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CONES OF HEEGNER DIVISORS

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Abstract

We show that the cone of primitive Heegner divisors is finitely generated for many orthogonal Shimura varieties, including the moduli space of polarized $K3$-surfaces. The proof relies on the growth of coefficients of modular forms.

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1. Introduction

Let X be a projective algebraic variety. Both the pseudo-effective cone of X (the closure of the cone $\mathrm{Eff}(X)$ of effective divisors) and dually (by [BDPP13]) the cone of movable curves are important geometric invariants of X that are notoriously hard to compute. The same claim can be made for the cone of curves, or dually for the cone of nef divisors. The pseudo-effective cone plays an important role in the computation of the Kodaira dimension of moduli spaces; see, e.g., [HM82] or [FP05] for the moduli space of curves and [GHS07] or [Pet15] for the moduli space of $K3$-surfaces.

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Mori's cone theorem is the most important result implying that extremal rays in the cone of curves in the half-space where K_X is negative can accumulate at most towards the plane $K_X^\perp$. On the other hand, abelian surfaces provide examples where all these cones are round. For varieties of general type the knowledge about these cones is very limited and the existing polyhedrality results (for example [KKL16]) have hypotheses that are restrictive or hard to verify. For example, it was shown recently by Mullane ([Mul17]) that the effective cone $\mathrm{Eff}(\overline{M}_{g,n})$ of the moduli space of curves is not finitely generated for $g \geq 2$ and $n \geq g+1$.

In this note we deduce from properties of modular forms the polyhedrality of a natural subcone of the pseudo-effective cone on the Baily-Borel compactification of orthogonal Shimura varieties. For concreteness, we work in the introduction with the coarse moduli space $\mathcal{F}_{2d}$ of quasi-polarized $K3$-surfaces of degree $2d$. For every integer d this space is the quotient $\mathcal{F}_{2d} = \widetilde{\mathrm{O}}^+(L)\backslash\mathcal{D}_{2d}$ of a 19-dimensional complex domain $\mathcal{D}_{2d}$ by an arithmetic lattice $\widetilde{\mathrm{O}}^+(L)$; see Section 2 for the background on lattices and this quotient. These moduli spaces carry an infinite collection of divisors, the Noether-Lefschetz divisors, geometrically defined as the loci of $K3$-surfaces with Picard group of rank ≥ 2. These divisors are also called Heegner divisors. They are not irreducible in general. We recall the structure of the irreducible components, called primitive Heegner divisors, in Section 4.

The structure of the Picard group $\mathrm{Pic}(\mathcal{F}_{2d})$ up to torsion is now completely understood. It is shown in [BLMM17] to be generated by Noether-Lefschetz divisors and the rank has been computed in [Bru02b]. A next step towards understanding the geometry of $\mathcal{F}_{2d}$ would be to compute the natural cones in $\mathrm{Pic}(\mathcal{F}_{2d})$, the ample cone and the pseudo-effective cone. The latter contains as a subcone the cone $\mathrm{Eff}^{\mathrm{NL}}(\mathcal{F}_{2d})$ generated by the primitive Heegner divisors. The theorem of [BLMM17] implies in this language that $\mathrm{Eff}^{\mathrm{NL}}(\mathcal{F}_{2d})$ is full-dimensional, but it does not imply that the two cones coincide. The following theorem answers a question raised by Peterson in [Pet15].

Theorem 1.1. *The cone $\mathrm{Eff}^{\mathrm{NL}}(\mathcal{F}_{2d})$ generated by the primitive Heegner divisors is rational polyhedral. In particular it is finitely generated.*

The methods can be applied to the more general situation where the lattice L is an even lattice of signature $(b^+, 2)$ with $b^+ \geq 3$. If L splits off a hyperbolic plane we show in Theorem 4.1 that the cone of Heegner divisors on the hermitian symmetric space $\mathcal{F}_L$ associated with L is rational polyhedral.

If the lattice L splits off two hyperbolic planes, we show in Theorem 4.4 generalizing Theorem 1.1 that the cone $\mathrm{Eff}^{\mathrm{H}}(\mathcal{F}_L)$ generated by primitive Heegner divisors is rational polyhedral. To some extent the results carry over to the case $b^+ = 2$ of Hilbert modular surfaces. The details are summarized in Section 5.

It is an interesting question whether rational polyhedrality also holds for the cone $\mathrm{Eff}(\mathcal{F}_L)$, or for which L the two cones differ at all. Even if they do, the cone $\mathrm{Eff}^{\mathrm{H}}(\mathcal{F}_L)$ is possibly at least as important. E.g., for moduli spaces of hyperkähler manifolds the ring generated by Noether-Lefschetz cycles is shown to coincide with the tautological subring of cohomology ([BL17], [PY16]).

2. Vector-valued modular forms and orthogonal Shimura varieties

This section gathers background material and notation on vector-valued modular forms associated with a lattice and on orthogonal Shimura varieties. It is well known that there are close connections between Heegner divisors on orthogonal Shimura varieties and such vector-valued modular forms. Some of these connections will be discussed and used in Section 4.

Vector-valued modular forms associated to a lattice. Suppose that $(L, (\cdot,\cdot))$ is an even lattice of signature (b^+, b^-) with quadratic form $Q(x) = \frac{1}{2}(x,x)$. We denote by $L^\vee$ the dual lattice of L and by $D_L = L^\vee/L$ the discriminant group. The order of D_L is given by the absolute value of the Gram determinant $\det(L)$, which is the determinant of any Gram matrix of the bilinear form $(\cdot,\cdot)$. We let N be the level of L, i.e., the smallest integer such that $NQ(x) \in \mathbb{Z}$ for every $x \in L^\vee$. We denote by $(\chi_\mu)_{\mu \in D_L}$ the standard basis of the group ring $\mathbb{C}[D_L]$. We realize the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ as the group of pairs (g, σ) where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and σ is a holomorphic square root of the automorphy factor $j(g,\tau) = c\tau + d$ on $\mathbb{H}$. Here the multiplication is defined as usual by

$$(g_1, \sigma_1(\tau))(g_2, \sigma_2(\tau)) = (g_1 g_2, \sigma_1(g_2\tau)\sigma_2(\tau)).$$

Recall that there is a Weil representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[D_L]$; see, e.g., [Bor98], [Bru02a].

A *vector-valued modular form of weight* $k \in \frac{1}{2}\mathbb{Z}$ for $\mathrm{Mp}_2(\mathbb{Z})$ and the Weil representation ρ_L is a holomorphic function $f : \mathbb{H} \to \mathbb{C}[D_L]$ that satisfies the transformation law

$$f(\gamma\tau) \;=\; \sigma(\tau)^{2k} \rho_L(\gamma) f(\tau)$$

for all $\gamma = (g, \sigma) \in \mathrm{Mp}_2(\mathbb{Z})$ and that is holomorphic at ∞. The vector space of such modular forms is denoted by $M_{k,L}$. Every $f \in M_{k,L}$ has a Fourier expansion

$$f(\tau) = \sum_{\mu\in D_L} \sum_{\substack{m\in Q(\mu)+\mathbb{Z}\\ m\geq 0}} a(m,\mu)\, q^m\, \chi_\mu, \quad \text{where} \quad q = e^{2\pi i\tau}\,.$$

