L with $\mathbb{Z}/12\mathbb{Z}$ together with the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form $r \mapsto -r^2/24$.

The function $\eta(\tau)^{-1}$ can be viewed as a component of a vector valued modular form in $M^!_{-1/2,\rho_L}$ as follows. (Note that the latter space is isomorphic to the space $J^{weak}_{0,6}$ of weak Jacobi forms of weight 0 and index 6.) We define

$$G(\tau) := \sum_{r\in\mathbb{Z}/12\mathbb{Z}} \chi_{12}(r)\eta(\tau)^{-1}\mathfrak{e}_r.$$

Here χ_{12} is the Kronecker character $\left(\frac{12}{\cdot}\right)$. Using the transformation law of the eta-function under $\tau \mapsto \tau+1$ and $\tau \mapsto -1/\tau$, it is easily checked that $G \in M^!_{-1/2,\rho_L}$. The principal part of G is equal to $q^{-1/24}(\mathfrak{e}_1 - \mathfrak{e}_5 - \mathfrak{e}_7 + \mathfrak{e}_{11})$.

On the other hand, G can be obtained as a theta lift. Let $F \in M^!_{-2}(6)$ be the function defined in (1.3). It is invariant under the Fricke involution W_6, and under the Atkin-Lehner involution W_3 it is taken to its negative. Hence, in terms of Poincaré series we have

$$F = F_1(\cdot,2,-2) - F_1(\cdot,2,-2) \mid W_2 - F_1(\cdot,2,-2) \mid W_3 + F_1(\cdot,2,-2) \mid W_6.$$

The function P is given by $\frac{1}{4\pi}R_{-2}(F)$. Using Corollary 3.4 and Proposition 3.1, we see that $\Lambda(\tau,F)$ is an element of $M^!_{-1/2,\rho_L}$ with principal part $-4Nq^{-1/24}(\mathfrak{e}_1 - \mathfrak{e}_5 - \mathfrak{e}_7 + \mathfrak{e}_{11})$. Consequently, we have

$$G = -\frac{1}{4N}\cdot\Lambda(\tau,F).$$

Now Theorem 3.6 tells us that for any positive integer n the coefficient of G with index $(\frac{24n-1}{24},1)$ is equal to

$$\frac{3}{N(24n-1)}\sum_{z\in Z\left(\frac{24n-1}{24},1\right)} P(z) = \frac{1}{24n-1}\sum_{Q\in\mathcal{Q}_n} P(\alpha_Q).$$

On the other hand, this coefficient is equal to $p(n)$ because

$$\frac{q^{\frac{1}{24}}}{\eta(z)} = \prod_{n=1}^{\infty}\frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)q^n.$$

4. Complex Multiplication and singular moduli

We have proved that $p(n) = \mathrm{Tr}(n)/(24n-1)$. To complete the proof of Theorem 1.1, we require results from the theory of complex multiplication, and some new general results which bound the denominators of singular moduli.

4.1. **Singular moduli for $j(z)$.** We first recall classical facts about Klein's j-function

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots. \tag{4.1}$$

A point $\tau \in \mathbb{H}$ is a CM point if it is a root of a quadratic equation over $\mathbb{Z}$. The *singular moduli* for $j(z)$, its values at such CM points, play a central role in the theory of complex multiplication. The following classical theorem (for example, see [6, 14]) summarizes some of the most important properties of these numbers.

Theorem 4.1. *Suppose that $Q = ax^2 + bxy + cy^2$ is a primitive positive definite binary quadratic form with discriminant $D = b^2 - 4ac < 0$, and let $\alpha_Q \in \mathbb{H}$ be the point for which $Q(\alpha_Q, 1) = 0$. Then the following are true:*

1. *We have that $j(\alpha_Q)$ is an algebraic integer, and its minimal polynomial has degree $h(D)$, the class number of discriminant D primitive positive definite binary quadratic forms.*
2. *The Galois orbit of $j(\alpha_Q)$ consists of the $j(z)$-singular moduli associated to the $h(D)$ classes of discriminant D forms.*
3. *If $K = \mathbb{Q}(\sqrt{D})$, then the discriminant D singular moduli are conjugate to one another over K. Moreover, $K(j(\alpha_Q))$ is the discriminant $-D$ Hilbert class field of K.*

Theorem 4.1 and the properties of the weight 2 nonholomorphic Eisenstein series

$$E_2^*(z) := -\frac{3}{\pi y} + E_2(z) = 1 - \frac{3}{\pi y} - 24 \sum_{n=1}^{\infty} \sum_{d|n} dq^n \tag{4.2}$$

will play a central role in the proof of Theorem 1.1.

