On the rank of Picard groups of modular varieties attached to orthogonal groups

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1 Introduction

Let L be an even lattice of signature $(2,l)$ with $l \geq 3$. Write $q(\cdot)$ for the quadratic form on L and $\mathcal{L}$ for the (finite) discriminant group of L.

Let $\mathrm{O}'(L \otimes_{\mathbb{Z}} \mathbb{R})$ be the spinor kernel of the real orthogonal group of L and denote the corresponding Hermitean symmetric domain by $\mathcal{H}_l$. We write $\mathrm{O}'(L)$ for the intersection of the integral orthogonal group of L with $\mathrm{O}'(L \otimes_{\mathbb{Z}} \mathbb{R})$. We consider the discriminant kernel $\Delta(L)$ of the group $\mathrm{O}'(L)$, that is the subgroup of those elements that act trivially on $\mathcal{L}$.

There is a natural notion of principal congruence groups for the group $\Delta(L)$: For any non-zero integer N we have the rescaled lattice $L(N)$, given by L as a $\mathbb{Z}$-module, but equipped with the quadratic form $Nq(\cdot)$. The discriminant kernel of $L(N)$ is a subgroup of $\Delta(L)$, defined by congruence conditions modulo N. We call it principal congruence subgroup of level N and denote it by $\Gamma(N)$.

We consider the arithmetic quotient $X(N) = \mathcal{H}_l/\Gamma(N)$. By the theory of Baily-Borel, it carries the structure of a quasiprojective algebraic variety. A fundamental geometric invariant is its algebraic Picard group $\mathrm{Pic}(X(N))$. Our assumption on l implies that this group is finitely generated. In the present paper we shall derive a nontrivial lower bound for the rank of $\mathrm{Pic}(X(N))$. In particular we are interested in the asymptotic behavior of the numbers

$$\mathrm{rank}\,(\mathrm{Pic}(X(N)))$$

as $N \to \infty$. Although this problem seems very natural, to the best of our knowledge, just partial results can be found in the literature. (See for instance [LW1, LW2] or [GN].) Certainly one would expect that the rank of $\mathrm{Pic}(X(N))$ tends to infinity as $N \to \infty$, reflecting the fact that the geometry of $X(N)$ gets more complicated as the level rises. However, even a result of this type seems not to be known in general.

Put $X = X(1)$. It is a consequence of the work of Borcherds [Bo1, Bo2] and the refinement given in [Br1] that there exists a homomorphism

$$S_{\kappa,L} \longrightarrow \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}/\, \mathbb{C}[E_{\mathrm{Hodge}}] \tag{1}$$

from a certain space $S_{\kappa,L}$ of $\mathbb{C}[\mathcal{L}]$-valued cusp forms of weight $\kappa = 1 + l/2$ to the quotient of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ modulo the span of the class of the Hodge line bundle E_{Hodge}.

If L splits two orthogonal hyperbolic planes over $\mathbb{Z}$, then the main result of [Br1] says that this map is injective (see also [Br2] or [BF] for related results). Hence in this case we can obtain a lower bound for $\mathrm{rank}(\mathrm{Pic}(X))$ by estimating the dimension of $S_{\kappa,L}$. By means of the Riemann-Roch theorem or the Selberg trace formula, the dimension of $S_{\kappa,L}$ can be computed. Thereby the original problem is reduced to estimating the different contributions in the dimension formula. Some of these are "strange" invariants of the discriminant group $\mathcal{L}$ and the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on it induced by q. They are studied in section 2, the technical heart of this paper.

Let us now assume that L splits two orthogonal hyperbolic planes over $\mathbb{Z}$, i.e. has the special shape $L = L_0 \perp H \perp H$, where L_0 is an even negative definite lattice. Then the above argument can be used to find a bound for the rank of $\mathrm{Pic}(X)$. Unfortunately, it cannot be applied directly to get a bound for $\mathrm{Pic}(X(N))$, since $L(N)$ does not split two hyperbolic planes over $\mathbb{Z}$.

Therefore we first consider the lattice

$$L[N] = L_0(N) \perp H \perp H$$

and its discriminant kernel $\Gamma[N] = \Delta(L[N])$. We write $X[N]$ for the quotient $\mathcal{H}_l/\Gamma[N]$. The group $\Gamma[N]$ can be viewed as a subgroup of the rational orthogonal group of L with the property that $\Gamma(N) \subset \Gamma[N]$. In the $\mathrm{O}(2,3)$-case of the Siegel modular group of genus 2 it is isomorphic to the paramodular group of level N. Using the injectivity of the map (1) and the estimate of section 2 for the dimension of $S_{\kappa,L}$, we obtain a bound for $\mathrm{rank}(\mathrm{Pic}(X[N]))$ (see Theorem 8). In particular we find that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ (which can be easily determined) such that

$$\mathrm{rank}\left(\mathrm{Pic}(X[N])\right) \geq \frac{l|\mathcal{L}|N^{l-2}}{48} - \begin{cases} C_\varepsilon N^{1/2+\varepsilon}, & \text{if } l = 3, \\ C_\varepsilon N^{l-3+\varepsilon}, & \text{if } l > 3, \end{cases} \tag{2}$$

for all $N \in \mathbb{N}$ (Corollary 9).

The projection $X(N) \to X[N]$ induces an injective homomorphism

$$\mathrm{Pic}\left(X[N]\right) \longrightarrow \mathrm{Pic}\left(X(N)\right).$$

Hence all bounds for the rank of $\mathrm{Pic}(X[N])$ give us also bounds for the rank of $\mathrm{Pic}(X(N))$. There are some reasons to believe that our estimate (2) actually describes the true asymptotic growth of $\mathrm{rank}(\mathrm{Pic}(X[N]))$, whereas the resulting bound for $\mathrm{Pic}(X(N))$ seems rather poor (see questions 1 and 2). Better results for $X(N)$ could be obtained by studying the injectivity properties of the map (1) more carefully for lattices which do not split two hyperbolic planes over $\mathbb{Z}$.