We suppose throughout that $2k \equiv b^+ - b^- \pmod 4$. Let $\Gamma'_\infty \subset \mathrm{Mp}_2(\mathbb{Z})$ be the stabilizer of the cusp ∞. For every half-integer $k > 2$ the Eisenstein series

$$E_{k,L}(\tau) = \sum_{(g,\sigma)\in\Gamma'_\infty\setminus \mathrm{Mp}_2(\mathbb{Z})} \sigma(\tau)^{-2k} \cdot \rho_L(g,\sigma)^{-1}\chi_0 \tag{1}$$

is a vector-valued modular form in $M_{k,L}$. There are two subspaces

$$S_{k,L} \subset M^0_{k,L} \subset M_{k,L}$$

of the space of modular forms that are relevant in what follows, namely, the subspace of cusp forms $S_{k,L}$ and the intermediate space $M^0_{k,L}$ of forms whose constant term is supported at the trivial element of D_L only. Following [Pet15] we refer to this space as *almost cusp forms*. Obviously

$$M^0_{k,L} = S_{k,L} \oplus \langle E_{k,L}\rangle\,. \tag{2}$$

Eisenstein series. The coefficients of the Fourier expansion of the Eisenstein series

$$E_{k,L}(\tau) = \sum_{\mu\in D_L}\sum_{m\geq 0} e_{k,L}(m,\mu) q^m \chi_\mu \tag{3}$$

have been computed explicitly in [BK01]. First, the $e_{k,L}$ are rational numbers and (compare [Bru17, Proposition 2.1] or [BK01, Theorem 4.6])

$$(-1)^{(2k-b^++b^-)/4}\, e_{k,L}(m,\mu) \geq 0 \quad \text{if} \quad (m,\mu) \neq (0,0)\,. \tag{4}$$

The constant term of $E_{k,L}$ is given by $\chi_0 \in \mathbb{C}[D_L]$.

We now specialize to the case $k = (b^+ + b^-)/2 = \mathrm{rank}(L)/2$, which is the relevant case for the geometric application. In this case, the Fourier coefficients of the Eisenstein series are given by the following formulas (see [BK01, Theorem 4.6]). For a discriminant $D \in \mathbb{Z}\setminus\{0\}$ we define the Dirichlet character $\chi_D = \left(\frac{D}{a}\right)$ and the divisor sums with character

$$\sigma_s(a,\chi) = \sum_{d\mid a} \chi(d) d^s\,.$$

For $\mu \in D_L$ we let $d_\mu = \min\{b \in \mathbb{Z}_{>0} :\ b\mu = 0\}$ be the order of μ. For $m \in \mathbb{Z} + Q(\mu)$ we denote by $N^L_{m,\mu}(a)$ the mod-a representation number

$$N^L_{m,\mu}(a) = |\{r \in L/aL :\ Q(r+\mu) \equiv m \pmod a\}|\,.$$

We will frequently drop the superscript L if it is clear from the context. Moreover, we define

$$w_p \;=\; w_p(m,\mu) \;=\; 1 + 2\,\mathrm{ord}_p(2d_\mu m)\,.$$

The reason for introducing this is that the normalized representation numbers $p^{\nu(1-2k)}N_{m,\mu}(p^\nu)$ are independent of ν if $\nu \geq w_p$ (see Hilfssatz 13 in [Sie35]).

Suppose that $b^+ + b^-$ is even. Then, for $\mu \in D_L$ and $m \in \mathbb{Z} + Q(\mu)$ positive, the coefficients are given by

$$e_{k,L}(m,\mu) = \frac{(2\pi)^k m^{k-1}(-1)^{b^-/2}}{\sqrt{|D_L|}\Gamma(k)} \cdot \frac{1}{L(k,\chi_{4D})} \cdot \varepsilon_{m,\mu}, \tag{5}$$

where D denotes the discriminant $D = (-1)^{(b^+ + b^-)/2}\det(L)$ and where

$$\varepsilon_{m,\mu} \;=\; \prod_{\substack{p \text{ prime}\\ p|2mN}} \frac{p^{w_p(1-2k)}N_{m,\mu}(p^{w_p})}{1-\chi_D(p)p^{-k}} \tag{6}$$

$$\;=\; \sigma_{1-k}(d_\mu^2 m, \chi_{4D}) \prod_{\substack{p \text{ prime}\\ p|2N}} \frac{N_{m,\mu}(p^{w_p})}{p^{(2k-1)w_p}}. \tag{7}$$

Suppose that $b^+ + b^-$ is odd. We write $md_\mu^2 = m_0 f^2$ for positive integers m_0 and f with $(f, 2N) = 1$ and $\mathrm{ord}_p(m_0) \in \{0,1\}$ for all primes p coprime to $2N$. In this case

$$e_{k,L}(m,\mu) \;=\; \frac{(2\pi)^k m^{k-1}(-1)^{b^-/2}}{\sqrt{|D_L|}\Gamma(k)} \cdot \frac{L(k-1/2,\chi_{D'})}{\zeta(2k-1)} \cdot \varepsilon_{m,\mu}, \tag{8}$$

where D' now denotes the discriminant $D' = 2(-1)^{(b^+ + b^- + 1)/2}m_0\det(L)$ and $\mu(\cdot)$ the Möbius function, and where

$$\varepsilon_{m,\mu} \;=\; \prod_{\substack{p \text{ prime}\\ p|2mN}} \frac{1-\chi_{D'}(p)p^{1/2-k}}{1-p^{1-2k}} p^{w_p(1-2k)}N_{m,\mu}(p^{w_p}) \tag{9}$$

$$\;=\; \sum_{g|f} \mu(g)\chi_{D'}(g)g^{1/2-k}\sigma_{2-2k}(f/g) \prod_{\substack{p \text{ prime}\\ p|2N}} \frac{N_{m,\mu}(p^{w_p})}{(1-p^{1-2k})p^{(2k-1)w_p}}\,. \tag{10}$$

Estimates for representation numbers. The following lemmas are used to give lower and upper bounds for the coefficients $e_{k,L}$. Note that the representation numbers $N_{m,\mu}(a)$ are weakly multiplicative in a.

We let U be the even unimodular lattice of signature $(1,1)$, realized as $\mathbb{Z}^2$ with the quadratic form $Q((x,y)) = xy$ (a hyperbolic plane). The following lemma is well known.

Lemma 2.1. *Let $m \in \mathbb{Z}$ and $\nu \in \mathbb{Z}_{\geq 0}$. Then*

$$N_{m,0}^{U}(p^\nu) = \begin{cases} (\operatorname{ord}_p(m)+1)(1-1/p)p^\nu & \text{if } \operatorname{ord}_p(m) < \nu, \\ \nu(1-1/p)p^\nu + p^\nu & \text{if } \operatorname{ord}_p(m) \geq \nu. \end{cases}$$

Corollary 2.2. *We have*

$$1 - 1/p \leq p^{-\nu} N_{m,0}^{U}(p^\nu) \leq \nu + 1.$$

Lemma 2.3. *Suppose that $L = L_1 \oplus U$ for an even lattice L_1 with* $\operatorname{rank}(L_1) = \operatorname{rank}(L) - 2$. *Let $\mu \in D_L$ and $m \in \mathbb{Z} + Q(\mu)$. Then*

$$1 - 1/p \leq p^{(1-2k)\nu} N_{m,\mu}^{L}(p^\nu) \leq \nu + 1\,. \tag{11}$$

Proof. Any lattice element $\lambda \in L$ can be written in the basis of $L_1 \oplus U$ as $\lambda = (\lambda_1, x, y)$ with $\lambda_1 \in L_1$ and $x, y \in \mathbb{Z}$. Then $Q(\lambda) = Q(\lambda_1) + xy$, and we may suppose that $\mu = (\mu_1, 0, 0)$ since U is self-dual. We have

$$\begin{aligned} N_{m,\mu}^{L}(p^\nu) &= |\{\lambda_1 \in L_1/p^\nu L_1, (x,y) \in U/p^\nu U \mid Q(\lambda_1+\mu_1)+xy \equiv m \pmod{p^\nu}\}| \\ &= \sum_{\lambda_1 \in L_1/p^\nu L_1} N_{m-Q(\lambda_1+\mu_1),0}^{U}(p^\nu). \end{aligned}$$

Hence the claimed bounds follow directly from Corollary 2.2. □

We now derive similar results for lattices which split two hyperbolic planes over $\mathbb{Z}$. It turns out that we get slightly stronger bounds in this case.