4.2. **Bounding the denominators.** Here we show that singular moduli like $6D \cdot P(\alpha_Q)$ are algebraic *integers*, where $-D$ denotes the discriminant of Q. We first introduce notation. For a positive integer N, we let ζ_N denote a primitive N-th root of unity. For a discriminant $-D < 0$ and $r \in \mathbb{Z}$ with $r^2 \equiv -D \pmod{4N}$ we let $\mathcal{Q}_{D,r,N}$ denote the set of positive definite integral binary quadratic forms $[a, b, c]$ of discriminant $-D$ with $N \mid a$ and $b \equiv r \pmod{2N}$. This notation is not to be confused with $\mathcal{Q}_n$ introduced earlier. This set-up is more natural in this section. For $Q = [a, b, c] \in \mathcal{Q}_{D,r,N}$ we let $\alpha_Q = \frac{-b+\sqrt{-D}}{2a}$ be the corresponding *Heegner point* in $\mathbb{H}$. We write $\mathcal{O}_D$ for the order of discriminant $-D$ in $\mathbb{Q}(\sqrt{-D})$.

Theorem 4.2. *Let $D > 0$ be coprime to 6 and $r \in \mathbb{Z}$ with $r^2 \equiv -D \pmod{24}$. If $Q \in \mathcal{Q}_{D,r,6}$ is primitive, then $6D \cdot P(\alpha_Q)$ is an algebraic integer contained in the ring class field corresponding to the order $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{-D})$.*

Remark. By Theorem 4.1, the multiset of values $P(\alpha_Q)$ is a union of Galois orbits. In a recent paper, Larson and Rolen [32] proved that each $P(\alpha_Q)$ is a 6-unit. Therefore, Theorem 4.2 then completes the proof of Theorem 1.1.

Theorem 4.2 will follow from Theorem 4.5 below, a general result on values of derivatives of weakly holomorphic modular forms at Heegner points. The following lemma is our key tool.

Lemma 4.3. *Let $\Gamma \subset \Gamma(1)$ be a level N congruence subgroup. Suppose that $f(z)$ is a weakly holomorphic modular function for Γ whose Fourier expansions at all cusps have coefficients in $\mathbb{Z}[\zeta_N]$. If $\tau_0 \in \mathbb{H}$ is a CM point, then $f(\tau_0)$ is an algebraic integer whose degree over $\mathbb{Q}(\zeta_N, j(\tau_0))$ is bounded by $[\Gamma(1) : \Gamma]$.*

Proof. We consider the polynomial

$$\Psi_f(X, z) = \prod_{\gamma \in \Gamma \backslash \Gamma(1)} (X - f(\gamma z)).$$

It is a monic polynomial in X of degree $[\Gamma(1):\Gamma]$ whose coefficients are weakly holomorphic modular functions in z for the group $\Gamma(1)$. Consequently, $\Psi_f(X,z)\in\mathbb{C}[j(z),X]$.

The assumption on the expansions of f at all cusps means that for every $\gamma\in\Gamma(1)$ the modular function $f\mid\gamma$ has a Fourier expansion with coefficients in $\mathbb{Z}[\zeta_N]$. So the coefficients of $\Psi_f(X,z)$ as a polynomial in X are weakly holomorphic modular functions for $\Gamma(1)$ with coefficients in $\mathbb{Z}[\zeta_N]$, and therefore they are elements of $\mathbb{Z}[\zeta_N,j(z)]$. Hence we actually have that $\Psi_f(X,z)\in\mathbb{Z}[\zeta_N,j(z),X]$.

Since $\Psi_f(f(z),z)=0$, we have, for every $z\in\mathbb{H}$, that $f(z)$ is integral over $\mathbb{Z}[\zeta_N,j(z)]$ with degree bounded by $[\Gamma(1):\Gamma]$. When τ_0 is a CM point, then $j(\tau_0)$ is an algebraic integer, and the claim follows. □

4.3. **Square-free level.** If the level N is square-free, then the group of Atkin-Lehner involutions acts transitively on the cusps of $\Gamma_0(N)$. Combining this fact with Lemma 4.3 leads to a handy criterion for the integrality of CM values. We begin by recalling some facts on Atkin-Lehner involutions (see e.g. [27] Chapter IX.7).