As an important example we consider the special case of the Siegel modular group of genus 2 in somewhat more detail. We take $L = \mathbb{Z}(-2) \perp H \perp H$ and use the exceptional isomorphism from $\mathrm{Sp}(4, \mathbb{R})$ to $\mathrm{O}(2, 3)$. Due to the work of Weissauer [We1, We2] we know a lot about the Picard groups $\mathrm{Pic}(X(N))$ in this case. For instance the Tate conjecture for algebraic divisors is proved in [We1]. However, lower bounds for the rank of $\mathrm{Pic}(X[N])$ or $\mathrm{Pic}(X(N))$ seem not to be known in general.

The group $\Gamma[N]$ is isomorphic to the paramodular group of level N (cf. [GrNi]). The quotient $X[N]$ is the moduli space of Abelian surfaces with a $(1, N)$-polarization. Our result implies that for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\mathrm{rank}\left(\mathrm{Pic}(X[N])\right) \geq N/8 - C_\varepsilon N^{1/2+\varepsilon}$$

for all $N \in \mathbb{N}$ (Corollary 10). The same estimate holds for the Siegel principal congruence subgroup of level N.

In the Appendix we apply some ideas of section 2 to derive certain class number identities. Together with the lemmas in section 2 they can be used to evaluate the formula for the dimension of $S_{\kappa,L}$ explicitly when L has the special shape $L = \mathbb{Z}(-2t_1) \perp \cdots \perp \mathbb{Z}(-2t_r)$ with nonzero integers $t_1, \dots, t_r$. Moreover, these identities might be of independent interest.

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2 The dimension formula

Let L be an even lattice of signature (b^+, b^-). We denote the bilinear form on L by $(\cdot, \cdot)$ and the associated quadratic form by $q(x) = \frac{1}{2}(x, x)$. We write L' for the dual lattice of L and $\mathcal{L} = L'/L$ for the (finite) discriminant group. Moreover, let $d = |\mathcal{L}/\{\pm 1\}|$, $r = b^+ + b^-$ be the rank of L, and denote by

$$D = \min\{n \in \mathbb{N}; \quad nq(\gamma) \in \mathbb{Z} \text{ for all } \gamma \in L'\} \tag{3}$$

the level of L.

We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic 2-fold cover of $\mathrm{SL}_2(\mathbb{R})$ and denote by $\mathrm{Mp}_2(\mathbb{Z})$ the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map. Recall that the elements of $\mathrm{Mp}_2(\mathbb{R})$ are pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, and ϕ denotes a holomorphic function on the upper complex half plane $\mathcal{H}$ with $\phi(\tau)^2 = c\tau + d$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \qquad \text{and} \qquad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

One has the relations $S^2 = (ST)^3 = Z$, where $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right), i \right)$ is the standard generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$.

There is a unitary representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[\mathcal{L}]$ of $\mathcal{L}$. If we denote the standard basis of $\mathbb{C}[\mathcal{L}]$ by $(\mathfrak{e}_\gamma)_{\gamma \in \mathcal{L}}$ then ρ_L can be defined by the action of the generators $S, T \in \mathrm{Mp}_2(\mathbb{Z})$ as follows (see also [Bo1], [Bo2], where the dual of ρ_L is used):

$$\rho_L(T)\mathfrak{e}_\gamma = e(-q(\gamma))\mathfrak{e}_\gamma, \tag{4}$$

$$\rho_L(S)\mathfrak{e}_\gamma = \frac{\sqrt{i}^{b^+ - b^-}}{\sqrt{|\mathcal{L}|}} \sum_{\delta \in \mathcal{L}} e((\gamma, \delta))\mathfrak{e}_\delta. \tag{5}$$

Here and throughout we abbreviate $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. This representation is essentially the Weil representation attached to the quadratic module $(\mathcal{L}, q)$ (see [No]).

Let $k \in \frac{1}{2}\mathbb{Z}$. We denote by $M_{k,L}$ the vector space of $\mathbb{C}[\mathcal{L}]$-valued modular forms of weight k with representation ρ_L for the group $\mathrm{Mp}_2(\mathbb{Z})$. The subspace of cusp forms is denoted by $S_{k,L}$. (See also [BF] or [Bo1].) It is easily seen that $M_{k,L} = 0$, if $2k \not\equiv b^- - b^+ \pmod 2$.

Since ρ_L factors through a finite quotient of $\mathrm{Mp}_2(\mathbb{Z})$, it is clear that the dimension of $M_{k,L}$ is finite. It can be computed using the Riemann-Roch theorem or the Selberg trace formula in a standard way. This is carried out in [Fi] in a more general situation. In our special case the following formula holds (see [Bo3], [Bo2] p. 228): Assume that $2k \equiv b^- - b^+ \pmod 4$ (we will only be interested in this case). Then the d-dimensional subspace $W = \mathrm{span}\{\mathfrak{e}_\gamma + \mathfrak{e}_{-\gamma};\quad \gamma \in \mathcal{L}\}$ of $\mathbb{C}[\mathcal{L}]$ is invariant under ρ_L. More precisely ρ_L acts by multiplication with $e(-k/2)$ on W. We denote by ρ the restriction of ρ_L to W. If M is a unitary matrix of size d with eigenvalues $e(\nu_j)$ and $0 \leq \nu_j < 1$ (for $j = 1, \ldots, d$), then we define

$$\alpha(M) = \sum_{j=1}^{d} \nu_j.$$

The dimension of $M_{k,L}$ is given by

$$\dim_{\mathbb{C}}(M_{k,L}) = d + dk/12 - \alpha\left(e^{\pi i k/2}\rho(S)\right) - \alpha\left(\left(e^{\pi i k/3}\rho(ST)\right)^{-1}\right) - \alpha(\rho(T)). \tag{6}$$

Furthermore, using Eisenstein series, it can be easily shown that the codimension of $S_{k,L}$ in $M_{k,L}$ is equal to the number of elements of the set

$$\{\gamma \in \mathcal{L}/\{\pm 1\};\quad q(\gamma) \in \mathbb{Z}\} \tag{7}$$

(see also [Br1] chapter 1.2.3).