Lemma 2.4. *Let $m \in \mathbb{Z}$ and $\nu \in \mathbb{Z}_{\geq 0}$. Then*

$$N_{m,0}^{U\oplus U}(p^\nu) = \begin{cases} p^{3\nu}(1+p^{-1})(1-p^{-\operatorname{ord}_p(m)-1}) & \text{if } \operatorname{ord}_p(m) < \nu, \\ p^{3\nu}(1+p^{-1}-p^{-\nu-1}) & \text{if } \operatorname{ord}_p(m) \geq \nu. \end{cases}$$

Proof. For the proof we briefly put $M = U \oplus U$. The statement for $\operatorname{ord}_p(m) < \nu$ follows from a result of Siegel; see, e.g., [BK01, Theorem 4.5]. The second statement can be deduced from the first, since

$$\begin{aligned} N_{0,0}^{M}(p^\nu) &= \#(M/p^\nu M) - \sum_{\substack{a \in \mathbb{Z}/p^\nu\mathbb{Z} \\ a \neq 0}} N_{a,0}^{M}(p^\nu) \\ &= p^{4\nu} - \sum_{j=0}^{\nu-1} \sum_{b \in (\mathbb{Z}/p^{\nu-j}\mathbb{Z})^\times} N_{p^j b,0}^{M}(p^\nu) \\ &= p^{4\nu} - \sum_{j=0}^{\nu-1} (p^{\nu-j} - p^{\nu-j-1}) p^{3\nu}(1+p^{-1})(1-p^{-j-1}). \end{aligned}$$

Computing the latter sum, we obtain the assertion. □

Corollary 2.5. *We have*

$$1 - p^{-2} \leq p^{-3\nu} N_{m,0}^{U\oplus U}(p^\nu) \leq 1 + p^{-1}.$$

Lemma 2.6. *Let L be an even lattice of rank $2k = b^+ + b^-$. Suppose that $L = L_0 \oplus U \oplus U$ for an even lattice L_0 of rank $2k-4$. Let $\mu \in D_L$ and $m \in \mathbb{Z} + Q(\mu)$. Then for all primes p and all $\nu \in \mathbb{Z}_{\geq 0}$ we have*

$$1 - p^{-2} \leq p^{(1-2k)\nu} N^L_{m,\mu}(p^\nu) \leq 1 + p^{-1}\,. \tag{12}$$

Proof. Any lattice element $\lambda \in L$ can be uniquely written as $\lambda = \lambda_0 + \lambda_1$ with $\lambda_0 \in L_0$ and $\lambda_1 \in U \oplus U$. Then $Q(\lambda) = Q(\lambda_0) + Q(\lambda_1)$ and

$$N^L_{m,\mu}(p^\nu) = \sum_{\lambda_0 \in L_0/p^\nu L_0} N^{U\oplus U}_{m-Q(\lambda_0+\mu),0}(p^\nu).$$

Hence the claimed bounds follow directly from Corollary 2.2. □

The period domain of orthogonal Shimura varieties and Heegner divisors. Let L be an even lattice of signature $(b^+, 2)$. The Hermitian symmetric domain $\mathcal{D}_L$ of the orthogonal group of this lattice can be realized as one of the two connected components of

$$\mathcal{D}_L \cup \overline{\mathcal{D}_L} \;=\; \{z \in L_{\mathbb{C}} \,:\, Q(z) = 0 \quad \text{and} \quad (z, \overline{z}) < 0\}/\mathbb{C}^\times\,.$$

Following [GHS07], we let $\mathrm{O}^+(L)$ be the index two subgroup of the orthogonal group $\mathrm{O}(L)$ which preserves the components, that is, the subgroup of elements of $\mathrm{O}(L)$ of positive spinor norm. We let $\widetilde{\mathrm{O}}(L)$ be the discriminant kernel of $\mathrm{O}(L)$, that is, the kernel of the natural homomorphism $\mathrm{O}(L) \to \mathrm{Aut}(D_L)$, and we put

$$\widetilde{\mathrm{O}}^+(L) = \widetilde{\mathrm{O}}(L) \cap \mathrm{O}^+(L).$$

The moduli spaces we are interested in are the locally symmetric spaces

$$\mathcal{F}_L(\Gamma) = \Gamma \backslash \mathcal{D}_L \quad \text{for} \quad \Gamma \subseteq \widetilde{\mathrm{O}}^+(L),$$

a subgroup of finite index. We abbreviate $\mathcal{F}_L = \mathcal{F}_L(\widetilde{\mathrm{O}}^+(L))$.

For any vector $v \in L^\vee$ with $Q(v) > 0$ the hyperplane $H_v \subset \mathcal{D}_L$ consists of the points z orthogonal to v. For $\mu \in D_L$ and $m \in Q(\mu) + \mathbb{Z}$ positive, the group $\widetilde{\mathrm{O}}^+(L)$ acts on vectors in $\mu + L$ of norm m with finitely many orbits. Consequently, for any $\Gamma \subseteq \widetilde{\mathrm{O}}^+(L)$ the (*reducible*) *Heegner divisors*, defined as

$$H_{m,\mu} \;=\; \Gamma \setminus \Bigg(\sum_{\substack{v \in \mu + L \\ Q(v) = m}} H_v \Bigg)\,,$$

are well-defined in $\mathcal{F}_L(\Gamma)$. These are in general neither reduced nor irreducible. In particular for $\Gamma \subseteq \widetilde{\mathrm{O}}^+(L)$ of large index, $H_{m,\mu}$ may have many components. Moreover, all the components have multiplicity two if $\mu = -\mu$ and they all have multiplicity one otherwise (see Lemma 4.2 below). We will discuss the passage to irreducible components in Section 4.

The tautological line bundle $\mathcal{O}(-1)$ on $\mathcal{D}_L$ descends to a line bundle λ on $\mathcal{F}_L$, called the *Hodge bundle*. (Thus λ is anti-ample in our notation.) The Hodge bundle plays no role in our calculation, but the intersection numbers with λ arise as coefficient extraction functionals, similar to the intersection numbers with Heegner divisors; see the proof of Theorem 4.1.

Moduli spaces of $K3$-surfaces and Noether-Lefschetz divisors. In the special case of the lattice $L = L_{2d}$ of signature $(19, 2)$ given by

$$L_{2d} \;=\; \langle 2d\rangle \oplus U^{\oplus 2} \oplus E_8^{\oplus 2} \tag{13}$$

the discriminant group is $D_L \cong \mathbb{Z}/2d$. The modular variety

$$\mathcal{F}_{2d} \cong \widetilde{\mathrm{O}}^+(L)\backslash\mathcal{D}_L$$

is closely related to the coarse moduli space of $2d$-polarized $K3$-surfaces. More precisely, an open subset $\mathcal{F}^\circ_{2d}$ of $\mathcal{F}_{2d}$, the complement of some Heegner divisors, is the coarse moduli space of polarized $K3$-surfaces of degree $2d$, while $\mathcal{F}_{2d}$ is the moduli space of quasi-polarized $K3$-surfaces. See [PSS71] and [Mor83] for the quasi-polarized case and see, e.g., [GHS13] for a survey.

