Borcherds products and Chern classes of Hirzebruch-Zagier divisors

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1 Introduction

In [Bo1, Bo2] Borcherds constructed a lifting from elliptic modular forms of weight $1-n/2$ with poles in the cusps to automorphic forms on the orthogonal group $\mathrm{O}(2,n)$ with known zeros and poles along Heegner divisors. These can be written as infinite products, so-called Borcherds products. The present paper was motivated by the question whether every principal Heegner divisor can be obtained as the divisor of such a Borcherds product.

We shall give an affirmative answer in the special $\mathrm{O}(2,2)$-case of Hilbert modular forms (under a technical condition on the underlying real quadratic field). We shall construct for each Heegner divisor H a certain "generalized Borcherds product" Ψ_H. Considering suitable finite products of these Ψ_H we find a new proof of [Bo2] Theorem 13.3 and [Bo3] in this particular case. Moreover it turns out that the function Ψ_H can be used to determine the Chern class of H explicitly. One obtains a lifting from elliptic modular forms into the cohomology. We shall show that it coincides with the Doi-Naganuma lifting.

We now describe the content of this paper in more detail. Let K be a real quadratic field of discriminant D with $D \equiv 1 \pmod 4$. Let $\mathcal{O}$ be the ring of integers and $\mathfrak{d}$ the different in K. Denote by $x \mapsto x'$ the conjugation and by $\mathrm{N}(x) = xx'$ the norm of an element $x \in K$. The Hilbert modular group $\Gamma_K = \mathrm{Sl}_2(\mathcal{O})$ acts on the product $\mathbb{H} \times \mathbb{H}$ of two upper half-planes, and the quotient $X_K = (\mathbb{H} \times \mathbb{H})/\Gamma_K$ is a quasi-projective algebraic variety. In this context Borcherds' Heegner divisors are essentially the well known Hirzebruch-Zagier divisors [HZ]. For a positive integer m the Hirzebruch-Zagier divisor $T(m)$ of discriminant m on $\mathbb{H} \times \mathbb{H}$ is given by

$$T(m) = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H}; \quad \exists (a, b, \lambda) \in \mathbb{Z} \times \mathbb{Z} \times \mathfrak{d}^{-1} \text{ with } ab - \mathrm{N}(\lambda) = m/D \text{ and } az_1z_2 + \lambda z_1 + \lambda' z_2 + b = 0\}.$$

It is the inverse image of an algebraic divisor on the quotient X_K, which will also be denoted by $T(m)$.

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The idea of our approach to Borcherds' theory is quite simple and can be described as follows: For each Hirzebruch-Zagier divisor $T(m)$ we consider the Poincaré series

$$\phi_m(z_1,z_2) = \sum_{\substack{a,b\in\mathbb{Z}\\ \lambda\in\mathfrak{d}^{-1}\\ ab-\mathrm{N}(\lambda)=m/D}} \log\left(\frac{|az_1\bar{z}_2+\lambda z_1+\lambda'\bar{z}_2+b|}{|az_1z_2+\lambda z_1+\lambda' z_2+b|}\right).$$

It is formally invariant under Γ_K and has a logarithmic singularity along $T(m)$. Therefore one could hope to obtain the absolute value of a generalized Borcherds product by taking $\exp(\phi_m(z_1,z_2))$.

Unfortunately $\phi_m(z_1,z_2)$ diverges. However, it can be regularized in the following way. For $s=\sigma+it\in\mathbb{C}$ with $\sigma>1$ we define

$$\Phi_m(z_1,z_2,s) = \sum_{\substack{a,b\in\mathbb{Z}\\ \lambda\in\mathfrak{d}^{-1}\\ ab-\mathrm{N}(\lambda)=m/D}} Q_{s-1}\left(1+\frac{|az_1z_2+\lambda z_1+\lambda' z_2+b|^2}{2y_1y_2m/D}\right),$$

where $Q_{s-1}(z)$ is the Legendre function of the second kind and $z_1=x_1+iy_1$, $z_2=x_2+iy_2$.

It can be easily seen that the series converges normally for $(z_1,z_2)\in\mathbb{H}\times\mathbb{H}-T(m)$ and $\sigma>1$ and therefore defines a Γ_K-invariant function. Since $Q_{s-1}(z)=-\frac{1}{2}\log(z-1)+O(1)$ for $t\to 1$, $\Phi_m(z_1,z_2,s)$ has a logarithmic singularity along $T(m)$. Note that for $D=m=1$ the function $\Phi_m(z_1,z_2,s)$ equals the resolvent kernel function for $\mathrm{Sl}_2(\mathbb{Z})$ (cf. [He]). The identity $Q_0(z)=\frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$ implies the formal equality $\Phi_m(z_1,z_2,1)=\phi_m(z_1,z_2)$.

In section 3.1 we determine the Fourier expansion of $\Phi_m(z_1,z_2,s)$ explicitly and prove that it can be continued to a holomorphic function on $\{s\in\mathbb{C};\ \sigma>3/4,\ s\neq 1\}$, which has a simple pole at $s=1$. We define the regularized Poincaré series $\Phi_m(z_1,z_2)$ to be the constant term of the Laurent expansion of $\Phi_m(z_1,z_2,s)$ at $s=1$.

Let $M_2(D,\chi_D)$ be the vector space of modular forms of weight 2 with character $\chi_D=\left(\frac{D}{\cdot}\right)$ with respect to $\Gamma_0(D)$, and denote the subspace of cusp forms by $S_2(D,\chi_D)$.

In section 3.3 we show that $\Phi_m(z_1,z_2)$ can be written as the sum of two real valued functions $\psi_m(z_1,z_2)$ and $\xi_m(z_1,z_2)$ with the following properties:

The function $\xi_m(z_1,z_2)$ has no singularities, i.e. is real analytic on the whole domain $\mathbb{H}\times\mathbb{H}$. In its Fourier expansion

$$\xi_m(z_1,z_2)=\tfrac{1}{2}q_0(m)\log(y_1y_2)+2\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu>0>\nu'}} p_{|D\nu\nu'|}(m)\log|1-e(\nu z_1+\nu'\bar{z}_2)|$$

the m-th coefficients $p_n(m)$ of certain Poincaré series $P_n\in S_2(D,\chi_D)$ occur. These P_n are essentially the linear combinations of Poincaré series attached to the cusps of $\Gamma_0(D)$ introduced by Zagier in [Za1]. The generating series $Q(z)=1+\sum_{n\geq 1}q_0(n)e^{2\pi i nz}$ of the $q_0(m)$ is an (analogous) linear combination of Eisenstein series in $M_2(D,\chi_D)$.

Moreover, $-\frac{1}{2}\psi_m(z_1,z_2)$ is the logarithm of the absolute value of a holomorphic function $\Psi_m(z_1,z_2)$ on $\mathbb{H}\times\mathbb{H}$. The only zeros of $\Psi_m(z_1,z_2)$ lie on $T(m)$, and in certain non-empty open subsets $W\subset\mathbb{H}\times\mathbb{H}$ the function $\Psi_m(z_1,z_2)$ has a Borcherds product expansion

$$\Psi_m(z_1,z_2)=e(\rho_W z_1+\rho_W' z_2)\prod_{\substack{\nu\in\mathfrak{d}^{-1}\\ (\nu,W)>0}}(1-e(\nu z_1+\nu' z_2))^{-q_{D\nu\nu'}(m)}.$$

Here ρ_W and ρ'_W are real constants, and the $q_n(m)$ ($n \neq 0$) denote the Fourier coefficients of certain Poincaré series of weight 2 with respect to $\Gamma_0(D)$ which have poles in the cusps.

Let a_r ($r \in \mathbb{N}$) be the functional in the dual space of $S_2(D, \chi_D)$ which takes a modular form f to its r-th Fourier coefficient. Write $A(D, \chi_D)$ for the $\mathbb{Z}$-module generated by the a_r. One can infer the following version of [Bo2] Theorem 13.3: If $c_1, \dots, c_N$ are integers with $\sum_{j=1}^N c_j a_j = 0$, then

$$\Psi(z_1, z_2) = \prod_{j=1}^N \Psi_j(z_1, z_2)^{c_j}$$

is an automorphic form of weight $-\frac{1}{2}\sum_{j=1}^N c_j q_0(j)$ with respect to Γ_K whose divisor is given by $\sum_{j=1}^N c_j T(j)$ (Theorem 5). In particular we find that $a_r \mapsto T(r)$ defines a homomorphism β from $A(D, \chi_D)$ to the (suitably modified) divisor class group $\widetilde{\mathrm{Cl}}(X_K)$ of X_K.

Furthermore, in section 5 we use the function ξ_m to determine the Chern class of the divisor $T(m)$ explicitly. (This can be done in a different way by means of results of Hirzebruch-Zagier [HZ] and Oda [Od], cf. [Ge].) The above properties of Ψ_m and ξ_m imply that $e^{\xi_m(z_1,z_2)/2}$ is a Hermitian metric on the sheaf $\mathcal{L}(T(m))$ attached to $T(m)$. Therefore the Chern class of $T(m)$ can be computed as

$$c(T(m)) = \frac{1}{2}\partial\bar{\partial}\xi_m(z_1, z_2)$$

(Theorem 7). For simplicity let us now assume that $\mathcal{O}$ contains a unit ε_0 of negative norm. Then one finds that the Chern class of $T(m)$ is essentially given by a certain Hilbert Poincaré series of weight 2 for Γ_K.

Hence over $\mathbb{C}$ the composition of β with the Chern class map can be considered as a map from $S_2(D, \chi_D)$ into the space $S_2(\Gamma_K)$ of Hilbert cusp forms of weight 2 with respect to Γ_K. In Theorem 8 we prove that it is equal to the Doi-Naganuma-lifting (cf. [DN, Na, Za1, As]).

As a consequence we may answer the question raised at the beginning. We may identify the kernel of β with the kernel of the Doi-Naganuma-lifting and deduce (Theorem 9):

Let f be an automorphic form with respect to Γ_K with an arbitrary character. Assume that its divisor has the form $(f) = \sum_{j=1}^N c_j T(j)$. Then up to a constant multiple f equals

$$\Psi(z_1, z_2) = \prod_{j=1}^N \Psi_j(z_1, z_2)^{c_j}.$$

In particular f has a Borcherds product expansion, and its weight is given in terms of the coefficients of the Eisenstein series $Q(z)$ by $-\frac{1}{2}\sum_{j=1}^N c_j q_0(j)$.