As already pointed out in the introduction, we need to find a lower bound for the dimension of $S_{k,L}$. In view of (6) and (7) we have to estimate the quantities

$$\begin{aligned}\alpha_1 &:= \alpha\left(e^{\pi i k/2}\rho(S)\right),\\ \alpha_2 &:= \alpha\left(\left(e^{\pi i k/3}\rho(ST)\right)^{-1}\right),\\ \alpha_3 &:= \alpha(\rho(T)),\\ \alpha_4 &:= \left|\,\{\gamma \in \mathcal{L}/\{\pm 1\};\quad q(\gamma)\in\mathbb{Z}\}\,\right|.\end{aligned}$$

This can easily be done for α_1, α_2, and α_4. However, for α_3 this problems turns out to be more difficult. In the appendix we will see that α_3 sometimes is related to class numbers of imaginary quadratic fields.

For the estimates we first need some facts on Gauss sums attached to L. Let $n \in \mathbb{Z}$. We define the Gauss sum $G(n, L)$ by

$$G(n, L) = \sum_{\gamma\in\mathcal{L}} e(nq(\gamma)). \tag{8}$$

Two basic but important properties of $G(n, L)$ are

$$G(-n, L) = \overline{G(n, L)}, \tag{9}$$
$$G(n + D, L) = G(n, L). \tag{10}$$

If n is an integer, we define

$$\mathcal{L}^n = \{\gamma \in \mathcal{L};\quad n\gamma = 0\}.$$

Observe that $|\mathcal{L}^2| = 2d - |\mathcal{L}|$. In general it follows from the theorem of elementary divisors that

$$|\mathcal{L}^n| \leq (D, n)^r, \tag{11}$$

where (D, n) denotes the greatest common divisor of D and n.

Lemma 1. *Let n be a positive integer. i) If $D|n$, then $G(n, L) = |\mathcal{L}|$. ii) The absolute value of $G(n, L)$ is given by*

$$|G(n, L)| = \sqrt{|\mathcal{L}|}\sqrt{|\mathcal{L}^n|}.$$

In particular $|G(n, L)| = \sqrt{|\mathcal{L}|}$, if $(n, D) = 1$.

The proof is left to the reader.

Lemma 2. *The quantities* α_1 *and* α_2 *can be expressed in terms of Gauss sums as follows:*

$$\alpha_1 = \frac{d}{4} - \frac{1}{4\sqrt{|\mathcal{L}|}} e\left((2k+b^+-b^-)/8\right) \Re\big(G(2,L)\big), \tag{12}$$

$$\alpha_2 = \frac{d}{3} + \frac{1}{3\sqrt{3|\mathcal{L}|}} \Re\Big(e\big((4k+3b^+-3b^- - 10)/24\big)\big(G(1,L)+G(-3,L)\big)\Big). \tag{13}$$

Proof. The idea of the proof was communicated to us by R. E. Borcherds. Let us first consider (12). In $\mathrm{Mp}_2(\mathbb{Z})$ we have the relation $S^2 = Z$. Since Z acts on $W \subset \mathbb{C}[\mathcal{L}]$ by multiplication with $e(-k/2)$, the identity

$$(e(k/4)\rho(S))^2 = e(k/2)\rho(Z) = \mathrm{id}$$

holds. Hence all eigenvalues of $e(k/4)\rho(S)$ equal ± 1. If b denotes the number of eigenvalues equal to -1, then

$$\mathrm{tr}_W\left(e(k/4)\rho(S)\right) = -b + (d-b) = d - 2b.$$

Thus

$$\alpha_1 = b/2 = \frac{d}{4} - \frac{1}{4}\mathrm{tr}_W\left(e(k/4)\rho(S)\right).$$

Note that $\mathrm{tr}_W(\rho(S)) = \frac{1}{2}\mathrm{tr}_{\mathbb{C}[\mathcal{L}]}(\rho(S)+\rho(S)X)$, where X denotes the map $\mathbb{C}[\mathcal{L}] \to \mathbb{C}[\mathcal{L}]$ given by $\mathfrak{e}_\gamma \mapsto \mathfrak{e}_{-\gamma}$. Hence it follows from (5) that

$$\mathrm{tr}_W\left(e(k/4)\rho(S)\right) = \frac{1}{\sqrt{|\mathcal{L}|}} e\left((2k+b^+-b^-)/8\right) \Re\left(G(2,L)\right).$$

This implies the assertion.

Equality (13) can be proved in the same way. Using the relation $(ST)^3 = Z$ we find

$$\alpha_2 = \frac{d}{3} + \frac{2}{3\sqrt{3}} \Re\big(e(-5/12 + k/6)\,\mathrm{tr}_W\left(\rho(ST)\right)\big).$$

Furthermore, by (5) and (4) we have

$$\mathrm{tr}_W\left(\rho(ST)\right) = \frac{1}{2\sqrt{|\mathcal{L}|}} e((b^+-b^-)/8)\left(G(1,L)+G(-3,L)\right).$$

□

From Lemma 2 we obtain the following corollary.