Let N be an integer and k a positive integer. If f is a complex valued function on the upper half plane $\mathbb{H}$ and $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2^+(\mathbb{R})$ then we put

$$(f\mid M)(z)=(f\mid_k M)(z)=\det(M)^{k/2}(cz+d)^{-k}f(Mz).$$

So scalar matrices act trivially. We write $M_k^!(N)$ for the space of weakly holomorphic modular forms of weight k for the group $\Gamma_0(N)$.

Let Q be an exact divisor of N (i.e. $Q|N$ and $(Q,N/Q)=1$), and let W_Q^N be an integral matrix of the form

$$W_Q^N=\begin{pmatrix}Q\alpha&\beta\\N\gamma&Q\delta\end{pmatrix}$$

with determinant Q. If $f\in M_k^!(N)$, then $f\mapsto f|W_Q^N$ is independent of the choices of $\alpha,\beta,\gamma,\delta$, and defines an involution of $M_k^!(N)$, called an Atkin-Lehner involution. If we write

$$R_Q^N=\begin{pmatrix}\alpha&\beta\\N\gamma/Q&Q\delta\end{pmatrix},$$

we have $W_Q^N=R_Q^N\left(\begin{smallmatrix}Q&0\\0&1\end{smallmatrix}\right)$, and $R_Q^N\in\Gamma_0(N/Q)$. For another exact divisor Q' of N, we have

$$f\mid W_Q^N\mid W_{Q'}^N=f\mid W_{Q*Q'}^N, \tag{4.3}$$

where $Q*Q'=QQ'/(Q,Q')^2$. If $(N',Q)=1$, then

$$f\mid W_Q^{NN'}=f\mid W_Q^N. \tag{4.4}$$

Clearly W_N^N acts as the usual Fricke involution W_N.

From now on we assume that N is square-free. Then the cusps of the group $\Gamma_0(N)$ are represented by $1/Q$, where Q runs through the divisors N. Two cusps a/c and a'/c' (where $a,c,a',c'\in\mathbb{Z}$ and $(a,c)=(a',c')=1$) are equivalent under $\Gamma_0(N)$ if and only if $(c,N)=(c',N)$ (for example, see [15], Prop. 3.8.3 and p. 103). In particular, a complete set of representatives for the cusps of $\Gamma_0(N)$ is given by $W_Q^N\infty$ with Q running though the divisors of N. Moreover, we have the disjoint left coset decomposition

$$\Gamma(1) = \bigcup_{Q|N} \bigcup_{j \ (Q)} \Gamma_0(N) R_Q^N \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}. \tag{4.5}$$

Lemma 4.4. *Let N be square-free, and suppose that $f \in M_0^!(N)$ has the property that $f \mid W_Q^N$ has coefficients in $\mathbb{Z}$ for every $Q \mid N$. If τ_0 is a level N Heegner point of discriminant $-D$, then $f(\tau_0)$ is an algebraic integer in the ring class field for the order $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{-D})$.*

Proof. The assumption on f implies that $f \in \mathbb{Q}(j, j_N)$. Therefore, by the theory of complex multiplication (see Theorem 4.1), $f(\tau_0)$ is contained in the claimed ring class field. Since N is square-free, the cusps of $\Gamma_0(N)$ are represented by $W_Q^N\infty$ with $Q \mid N$. Consequently, Lemma 4.3 implies that $f(\tau_0)$ is an algebraic integer. □

4.4. **CM values of derivatives of weakly holomorphic modular forms.** The goal of this section is to prove the following theorem which easily implies Theorem 4.2.

Theorem 4.5. *Let N be a square-free integer, and suppose that $f \in M_{-2}^!(N)$ has the property that $f \mid_{-2} W_Q^N$ has coefficients in $\mathbb{Z}$ for every $Q \mid N$. Define $\partial f = \frac{1}{4\pi} R_{-2} f$. Let $D > 0$ be coprime to $2N$ and $r \in \mathbb{Z}$ with $r^2 \equiv -D \pmod{4N}$. If $Q \in \mathcal{Q}_{D,r,N}$ is primitive, then $6D \cdot \partial f(\alpha_Q)$ is an algebraic integer in the ring class field for the order $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{-D})$.*

To prove the theorem we need two lemmas.