Certainly the Poincaré series $\Phi_m(z_1, z_2)$ can be generalized to arbitrary orthogonal groups $\mathrm{O}(2, n)$. One should obtain generalized Borcherds products and a lifting into the cohomology in a similar way. However, new technical difficulties arise due to the fact that the cohomology of the underlying complex space is more difficult to describe. We hope to come back to this topic in a future paper.

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2 Eisenstein series and Poincaré series for $\Gamma_0(D)$

In the following section we briefly recall some basic facts concerning Eisenstein series and Poincaré series for $\Gamma_0(D)$. This mainly serves to fix our notation. We omit the proofs which are either given in [Za1] or can be obtained by slight modifications of the arguments in [Za1] and [Sc] chapter III.

2.1 Eisenstein series for $\Gamma_0(D)$

Recall from the introduction that D denotes a positive fundamental discriminant with $D \equiv 1 \pmod 4$. Let χ_D be the primitive character modulo D defined by the Kronecker symbol $x \mapsto \left(\frac{D}{x}\right)$. As usual we put

$$\Gamma_0(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}); \quad c \equiv 0 \pmod D \right\}$$

and write $M_2(D, \chi_D)$ for the space of modular forms of weight 2 and Nebentypus character χ_D with respect to $\Gamma_0(D)$. The subspace of cusp forms is denoted by $S_2(D, \chi_D)$.

The cusps of $\Gamma_0(D)$ correspond one-to-one to the positive divisors of D via $k/l \mapsto (l, D)$. For $D_1 > 0$, $D_1 | D$ and $D_2 = D/D_1$ we choose $p, q \in \mathbb{Z}$ with $pD_1 + qD_2 = 1$ and put

$$A_{D_1} = \begin{pmatrix} D_2 & -p \\ D_1 & q \end{pmatrix}.$$

Then $\{A_{D_1}^{-1}\infty;\ D_1 > 0,\ D_1|D\}$ is precisely the set of cusps of $\Gamma_0(D)$. Throughout we will simply write D_1 for the cusp $A_{D_1}^{-1}\infty$.

For $z = x + iy$ in the upper half-plane $\mathbb{H}$ and $s = \sigma + it \in \mathbb{C}$, $\sigma > 0$ we define the Eisenstein series (with parameter s) of weight 2 in the cusp D_1 by

$$E^{D_1}(z, s) = \frac{1}{2} \sum_{\substack{A \in \Gamma_{D_1} \backslash A_{D_1}\Gamma_0(D) \\ A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} \chi_D(A_{D_1}^{-1} A) \frac{1}{(cz+d)^2} \frac{y^s}{|cz+d|^{2s}}, \tag{1}$$

where

$$\Gamma_{D_1} := A_{D_1}\Gamma_0(D)A_{D_1}^{-1} \cap \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix};\ b \in \mathbb{Z} \right\}.$$

This series converges normally for $z \in \mathbb{H}$ and $\sigma > 0$ and therefore transforms according to $E^{D_1}(\gamma z, s) = \chi_D(\gamma)(cz+d)^2 E^{D_1}(z, s)$ under $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(D)$.

As usual, calculating the Fourier expansion, one can show that $E^{D_1}(z, s)$ has a meromorphic continuation in s which is holomorphic in a neighborhood of $s = 0$. Hence we may define the Eisenstein series of weight 2 in the cusp D_1 by $E^{D_1}(z) = E^{D_1}(z, 0)$. It can be proved that E^{D_1} lies in $M_2(D, \chi_D)$ and has the Fourier expansion

$$E^{D_1}(z) = \delta_{D_1 D} - 4\pi^2 \sum_{n \geq 1} \sum_{\substack{c \geq 1 \\ (c,D) = D_1}} \frac{1}{c} H^{D_1}(0, n) n e(nz).$$

Here

$$H_c^{D_1}(0,n) = \frac{1}{c}\left(\frac{c}{D_2}\right)\sum_{d(c)^*}\left(\frac{-d}{D_1}\right)e\left(\frac{nd}{c}\right) \qquad (n\in\mathbb{Z}), \tag{2}$$

$e(z) := e^{2\pi i z}$, and $\delta_{D_1 D}$ denotes the Kronecker delta. The sum in (2) runs over all primitive residues d modulo c.

We now introduce some linear combinations of the E^{D_1} (similar to [Za1] p. 22) which will be needed later. We define

$$E(z) = \sum_{\substack{D_1>0\\ D_1D_2=D}} \frac{\overline{\psi}(D_2)}{D_2^2} E^{D_1}(z), \tag{3}$$

where

$$\psi(D_2) = \begin{cases} \left(\frac{D_1}{D_2}\right)\sqrt{D_2}, & \text{if } D_1 \equiv 1 \pmod 4,\\ -i\left(\frac{D_1}{D_2}\right)\sqrt{D_2}, & \text{if } D_1 \equiv 3 \pmod 4 \end{cases} \tag{4}$$

for $D_1D_2 = D$.

Proposition 1. *The Eisenstein series $E(z)$ lies in $M_2(D,\chi_D)$ and has the Fourier expansion*

$$E(z) = 1 - \frac{4\pi^2}{D}\sum_{n\geq 1}\sum_{b\geq 1}\frac{1}{b}H_b(0,-n)ne(nz)$$

with

$$H_b(0,n) = \sum_{\substack{D_1D_2=D\\ (b,D_2)=1}} \frac{\psi(D_2)}{D_2} H_{bD_1}^{D_1}(0,n).$$

2.2 Poincaré series for $\Gamma_0(D)$

As usual we write J_ν for the Bessel function of the first kind, I_ν for the modified Bessel function of the first kind, and K_ν for the modified Bessel function of the third kind (cf. [AbSt] §9). Let r be a non-zero integer and $D_1, D_2 \in \mathbb{N}$ with $D_1D_2 = D$.

We define the r-th Poincaré series (with parameter s) of weight 2 in the cusp D_1 by

$$G_r^{D_1}(z,s) = \frac{1}{2}\sum_{\substack{A\in\Gamma_{D_1}\backslash A_{D_1}\Gamma_0(D)\\ A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}} \chi_D(A_{D_1}^{-1}A)e\left(\frac{r}{D_2}Az\right)\frac{1}{(cz+d)^2}\frac{y^s}{|cz+d|^{2s}} \qquad (\sigma>0). \tag{5}$$

(Note that D_2 is the width of the cusp D_1.) Again, it can be shown that $G_r^{D_1}(z,s)$ has a holomorphic continuation in s to a neighborhood of $s=0$. One defines $G_r^{D_1}(z) = G_r^{D_1}(z,0)$.

In the case $r>0$ it is proved in [Za1] Appendix 1 that $G_r^{D_1}(z)\in S_2(D,\chi_D)$ and

$$G_r^{D_1}(z)=\delta_{D_1D}e(rn)-2\pi\sum_{n=1}^{\infty}\sqrt{nD_2/r}\sum_{\substack{c\geq 1\\(c,D)=D_1}}H_c^{D_1}(r,n)J_1\left(\frac{4\pi}{c}\sqrt{nr/D_2}\right)e(nz),$$

where $H_c^{D_1}(r,n)$ is given by

$$H_c^{D_1}(m,n)=\frac{1}{c}\left(\frac{c}{D_2}\right)\sum_{d(c)^*}\left(\frac{-d}{D_1}\right)e\left(\frac{nd+m\bar{D}_2\bar{d}}{c}\right)\qquad (m,n\in\mathbb{Z}). \tag{6}$$

The sum in (6) runs over all primitive residues d modulo c; and $\bar{D}_2$, $\bar{d}$ are determined by $\bar{D}_2D_2\equiv 1 \pmod c$ respectively $\bar{d}d\equiv 1 \pmod c$. For $r<0$ one can calculate the Fourier expansion of $G_r^{D_1}$ analogously and obtains

$$G_r^{D_1}(z)=\delta_{D_1D}e(rz)-2\pi\sum_{n\geq 1}\sqrt{nD_2/|r|}\sum_{\substack{c\geq 1\\(c,D)=D_1}}H_c^{D_1}(r,n)I_1\left(\frac{4\pi}{c}\sqrt{n|r|/D_2}\right)e(nz).$$

In the same way as above we consider suitable linear combinations of the $G_r^{D_1}$. For $r\in\mathbb{Z}-\{0\}$ we define

$$P_r(z)=\sum_{\substack{D_2|r,\ D_2>0\\D_1D_2=D}}\frac{\overline{\psi}(D_2)}{D_2^2}G_{r/D_2}^{D_1}(z), \tag{7}$$

where $\psi(D_2)$ is given by (4). Using $\overline{\psi}(D_2)=\left(\frac{-1}{D_1}\right)\psi(D_2)$ and the obvious identity $H_c^{D_1}(-r,-n)=\left(\frac{-1}{D_1}\right)H_c^{D_1}(r,n)$, we find

Proposition 2. *Let $r>0$. Then P_r is an element of $S_2(D,\chi_D)$ and has the Fourier expansion $P_r(z)=\sum_{n\geq 1}p_r(n)e(nz)$ with*

$$p_r(n)=\delta_{rn}-2\pi\sqrt{n/r}\sum_{b\geq 1}H_b(-r,-n)J_1\left(\frac{4\pi}{bD}\sqrt{nr}\right) \tag{8}$$

and

$$H_b(m,n)=\sum_{\substack{D_1D_2=D\\D_2|m\\(b,D_2)=1}}\frac{\psi(D_2)}{D_2}H_{bD_1}^{D_1}(m/D_2,n). \tag{9}$$

In the case $r<0$ the Poincaré series P_r is not contained in $S_2(D,\chi_D)$ but still transforms in the same way under $\Gamma_0(D)$. Its Fourier expansion has the form

$$P_r(z)=e(rz)-2\pi\sum_{n\geq 1}\sqrt{n/|r|}\sum_{b\geq 1}H_b(-r,-n)I_1\left(\frac{4\pi}{bD}\sqrt{n|r|}\right)e(nz). \tag{10}$$

3 Poincaré series attached to Hirzebruch-Zagier divisors

Recall that K is a real quadratic field of discriminant D. We write $x \mapsto x'$ for the conjugation, $\mathrm{N}(x) = xx'$ for the norm, $\mathrm{tr}(x) = x + x'$ for the trace, and $\mathfrak{d} = (\sqrt{D})$ for the different in K. We denote by $\mathcal{O}$ the ring of integers and by $\Gamma_K = \mathrm{Sl}_2(\mathcal{O})$ the Hilbert modular group of K.