Corollary 3. *The quantities* α_1 *and* α_2 *satisfy the estimates*

$$|\alpha_1 - d/4| \le \frac{1}{4}\sqrt{|\mathcal{L}^2|}, \tag{14}$$

$$|\alpha_2 - d/3| \le \frac{1}{3\sqrt{3}}\left(1 + \sqrt{|\mathcal{L}^3|}\right). \tag{15}$$

We now derive an estimate for α_4. If n is a positive integer, we define the divisor sum $\sigma_t(n) = \sum_{a|n} a^t$.

Lemma 4. *We have*

$$|\alpha_4| \leq \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2} \sigma_{r/2-1}(D).$$

Proof. We write α_4 as

$$\alpha_4 = \frac{1}{2} \sum_{\substack{\gamma \in \mathcal{L}^2 \\ q(\gamma) \in \mathbb{Z}}} 1 + \frac{1}{2} \sum_{\substack{\gamma \in \mathcal{L} \\ q(\gamma) \in \mathbb{Z}}} 1.$$

The second term on the right hand side is equal to

$$\frac{1}{2D} \sum_{\gamma \in \mathcal{L}} \sum_{\nu\,(D)} e(q(\gamma)\nu) = \frac{1}{2D} \sum_{\nu\,(D)} G(\nu, L).$$

Thus, using Lemma 1, we obtain

$$\begin{aligned}
|\alpha_4| &\leq \frac{|\mathcal{L}^2|}{2} + \frac{1}{2D} \sum_{\nu\,(D)} \sqrt{|\mathcal{L}|}\sqrt{|\mathcal{L}^\nu|} \\
&\leq \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2D} \sum_{\nu\,(D)} (\nu, D)^{r/2} \\
&\leq \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2D} \sum_{a|D} \sum_{\substack{\mu=1 \\ (\mu, D/a)=1}}^{D/a} a^{r/2} \\
&\leq \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2D} \sum_{a|D} \frac{D}{a} a^{r/2} \\
&\leq \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2} \sigma_{r/2-1}(D).
\end{aligned}$$

□

Before we consider α_3 we introduce some more notation. If $x \in \mathbb{R}$, then we write $[x]$ for the greatest-integer function $\max\{n \in \mathbb{Z};\ n \leq x\}$. Moreover, we define

$$\mathbb{B}(x) = x - \tfrac{1}{2}([x] - [-x]). \tag{16}$$

Thus $\mathbb{B}(x)$ is the 1-periodic function on $\mathbb{R}$ with $\mathbb{B}(x) = 0$ for $x = 0, 1$ and $\mathbb{B}(x) = x - 1/2$ for $0 < x < 1$. By definition

$$\alpha_3 = \sum_{\gamma \in \mathcal{L}/\{\pm 1\}} \left(-q(\gamma) - [-q(\gamma)]\right).$$

Using $\mathbb{B}(x)$ and α_4 we may rewrite this in the form

$$\alpha_3 = \frac{d}{2} - \frac{\alpha_4}{2} - \sum_{\gamma \in \mathcal{L}/\{\pm 1\}} \mathbb{B}(q(\gamma)).$$

Hence, to obtain information on α_3, it suffices to consider the invariants

$$\alpha_5 = \sum_{\gamma \in \mathcal{L}/\{\pm 1\}} \mathbb{B}(q(\gamma)), \tag{17}$$

$$\alpha_5' = \sum_{\gamma \in \mathcal{L}} \mathbb{B}(q(\gamma)) \tag{18}$$

of L. Obviously the relation

$$\alpha_5 = \frac{1}{2} \sum_{\gamma \in \mathcal{L}^2} \mathbb{B}(q(\gamma)) + \frac{\alpha_5'}{2}$$

holds. For $\gamma \in \mathcal{L}^2$, we have $q(\gamma) \in \frac{1}{4}\mathbb{Z}$ and thereby $|\mathbb{B}(q(\gamma))| \leq 1/4$. Hence

$$|\alpha_5| \leq |\mathcal{L}^2|/8 + |\alpha_5'|/2 \qquad \text{and}$$
$$|\alpha_3 - d/2 + \alpha_4/2| \leq |\mathcal{L}^2|/8 + |\alpha_5'|/2. \tag{19}$$

The main result of this section is the following estimate for α_5'.

Lemma 5. *The invariant α_5' satisfies*

$$|\alpha_5'| \leq \frac{\sqrt{|\mathcal{L}|}}{\pi} (3/2 + \ln(D)) \left(\sigma_{r/2-1}(D) - D^{r/2-1}\right).$$

Proof. The 1-periodic function $\mathbb{B}(x)$ has the pointwise convergent Fourier expansion

$$\mathbb{B}(x) = -\frac{1}{2\pi i} \sum_{n \in \mathbb{Z}-\{0\}} \frac{e(nx)}{n}. \tag{20}$$

Inserting this into the definition of α_5' we find

$$\begin{aligned}
\alpha_5' &= -\frac{1}{2\pi i} \sum_{n \in \mathbb{Z}-\{0\}} \frac{1}{n} \sum_{\gamma \in \mathcal{L}} e(nq(\gamma)) \\
&= -\frac{1}{2\pi i} \sum_{n \in \mathbb{Z}-\{0\}} \frac{1}{n} G(n, L) \\
&= -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \Im(G(n, L)).
\end{aligned}$$

We use (9) and (10) and the fact $\Im(G(D\nu, L)) = 0$ to rewrite this as follows:

$$\begin{aligned}
\alpha_5' &= -\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{\nu=1}^{D-1}\left(\frac{\Im\big(G(Dn+\nu, L)\big)}{Dn+\nu} + \frac{\Im\big(G(D(n+1)-\nu, L)\big)}{D(n+1)-\nu}\right)\\
&= -\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{\nu=1}^{D-1}\left(\frac{1}{Dn+\nu} - \frac{1}{D(n+1)-\nu}\right)\Im(G(\nu, L))\\
&= -\frac{1}{\pi}\sum_{\nu=1}^{D-1}\frac{1}{\nu}\Im(G(\nu, L)) - \frac{1}{2\pi}\sum_{n=1}^{\infty}\sum_{\nu=1}^{D-1}\frac{D-2\nu}{D^2n(n+1)+D\nu-\nu^2}\Im(G(\nu, L)).
\end{aligned}$$