Note that in our definition L is isomorphic to the orthogonal complement of the polarization class H with $Q(H) = d$ in middle cohomology lattice

$$L_{K3} \;=\; U^{\oplus 3} \oplus E_8^{\oplus 2}$$

of the $K3$-surface with the *negative* of the intersection pairing. We let ω be a fixed generator of the first summand of L in (13). Hence, D_L is generated by $\omega/2d$.

The generic algebraic $K3$-surface has a Picard group of rank one. The (reducible) *Noether-Lefschetz divisors* $D_{h,a}$ are the closures in $\mathcal{F}_{2d}$ of the loci where the Picard group of the polarized $K3$-surfaces (S, H) have a class β not in the linear space of H with $Q(\beta) = h - 1$ and $(\beta, H) = a$. We may assume that $0 \le a < 2d$. These are also the images in $\mathcal{F}_{2d}$ of certain hyperplanes in $\mathcal{D}_{2d}$ (see, e.g., [MP13, Section 4.4]). More precisely, the projection onto $H^\perp$ given by $v \mapsto \beta - \frac{a}{2d}H$ induces a bijection of the Noether-Lefschetz divisors $D_{h,a}$ and the Heegner divisor $H_{m,\mu}$ with invariants related by

$$m \;=\; \frac{a^2}{4d} - (h-1) \qquad \mu = a \cdot \omega/2d \bmod L\,. \tag{14}$$

(The positivity of m is guaranteed by the Hodge index theorem.) We thus use the terms Heegner divisors and Noether-Lefschetz divisors interchangeably in the $K3$ case.

3. Cones of coefficients of modular forms

Our goal in this section is to show (in Theorem 3.4 below) that the cone generated by the coefficient functionals of Fourier expansions of vector-valued modular forms for a lattice L is rational polyhedral on the space of almost-cusp forms. Our main criterion for rational polyhedrality is the following geometric observation.

Lemma 3.1. *Suppose that V is a finite-dimensional $\mathbb{Q}$-vector space and consider the cone*

$$\mathcal{C} \;=\; \Big\{ \sum_{n \geq 0} \lambda_n c_n \;\Big|\; \textit{all } \lambda_n \in \mathbb{R}_{\geq 0}, \textit{ and almost all } \lambda_n \textit{ vanish} \Big\} \subseteq V_{\mathbb{R}} = V \otimes \mathbb{R}$$

generated by a countable collection of non-zero vectors $(c_n) \subset V$. Suppose that there exists a codimension one subspace $S \subset V$ and an element $e \in V \setminus S$ with the following properties:

(i) *Writing $c_n = \gamma_n e + s$ with $s \in S$, the coefficient γ_n is positive for all n.*
(ii) *The vectors c_n converge $\mathbb{R}_{>0}$-projectively to e, i.e., $c_n/\gamma_n - e \to 0 \in S_{\mathbb{R}}$.*
(iii) *Among the c_n there exist elements $c_{n_1}, \ldots, c_{n_s}$ such that a linear combination $\sum_{i=1}^{s} \lambda_i c_{n_i}$ with all $\lambda_i \in \mathbb{R}_{>0}$ strictly positive lies in $\langle e \rangle$ and such that the classes of $c_{n_i} \in V/\langle e \rangle \cong S$ span S.*

Then the cone $\mathcal{C}$ is rational polyhedral.

Proof. Due to the first condition the cone lies in the half-space of V where to e-coefficient is positive. It thus suffices to show that the convex body $\mathcal{C}_S$ defined as the intersection of $\mathcal{C}$ with the affine hyperplane $e + S_{\mathbb{R}}$ is rational polyhedron. We view $\mathcal{C}_S \subset S_{\mathbb{R}}$ by projection along e. Condition iii) now implies immediately that this polyhedron $\mathcal{C}_S$ contains an open neighborhood of zero. Condition ii) implies that the projections of c_n to $\mathcal{C}_S$ converge to zero in $S_{\mathbb{R}}$, which is an interior point of $\mathcal{C}_S$. Consequently, $\mathcal{C}_S$ is the convex hull of finitely many points $\mathbb{R}_{>0} \cdot c_n \cap (e + S_{\mathbb{R}})$. Since these are rational, the claim follows. $\square$

We suppose *in the remainder of this section that $k \geq 2$, with $2k - b^+ + b^- \equiv 4 \pmod 8$, and that $L = L_1 \oplus U$ splits off a hyperbolic plane U.* Here we want to apply Lemma 3.1 to $V = (M^0_{k,L}(\mathbb{Q}))^\vee$, the dual of the space of almost-cusp forms of half-integral weight k with rational coefficients. This rationality statement and the rationality hypothesis in Lemma 3.1 is a restatement of the fact ([McG03]) that $M_{k,L}$ has a basis with rational coefficients.

The direct sum decomposition (2) implies that V decomposes as

$$V \;=\; \langle e \rangle \oplus S\,,$$

where e is defined by $e(E_{k,L}) = -1$ and $e(S_{k,L}) = 0$, and where S is the subspace of functionals that are zero on $E_{k,L}$. We want to apply this lemma to the vectors c_n being the coefficient extraction functionals

$$c_{m,\mu} : M^0_{k,L}(\mathbb{Q}) \to \mathbb{Q}, \quad f = \sum_{\mu\in D_L}\sum_{n\geq 0} a_{m,\mu}q^m e_\mu \mapsto c_{m,\mu}(f) = a_{m,\mu}$$

for $\mu \in D_L$ and $m \in (Q(\mu)+\mathbb{Z})\cap\mathbb{Q}_{>0}$; i.e., the index set consists of pairs $n = (m,\mu)$.

Condition i) of the lemma is simply a restatement of (4). Note that the strict positivity follows from the fact that L splits off a hyperbolic plane over $\mathbb{Z}$. Condition ii) of this lemma is a consequence of the following proposition.

Proposition 3.2. *Assume $k \geq 5/2$. For $\mu \in D_L$ and $m \in Q(\mu)+\mathbb{Z}$ positive, the coefficients $e_{k,L}(m,\mu)$ of the Eisenstein series $E_{k,L}$ are negative and satisfy*

$$-e_{k,L}(m,\mu) \geq C_{k,L}\cdot m^{k-1}$$

for some positive constant $C_{k,L}$ depending on the weight and the lattice.

If $k = 2$ coefficients $e_{k,L}(m,\mu)$ of the Eisenstein series $E_{k,L}$ are negative and for every $\varepsilon > 0$ there is a positive constant $C_{L,\varepsilon}$ depending on the lattice such that

$$-e_{2,L}(m,\mu) \geq C_{L,\varepsilon}\cdot m^{1-\varepsilon}.$$

For any $k \geq 2$ the coefficients $a_{m,\mu}$ of any cusp form $f = \sum a_{m,\mu}q^m\chi_\mu \in S_{k,L}$ are bounded above in absolute value by

$$|a_{m,\mu}| \leq C_{f,\varepsilon}\, m^{\frac{k}{2}-\frac{1}{4}+\varepsilon} \tag{15}$$

for some positive constant $C_{f,\varepsilon}$ depending on f and ε.