Let m be a positive integer. We consider the Hirzebruch-Zagier divisor

$$T(m) = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H}; \quad \exists (a, b, \lambda) \in \mathbb{Z} \times \mathbb{Z} \times \mathfrak{d}^{-1} \text{ with } ab - \mathrm{N}(\lambda) = m/D \\ \text{and } a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b = 0\} \quad (11)$$

of discriminant m on $\mathbb{H} \times \mathbb{H}$. It is the inverse image under the canonical projection of an algebraic divisor on the quotient $X_K = (\mathbb{H} \times \mathbb{H})/\Gamma_K$.

Definition 1. *Let $(z_1, z_2) \in \mathbb{H} \times \mathbb{H} - T(m)$ and $s = \sigma + it \in \mathbb{C}$, $\sigma > 1$. The Poincaré series $\Phi_m(z_1, z_2, s)$ attached to $T(m)$ is defined by*

$$\Phi_m(z_1, z_2, s) = \sum_{\substack{a,b \in \mathbb{Z} \\ \lambda \in \mathfrak{d}^{-1} \\ ab - \mathrm{N}(\lambda) = m/D}} Q_{s-1}\left(1 + \frac{|a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b|^2}{2 y_1 y_2 m/D}\right). \quad (12)$$

Here $Q_{s-1}(z)$ is the Legendre function of the second kind (cf. [AbSt] §8), defined by

$$Q_{s-1}(z) = \int_0^\infty (z + \sqrt{z^2 - 1} \cosh u)^{-s} du \qquad (z > 1,\ s > 0), \quad (13)$$

and $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Note that $T(m)$ is empty if m is not a quadratic residue modulo D. In this case we put $\Phi_m(z_1, z_2, s) = 0$.

The following argument shows that $\Phi_m(z_1, z_2, s)$ converges normally for $\sigma > 1$ and $(z_1, z_2) \in \mathbb{H} \times \mathbb{H} - T(m)$. It is well known that for any compact set $B \subset \mathbb{H} \times \mathbb{H}$ and for any $C \geq 0$ the set

$$\mathcal{M}_{B,m,C} = \{(a, b, \lambda) \in \mathbb{Z} \times \mathbb{Z} \times \mathfrak{d}^{-1}; \quad ab - \mathrm{N}(\lambda) = m/D, \\ |a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b| \leq C \text{ for a } (z_1, z_2) \in B\} \quad (14)$$

is finite. This, together with the asymptotic property $Q_{s-1}(z) = O(z^{-s}) \quad (z \to \infty)$ of the Legendre function, implies that $\Phi_m(z_1, z_2, s)$ has the local majorant

$$\sum_{\substack{a,b \in \mathbb{Z} \\ \lambda \in \mathfrak{d}^{-1} \\ ab - \mathrm{N}(\lambda) = m/D}} \frac{1}{|a z_1 \bar{z}_2 + \lambda z_1 + \lambda' \bar{z}_2 + b|^{2\sigma}}.$$

The normal convergence of the latter series for $\sigma > 1$ is well known (cp. [Za1]).

For $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_K$ and $(a, b, \lambda) \in \mathbb{Z} \times \mathbb{Z} \times \mathfrak{d}^{-1}$ we have the identity

$$a\left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}\right)\left(\frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'}\right) + \lambda\left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}\right) + \lambda'\left(\frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'}\right) + b = \frac{\tilde{a} z_1 z_2 + \tilde{\lambda} z_1 + \tilde{\lambda}' z_2 + \tilde{b}}{(\gamma z_1 + \delta)(\gamma' z_2 + \delta')}$$

with uniquely determined $(\tilde{a}, \tilde{b}, \tilde{\lambda}) \in \mathbb{Z} \times \mathbb{Z} \times \mathfrak{d}^{-1}$ and $\tilde{a}\tilde{b} - \mathrm{N}(\tilde{\lambda}) = ab - \mathrm{N}(\lambda)$. Hence the function $\Phi_m(z_1, z_2, s)$ is Γ_K-invariant.

In the following section we will determine the Fourier expansion of $\Phi_m(z_1, z_2, s)$ explicitly. Thereby one can show that $\Phi_m(z_1, z_2, s)$ has a meromorphic continuation in s to $\{s \in \mathbb{C};\ \sigma > 3/4\}$ which is holomorphic up to a simple pole at $s = 1$. As explained in the introduction, we define $\Phi_m(z_1, z_2)$ to be the constant term of the Laurent expansion of $\Phi_m(z_1, z_2, s)$ at $s = 1$.

The function $\Phi_m(z_1, z_2)$ has a logarithmic singularity along $T(m)$. It can be written as the sum of two functions $\psi_m(z_1, z_2) + \xi_m(z_1, z_2)$ where $-\psi_m/2$ is the logarithm of the absolute value of a "generalized Borcherds product" Ψ_m and ξ_m has a cohomological interpretation.

3.1 Fourier expansion and meromorphic continuation

We write $\Phi_m(z_1, z_2, s)$ in the form

$$\Phi_m(z_1, z_2, s) = \Phi_m^0(z_1, z_2, s) + 2\sum_{a=1}^{\infty} \Phi_m^a(z_1, z_2, s) \tag{15}$$

with

$$\Phi_m^a(z_1, z_2, s) = \sum_{\substack{b \in \mathbb{Z} \\ \lambda \in \mathfrak{d}^{-1} \\ ab - \mathrm{N}(\lambda) = m/D}} Q_{s-1}\left(1 + \frac{|a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b|^2}{2 y_1 y_2 m/D}\right). \tag{16}$$

Note that the partial sums $\Phi_m^a(z_1, z_2, s)$ converge normally for $\sigma > 1/2$.

For the calculation of the Fourier expansion of the functions $\Phi_m^a(z_1, z_2, s)$ we distinguish the cases $a = 0$ and $a > 0$. In the latter case it is useful to write

$$\Phi_m^a(z_1, z_2, s) = \sum_{\lambda \in R} \sum_{\theta \in \mathcal{O}} Q_{s-1}\left(1 + \frac{|(z_1 + \theta + \lambda'/a)(z_2 + \theta' + \lambda/a) + A/a^2|^2}{2 y_1 y_2 A/a^2}\right)$$

similar as in [Za1] §2. Here we have put $A = m/D$, and R denotes a set of representatives for

$$\{\lambda \in \mathfrak{d}^{-1}/a\mathcal{O};\quad \mathrm{N}(\lambda\sqrt{D}) \equiv m \pmod{aD}\}.$$

Let H_s^A be the function

$$H_s^A(z_1, z_2) = \sum_{\theta \in \mathcal{O}} Q_{s-1}\left(1 + \frac{|(z_1 + \theta)(z_2 + \theta') + A|^2}{2 y_1 y_2 A}\right) \tag{17}$$

and denote its Fourier expansion in the form

$$H_s^A(z_1,z_2) = \sum_{\nu\in\mathfrak{d}^{-1}} b_s^A(\nu,y_1,y_2)e(\nu x_1+\nu' x_2), \qquad y_1y_2>A.$$

Then we have

$$\Phi_m^a(z_1,z_2,s) = \sum_{\nu\in\mathfrak{d}^{-1}} G_a(m,\nu) b_s^{A/a^2}(\nu,y_1,y_2)e(\nu z_1+\nu' z_2)$$

with the finite exponential sum

$$G_a(m,\nu) = \sum_{\substack{\lambda\in\mathfrak{d}^{-1}/a\mathcal{O}\\ \mathrm{N}(\lambda)\equiv -m/D \quad (a\mathbb{Z})}} e\left(\frac{\mathrm{tr}(\nu\lambda)}{a}\right). \tag{18}$$

Note that our $G_a(m,\nu)$ equals $G_a(-m,\nu)$ in the notation of [Za1]. We finally obtain

$$\Phi_m(z_1,z_2,s) = \Phi_m^0(z_1,z_2,s) + 2\sum_{\nu\in\mathfrak{d}^{-1}}\left[\sum_{a=1}^{\infty} G_a(m,\nu) b_s^{A/a^2}(\nu,y_1,y_2)\right] e(\nu x_1+\nu' x_2). \tag{19}$$

Therefore it suffices to calculate the Fourier expansions of $\Phi_m^0(z_1,z_2,s)$ and $H_s^A(z_1,z_2)$.