By means of Lemma 1 we obtain

$$\begin{aligned}
|\alpha_5'| &\le \frac{1}{\pi}\sum_{\nu=1}^{D-1}\frac{1}{\nu}|G(\nu, L)| + \frac{1}{2\pi}\sum_{n=1}^{\infty}\sum_{\nu=1}^{D-1}\frac{D-2}{D^2n(n+1)}|(G(\nu, L))|\\
&\le \frac{\sqrt{|\mathcal{L}|}}{\pi}\sum_{\nu=1}^{D-1}\frac{1}{\nu}\sqrt{|\mathcal{L}^\nu|} + \frac{\sqrt{|\mathcal{L}|}}{2\pi D}\sum_{\nu=1}^{D-1}\sum_{n=1}^{\infty}\sqrt{|\mathcal{L}^\nu|}\frac{1}{n(n+1)}.
\end{aligned}$$

The latter sum over n equals 1. We apply (11) and rewrite the sum over ν. We get

$$\begin{aligned}
|\alpha_5'| &\le \frac{\sqrt{|\mathcal{L}|}}{\pi}\sum_{\substack{a|D\\ a\ne D}}\sum_{\substack{\mu=1\\ (\mu,D/a)=1}}^{D/a}\frac{1}{a\mu}a^{r/2} + \frac{\sqrt{|\mathcal{L}|}}{2\pi D}\sum_{\substack{a|D\\ a\ne D}}\sum_{\substack{\mu=1\\ (\mu,D/a)=1}}^{D/a}a^{r/2}\\
&\le \frac{\sqrt{|\mathcal{L}|}}{\pi}\sum_{\substack{a|D\\ a\ne D}}(1+\ln(D/a))a^{r/2-1} + \frac{\sqrt{|\mathcal{L}|}}{2\pi D}\sum_{\substack{a|D\\ a\ne D}}\frac{D}{a}a^{r/2}\\
&\le \frac{\sqrt{|\mathcal{L}|}}{\pi}(3/2+\ln(D))\left(\sigma_{r/2-1}(D) - D^{r/2-1}\right).
\end{aligned}$$

Here we have also used the estimate $\sum_{\nu=1}^{n}\frac{1}{\nu} \le 1+\ln(n)$. ☐

If we put the above lemmas together we finally obtain the desired estimate for the dimension of $S_{k,L}$.

Theorem 6. *Assume that* $2k \equiv b^- - b^+ \pmod 4$. *Then*

$$\begin{aligned}
\left|\dim(S_{k,L}) - \frac{(k-1)d}{12}\right| &\le \frac{\sqrt{|\mathcal{L}^2|}}{4} + \frac{1+\sqrt{|\mathcal{L}^3|}}{3\sqrt{3}} + \frac{3}{8}|\mathcal{L}^2| + \frac{\sqrt{|\mathcal{L}|}}{4}\sigma_{r/2-1}(D)\\
&\quad + \frac{\sqrt{|\mathcal{L}|}}{2\pi}(3/2+\ln(D))\left(\sigma_{r/2-1}(D) - D^{r/2-1}\right).
\end{aligned}$$

This estimate could be further improved by using the theorem of elementary divisors more carefully in the proof of Lemma 4 and 5. However, since we are mainly interested in asymptotic questions, the above result suffices for our purposes. Recall that the quantities $|\mathcal{L}^\nu|$ are bounded by (11).

3 Picard groups

For any lattice (L, q) and any non-zero integer N, we may consider the rescaled lattice $L(N)$. It is given by L as a $\mathbb{Z}$-module, but equipped with the rescaled quadratic form $Nq(\cdot)$. The dual is given by $L(N)' = \frac{1}{N}L'$.

From now on we suppose that L has signature $(2, l)$ with $l \geq 3$. The orthogonal group $\mathrm{O}(L)$ of L is a discrete subgroup of the real orthogonal group $\mathrm{O}(L \otimes_{\mathbb{Z}} \mathbb{R}) \cong \mathrm{O}(2, l)$. Let $\mathrm{O}'(L \otimes_{\mathbb{Z}} \mathbb{R})$ be the spinor kernel of $\mathrm{O}(L \otimes_{\mathbb{Z}} \mathbb{R})$ and $\mathrm{O}'(L) = \mathrm{O}'(L \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathrm{O}(L)$. We denote by $\Delta(L)$ the discriminant kernel of the group $\mathrm{O}'(L)$. By definition, this is the subgroup of those elements of $\mathrm{O}'(L)$, which act trivially on the discriminant group $\mathcal{L}$.

Let us briefly recall the construction of the Hermitean symmetric domain $\mathcal{H}_l$ associated to $\mathrm{O}'(L \otimes_{\mathbb{Z}} \mathbb{R})$. We extend the bilinear form $(\cdot, \cdot)$ on L to a $\mathbb{C}$-bilinear form on the complexification $L \otimes_{\mathbb{Z}} \mathbb{C}$ and consider the following chain of subsets of the associated projective space $P(L \otimes_{\mathbb{Z}} \mathbb{C})$:

$$\mathcal{H}_l \subset \mathcal{K} \subset \mathcal{N} \subset P(L \otimes_{\mathbb{Z}} \mathbb{C}).$$

Here $\mathcal{N}$ denotes the zero quadric, i.e. the subset of $P(L \otimes_{\mathbb{Z}} \mathbb{C})$ represented by vectors z of norm zero $(z, z) = 0$. The open subset $\mathcal{K}$ is defined by the condition $(z, \bar{z}) > 0$. It has two connected components. We choose one of them and denote it by $\mathcal{H}_l$. The real orthogonal group of L acts on $L \otimes_{\mathbb{Z}} \mathbb{C}$, $P(L \otimes_{\mathbb{Z}} \mathbb{C})$, $\mathcal{N}$, and $\mathcal{K}$. The spinor kernel acts on $\mathcal{H}_l$.