Proof. The last statement is the Weil bound for the coefficients of cusp forms, see, e.g., [Sar90, Propositions 1.5.3 and 1.5.5].

The negativity in the first statement is a consequence of (4) and our congruence condition on k. To prove the lower bound, given the factor m^{k-1} in both (5) and (8), we have to bound the other terms that depend on m uniformly from below.

In the case b^+ odd we use the expression in (10). Lemma 2.3 gives a lower bound for $N_{m,\mu}(p^{w_p})/p^{(2k-1)w_p}$ for the finitely many primes dividing $2N$. For the remaining terms we use the estimate that for $k \geq 5/2$,

$$\begin{aligned}\sum_{g|f}\mu(g)\chi_{D'}(g)g^{1/2-k}\sigma_{2-2k}(f/g) &\geq 1-\sum_{\substack{g|f\\ g>1}} g^{1/2-k}\sigma_{2-2k}(f/g)\\ &\geq 1-(\zeta(2)-1)\zeta(3) \geq 1/5.\end{aligned}$$

In the case b^+ even and $k \geq 3$ we use the expression (7). Again, Lemma 2.3 gives a lower bound for the normalized representation numbers and together with

$$\sigma_{1-k}(d_\mu^2 m, \chi_{4D}) \geq 1 - (\zeta(2) - 1) > 0$$

we obtain a uniform lower bound for $\varepsilon_{m,\mu}$. Finally, in the case $k = 2$ we split off the contribution of the divisor 1 to $\sigma_{1-k}(d_\mu^2 m, \chi_{4D})$ and use the estimate

$$\sigma_{1-k}(d_\mu^2 m, \chi_{4D}) \leq \sigma_1(d_\mu^2 m) = O(\log(m))$$

for the remaining terms. □

Because of the direct sum decomposition $V = \langle e \rangle \oplus S$ provided by the conditions in Lemma 3.1, its condition iii) can be formulated equivalently in terms of the restriction of the coefficient functionals

$$\bar{c}_{m,\mu} = c_{m,\mu}|_{S_{k,L}(\mathbb{Q})} : S_{k,L}(\mathbb{Q}) \to \mathbb{Q}$$

to the subspace of cusp forms with rational coefficients. The statement is precisely the content of the following proposition.

Proposition 3.3. *There exist indices $((m_i, \mu_i))_{i=1}^s$ and real numbers $\lambda_i > 0$ such that*

$$\sum_{i=1}^{s} \lambda_i \, \bar{c}_{m_i,\mu_i} = 0$$

in $S_{k,L}(\mathbb{R})^\vee$ and such that the functionals $(\bar{c}_{m_i,\mu_i})_{i=1}^s$ span $S_{k,L}(\mathbb{R})^\vee$.

Proof. As in [Bru17] we write L^- for the lattice $(L, -Q)$. We identify the Weil representation ρ_{L^-} with the dual representation of ρ_L. The product of a weakly holomorphic modular form h of weight $2 - k$ for the representation ρ_{L^-} and any element $g \in S_{k,L}$ is a weakly holomorphic modular form of weight 2 for $\mathrm{Mp}_2(\mathbb{Z})$, i.e., a meromorphic differential form on the modular curve $X(1)$. The residue at the cusp ∞ of hg vanishes by the residue theorem. The idea is to construct a weakly holomorphic modular form h for ρ_{L^-} whose principal part at ∞ has non-negative coefficients only and whose principal part has sufficiently many non-vanishing terms (that will be the λ_i of the proposition) so that the residue pairing $g \mapsto \mathrm{Res}(hg)$ involves a spanning system of $S_{k,L}(\mathbb{R})^\vee$.

This follows from [Bru17, Lemma 3.5 and Proposition 3.2]. In fact, let

$$t_\mu = \min\{-Q(\lambda) \mid \lambda \in \mu + L, \ -Q(\lambda) > 0\} \in \tfrac{1}{N}\mathbb{Z}_{>0} \quad \text{for} \quad \mu \in D_L,$$

and let $T = \max\{t_\mu \mid \mu \in D_L\}$. Choose $B \in \mathbb{Z}_{>0}$ sufficiently large such that the weight $k' := 2 - k + 12B > 2$ and such that the functionals $\bar{c}_{\mu,\ell}$ for $\ell < B - T$ generate $S_{k,L}(\mathbb{R})^\vee$.

We claim that the weakly holomorphic modular form

$$h(\tau) = \Delta(\tau)^{-B} E_{k',L^-}(\tau) \in M^!_{2-k,L^-}$$

has non-negative Fourier coefficients $c_{\mu,\ell}(h)$ and moreover

$$c_{\ell,\mu}(h) > 0 \quad \text{for all} \quad \mu \in D_L, \quad \ell \in \mathbb{Z} - Q(\mu), \quad \ell \geq T - B\,.$$

To see this, note that the Fourier expansion of $\Delta(\tau)^{-B}$ is q^{-B} times a positive power of the generating function $\prod_{j\geq 1}(1-q^j)^{-1}$ of the partition function. Consequently, the coefficient extraction $c_j(\Delta^{-B})$ is positive for integral $j \geq -B$ and zero otherwise. If $\mu \in D_L$ and $\ell \in \mathbb{Z} - Q(\mu)$, then

$$c_{\ell,\mu}(h) \;=\; c_{\ell - t_\mu}(\Delta^{-B}) \cdot e_{k',L^-}(t_\mu, \mu) + \sum_{\ell - t_\mu < j \in \mathbb{Z}} c_j(\Delta^{-B}) \cdot e_{k',L^-}(\ell - j, \mu)\,.$$

The congruence hypothesis on k implies that

$$2k' - b^+(L^-) + b^-(L^-) \;=\; 2k' - b^- + b^+ \;\equiv\; 0\,(8)$$

and hence that by (4) all the coefficients $e_{k',L^-}(\cdot\,,\cdot)$ are positive. Consequently, the first summand of the right hand side is positive by the definition of t_μ and the hypothesis on ℓ, and the other summands are non-negative. This implies the claim.

Let $((m_i,\mu_i))_{i=1}^s$ be some enumeration of the pairs (m,μ) for $\mu \in D_L$ and $m \in \mathbb{Z}+Q(\mu)$ with $0 < m < B-T$. By our choice of B, the functionals $\bar{c}_{m_i,\mu_i}$ span $S_{k,L}(\mathbb{R})^\vee$. If we let $\lambda_i = c_{-m_i,\mu_i}(h)$, then the residue theorem applied to gh implies that

$$\sum_{i=1}^s \lambda_i\, \bar{c}_{m_i,\mu_i}(g) \;=\; 0$$

for any $g \in S_{k,L}$. □

Not only does this proof break down if the congruence hypothesis on k is violated but also the statement is wrong in this case, as pairing the weakly holomorphic form h with the Eisenstein series $E_{k,L}$ shows. In fact, all its coefficients are positive, including the constant term.

We summarize the results of this section in the following statement.

Theorem 3.4. *Let L be a lattice of signature (b^+,b^-) that splits off a hyperbolic plane. We suppose that $k \geq 2$ and $2k - b^+ + b^- \equiv 4 \pmod 8$. Then the cone $\mathcal{C}$ generated by the coefficient functionals $c_{m,\mu}$ on the space of weight k almost-cusp forms $M^0_{k,L}$ for the lattice L (where $\mu \in D_L$ and $m \in (\mathbb{Z}+Q(\mu)) \cap \mathbb{Q}_{>0}$) is a rational polyhedral cone. In particular, the cone $\mathcal{C}$ is finitely generated.*

4. Cones of primitive Heegner divisors

We now translate the results of the previous section into geometric statements. We suppose for the rest of this paper that $b^- = 2$ and put $k = 1+b^+/2$.