We put

$$S(m) = \{(z_1,z_2)\in\mathbb{H}\times\mathbb{H};\quad \exists\lambda\in\mathfrak{d}^{-1} \text{ with } \mathrm{N}(\lambda)=-m/D \text{ and } \lambda y_1+\lambda' y_2 = 0\}. \tag{20}$$

Furthermore, if r_1 and r_2 are real numbers we abbreviate

$$\alpha(r_1,r_2) := \max(|r_1|,|r_2|),$$
$$\beta(r_1,r_2) := \min(|r_1|,|r_2|).$$

Lemma 1. *The series*

$$\Phi_m^0(z_1,z_2,s) = \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \mathrm{N}(\lambda)=-m/D}} \sum_{b\in\mathbb{Z}} Q_{s-1}\left(1+\frac{|\lambda z_1+\lambda' z_2+b|^2}{2y_1y_2m/D}\right)$$

converges normally for $(z_1,z_2)\in\mathbb{H}\times\mathbb{H}-T(m)$ and $\sigma>1/2$. Moreover, on $\mathbb{H}\times\mathbb{H}-S(m)$ one has the Fourier expansion

$$\begin{aligned}\Phi_m^0(z_1,z_2,s) = \frac{2\pi}{(2s-1)} &\sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \mathrm{N}(\lambda)=-m/D}} \alpha(\lambda y_1,\lambda' y_2)^{1-s}\beta(\lambda y_1,\lambda' y_2)^s\\ &+4\pi\sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \mathrm{N}(\lambda)=-m/D}}\sum_{n\geq1} |\lambda\lambda' y_1y_2|^{1/2} I_{s-1/2}(2\pi n\beta(\lambda y_1,\lambda' y_2))\\ &\times K_{s-1/2}(2\pi n\alpha(\lambda y_1,\lambda' y_2))e(n\lambda x_1+n\lambda' x_2).\end{aligned} \tag{21}$$

Proof. The convergence statement can be easily verified. The function $\Phi_m^0(z_1, z_2, s)$ can also be written in the form

$$\Phi_m^0(z_1, z_2, s) = \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \mathrm{N}(\lambda) = -m/D}} \sum_{b \in \mathbb{Z}} Q_{s-1}\left(\frac{(\lambda x_1 + \lambda' x_2 + b)^2 + \lambda^2 y_1^2 + \lambda'^2 y_2^2}{2|\lambda\lambda'| y_1 y_2} \right).$$

For $\alpha > \beta > 0$, $\sigma > 1/2$ and $x \in \mathbb{R}$ we consider the function

$$h_{\alpha,\beta}(x) = \sum_{b \in \mathbb{Z}} Q_{s-1}\left(\frac{(x+b)^2 + \alpha^2 + \beta^2}{2\alpha\beta} \right).$$

It is periodic and has a Fourier expansion

$$h_{\alpha,\beta}(x) = \sum_{n \in \mathbb{Z}} a_{\alpha,\beta}(n) e(nx).$$

Using Poisson summation we find

$$a_{\alpha,\beta}(n) = \int_{-\infty}^{\infty} Q_{s-1}\left(\frac{x^2 + \alpha^2 + \beta^2}{2\alpha\beta} \right) e(-nx)\, dx. \tag{22}$$

First suppose that $n \neq 0$. By virtue of the identity

$$\int_0^{\infty} K_\nu(\alpha x) I_\nu(\beta x) \cos(xy)\, dx = \frac{1}{2}(\alpha\beta)^{-1/2} Q_{\nu-1/2}\left(\frac{y^2 + \alpha^2 + \beta^2}{2\alpha\beta} \right)$$

([EMOT] p. 49 (47)), which is valid for $y > 0$, the right-hand side of (22) can be evaluated by the Fourier inversion formula:

$$\int_0^{\infty} Q_{\nu-1/2}\left(\frac{x^2 + \alpha^2 + \beta^2}{2\alpha\beta} \right) \cos(xy)\, dx = \pi\sqrt{\alpha\beta} K_\nu(\alpha y) I_\nu(\beta y), \tag{23}$$

$$a_{\alpha,\beta}(n) = 2\pi\sqrt{\alpha\beta} K_{s-1/2}(2\pi|n|\alpha) I_{s-1/2}(2\pi|n|\beta).$$

Since the integral (22) is a continuous function in $n \in \mathbb{R}$, we have

$$a_{\alpha,\beta}(0) = \lim_{n \to 0}(a_{\alpha,\beta}(n)).$$

Using the asymptotic properties

$$I_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)}, \qquad K_\nu(z) \sim \frac{\Gamma(\nu)}{2}(z/2)^{-\nu} \tag{24}$$

$(z \to 0)$ of the Bessel functions, we find

$$a_{\alpha,\beta}(0) = \frac{2\pi}{2s-1}\alpha^{1-s}\beta^s.$$

Thus the Fourier expansion of $h_{\alpha,\beta}(x)$ is given by

$$h_{\alpha,\beta}(x) = \frac{2\pi}{2s-1}\alpha^{1-s}\beta^s + 2\pi\sqrt{\alpha\beta}\sum_{n\in\mathbb{Z}-\{0\}} I_{s-1/2}(2\pi|n|\beta)K_{s-1/2}(2\pi|n|\alpha)e(nx).$$

The assertion now follows from the identity

$$\Phi_m^0(z_1,z_2,s) = \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \mathrm{N}(\lambda)=-m/D}} h_{\alpha(\lambda y_1,\lambda' y_2),\beta(\lambda y_1,\lambda' y_2)}(\lambda x_1+\lambda' x_2).$$

$\square$

Lemma 2. *Let $A>0$. The function $H_s^A(z_1,z_2)$ defined by (17) converges normally for $\sigma>1/2$. For $y_1y_2>A$ it has the Fourier expansion*

$$H_s^A(z_1,z_2) = \sum_{\nu\in\mathfrak{d}^{-1}} b_s^A(\nu,y_1,y_2)e(\nu x_1+\nu' x_2)$$

with

$$b_s^A(0,y_1,y_2) = \frac{\pi\Gamma(s-1/2)^2}{2\sqrt{D}\,\Gamma(2s)}(4A)^s(y_1y_2)^{1-s},$$
$$b_s^A(\nu,y_1,y_2) = 4\pi\sqrt{\frac{Ay_1y_2}{D}}\,I_{2s-1}(4\pi\sqrt{A|\nu\nu'|})K_{s-1/2}(2\pi|\nu|y_1)K_{s-1/2}(2\pi|\nu'|y_2), \quad \nu\nu'>0,$$
$$b_s^A(\nu,y_1,y_2) = 4\pi\sqrt{\frac{Ay_1y_2}{D}}\,J_{2s-1}(4\pi\sqrt{A|\nu\nu'|})K_{s-1/2}(2\pi|\nu|y_1)K_{s-1/2}(2\pi|\nu'|y_2), \quad \nu\nu'<0.$$

Proof. The convergence statement can be easily verified. By Poisson summation we have

$$b_s^A(\nu,y_1,y_2) = \frac{1}{\sqrt{D}}\int\limits_{x_2=-\infty}^{\infty}\int\limits_{x_1=-\infty}^{\infty} Q_{s-1}\left(1+\frac{|z_1z_2+A|^2}{2y_1y_2A}\right)e(-\nu x_1-\nu' x_2)\,dx_1dx_2. \tag{25}$$

We write

$$|z_1z_2+A|^2 = |z_2|^2\left(\left(x_1+\frac{Ax_2}{|z_2|^2}\right)^2+\left(y_1-\frac{Ay_2}{|z_2|^2}\right)^2\right)$$

and substitute $x_1\mapsto x_1-Ax_2/|z_2|^2$ to obtain

$$b_s^A(\nu,y_1,y_2) = \frac{1}{\sqrt{D}}\int\limits_{x_2=-\infty}^{\infty} e(\nu Ax_2/|z_2|^2)e(-\nu' x_2)$$
$$\times\int\limits_{x_1=-\infty}^{\infty} Q_{s-1}\left(\frac{x_1^2+y_1^2+(Ay_2/|z_2|^2)^2}{2Ay_1y_2/|z_2|^2}\right)e(-\nu x_1)\,dx_1dx_2.$$

Now suppose $\nu \neq 0$. The assumption $y_1y_2 > A$ implies $y_1 > Ay_2/|z_2|^2$. Therefore we may apply formula (23) to evaluate the inner integral:

$$b_s^A(\nu, y_1, y_2) = 2\pi\sqrt{\frac{Ay_1y_2}{D}}K_{s-1/2}(2\pi|\nu|y_1)$$
$$\times \int_{x_2=-\infty}^{\infty} e\left(x_2\left(-\nu' + \frac{\nu A}{x_2^2+y_2^2}\right)\right)\left(\frac{1}{x_2^2+y_2^2}\right)^{1/2} I_{s-1/2}\left(\frac{2\pi|\nu|Ay_2}{x_2^2+y_2^2}\right)dx_2.$$

In the latter integral I we substitute $u = -2\pi\nu' x_2$, $y' = 2\pi|\nu'|y_2$ and find

$$I = (4\pi^2|\nu\nu'|Ay')^{-1/2}\int_{u=-\infty}^{\infty} \exp\left(iu\left(1 - \frac{4\pi^2\nu\nu' A}{u^2+y'^2}\right)\right)$$
$$\times\left(\frac{4\pi^2|\nu\nu'|y'A}{u^2+y'^2}\right)^{1/2} I_{s-1/2}\left(\frac{4\pi^2|\nu\nu'|Ay'}{u^2+y'^2}\right)du.$$

Putting $\kappa = 4\pi^2|\nu\nu'|A$ and $\varepsilon = -\frac{\nu\nu'}{|\nu\nu'|}$ we obtain

$$I = (\kappa y')^{-1/2}\int_{-\infty}^{\infty} \exp\left(iu\left(1 + \frac{\varepsilon\kappa}{u^2+y'^2}\right)\right)\left(\frac{\kappa y'}{u^2+y'^2}\right)^{1/2} I_{s-1/2}\left(\frac{\kappa y'}{u^2+y'^2}\right)du$$
$$= (\kappa y')^{-1/2} I'(y', \kappa, \varepsilon),$$

where $I'(y', \kappa, \varepsilon)$ denotes the latter integral. This was computed by Niebur in [Ni] (where it is denoted by $G_1(y', \kappa)$). One has

$$I'(y', \kappa, \varepsilon) = 2\sqrt{\kappa y'}\, K_{s-1/2}(y') \cdot \begin{cases} I_{2s-1}(2\sqrt{\kappa}), & \text{if } \varepsilon < 0, \\ J_{2s-1}(2\sqrt{\kappa}), & \text{if } \varepsilon > 0. \end{cases}$$

(Note that in [Ni] the factor 2 is missing. Furthermore, in the statement of Theorem 1 [Ni] $K_{s-1/2}(2\pi|n|y)$ should be replaced by $K_{s-1/2}(2\pi|m|y)$ and $M_{2s-1}(4\pi(mn)^{1/2}c)$ by $M_{2s-1}(4\pi(mn)^{1/2}/c)$.) We finally find

$$I = 2K_{s-1/2}(2\pi|\nu'|y_2) \cdot \begin{cases} I_{2s-1}(4\pi\sqrt{A|\nu\nu'|}), & \text{if } \nu\nu' > 0, \\ J_{2s-1}(4\pi\sqrt{A|\nu\nu'|}), & \text{if } \nu\nu' < 0. \end{cases}$$

This proves the assertion in the case $\nu \neq 0$.