Let $\Gamma = \Delta(L)$ and X be the quotient $\mathcal{H}_l/\Gamma$. By the theory of Baily-Borel, X is a quasi-projective algebraic variety.

If Γ acts freely on $\mathcal{H}_l$, then X is smooth. In this case we denote by $\mathrm{Pic}(X)$ the usual algebraic Picard group, i.e. the group of isomorphism classes of algebraic holomorphic line bundles on X. If Γ does not act freely, then we choose a normal subgroup Γ' of finite index which acts freely. We define the Picard group of X by

$$\mathrm{Pic}(X) = \mathrm{Pic}(\mathcal{H}_l/\Gamma')^{\Gamma/\Gamma'},$$

i.e. as the subgroup of $\mathrm{Pic}(\mathcal{H}_l/\Gamma')$, which is invariant under the action of the finite group Γ/Γ'. Our assumption on l implies that these Picard groups are finitely generated.

In the same way we define the divisor class group $\mathrm{Cl}(X)$ of X. (See also [Bo2] and [Br1].) Moreover, we write $\widetilde{\mathrm{Cl}}(X)$ for the quotient of $\mathrm{Cl}(X)$ modulo the subgroup $A(X)$ of divisor classes coming from meromorphic automorphic forms (of generally non-zero weight with a character of finite order) for the group Γ. There is the usual injective map

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X),$$

which assigns to a divisor class its associated class of line bundles. (By our definition of Cl and Pic this map also makes sense if Γ does not act freely. Since X is quasi-projective, this map is in fact an isomorphism.) Thus the rank of $\mathrm{Pic}(X)$ is bounded by $\dim_{\mathbb{C}}(\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{C})$. It follows from the Koecher boundedness principle (which holds since $l \geq 3$) that $\dim(A(X) \otimes_{\mathbb{Z}} \mathbb{C}) = 1$ and thereby

$$\mathrm{rank}(\mathrm{Pic}(X)) \geq 1 + \dim_{\mathbb{C}}(\widetilde{\mathrm{Cl}}(X) \otimes_{\mathbb{Z}} \mathbb{C}). \tag{21}$$

Put $\kappa = 1 + l/2$. It is a consequence of the existence of Borcherds' lifting from modular forms of negative weight $1 - l/2$ to automorphic products for the group Γ and Serre duality that there exists a homomorphism from the space of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-conjugates of $S_{\kappa,L}$ to $\widetilde{\mathrm{Cl}}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ (cf. [Bo1, Bo2]). By the refinement given in [Br1] chapter 5.1, we more precisely know that there is a homomorphism

$$\eta : S_{\kappa,L} \longrightarrow \widetilde{\mathrm{Cl}}(X) \otimes_{\mathbb{Z}} \mathbb{C}. \tag{22}$$

We may infer the following fundamental proposition.

Proposition 7. *Suppose that the map η is injective. Then*

$$\mathrm{rank}(\mathrm{Pic}(X)) \geq 1 + \dim_{\mathbb{C}}(S_{\kappa,L}).$$

Recall that a hyperbolic plane is a lattice H which is isomorphic to the lattice $\mathbb{Z}^2$ equipped with the quadratic form $q((a,b)) = ab$. For the rest of this section we assume that L splits two orthogonal hyperbolic planes over $\mathbb{Z}$, i.e. has the special shape $L = L_0 \perp H \perp H$, where L_0 is an even negative definite lattice of rank $l-2$.

Let N be a positive integer. We consider the lattice

$$L[N] = L_0(N) \perp H \perp H,$$

its discriminant kernel $\Gamma[N] = \Delta(L[N])$, and the associated modular variety $X[N] = \mathcal{H}_l/\Gamma[N]$. We may view $\Gamma[N]$ as a subgroup of $\mathrm{O}(L \otimes_{\mathbb{Z}} \mathbb{Q})$ which is commensurable with $\Gamma = \Delta(L)$.

Theorem 8. *Let L be a lattice as above and $\mathcal{L}$ its discriminant group. Let D be the level of L as defined in (3). Then*

$$\begin{aligned}\mathrm{rank}\left(\mathrm{Pic}(X[N])\right) \geq\ & \frac{l|\mathcal{L}|N^{l-2}}{48} + l/48 + 1 - 2^{l/2-3} - 3 \cdot 2^{l-5} - 3^{-3/2} - 3^{l/2-5/2} \\ & - \frac{\sqrt{|\mathcal{L}|}}{4} N^{l/2-1} \sigma_{l/2-2}(DN) \\ & - \frac{\sqrt{|\mathcal{L}|}}{2\pi} N^{l/2-1}(3/2 + \ln(DN)) \left(\sigma_{l/2-2}(DN) - (DN)^{l/2-2}\right).\end{aligned}$$

Proof. By construction the lattice $L[N]$ splits two hyperbolic planes over $\mathbb{Z}$. The main result of [Br1] chapter 5.2 says that the map (22) is injective in this case. By Proposition 7 we find

$$\mathrm{rank}\left(\mathrm{Pic}(X[N])\right) \geq 1 + \dim(S_{\kappa,L[N]}) = 1 + \dim(S_{\kappa,L_0(N)}).$$