Our motivation for studying cones of coefficient functionals in spaces of modular forms comes from the following transport of structure to the rational Picard group of orthogonal Shimura varieties. By [Bor99] or [Bru02a, Theorem 0.4] and [Pet15, Theorem 4.3.2] the map

$$\psi : M^0_{k,L}(\mathbb{Q})^\vee \;\to\; \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_L(\Gamma)), \qquad c_{m,\mu} \mapsto H_{m,\mu}\,, \tag{16}$$

sending the coefficient extraction functional $c_{m,\mu}$ to the Heegner divisor $H_{m,\mu}$ and the coefficient extraction functional $-c_{0,0}$ to the Hodge class λ, is a homomorphism. We use this to show the following.

Theorem 4.1. *Let $\Gamma \subseteq \widetilde{\mathrm{O}}^+(L)$ be a finite index subgroup. Suppose that $b^+ \geq 3$ and that L splits off a hyperbolic plane. Then the cone generated by the* (*reducible*) *Heegner divisors $H_{m,\mu}$ on $\mathcal{F}_L(\Gamma)$ is rational polyhedral.*

Proof. Since $b^- = 2$ and $k = 1 + b^+/2$, the congruence condition for k in Theorem 3.4 holds. The claim follows from this theorem and the fact that the image of a rational polyhedral cone under a linear map is still rational polyhedral. □

The map ψ is injective in many situations (e.g., if L splits off two hyperbolic planes, [Bru02a]), and we will use this below. It is moreover surjective ([BLMM17]) under the hypotheses made here. This implies that the image cone is full-dimensional.

The second goal of this section is to discuss the passage from primitive to (reducible) Heegner divisors and to prove Theorem 1.1 stated in the introduction, and the generalization Theorem 4.4 below.

Primitive Heegner divisors. The Heegner divisors are in general not irreducible. The divisibility of the defining lattice element $v \in L^\vee$ with $Q(v) = m$ and $v \equiv \mu \mod D_L$ is an obvious invariant distinguishing irreducible components. Since divisibility is preserved by the action of $\widetilde{\mathrm{O}}^+(L)$, the definition

$$P_{\Delta,\delta} \;=\; \widetilde{\mathrm{O}}^+(L) \setminus \Bigg(\sum_{\substack{L+\delta \ni v \text{ primitive},\\ Q(v)=\Delta}} H_v \Bigg)$$

for $\delta \in D_L$ and $\Delta \in \mathbb{Z}+Q(\delta)$ gives well-defined divisors in $\mathcal{F}_L$, called *primitive Heegner divisors*. By definition (and the fact that any lattice vector can be written uniquely as a positive multiple of a primitive lattice vector)

$$H_{m,\mu} \;=\; \sum_{\substack{r\in\mathbb{Z}_{>0}\\ r^2\mid m}} \sum_{\substack{\delta\in D_L\\ r\delta=\mu}} P_{m/r^2,\delta}\,. \tag{17}$$

Here and in what follows we say that $r^2 \mid m$ with $m \in Q(\mu)+\mathbb{Z}$ if there exists $\delta \in D_L$ with $m/r^2 \in Q(\delta)+\mathbb{Z}$. We could drop this condition, since $P_{m/r^2,\delta}$ is empty otherwise.

The converse to (17) follows from a variant of Möbius inversion.

Lemma 4.2. *The primitive divisors $P_{\Delta,\delta}$ can be written in terms of the Heegner divisors $H_{m,\sigma}$ as*

$$P_{\Delta,\delta} = \sum_{\substack{r\in\mathbb{Z}_{>0}\\ r^2|\Delta}} \mu(r) \sum_{\substack{\sigma\in D_L\\ r\sigma=\delta}} H_{\Delta/r^2,\sigma}\,. \tag{18}$$

Proof. The statement for Δ without quadratic divisors is obvious and from the definition in (17) and induction on the number of quadratic divisors we obtain

$$\begin{aligned} P_{\Delta,\delta} &= H_{\Delta,\delta} - \sum_{\substack{s\neq 1\\ s^2|\Delta}} \sum_{\tau s=\delta} \Bigg(\sum_{\substack{t\in\mathbb{Z}\\ t^2|\Delta/s^2}} \mu(t) \sum_{\sigma t=\tau} H_{\Delta/s^2t^2,\sigma} \Bigg) \\ &= H_{\Delta,\delta} - \sum_{\substack{s\neq 1\\ s^2|\Delta}} \Bigg(\sum_{\substack{t\in\mathbb{Z}\\ t^2|\Delta/s^2}} \mu(t) \sum_{\tau s=\delta} \sum_{\sigma t=\tau} H_{\Delta/s^2t^2,\sigma} \Bigg). \end{aligned} \tag{19}$$

We can group the interior double sum as a single sum over all σ with $\sigma\cdot st=\delta$. We let $r=st$ and consider the summands contributing to $H_{\Delta/r^2,\sigma}$. It remains to show that for given r the exterior double sum including the factor $\mu(t)$ adds up to $\mu(r)$. If some prime divides r more than once, the claimed contribution follows since $\sum_{I\subseteq P}(-1)^{|I|}=0=\mu(h)$ for any finite set (of primes) P. In the remaining cases, one summand is missing in the subset summation since $1\neq s$, and with the global minus sign we obtain the coefficient $\mu(h)$ we want. □

Primitive Noether-Lefschetz divisors. In the case of $K3$-lattice $L=L_{2d}$ the decomposition of Heegner divisors into irreducible components can also be described by a geometric decomposition of Noether-Lefschetz divisors. The Picard group of a generic member of the Noether-Lefschetz divisors contains a rank two lattice Λ with signature $(1,1)$. Conversely, the intersection matrix Λ with respect to some basis $\{H,\beta\}$ starting with the polarization class H with $Q(H)=2d$ has the form

$$M_\Lambda = \begin{pmatrix} 2d & y \\ y & 2x \end{pmatrix}.$$

The discriminant $\Delta(\Lambda)=\det(M_\Lambda)\in\mathbb{Z}$ and the coset $\delta=y \mod 2d$ are invariants of such a lattice and it is easy to show that the pair (Δ,δ) is a classifying invariant of such rank two lattices. We now define the primitive Noether-Lefschetz divisors $\mathcal{P}_{\Delta,\delta}$ to be the closure of the locus of $K3$-surfaces that have a sublattice $\Lambda\subset L_{K3}$ of signature $(1,1)$, containing H and we provide them with multiplicity one or two depending on $2\delta\neq 0$ or not modulo $2d$.

The similarity in notation to primitive Heegner divisors is intentional, since we claim that

$$\mathcal{P}_{\Delta,\delta} \;=\; P_{\Delta/4d,\delta(\omega/2d)}\,. \tag{20}$$

This can be seen from the definitions, tracing the definitions along the bijection $v \mapsto \beta - \frac{1}{2d}H$ between Heegner and Noether-Lefschetz divisors given along with equation (14).

The main result on $\mathrm{Eff}^{\mathrm{H}}(\mathcal{F}_L)$**.** We suppose in the remainder of this section that $b^+ \geq 3$ and that $L = L_0 \oplus U^{\oplus 2}$ splits off **two** copies of a hyperbolic plane U. We put $k = 1 + b^+/2$ and $\Gamma = \widetilde{\mathrm{O}}^+(L)$; i.e., we work on the Shimura varieties $\mathcal{F}_L$.