Since the integral on the right-hand side of (25) is continuous in ν, we may determine $b_s^A(0, y_1, y_2)$ as

$$b_s^A(0, y_1, y_2) = \lim_{\nu\to 0}\left(b_s^A(\nu, y_1, y_2)\right).$$

By (24) we deduce

$$b_s^A(0, y_1, y_2) = \frac{\pi\Gamma(s-1/2)^2}{2\sqrt{D}\,\Gamma(2s)}(4A)^s(y_1y_2)^{1-s}.$$

□

We now show that $\Phi_m(z_1, z_2, s)$ has a meromorphic continuation in s to $\{s \in \mathbb{C};\ \sigma > 3/4\}$. The following lemma due to Zagier ([Za1] §4 Proposition) is important for our argument.

Lemma 3. *Let $a \in \mathbb{N}$, $m \in \mathbb{Z}$, and $\nu \in \mathfrak{d}^{-1}$. Then*

$$\frac{1}{a\sqrt{D}} G_a(m, \nu) = \sum_{\substack{r|\nu \\ r|a}} H_{a/r}\left(\frac{D\nu\nu'}{r^2}, -m\right),$$

where the finite exponential sums $G_a(m, \nu)$ resp. $H_b(m, n)$ are defined by (18) resp. (9).

Corollary 1. *There exists a constant $C > 0$ such that*

$$|G_a(m, \nu)| \leq C d(a) \sqrt{a|\nu\nu'|}$$

for all $m \in \mathbb{Z}$, $\nu \in \mathfrak{d}^{-1} - \{0\}$, and $a \in \mathbb{N}$. Here $d(a)$ denotes the number of positive divisors of a.

Proof. It is known that there is a $C > 0$ with

$$|H_c^{D_1}(n, m)| < C\sqrt{|n|/c}$$

for all $c \in \mathbb{N}$, $n \in \mathbb{Z} - \{0\}$, and $m \in \mathbb{Z}$ (cf. [Br] Lemma 3.2, notice the different normalization). This implies an analogous estimate for the $H_b(n, m)$. Now the assertion can be deduced by Lemma 3. □

Theorem 1. *Let $(z_1, z_2) \in \mathbb{H} \times \mathbb{H} - T(m)$. Then the function $\Phi_m(z_1, z_2, s)$ has a meromorphic continuation in s to $\{s \in \mathbb{C};\ \sigma > 3/4\}$. Up to a simple pole in $s = 1$ it is holomorphic in this domain.*

Proof. Let a_0 be an arbitrary positive integer and put $A = m/D$. It suffices to prove that the function

$$\tilde{\Phi}_m(z_1, z_2, s) = 2 \sum_{a \geq a_0} \Phi_m^a(z_1, z_2, s)$$

has for $y_1 y_2 > A/a_0^2$ a holomorphic continuation to $\{s \in \mathbb{C};\ \sigma > 3/4, s \neq 1\}$ with a simple pole at $s = 1$. Since the $\Phi_m^a(z_1, z_2, s)$ $(a \in \mathbb{N}_0)$ are holomorphic functions in s for $\sigma > 1/2$, the assertion then follows from (15).

We will use the Fourier expansion

$$\tilde{\Phi}_m(z_1, z_2, s) = 2 \sum_{\nu \in \mathfrak{d}^{-1}} \left[\sum_{a \geq a_0} G_a(m, \nu) b_s^{A/a^2}(\nu, y_1, y_2)\right] e(\nu x_1 + \nu' x_2)$$

of $\tilde{\Phi}_m(z_1, z_2, s)$ to establish its continuation. It suffices to show:

i) The sum

$$\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\neq 0}} \left[\sum_{a\geq a_0} G_a(m,\nu) b_s^{A/a^2}(\nu,y_1,y_2) \right] e(\nu x_1+\nu' x_2)$$

converges normally for $\sigma > 3/4$. Here the $b_s^{A/a^2}(\nu, y_1, y_2)$ are given by Lemma 2.

ii) The function

$$\sum_{a\geq a_0} G_a(m,0) b_s^{A/a^2}(0,y_1,y_2)$$

has a holomorphic continuation to $\{s \in \mathbb{C};\ \sigma > 3/4, s \neq 1\}$ with a simple pole at $s = 1$.

Assertion (i) can be proved using Corollary 1 and some standard estimates for the Bessel functions. For (ii) it suffices to consider the function

$$f(s) := \sum_{a\geq 1} G_a(m,0) b_s^{A/a^2}(0,y_1,y_2).$$

According to Lemma 2 and Lemma 3 we have

$$\begin{aligned} f(s) &= \frac{\pi\Gamma(s-1/2)^2}{2\sqrt{D}\,\Gamma(2s)}(4A)^s(y_1y_2)^{1-s}\sum_{a\geq 1} G_a(m,0)a^{-2s} \\ &= \frac{\pi\Gamma(s-1/2)^2}{2\Gamma(2s)}(4A)^s(y_1y_2)^{1-s}\sum_{a\geq 1}\sum_{r\mid a} a^{1-2s}H_{a/r}(0,-m) \\ &= \frac{\pi\Gamma(s-1/2)^2}{2\Gamma(2s)}(4A)^s(y_1y_2)^{1-s}\zeta(2s-1)\sum_{a\geq 1} a^{1-2s}H_a(0,-m). \end{aligned}$$

The assertion now follows from $H_a(0,-m) = O(a^{-1/2})$ for $a \to \infty$ and the properties of the Riemann zeta-function $\zeta(s)$. □

Definition 2. *Let $D \subset \mathbb{C}$ be an open subset, $a \in D$, and f a meromorphic function on D. We denote the constant term of the Laurent expansion of f at $s = a$ by $\mathcal{C}_{s=a}[f(s)]$.*

Definition 3. *Using the above notation we define*

$$\Phi_m(z_1,z_2) = \mathcal{C}_{s=1}\left[\Phi_m(z_1,z_2,s)\right].$$

By construction $\Phi_m(z_1, z_2)$ is a Γ_K-invariant function. We now consider its Fourier expansion.

Definition 4. *For a positive integer m we put*

$$q_n(m) = \begin{cases} -\delta_{-m,n}, & \text{if } n < 0, \\ -\dfrac{4\pi^2 m}{D} \displaystyle\sum_{b\geq 1} H_b(0,-m) b^{-1}, & \text{if } n = 0, \\ -2\pi\sqrt{m/n} \displaystyle\sum_{b\geq 1} H_b(n,-m) I_1\left(\dfrac{4\pi}{bD}\sqrt{nm}\right), & \text{if } n > 0. \end{cases} \tag{26}$$

According to Proposition 1, $q_0(m)$ is precisely the m-th Fourier coefficient of the Eisenstein series $E(z) \in M_2(D,\chi_D)$. If $n > 0$ then $q_n(m)$ equals the m-th coefficient of the Poincaré series P_{-n} of weight 2 (see (10)). Since $H_b(n,m) = \overline{H_b(n,m)}$, all $q_n(m)$ are real numbers.

Note that the $q_n(m)$ $(n \neq 0)$ could also be viewed as the Fourier coefficients of suitable non-holomorphic Poincaré series of weight 0.

Lemma 4. *For $(z_1,z_2) \in \mathbb{H}\times\mathbb{H} - S(m)$ with $y_1 y_2 > m/D$ we have the identity*

$$\begin{aligned}
\Phi_m(z_1,z_2) &= L + \frac{q_0(m)}{2}\log(y_1y_2) + \pi \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \mathrm{N}(\lambda)=-m/D}} \left(|\lambda y_1 - \lambda' y_2| - |\lambda y_1 + \lambda' y_2|\right) \\
&+ \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \mathrm{N}(\lambda)=-m/D}} \sum_{n\geq 1} \frac{1}{n}\left(e^{-2\pi n|\lambda y_1 + \lambda' y_2|} - e^{-2\pi n|\lambda y_1 - \lambda' y_2|}\right) e(n\lambda x_1 + n\lambda' x_2) \\
&+ \frac{2\pi}{D} \sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu>0\\ \nu'>0}} \sqrt{\frac{m}{\nu\nu'}} \sum_{a\geq 1} \frac{1}{a} G_a(m,\nu) I_1\left(\frac{4\pi}{aD}\sqrt{mD\nu\nu'}\right)\left(e(\nu z_1 + \nu' z_2) + e(-\nu\bar z_1 - \nu'\bar z_2)\right) \\
&+ \frac{2\pi}{D} \sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu>0\\ \nu'<0}} \sqrt{\frac{m}{|\nu\nu'|}} \sum_{a\geq 1} \frac{1}{a} G_a(m,\nu) J_1\left(\frac{4\pi}{aD}\sqrt{mD|\nu\nu'|}\right)\left(e(\nu z_1 + \nu' \bar z_2) + e(-\nu\bar z_1 - \nu' z_2)\right).
\end{aligned}$$

Here L denotes a suitable real constant which is independent of (z_1,z_2).