Lemma 4.6. *Let N be a square-free integer. Let $f \in M_{-2}^!(N)$ and assume that $f \mid_{-2} W_Q^N$ has integral Fourier coefficients for all $Q \mid N$. Define*

$$A_f = \frac{1}{4\pi} R_{-2} f - \frac{1}{6} f E_2^*.$$

Let $D > 0$ and $r \in \mathbb{Z}$ with $r^2 \equiv -D \pmod{4N}$. If $Q \in \mathcal{Q}_{D,r,N}$ is primitive, then $6 \cdot A_f(\alpha_Q)$ is an algebraic integer in the ring class field for the order $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{-D})$.

Proof. First, computing the Fourier expansion, we notice that $A_f \in M_0^!(N)$. Then, using the fact that $R_{-2}(f \mid_{-2} W_Q^N) = (R_{-2} f) \mid_0 W_Q^N$, we see that $6A_f \mid W_Q^N$ has integral coefficients for all $Q \mid N$. Consequently, the assertion follows from Lemma 4.4. □

Lemma 4.7. *Let N be a square-free integer. Let $f \in M_{-2}^!(N)$ and assume that $f \mid W_Q^N$ has integral Fourier coefficients for all $Q \mid N$. Define $\hat{f} = f \cdot E_2^*$. Let $D > 0$ be coprime to $2N$ and $r \in \mathbb{Z}$ with $r^2 \equiv -D \pmod{4N}$. If $Q \in \mathcal{Q}_{D,r,N}$ is primitive, then $D \cdot \hat{f}(\alpha_Q)$ is an algebraic integer in the ring class field for the order $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{-D})$.*

Proof. We write $Q = [a, b, c]$. Since $-D = b^2 - 4ac$ is odd, b is odd. Hence

$$M = \begin{pmatrix} -b & -2c \\ 2a & b \end{pmatrix}$$

is a primitive integral matrix of determinant D, satisfying $M\alpha_Q = \alpha_Q$. By the elementary divisor theorem there exist $\gamma_1, \gamma_2 \in \Gamma_0(N)$ such that

$$M = \gamma_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \gamma_2. \tag{4.6}$$

We put

$$r_D(z) = E_2^*(z) - DE_2^*(Dz) = E_2^*(z) - (E_2^* \mid W_D)(z).$$

Because of (4.6), we have

$$E_2^* \mid M = r_D \mid \gamma_1 M + E_2^*.$$

Using the fact that $(2a\alpha_Q + b)^2 = -D$, we find that that

$$\begin{aligned} E_2^*(\alpha_Q) &= \frac{1}{2}(r_D \mid_2 \gamma_1)(\alpha_Q), \\ \hat{f}(\alpha_Q) &= \frac{1}{2}(f \cdot r_D)(\gamma_1 \alpha_Q). \end{aligned}$$

Arguing as in the proof of Proposition 3.1 of [33], we see that $\hat{f}(\alpha_Q)$ is contained in the claimed ring class field. Hence, replacing Q by $\gamma_1 Q \in \mathcal{Q}_{D,r,N}$, it suffices to prove that $\frac{D}{2}(f \cdot r_D)(\alpha_Q)$ is an algebraic integer.

In view of Lemma 4.3 it suffices to show that for any $\gamma \in \Gamma(1)$, the weakly holomorphic modular form $\frac{D}{2}(f \cdot r_D) \mid \gamma$ has Fourier coefficients in $\mathbb{Z}[\zeta_{ND}]$. According to (4.5), there exists a $\gamma' \in \Gamma_0(N)$, a divisor $Q \mid N$ and $j \in \mathbb{Z}$ such that

$$\gamma = \gamma' R_Q^N \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \gamma' W_Q^N \begin{pmatrix} 1/Q & j/Q \\ 0 & 1 \end{pmatrix}.$$

Consequently, we have

$$(f \mid_{-2} \gamma)(z) = Q \cdot \left(f \mid_{-2} W_Q^N\right)\left(\frac{z+j}{Q}\right) \in Q \cdot \mathbb{Z}[\zeta_Q]((q^{1/Q})).$$

To analyze the situation for r_D, we write the integral matrix $W_D\gamma$ of determinant D as

$$W_D \gamma = \gamma'' \begin{pmatrix} D_1 & k \\ 0 & D_2 \end{pmatrix}$$

with $\gamma'' \in \Gamma(1)$ and positive integers D_1, D_2, k satisfying $D_1 D_2 = D$. Then we have

$$\begin{aligned} (r_D \mid_2 \gamma)(z) &= (E_2^* \mid \gamma)(z) - (E_2^* \mid W_D \gamma)(z) \\ &= E_2^*(z) - \frac{D}{D_2^2} E_2^*\left(\frac{D_1 z + k}{D_2}\right). \end{aligned}$$