Lemma 4.3. *Under these conditions the primitive Heegner divisors* $P_{\Delta,\delta}$ *are irreducible if* $2\delta \neq 0 \in D_L$*. If* $2\delta = 0 \in D_L$*, then* $P_{\Delta,\delta}$ *is (if non-empty) an irreducible divisor with multiplicity two.*

Proof. The multiplicity two stems from the fact that v and $-v$ give the same divisor if $2\delta = 0 \in D_L$. It remains to show that any two primitive elements in $L^\vee$ with the same norm and the same D_L-coset lie in the same $\widetilde{\mathrm{O}}^+(L)$-orbit. This can be done using Eichler transformations; see Lemma 4.4 in [FH00]. □

The primitive Heegner divisors in our main result are thus irreducible.

Theorem 4.4. *The cone* $\mathrm{Eff}^{\mathrm{H}}(\mathcal{F}_L)$ *generated by the primitive Heegner divisors* $P_{\Delta,\delta}$ *is rational polyhedral. In particular it is finitely generated.*

We prove this theorem in the same way as Theorem 3.4, using a refinement of the estimate in Proposition 3.2. Together with the main observation of the proof of Lemma 3.1, the following proposition implies that the vectors corresponding to primitive Heegner divisors still converge to an interior point of the cone $\mathcal{C} = \mathrm{Eff}^{\mathrm{H}}(\mathcal{F}_L)$. Theorems 1.1 and 4.4 follow immediately from the following statement. Let $\varphi = \psi^{-1}$ be the inverse of the map ψ defined in (16).

Proposition 4.5. *Under the identification* φ *any infinite sequence of pairwise different primitive Heegner divisor* $P_{\Delta,\delta}$ *converges* $\mathbb{R}_{>0}$*-projectively to the functional* e*.*

Proof. As in Proposition 3.2, we show that there is a constant $C > 0$ and for any $\varepsilon > 0$ and any cusp form f of weight k there is a constant $C_{f,\varepsilon} > 0$ such that the bounds

$$\varphi(P_{\Delta,\delta})(E_{k,L}) \;\geq\; C \cdot \Delta^{k-1}$$

for the coefficients functional evaluated at Eisenstein series, and

$$|\varphi(P_{\Delta,\delta})(f)| \;\leq\; C_{f,\varepsilon} \cdot \Delta^{\frac{k}{2}-\frac{1}{4}+\varepsilon} \quad \text{if} \quad k \in \mathbb{Z}$$

hold for any $\delta \in D_L$ and any positive $\Delta \in Q(\delta) + \mathbb{Z}$.

The claim about cusp forms follows immediately from (15), since the number of summands in (18) contributing to $P_{\Delta,\delta}$ is at most D_L times the number of square divisors of Δ, which is $O(\Delta^{\varepsilon})$ for any ε.

In order to estimate the Eisenstein series contribution,we define $K_r \subseteq D_L$ to be the kernel of the multiplication by r and we observe that $1 \leq |K_r| \leq r^{2k-4}$, since the lattice L_0 is of rank $2k-4$.

In the case b^+ odd, we deduce from (18) and the formula (8) for the coefficients that $\varphi(P_{\Delta,\delta})$ evaluated at the Eisenstein series is equal to Δ^{k-1} times some constants independent of Δ times

$$Q(\Delta,\delta) \;=\; \sum_{r:r^2|\Delta} \mu(r) \sum_{\mu r=\delta} \frac{1}{r^{2(k-1)}} \varepsilon_{\Delta/r^2,\mu}\,, \tag{21}$$

where $\varepsilon_{m,\mu}$ was defined in (9). We want to show that there is some $C > 0$ such that $Q(\Delta,\delta) > C$ for all Δ. By definition of the Möbius function, $Q(\Delta,\delta)$ is greater than or equal to $\varepsilon_{\Delta,\delta}$ (stemming from $r=1$) minus the sum over subsets P of odd cardinality of the set of prime divisors of Δ. In order to estimate these negative contributions from above, we compare $\varepsilon_{\Delta,\delta}$ with $\varepsilon_{\Delta/r^2,\mu}$ using (9). First we remark that we can arbitrarily enlarge (by Theorem 7 in [BK01]) the set of primes over which the product runs. Hence we can suppose that the product runs over the same set of primes when computing $\varepsilon_{\Delta,\delta}$ and $\varepsilon_{\Delta/r^2,\mu}$. The terms $1-p^{1-2k}$ obviously cancel and we claim that the same holds for the terms $1-\chi_{D'}(p)p^{1/2-k}$. Here $D' = 2(-1)^{(b^++b^-+1)/2}\Delta|D_L|$ and the corresponding discriminant associated with Δ/r^2 is by definition $\widetilde{D}' = D'/r_0^2$, where we have written $r = r_0 r_1$ with r_1 the largest factor in r coprime to $2N$. Said differently, $\widetilde{D}'$ and D' differ only in prime factors p dividing $2N$ and for those $\chi_{D'}(p) = 0 = \chi_{\widetilde{D}'}(p)$ since $2|D_L|$ divides $\widetilde{D}'$.

As a quotient of the remaining factors we obtain

$$\begin{aligned}\frac{\varepsilon_{\Delta/r^2,\mu}}{\varepsilon_{\Delta,\delta}} &= \prod_{p|r} \frac{p^{w_p(\Delta/r^2,\mu)(1-2k)} N_{\Delta/r^2,\mu}(p^{w_p(\Delta/r^2,\mu)})}{p^{w_p(\Delta,\delta)(1-2k)} N_{\Delta,\delta}(p^{w_p(\Delta,\delta)})} \\ &= \prod_{p|r} \frac{p^{w_p(\Delta,\delta)(1-2k)} N_{\Delta/r^2,\mu}(p^{w_p(\Delta,\delta)})}{p^{w_p(\Delta,\delta)(1-2k)} N_{\Delta,\delta}(p^{w_p(\Delta,\delta)})}.\end{aligned} \tag{22}$$

According to Lemma 2.6, we get the bound

$$\frac{\varepsilon_{\Delta/r^2,\mu}}{\varepsilon_{\Delta,\delta}} \leq \prod_{p|r} \frac{1+p^{-1}}{1-p^{-2}} = \prod_{p|r} \frac{1}{1-p^{-1}}. \tag{23}$$

Using $|K_r| \le r^{2k-4}$, we obtain

$$\begin{aligned} \frac{Q(\Delta,\delta)}{\varepsilon_{\Delta,\delta}} &\ge 1 - \sum_{|P|\,\text{odd}} \prod_{p\in P} \frac{1}{p(p-1)} \\ &\ge 1 - \frac{1}{2}\Big(\prod_{p\,\text{prime}} \Big(1+\frac{1}{p(p-1)}\Big) - \prod_{p\,\text{prime}} \Big(1-\frac{1}{p(p-1)}\Big)\Big). \end{aligned} \tag{24}$$

The first Euler product appearing on the right hand side is known as Landau's totient constant

$$\prod_{p\,\text{prime}} \Big(1+\frac{1}{p(p-1)}\Big) = \frac{315}{2\pi^4}\zeta(3) = 1.943596\dots.$$

Inserting the numerical values, we see that

$$\frac{Q(\Delta,\delta)}{\varepsilon_{\Delta,\delta}} \ge 0.02820178\dots > 0, \tag{25}$$

which can be rigorously proven to be positive by standard remainder term estimates for zeta-functions. (The estimate can even be improved using Artin's constant $\prod_{p\,\text{prime}}\Big(1-\frac{1}{p(p-1)}\Big) = 0.373955\dots$.)