Proof. For brevity we put $A = m/D$. By definition and (15) one has

$$\begin{aligned}
\Phi_m(z_1,z_2) &= \mathcal{C}_{s=1}\left[\Phi_m^0(z_1,z_2,s)\right] + \mathcal{C}_{s=1}\left[2\sum_{a=1}^{\infty}\Phi_m^a(z_1,z_2,s)\right] \\
&= \Phi_m^0(z_1,z_2,1) + 2\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\neq 0}}\left[\sum_{a\geq 1} G_a(m,\nu) b_1^{A/a^2}(\nu,y_1,y_2)\right] e(\nu x_1 + \nu' x_2) \\
&\quad + \mathcal{C}_{s=1}\left[2\sum_{a\geq 1} G_a(m,0) b_s^{A/a^2}(0,y_1,y_2)\right].
\end{aligned} \tag{27}$$

Putting $s = 1$ into the Fourier expansion (21) and using the identities

$$\sqrt{\frac{\pi z}{2}} I_{1/2}(z) = \sinh(z) = \frac{e^z - e^{-z}}{2}, \tag{28}$$

$$\sqrt{\frac{2z}{\pi}} K_{1/2}(z) = e^{-z}, \tag{29}$$

we obtain

$$\Phi_m^0(z_1, z_2, 1) = 2\pi \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \mathrm{N}(\lambda) = -m/D}} \beta(\lambda y_1, \lambda' y_2) + \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \mathrm{N}(\lambda) = -m/D}} \sum_{n \geq 1} \frac{1}{n}$$
$$\times \left(e^{-2\pi n (\alpha(\lambda y_1, \lambda' y_2) - \beta(\lambda y_1, \lambda' y_2))} - e^{-2\pi n (\alpha(\lambda y_1, \lambda' y_2) + \beta(\lambda y_1, \lambda' y_2))} \right) e(n\lambda x_1 + n\lambda' x_2).$$

For $r_1, r_2 \in \mathbb{R}$, $r_1 r_2 < 0$ one has the relations

$$\alpha(r_1, r_2) + \beta(r_1, r_2) = |r_1 - r_2|,$$
$$\alpha(r_1, r_2) - \beta(r_1, r_2) = |r_1 + r_2|,$$
$$|r_1 - r_2| - |r_1 + r_2| = 2\beta(r_1, r_2),$$

which make it possible to write $\Phi_m^0(z_1, z_2, 1)$ in the form

$$\Phi_m^0(z_1, z_2, 1) = \pi \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \mathrm{N}(\lambda) = -m/D}} (|\lambda y_1 - \lambda' y_2| - |\lambda y_1 + \lambda' y_2|)$$
$$+ \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \mathrm{N}(\lambda) = -m/D}} \sum_{n \geq 1} \frac{1}{n} \left(e^{-2\pi n |\lambda y_1 + \lambda' y_2|} - e^{-2\pi n |\lambda y_1 - \lambda' y_2|} \right) e(n\lambda x_1 + n\lambda' x_2).$$

The second term in the right-hand side of (27) can be evaluated in a similar way by Lemma 2 and (29).

To compute the remaining term, we write as in the proof of Theorem 1

$$\frac{1}{\sqrt{D}} \sum_{a \geq 1} G_a(m, 0) a^{-2s} = \zeta(2s - 1) \sum_{a \geq 1} a^{1-2s} H_a(0, -m),$$

and obtain

$$\mathcal{C}_{s=1} \left[2 \sum_{a \geq 1} G_a(m, 0) b_s^{A/a^2}(0, y_1, y_2) \right]$$
$$= \mathcal{C}_{s=1} \left[(y_1 y_2)^{1-s} \zeta(2s - 1) \frac{\pi \Gamma(s - 1/2)^2}{\Gamma(2s)} \left(\frac{4m}{D} \right)^s \sum_{a \geq 1} a^{1-2s} H_a(0, -m) \right].$$

Observe that

$$\mathcal{C}_{s=1} \left[\frac{\pi \Gamma(s - 1/2)^2}{\Gamma(2s)} \left(\frac{4m}{D} \right)^s \sum_{a \geq 1} a^{1-2s} H_a(0, -m) \right]$$

is holomorphic at $s = 1$ with value $-q_0(m)$. Now by virtue of the Laurent expansions at $s = 1$

$$\begin{aligned}(y_1 y_2)^{1-s} &= 1 - \log(y_1 y_2)(s-1) + \dots,\\ \zeta(2s-1) &= \frac{1}{2}(s-1)^{-1} - \Gamma'(1) + \dots,\end{aligned}$$

the assertion can be deduced.

□

Theorem 2. *On* $\{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} - S(m);\ y_1 y_2 > m/D\}$ *the function* $\Phi_m(z_1, z_2)$ *has the Fourier expansion*

$$\begin{aligned}\Phi_m(z_1, z_2) &= L + \pi \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \mathrm{N}(\lambda) = -m/D}} \left(|\lambda y_1 - \lambda' y_2| - |\lambda y_1 + \lambda' y_2| \right) \\ &\quad + 2 \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu > 0}} q_{D\nu\nu'}(m) \log |1 - e(\nu x_1 + \nu' x_2 + i|\nu y_1 + \nu' y_2|)| \\ &\quad + \frac{q_0(m)}{2} \log(y_1 y_2) + 2 \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu > 0 \\ \nu' < 0}} p_{|D\nu\nu'|}(m) \log |1 - e(\nu z_1 + \nu' \bar{z}_2)| .\end{aligned}$$

Here L *is a real constant (as in Lemma 4), and the* $p_r(n)$ *denote the Fourier coefficients of* $P_r \in S_2(D, \chi_D)$ *as in Proposition 2.*

Proof. We may use Lemma 3 to rewrite the expression given in Lemma 4. For instance one has

$$\begin{aligned}T_1 &:= \frac{2\pi}{D} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu > 0 \\ \nu' < 0}} \sqrt{\frac{m}{|\nu\nu'|}} \sum_{a \geq 1} \frac{1}{a} G_a(m, \nu) J_1\left(\frac{4\pi}{aD} \sqrt{mD|\nu\nu'|} \right) \left(e(\nu z_1 + \nu' \bar{z}_2) + e(-\nu \bar{z}_1 - \nu' z_2) \right) \\ &= \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu > 0 \\ \nu' < 0}} \sum_{n \geq 1} \frac{1}{n} \left[2\pi \sqrt{\frac{m}{|D\nu\nu'|}} \sum_{a \geq 1} H_a(D\nu\nu', -m) J_1\left(\frac{4\pi}{aD} \sqrt{mD|\nu\nu'|} \right) \right] \\ &\qquad \times \left(e(n\nu z_1 + n\nu' \bar{z}_2) + e(-n\nu \bar{z}_1 - n\nu' z_2) \right) \\ &= -2 \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu > 0 \\ \nu' < 0}} \left[2\pi \sqrt{\frac{m}{|D\nu\nu'|}} \sum_{a \geq 1} H_a(D\nu\nu', -m) J_1\left(\frac{4\pi}{aD} \sqrt{mD|\nu\nu'|} \right) \right] \log |1 - e(\nu z_1 + \nu' \bar{z}_2)|\end{aligned}$$

and

$$\begin{aligned}T_2 :&= - \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \mathrm{N}(\nu) = -m/D}} \sum_{n \geq 1} \frac{1}{n} e^{-2\pi n |\nu y_1 - \nu' y_2|} e(n\nu x_1 + n\nu' x_2) \\ &= 2 \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu > 0 \\ \mathrm{N}(\nu) = -m/D}} \log |1 - e(\nu z_1 + \nu' \bar{z}_2)| .\end{aligned}$$

Hence we find

$$T_1 + T_2 = 2 \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu > 0 \\ \nu' < 0}} p_{|D\nu\nu'|}(m) \log |1 - e(\nu z_1 + \nu' \bar{z}_2)| \, .$$

By an analogous argument for the remaining terms, the assertion can be inferred. ☐

3.2 The singularities of $\Phi_m(z_1, z_2)$

It is well known that for any open subset $V \subset \mathbb{H} \times \mathbb{H}$ with compact closure $\overline{V} \subset \mathbb{H} \times \mathbb{H}$ the set

$$\mathcal{M}_{V,m} = \{(a, b, \lambda) \in \mathbb{Z} \times \mathbb{Z} \times \mathfrak{d}^{-1}; \quad ab - \mathrm{N}(\lambda) = m/D, \\ az_1 z_2 + \lambda z_1 + \lambda' z_2 + b = 0 \text{ for a } (z_1, z_2) \in V\} \quad (30)$$

is finite.

Theorem 3. *Let V be an open subset with compact closure $\overline{V} \subset \mathbb{H} \times \mathbb{H}$. Then*

$$\Phi_m(z_1, z_2) + \sum_{(a,b,\lambda) \in \mathcal{M}_{V,m}} \log |az_1 z_2 + \lambda z_1 + \lambda' z_2 + b|$$

is a real analytic function on $(\mathbb{H} \times \mathbb{H} - T(m)) \cap V$, which can be continued analytically to V. In other words: $\Phi_m(z_1, z_2)$ is a real analytic function on $\mathbb{H} \times \mathbb{H} - T(m)$ with a logarithmic singularity along $T(m)$.

Proof. Let $A = m/D$. Choose an $a_0 \in \mathbb{N}$ such that V is contained in $\{(z_1, z_2) \in \mathbb{H} \times \mathbb{H}; \ y_1 y_2 > A/a_0^2\}$. By definition and (15) we have

$$\Phi_m(z_1, z_2) = \Phi_m^0(z_1, z_2, 1) + 2 \sum_{a=1}^{a_0 - 1} \Phi_m^a(z_1, z_2, 1) + \mathcal{C}_{s=1} \left[\sum_{a \geq a_0} \Phi_m^a(z_1, z_2, s) \right]. \quad (31)$$

Using the Fourier expansion (cf. Lemma 4) one easily sees that $\mathcal{C}_{s=1} \left[\sum_{a \geq a_0} \Phi_m^a(z_1, z_2, s) \right]$ is a real analytic function on $\{(z_1, z_2) \in \mathbb{H} \times \mathbb{H}; \ y_1 y_2 > A/a_0^2\}$. Hence it suffices to consider the first two terms in the right-hand side of (31). Since

$$Q_0(z) = \frac{1}{2} \log \left(\frac{z+1}{z-1} \right),$$

these can be written as

$$\Phi_m^0(z_1, z_2, 1) + 2 \sum_{a=1}^{a_0 - 1} \Phi_m^a(z_1, z_2, 1) = \sum_{\substack{a, b \in \mathbb{Z}, |a| < a_0 \\ \lambda \in \mathfrak{d}^{-1} \\ ab - \mathrm{N}(\lambda) = m/D \\ (a,b,\lambda) \notin \mathcal{M}_{V,m}}} \log \left(\frac{|az_1 \bar{z}_2 + \lambda z_1 + \lambda' \bar{z}_2 + b|}{|az_1 z_2 + \lambda z_1 + \lambda' z_2 + b|} \right)$$
$$+ \sum_{(a,b,\lambda) \in \mathcal{M}_{V,m}} \log |az_1 \bar{z}_2 + \lambda z_1 + \lambda' \bar{z}_2 + b|$$
$$- \sum_{(a,b,\lambda) \in \mathcal{M}_{V,m}} \log |az_1 z_2 + \lambda z_1 + \lambda' z_2 + b|. \quad (32)$$

For $ab - \mathrm{N}(\lambda) = m/D$ the function $az_1 \bar{z}_2 + \lambda z_1 + \lambda' \bar{z}_2 + b$ has no zeros in $\mathbb{H} \times \mathbb{H}$. ☐

3.3 An infinite product that vanishes on $T(m)$

We now write $\Phi_m(z_1,z_2)$ as the sum of two functions ξ_m and ψ_m, where ξ_m has no singularities, and $-\frac{1}{2}\psi_m$ is the logarithm of the absolute value of a holomorphic infinite product Ψ_m whose only zeros lie on $T(m)$.