We apply Theorem 6 to estimate the dimension of $S_{\kappa,L_0(N)}$. The rank of $L_0(N)$ is $l-2$, the level of $L_0(N)$ is DN, and

$$|L_0(N)'/L_0(N)| = N^{l-2}|\mathcal{L}|,$$
$$|(L_0(N)'/L_0(N))/\{\pm 1\}| \geq \tfrac{1}{2}(1 + N^{l-2}|\mathcal{L}|).$$

If we also take into account (11) we obtain the assertion. $\square$

Corollary 9. *Let $\varepsilon > 0$. Then there exist positive constants $C_1 = C_1(L, \varepsilon)$ and $C_2 = C_2(L)$ (which can be easily determined explicitly) such that*

$$\operatorname{rank}(\operatorname{Pic}(X[N])) \geq \frac{l|\mathcal{L}|N^{l-2}}{48} - C_2 - \begin{cases} C_1 N^{1/2+\varepsilon}, & \text{if } l = 3, \\ C_1 N^{l-3+\varepsilon}, & \text{if } l > 3, \end{cases}$$

for all $N \in \mathbb{N}$.

In the above situation the map (22) induces in fact an isomorphism from $S_{\kappa,L[N]}$ to the subspace of $\widetilde{\operatorname{Cl}}(X[N]) \otimes_{\mathbb{Z}} \mathbb{C}$, which is generated by algebraic divisors $\lambda^\perp$, where $\lambda \in L[N]'$ is a negative norm vector and the orthogonal complement is taken is $\mathcal{H}_l$. According to the Tate conjecture one should expect that the codimension of this subspace in $\widetilde{\operatorname{Cl}}(X[N]) \otimes_{\mathbb{Z}} \mathbb{C}$ is small. This leads us to the following

Question 1. *Is it true that*

$$\operatorname{rank}(\operatorname{Pic}(X[N])) \sim l|\mathcal{L}|N^{l-2}/48, \qquad N \to \infty?$$

Let N be a positive integer. It is natural to define the *principal congruence subgroup* of level N of $\Gamma = \Delta(L)$ by

$$\Gamma(N) = \Delta(L(N)).$$

We now consider the Picard groups of the modular varieties $X(N) = \mathcal{H}_l/\Gamma(N)$. In the same way as in [Fr] (chapter 2.6 Hilfssatz 6.5) it can be proved that for $N \geq 3$ the group $\Gamma(N)$ acts freely on $\mathcal{H}_l$. Thus $X(N)$ is smooth in this case.

To obtain an estimate for the rank of $\operatorname{Pic}(X(N))$ we cannot argue as above. Since $L(N)$ does not split two hyperbolic planes over $\mathbb{Z}$, we do not have the result of [Br1] saying that the map η (22) is injective.

However, we can still get an estimate for the rank of $\operatorname{Pic}(X(N))$ in the following way. There exists a lattice $\tilde{L}$, which is isomorphic to $L[N]$ and contains

$$L(N) = L_0(N) \perp H(N) \perp H(N)$$

as a sub-lattice. It is easily seen that

$$\Gamma(N) = \Delta(L(N)) \subset \Delta(\tilde{L}).$$

(In fact, taking the discriminant kernel of a lattice is functorial.) Therefore we may view $\Gamma(N)$ as a subgroup of $\Gamma[N]$. The natural projection $X(N) \to X[N]$ induces an injective map of Picard groups

$$\operatorname{Pic}(X[N]) \longrightarrow \operatorname{Pic}(X(N)).$$

Thus Theorem 8 gives us a lower bound for $\operatorname{rank}(\operatorname{Pic}(X(N)))$, too. The asymptotic bound of corollary 9 also holds.

It is clear that these bounds for the rank of $\operatorname{Pic}(X(N))$ are probably not optimal. Here it is natural to ask

Question 2. *What is the asymptotic behavior of the numbers* $\operatorname{rank}(\operatorname{Pic}(X(N)))$ *for* $N \to \infty$*?*

3.1 The Siegel modular group of genus 2

If R is a subring of $\mathbb{C}$, then we denote by

$$\mathrm{Sp}(2,R)=\{M\in \mathrm{GL}(4,R);\quad M^tIM=I\}$$

the symplectic group of genus 2 with coefficients in R. Here I denotes the matrix $\left(\begin{smallmatrix}0 & E\\ -E & 0\end{smallmatrix}\right)$ and E the 2×2 identity matrix. The group $\mathrm{Sp}(2,\mathbb{R})$ acts on the Siegel half plane $\mathbb{H}_2$. Let N be a positive integer. The paramodular group $\Gamma_S[N]$ of level N is the subgroup of $\mathrm{Sp}(2,\mathbb{Q})$ given by matrices of the form

$$\begin{pmatrix} * & N* & * & * \\ * & * & * & N^{-1}* \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix},$$

where the $*$ are all integral. The quotient $\mathbb{H}_2/\Gamma_S[N]$ is the moduli space of Abelian surfaces with a $(1,N)$-polarization.

Let L be the lattice $H\perp H\perp \mathbb{Z}(-2)$ of signature $(2,3)$. It is well known that there exists an isomorphism $\mathrm{Sp}(2,\mathbb{R})/\{\pm 1\}\to \mathrm{O}'(L[N]\otimes\mathbb{R})/\{\pm1\}$, which commutes with the action of $\mathrm{Sp}(2,\mathbb{R})$ on $\mathbb{H}_2$ and the action of $\mathrm{O}'(L\otimes\mathbb{R})$ on $\mathcal{H}_3$, and which induces an isomorphism

$$\Gamma_S[N]/\{\pm1\}\longrightarrow \Gamma[N]/\{\pm1\}=\Delta(L[N])/\{\pm1\}$$

(see [GN]). Hence Corollary 9 implies

Corollary 10. *Let $\varepsilon>0$. Then there exist positive constants $C_1=C_1(\varepsilon)$ and $C_2<0.6$ (which can be easily determined) such that*

$$\mathrm{rank}\,(\mathrm{Pic}(\mathbb{H}_2/\Gamma_S[N]))\geq N/8-C_2-C_1N^{1/2+\varepsilon}$$

for all $N\in\mathbb{N}$.