Taking into account that D is odd, we see that $\frac{D}{2}(r_D \mid_2 \gamma) \in \mathbb{Z}[\zeta_D]((q^{1/D}))$. This concludes the proof of the lemma. □

Proof of Theorem 4.5. Using the notation of Lemma 4.6 and Lemma 4.7, we have

$$\partial f(z) = A_f(z) + \frac{1}{6}\hat{f}(z).$$

Consequently, the assertion follows from these lemmas. □

Proof of Theorem 4.2. We apply Theorem 4.5 to the function $F \in M_{-2}^!(6)$ defined in (1.3). Note that $F \mid W_6^6 = F$ and $F \mid W_3^6 = -F$. Moreover, we have $P = \frac{1}{4\pi} R_{-2} F = \partial F$. □

5. Examples

To compute $p(n)$ using Theorem 1.1, one first finds representatives for $\mathcal{Q}_n = Q_{24n-1,1,6}/\Gamma_0(6)$, a set which has $h(-24n+1)$ many elements. Gross, Kohnen, and Zagier (see pages 504–505 of [22]) establish a one to one correspondence between representatives of $\mathcal{Q}_n$ and positive definite binary quadratic forms under $\mathrm{SL}_2(\mathbb{Z})$ with discriminant $-24n+1$. Therefore, to determine representatives for $\mathcal{Q}_n$, it suffices to use the theory of *reduced forms* (for example, see page 29 of [14]) to determine representatives for the $\mathrm{SL}_2(\mathbb{Z})$ equivalence classes, and to then apply the Gross-Kohnen-Zagier correspondence (see the Proposition on page 505 of [22]).

For example, if $n = 2$, then $-24n+1 = -47$, and we have that $h(-47) = 5$ and $\mathcal{Q}_2 = \{[6,1,2],\ [12,1,1],\ [18,13,3],\ [24,25,7],\ [36,49,17]\}$, where $[a,b,c] := ax^2+bxy+cz^2$. Calculating $p(n)$, by Theorem 1.1, now follows from sufficiently accurate numerical approximations of the algebraic integers $6(24n-1)P(\alpha_Q)$.

We used this method to compute the first few "partition polynomials" $H_n(x)$.

n	$(24n-1)p(n)$	$H_n(x)$
1	23	$x^3 - 23x^2 + \frac{3592}{23}x - 419$
2	94	$x^5 - 94x^4 + \frac{169659}{47}x^3 - 65838x^2 + \frac{1092873176}{47^2}x + \frac{1454023}{47}$
3	213	$x^7 - 213x^6 + \frac{1312544}{71}x^5 - 723721x^4 + \frac{44648582886}{71^2}x^3$ $+\frac{9188934683}{71}x^2 + \frac{166629520876208}{71^3}x + \frac{2791651635293}{71^2}$
4	475	$x^8 - 475x^7 + \frac{9032603}{95}x^6 - 9455070x^5 + \frac{3949512899743}{95^2}x^4$ $-\frac{97215753021}{19}x^3 + \frac{9776785708507683}{95^3}x^2$ $-\frac{53144327916296}{19^2}x - \frac{134884469547631}{5^4\cdot 19}.$

We conclude with some natural questions which merit further investigation.

1. What can be said about the irreducibility of the $H_n(x)$?
2. Is there a "closed formula" for the constant terms of the $H_n(x)$ which is analogous to the formula of Gross and Zagier [23] on norms of differences of $j(z)$-singular moduli?
3. Do the singular moduli $P(\alpha_Q)$ enjoy special congruence properties? If so, do such congruences imply Ramanujan's congruences modulo 5, 7, and 11? Moreover, can one obtain an alternative proof of the well-known theorem of Ahlgren and Boylan [2] that the only congruences of the form
$$p(\ell n + a) \equiv 0 \pmod{\ell},$$
where ℓ is prime and $0 \leq a < \ell$, correspond to the pairs $(\ell, a) \in \{(5,4),(7,5),(11,6)\}$?

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Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, D-64289, Darmstadt, Germany
E-mail address: `bruinier@mathematik.tu-darmstadt.de`

Department of Math and Computer Science, Emory University, Atlanta, Georgia 30322
E-mail address: `ono@mathcs.emory.edu`