Definition 5. *For a positive integer m we put*

$$\xi_m(z_1,z_2) = \frac{q_0(m)}{2}\log(y_1y_2) + 2\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu>0\\ \nu'<0}} p_{|D\nu\nu'|}(m)\log|1-e(\nu z_1+\nu'\bar{z}_2)|$$

and

$$\psi_m(z_1,z_2) = \Phi_m(z_1,z_2) - \xi_m(z_1,z_2).$$

Using some standard estimates for the J_1-Bessel function one easily finds $p_r(m) = O(r)$ as $r\to\infty$. Thus the series defining $\xi_m(z_1,z_2)$ converges normally on $\mathbb{H}\times\mathbb{H}$. This implies that $\xi_m(z_1,z_2)$ is real analytic on $\mathbb{H}\times\mathbb{H}$.

Let W be a fixed connected component of $\mathbb{H}\times\mathbb{H}-S(m)$. For $\lambda\in\mathfrak{d}^{-1}$ we write $(\lambda,W)>0$, if $\lambda y_1+\lambda' y_2>0$ for all $(z_1,z_2)\in W$. Let ρ_W, ρ_W' be the uniquely determined real numbers such that

$$4(\rho_W y_1+\rho_W' y_2) = \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \mathrm{N}(\lambda)=-m/D}} (|\lambda y_1-\lambda' y_2| - |\lambda y_1+\lambda' y_2|)$$

for all $(z_1,z_2)\in W$. Then, for $(z_1,z_2)\in W$ with $y_1y_2>m/D$, the Fourier expansion of ψ_m can be written as

$$\psi_m(z_1,z_2) = L + 4\pi(\rho_W y_1+\rho_W' y_2) + 2\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ (\nu,W)>0}} q_{D\nu\nu'}(m)\log|1-e(\nu z_1+\nu' z_2)|. \tag{33}$$

Definition 6. *For $(z_1,z_2)\in\{(z_1,z_2)\in W;\ y_1y_2>m/D\}$ we define*

$$\Psi_m(z_1,z_2) = e(\rho_W z_1+\rho_W' z_2)\prod_{\substack{\nu\in\mathfrak{d}^{-1}\\ (\nu,W)>0}} (1-e(\nu z_1+\nu' z_2))^{-q_{D\nu\nu'}(m)}. \tag{34}$$

Note that the convergence of the infinite product (34) follows from the normal convergence of (33). We may also write

$$\Psi_m(z_1,z_2) = e(\rho_W z_1+\rho_W' z_2)\prod_{\substack{\nu\in\mathfrak{d}^{-1}\\ (\nu,W)>0\\ \mathrm{N}(\nu)=-m/D}} (1-e(\nu z_1+\nu' z_2))\prod_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu>0\\ \nu'>0}} (1-e(\nu z_1+\nu' z_2))^{-q_{D\nu\nu'}(m)}.$$

Therefore $\Psi_m(z_1,z_2)$ is holomorphic on $\{(z_1,z_2)\in W;\ y_1y_2>m/D\}$. It obviously satisfies

$$\log|\Psi_m(z_1,z_2)| = -\frac{1}{2}(\psi_m(z_1,z_2)-L). \tag{35}$$

Theorem 4. *The function $\Psi_m(z_1,z_2)$ has a holomorphic continuation to $\mathbb{H}\times\mathbb{H}$, and (35) holds on $\mathbb{H}\times\mathbb{H}-T(m)$. Let $V\subset\mathbb{H}\times\mathbb{H}$ be an open subset with compact closure $\overline{V}\subset\mathbb{H}\times\mathbb{H}$ and denote by $\mathcal{M}_{V,m}^+$ the finite set*

$$\mathcal{M}_{V,m}^+=\{(a,b,\lambda)\in\mathcal{M}_{V,m};\quad a>0 \text{ or } (a=0 \text{ and } \lambda>0)\}.$$

Then

$$\Psi_m(z_1,z_2)\prod_{(a,b,\lambda)\in\mathcal{M}_{V,m}^+}(az_1z_2+\lambda z_1+\lambda' z_2+b)^{-1} \tag{36}$$

is a holomorphic function without any zeros on V.

Proof. It suffices to show that Ψ_m can be continued holomorphically to any set

$$V=(a_1,b_1)\times i(a_2,b_2)\times(a_3,b_3)\times i(a_4,b_4)\subset\mathbb{H}\times\mathbb{H}$$

with compact closure $\overline{V}\subset\mathbb{H}\times\mathbb{H}$ and non-empty intersection $V\cap\{(z_1,z_2)\in W;\ y_1y_2>m/D\}$, and that (35), (36) hold on V.

According to Theorem 3 we may consider

$$\psi_m+2\sum_{(a,b,\lambda)\in\mathcal{M}_{V,m}^+}\log|az_1z_2+\lambda z_1+\lambda' z_2+b| \tag{37}$$

as a real analytic function on V. Moreover, the Fourier expansion (33) of ψ_m on $\{(z_1,z_2)\in W;\ y_1y_2>m/D\}$ implies that (37) is even pluriharmonic, i.e. is annihilated by the matrix differential operator

$$\begin{pmatrix}\frac{\partial^2}{\partial z_1\partial\bar z_1} & \frac{\partial^2}{\partial z_1\partial\bar z_2}\\ \frac{\partial^2}{\partial z_2\partial\bar z_1} & \frac{\partial^2}{\partial z_2\partial\bar z_2}\end{pmatrix}.$$

Hence there exists a holomorphic function $f:V\to\mathbb{C}$ with

$$\psi_m+2\sum_{(a,b,\lambda)\in\mathcal{M}_{V,m}^+}\log|az_1z_2+\lambda z_1+\lambda' z_2+b|=\Re(f(z_1,z_2))$$

(cf. [GR] ch. IX sect. C). On the non-empty intersection of the open sets V and $\{(z_1,z_2)\in W;\ y_1y_2>m/D\}$ we therefore have

$$\Re\left(\operatorname{Log}\Psi_m(z_1,z_2)-L/2-\sum_{(a,b,\lambda)\in\mathcal{M}_{V,m}^+}\operatorname{Log}(az_1z_2+\lambda z_1+\lambda' z_2+b)\right)=-\frac{1}{2}\Re(f(z_1,z_2)).$$

But now f can only differ by an additive constant from the expression in brackets on the left-hand side. We may assume that this constant equals zero and find

$$\Psi_m(z_1,z_2)\prod_{(a,b,\lambda)\in\mathcal{M}_{V,m}^+}(az_1z_2+\lambda z_1+\lambda' z_2+b)^{-1}=e^{L/2}e^{-f(z_1,z_2)/2}.$$

Since $e^{-f(z_1,z_2)/2}$ is a holomorphic function without any zeros on V, we obtain the assertion. □

According to (35), $\Psi_m(z_1,z_2)$ is independent from the choice of W up to multiplication by a constant of absolute value 1.

4 Borcherds products on Hilbert modular surfaces

In the previous section we constructed for each $T(m)$ a holomorphic infinite product Ψ_m vanishing on $T(m)$. This is not necessarily automorphic; by construction we only know that

$$|\Psi_m(z_1, z_2)|e^{-\xi_m(z_1,z_2)/2} = e^{L/2}e^{-\Phi_m(z_1,z_2)/2} \tag{38}$$

is Γ_K-invariant. However, taking suitable finite products of the Ψ_m one can attain that the main parts of the ξ_m cancel out. Thereby one finds a new proof of [Bo2] Theorem 13.3 and [Bo3] in the special case of Hilbert modular surfaces.

Lemma 5. *(cp. [Bo2] Lemma 13.1.) Let Ψ be a meromorphic function on $\mathbb{H} \times \mathbb{H}$ and $k \in \mathbb{R}$. Suppose that $|\Psi(z_1, z_2)|(y_1 y_2)^{k/2}$ is invariant under Γ_K. Then there is a character χ of Γ_K such that Ψ is an automorphic form of weight k and character χ with respect to Γ_K, i.e. satisfies*

$$\Psi(\gamma z_1, \gamma' z_2) = \chi(\gamma)(cz_1 + d)^k (c' z_2 + d')^k \Psi(z_1, z_2) \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K.$$

Proof. Let $\gamma \in \Gamma_K$. According to the assumption we have

$$|\Psi(\gamma z_1, \gamma' z_2)|(\Im(\gamma z_1)\Im(\gamma' z_2))^{k/2} = |\Psi(z_1, z_2)|(y_1 y_2)^{k/2},$$
$$\left|\frac{\Psi(\gamma z_1, \gamma' z_2)}{\Psi(z_1, z_2)}\right| |(cz_1 + d)^{-k}(c' z_2 + d')^{-k}| = 1.$$

By the maximum modulus principle there exists a constant $\chi(\gamma)$ of absolute value 1 with

$$\frac{\Psi(\gamma z_1, \gamma' z_2)}{\Psi(z_1, z_2)}(cz_1 + d)^{-k}(c' z_2 + d')^{-k} = \chi(\gamma).$$

One easily checks that $\chi(\gamma)$ is multiplicative. □

Lemma 6. *The space $S_2(D, \chi_D)$ has a basis of cusp forms with integral rational Fourier coefficients.*