Note that $\dim(S_{\kappa,L[N]})$ can be computed explicitly in this case. By Lemma 2 the quantities α_1 and α_2 can be expressed in terms of standard Gauss sums $G(n,a)=\sum_{\nu(a)}e(n\nu^2/a)$. Moreover, α_4 is equal to $[1+b/2]$, where b is the largest integer whose square divides N. Finally, using Theorem 11 of the appendix, α_5 can be written as a sum of class numbers. Therefore we could obtain a sharper estimate than in Theorem 8. However, in the asymptotic estimate Corollary 10 this would only improve the constants C_1 and C_2.

Let $\Gamma_S(N)\subset \mathrm{Sp}(2,\mathbb{Z})$ be the principal congruence subgroup of level N, i.e. the kernel of the reduction homomorphism $\mathrm{Sp}(2,\mathbb{Z})\to\mathrm{Sp}(2,\mathbb{Z}/N\mathbb{Z})$. Since $\Gamma_S(N)\subset\Gamma_S[N]$, the above estimate also holds for the group $\Gamma_S(N)$. (To see this we could have also used the fact that the orthogonal principal congruence subgroup $\Gamma(N)$ is isomorphic to a group G with $\Gamma_S(2N)\subset G\subset\Gamma_S(N)$.)

Appendix

In section 2 we saw that the quantities $\alpha_1, \alpha_2, \alpha_4$ can all be expressed in terms of Gauss sums. We now indicate, how the idea of the proof of Lemma 5 can sometimes be used to obtain a closed formula for α_5' (and thereby for α_3) in terms of class numbers.

Let L be the negative definite lattice of rank r given by

$$L = \mathbb{Z}(-2N) \perp \cdots \perp \mathbb{Z}(-2N).$$

Define

$$A_r(N) = \sum_{\nu_1,\dots,\nu_r\,(N)} \mathbb{B}\left(\frac{\nu_1^2}{N} + \cdots + \frac{\nu_r^2}{N}\right),$$

where $\nu_1, \dots, \nu_r$ run through a set of representatives of $\mathbb{Z}/N\mathbb{Z}$. Then for our particular lattice L we have $\alpha_5' = -\frac{1}{2}A_r(4N)$.

We denote by $H(a)$ for $a \neq -3, -4$ the class number of positive definite binary quadratic forms of discriminant a and put $H(-3) = 1/3$, $H(-4) = 1/2$. Then $H(a) = 0$, if $a > 0$ or $a \not\equiv 0, 1 \pmod 4$. Moreover, we write χ_a for the Dirichlet character defined by the Kronecker symbol $x \mapsto \left(\frac{a}{x}\right)$.

Theorem 11. *Suppose that r is odd. Then*

$$A_r(N) = -\chi_{-4}(r)N^{r-1} \sum_{\substack{a|N \\ a\equiv -1\,(4)}} a^{\frac{1-r}{2}} H(-a) - \chi_{-8}(r)\left(\sqrt{2}N\right)^{r-1} \sum_{\substack{a|N \\ a\equiv 0\,(4)}} a^{\frac{1-r}{2}} H(-a).$$

Here the sums run through the positive divisors of N satisfying the indicated conditions.

Proof. If $n \in \mathbb{Z}$ and $a \in \mathbb{N}$, then we denote by $G(n, a) = \sum_{\nu(a)} e(n\nu^2/a)$ the standard Gauss sum. By means of the Fourier expansion (20) of the function $\mathbb{B}$, we can rewrite $A_r(N)$ as a Dirichlet series:

$$A_r(N) = -\frac{1}{2\pi i} \sum_{n\in\mathbb{Z}-\{0\}} \frac{1}{n} G(n, N)^r = -\frac{1}{\pi} \sum_{n\geq 1} \frac{1}{n} \Im(G(n, N)^r).$$

Using the fact $G(n, N) = aG(n/a, N/a)$ for $a|(n, N)$, we find

$$A_r(N) = -\frac{N^{r-1}}{\pi} \sum_{a|N} \sum_{\substack{m\geq 1 \\ (m,a)=1}} \frac{1}{m} a^{1-r} \Im(G(m, a)^r).$$

If we insert the explicit formula for $G(m, a)$ (cf. [La] chapter 4.3), we obtain by a lengthy but straightforward calculation

$$A_r(N) = -\frac{N^{r-1}}{\pi} \sum_{a|N} a^{1-r/2} L(\chi_{-a}, 1) \cdot \begin{cases} 0, & \text{if } a \equiv 1, 2 \pmod 4, \\ \chi_{-4}(r), & \text{if } a \equiv -1 \pmod 4, \\ 2^{(r-1)/2}\chi_{-8}(r), & \text{if } a \equiv 0 \pmod 4. \end{cases}$$

Here $L(\chi_a, s)$ denotes the Dirichlet series associated to the Dirichlet character χ_a. Since $L(\chi_{-a}, 1) = \pi H(-a)/\sqrt{a}$ (cf. [Za] §8), this implies the assertion. □

By virtue of the above argument, A_r can also be evaluated for even r. In this case class numbers do not show up. For instance for $r \equiv 0 \pmod 4$ one finds that $A_r(N) = 0$. More generally α_5' can be computed for any lattice of the form $\mathbb{Z}(-2N_1) \perp \cdots \perp \mathbb{Z}(-2N_r)$ with $N_1, \dots, N_r \in \mathbb{N}$. Note that for $r = 1$ the above formula is already contained in the book [EZ] in §10 (but with a different proof).

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