The case b^+ even is similar, but easier. Again we need to estimate $\frac{\varepsilon_{\Delta/r^2,\mu}}{\varepsilon_{\Delta,\delta}}$ from (6) uniformly from below. By Theorem 7 in [BK01] we may again suppose that the product runs over the same set of primes for Δ and Δ/r^2. This time, the discriminant involved in $1-\chi_D(p)p^{-k}$ does not depend on m. Hence the corresponding factors cancel. The remaining expression is the same as in (22) and can be estimated as in (23) above. □

Algorithmic aspects. The proof of Theorem 4.4 is effective and can be turned into an algorithm to compute extremal rays of $\mathrm{Eff}^{\mathrm{H}}(\mathcal{F}_L)$ for any L as follows.

The first step is the computation of functionals that span $S_{k,L}(\mathbb{R})^\vee$ and whose convex combination contains zero (Proposition 3.3). The bound in this proof is effective, as it depends on the dimension of $S_{k,L}(\mathbb{R})^\vee$ and the values of the quadratic form on D_L. For practical purposes one computes a ball B_r around zero that is contained in the span of these functionals.

The second step consists of computing m_0 such that for $m \ge m_0$ the functionals $c_{m,\mu}$ belong to the tube $B_r + \langle e\rangle$. This requires us to make the constants $C_{f,\varepsilon}$ and $C_{L,\varepsilon}$ of Proposition 3.2 effective, which is possible as an inspection of the corresponding proofs shows. So far, we have made Theorem 3.4 effective. In order to do the same for Theorem 4.4, one has to use the slightly worse bound for the Eisenstein series given in the proof of Proposition 4.5.

Finally, $\mathrm{Eff}^{\mathrm{H}}(\mathcal{F}_L)$ can be computed among the functional $c_{m,\mu}$ for $m \le m_0$ by linear programming.

5. The case of Hilbert modular surfaces

Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant $D > 0$ with ring of integers $\mathcal{O}_F$. Denote the conjugation in F by $\nu \mapsto \nu'$. The Hilbert modular surface associated with F and an ideal $\mathfrak{b} \subset \mathcal{O}_F$ is the quotient $X_{F,\mathfrak{b}} = \mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{b})\backslash \mathbb{H}^2$; see, e.g., [vdG88] for a textbook reference. In this section we show that the results of the preceding sections partially also apply to Hilbert modular surfaces.

We first remark that the cone generated by Heegner divisors (also called Hirzebruch-Zagier cycles in this case) is no longer full-dimensional on $X_{F,\mathfrak{b}}$, contrary to the case of $b^+ \geq 3$ (see [BLMM17]). First, the two foliations on Hilbert modular surfaces define two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $X_{F,\mathfrak{b}}$. Heegner divisors always lie in the subspace whose intersection with $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ is zero. Second, even in this subspace, the cone is not always full-dimensional; see [HLR86].

Hilbert modular surfaces nevertheless fall into the scope of the preceding sections. In fact, let $B = \mathrm{N}(\mathfrak{b})$, and consider the lattice

$$L_{\mathfrak{b}} = \left\{ \begin{pmatrix} x & \nu' \\ \nu & y \end{pmatrix} :\; x \in \mathbb{Z},\, y \in B\mathbb{Z},\, \nu \in \mathfrak{b} \right\}$$

with the integral quadratic form $Q(X) = -\frac{1}{B}\det(X)$. The dual lattice of $L_{\mathfrak{b}}$ is given by

$$L_{\mathfrak{b}}^\vee = \left\{ \begin{pmatrix} x & \nu' \\ \nu & y \end{pmatrix} :\; x \in \mathbb{Z},\, y \in B\mathbb{Z},\, \nu \in \mathfrak{d}_F^{-1}\mathfrak{b} \right\},$$

where $\mathfrak{d}_F \subset \mathcal{O}_F$ is the different ideal. In particular, we have $L_{\mathfrak{b}}^\vee/L_{\mathfrak{b}} \cong \mathfrak{d}_F^{-1}/\mathcal{O}_F$. The Hilbert modular group $\mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{b})$ acts on $L_{\mathfrak{b}}$ by

$$(g, X) \mapsto gX\,{}^t g'$$

for $g \in \mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{b})$ and $X \in L_{\mathfrak{b}}$. This action preserves the quadratic form Q, and according to [Bru08, Section 2.7] it induces an isomorphism

$$\mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{b}) \cong \mathrm{Spin}(L_{\mathfrak{b}}).$$

On the other hand, according to [MP16, Lemma 2.6], the image of the spin group of a lattice in the orthogonal group is given by the intersection of the stable special orthogonal group with the subgroup of elements with positive spinor norm, that is,

$$\mathrm{Spin}(L_{\mathfrak{b}})/\{\pm 1\} \cong \widetilde{\mathrm{SO}}^+(L_{\mathfrak{b}}).$$

Consequently, we have $\mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{b})/\{\pm 1\} \cong \widetilde{\mathrm{SO}}^+(L_{\mathfrak{b}})$, and

$$X_{F,\mathfrak{b}} \cong \mathcal{F}_{L_{\mathfrak{b}}}(\widetilde{\mathrm{SO}}^+(L_{\mathfrak{b}})).$$

The explicit identification of $X_{F,\mathfrak{b}}$ with the orthogonal Shimura variety on the right hand side is given by [Bru08, Equation (2.33)]. The Heegner divisors on the right hand side can be identified with Hirzebruch-Zagier divisors on $X_{F,\mathfrak{b}}$.

To describe the symmetric Hilbert modular group, we consider the vector $\lambda = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -B \end{smallmatrix}\right) \in L_{\mathfrak{b}}$ with $Q(\lambda) = 1$. The reflection $\tau_\lambda \in \mathrm{O}(L_{\mathfrak{b}})$ taking λ to its negative and fixing its orthogonal complement belongs to $\widetilde{\mathrm{O}}^+(L_{\mathfrak{b}})$ and has determinant -1. On $\mathbb{H} \times \mathbb{H}$ it induces the transformation

$$(z_1, z_2) \mapsto \left(-\frac{1}{Bz_2}, -\frac{1}{Bz_1}\right). \tag{26}$$

Hence, the projective symmetric Hilbert modular group is isomorphic to $\widetilde{\mathrm{O}}^+(L_{\mathfrak{b}})$, and the corresponding symmetric Hilbert modular surface is given by

$$X_{F,\mathfrak{b}}^{\mathrm{symm}} \cong \mathcal{F}_{L_{\mathfrak{b}}}(\widetilde{\mathrm{O}}^+(L_{\mathfrak{b}})).$$

Since $L_{\mathfrak{b}}$ splits one hyperbolic plane over $\mathbb{Z}$, we may apply Theorem 4.1 in this situation.

Corollary 5.1. *The cone generated by the* (*reducible*) *Hirzebruch-Zagier cycles on the Hilbert modular surface $X_{F,\mathfrak{b}}$ is rational polyhedral. The same statement holds on the symmetric Hilbert modular surface $X_{F,\mathfrak{b}}^{\mathrm{symm}}$.*

It seems quite plausible that the rational polyhedrality can be extended in the case of Hilbert modular surfaces to the cone generated by the irreducible components of Hirzebruch-Zagier cycles. The description is more complicated than in the case when L splits off two hyperbolic planes. It has been given in many cases by Hirzebruch, Franke, and Hausmann; see, e.g., the survey and references in [MZ16, Section 5.2] or [vdG88, Section 5.3].

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