Proof. Cf. [DI] Corollary 12.3.8, Proposition 12.3.11. □

Theorem 5. *Let $f_1, \dots, f_d$ be a basis of $S_2(D, \chi_D)$ with integral rational Fourier coefficients and write $f_j(z) = \sum_{n \geq 1} a_n(f_j)e(nz)$. Assume that $N \in \mathbb{N}$ and $c_1, \dots, c_N \in \mathbb{Z}$ with*

$$c_1 a_1(f_j) + \ldots + c_N a_N(f_j) = 0$$

for $j = 1, \dots, d$. Then

$$\Psi(z_1, z_2) = \prod_{j=1}^{N} \Psi_j(z_1, z_2)^{c_j}$$

is a meromorphic function on $\mathbb{H} \times \mathbb{H}$ with the following properties:

i) Ψ *is an automorphic form of weight* $-\frac{1}{2}\sum_{j=1}^{N} c_j q_0(j)$ *with a certain character* χ *with respect to* Γ_K*. Here* $q_0(j)$ *is the* j*-th Fourier coefficient of the Eisenstein series* $E \in M_2(D, \chi_D)$ *(cf. Prop. 1).*

ii) *For any open subset* V *with compact closure* $\overline{V} \subset \mathbb{H} \times \mathbb{H}$ *the function*

$$\Psi(z_1, z_2) \prod_{\substack{j=1,\dots,N \\ (a,b,\lambda)\in\mathcal{M}_{V,j}^{+}}} (az_1z_2 + \lambda z_1 + \lambda' z_2 + b)^{-c_j}$$

is holomorphic without any zeros on V*.*

iii) *Let* W *be a connected component of* $\mathbb{H} \times \mathbb{H} - \bigcup_{j=1}^{N} S(j)$*. Then there exist real numbers* ρ_W*,* ρ'_W *and a constant* C *of modulus* 1 *such that* Ψ *has the Borcherds product expansion*

$$\Psi(z_1, z_2) = Ce(\rho_W z_1 + \rho'_W z_2) \prod_{j=1}^{N} \prod_{\substack{\nu\in\mathfrak{d}^{-1} \\ (\nu,W)>0}} (1 - e(\nu z_1 + \nu' z_2))^{-q_{D\nu\nu'}(j)}$$

on $\{(z_1, z_2) \in W;\ y_1y_2 > N/D\}$*. Here* $q_n(j)$ $(n > 0)$ *denotes the* j*-th Fourier coefficient of the Poincaré series* P_{-n} *of weight* 2 *(cf. (10)). For* $n < 0$ *we simply have* $q_n(j) = -\delta_{-j,n}$ *(cf. Def. 4).*

Proof. i). The assumption implies in particular

$$c_1p_r(1) + \ldots + c_Np_r(N) = 0$$

for all $r \in \mathbb{N}$. Thus we have

$$\xi(z_1, z_2) := \sum_{j=1}^{N} c_j\xi_j(z_1, z_2) = \frac{1}{2}\log(y_1y_2) \sum_{j=1}^{N} c_j q_0(j).$$

Since

$$|\Psi(z_1, z_2)|e^{-\xi(z_1,z_2)/2} = |\Psi(z_1, z_2)|(y_1y_2)^{-\frac{1}{4}\sum_{j=1}^{N} c_j q_0(j)}$$

is invariant under Γ_K, we find by Lemma 5 that $\Psi(z_1, z_2)$ is an automorphic form of weight

$$-\frac{1}{2}\sum_{j=1}^{N} c_j q_0(j).$$

Assertion (ii) follows by Theorem 4 and (iii) by the definition of Ψ_m. □

Remark. The coefficients $q_0(m)$ of the Eisenstein series $E(z)$ can be computed explicitly. For instance suppose that $D = p$ is a prime with $p \equiv 1 \pmod 4$ and that m also is a prime. Then $q_0(m) = 0$, if $\left(\frac{m}{p}\right) = -1$, and

$$q_0(m) = 4\frac{m+1}{L(-1, \chi_p)},$$

if $\left(\frac{m}{p}\right) = +1$.

5 Chern classes of Hirzebruch-Zagier divisors

Let X be a normal irreducible complex space. By a divisor on X we mean a formal linear combination $D = \sum n_Y Y$ ($n_Y \in \mathbb{Z}$) of irreducible closed analytic subsets Y of codimension 1 such that the support $\bigcup_{n_Y \neq 0} Y$ is a closed analytic subset of everywhere pure codimension 1. We denote the group of divisors on X by $D(X)$.

Now let $X = \mathbb{H} \times \mathbb{H}$ and Γ be the Hilbert modular group Γ_K of the real quadratic field K or a subgroup of finite index. For each positive integer m the Hirzebruch-Zagier divisor of discriminant m on X/Γ is defined as follows: The support of its inverse image under the canonical projection $\pi : X \to X/\Gamma$ is precisely the set $T(m)$ (see (11)), and the multiplicities of all irreducible components equal 1. (There is no ramification in codimension 1.) For simplicity we denote this divisor on X/Γ by $T(m)$, too. It is well known that $T(m)$ is algebraic.

We will use the following *modified divisor class group* $\widetilde{\mathrm{Cl}}(X/\Gamma)$: Each automorphic form f with character χ with respect to Γ defines via its zeros and poles a Γ-invariant divisor in $D(X)$, which is the inverse image of an algebraic divisor (f) in $D(X/\Gamma)$. We denote the subgroup generated by these divisors (f) by $\tilde{H}(X/\Gamma)$ and put

$$\widetilde{\mathrm{Cl}}(X/\Gamma) = D(X/\Gamma)/\tilde{H}(X/\Gamma).$$

We write $\mathrm{Cl}(X/\Gamma)$ for the usual divisor class group of X/Γ, i.e. the quotient $D(X/\Gamma)$ modulo the group of divisors coming from automorphic forms of weight 0 with trivial character.

We now give an algebraic interpretation of Theorem 5. First we need some more notation. Let $\mathcal{S}(D, \chi_D)$ be the $\mathbb{Z}$-module of all cusp forms in $S_2(D, \chi_D)$ whose Fourier coefficients lie in $\mathbb{Z}$, and put $d = \dim S_2(D, \chi_D)$. By virtue of Lemma 6 we have

$$\mathcal{S}(D, \chi_D) \cong \mathbb{Z}^d \quad \text{and} \quad \mathcal{S}(D, \chi_D) \otimes_{\mathbb{Z}} \mathbb{C} = S_2(D, \chi_D).$$

Special elements of the dual $\mathbb{Z}$-module $\mathcal{S}^*(D, \chi_D)$ of $\mathcal{S}(D, \chi_D)$ are the functionals

$$a_r : \mathcal{S}(D, \chi_D) \to \mathbb{Z}, \quad f = \sum_{n \geq 1} b_n e(nz) \mapsto a_r(f) = b_r \qquad (r \in \mathbb{N}). \tag{39}$$

The usual unfolding argument shows that a_r can also be described by means of the Petersson scalar product as $f \mapsto a_r(f) = 4\pi r \langle f, G_r^D \rangle$, where G_r^D denotes the Poincaré series defined in (5). Denote the $\mathbb{Z}$-submodule of $\mathcal{S}^*(D, \chi_D)$ generated by the a_r by $A(D, \chi_D)$ and the $\mathbb{C}$-dual space of $S_2(D, \chi_D)$ by $S_2^*(D, \chi_D)$. Then we have

$$A(D, \chi_D) \cong \mathbb{Z}^d \quad \text{and} \quad A(D, \chi_D) \otimes_{\mathbb{Z}} \mathbb{C} = S_2^*(D, \chi_D).$$

Now Theorem 5 obviously implies:

Theorem 6. *By $a_r \mapsto T(r)$ ($r \in \mathbb{N}$) a homomorphism*

$$\beta : A(D, \chi_D) \longrightarrow \widetilde{\mathrm{Cl}}(X_K) \tag{40}$$

is defined.

Let us briefly recall some basic facts on Chern classes and the cohomology of X_K. For any divisor D on X/Γ one has a corresponding sheaf $\mathcal{L}(D)$. The sections of $\mathcal{L}(D)$ over an open subset $U \subset X/\Gamma$ are meromorphic functions f with $(f) \geq -D$ on U.

We now temporarily assume that Γ acts fixed point freely on X. Then X/Γ is an analytic manifold and every divisor D on X/Γ a Cartier divisor, i.e. $\mathcal{L}(D)$ is a line bundle. The *Chern class*

$$c(D) = c(\mathcal{L}(D)) \in H^2(X/\Gamma, \mathbb{C})$$

of $\mathcal{L}(D)$ can be constructed as follows: One chooses a meromorphic function f on X such that (f) equals the inverse image $\pi^*(D)$ of D under the canonical projection π. Then

$$J(\gamma, z) = \frac{f(\gamma(z))}{f(z)}, \qquad \gamma \in \Gamma,$$

is a 1-cocycle of Γ in the ring of holomorphic invertible functions on X. Hence a Hermitean metric on the bundle $\mathcal{L}(D)$ is given by a positive C^∞-function $h : X \to \mathbb{R}$ with

$$h(\gamma z) = |J(\gamma, z)| h(z) \qquad \text{for all } \gamma \in \Gamma.$$

Then $\omega = \partial\bar{\partial}\log(h)$ defines a cohomology class in $H^2(X/\Gamma, \mathbb{C})$. This is the Chern class of D in the case that Γ acts fixed point freely on X.

In the general case one chooses a normal subgroup $\Gamma_0 \leq \Gamma$ of finite index and obtains the Chern class $c(D)$ by the isomorphism $H^2(X/\Gamma, \mathbb{C}) \cong H^2(X/\Gamma_0, \mathbb{C})^{\Gamma/\Gamma_0}$.

The construction of the Chern class gives rise to a homomorphism

$$c : \mathrm{Cl}(X_K) \longrightarrow H^2(X_K, \mathbb{C})$$

into the second cohomology. We will determine the images of the $T(m)$ explicitly. This has been done in a different way by Hirzebruch, Zagier and Oda (cf. [HZ] Conjecture 2, 2', [Od]).

According to the theory of Harder on the cohomology of X_K [Ha], the space $H^2(X_K, \mathbb{C})$ decomposes into a direct sum

$$H^2(X_K, \mathbb{C}) = H^2_{\text{Eis}}(X_K, \mathbb{C}) \oplus H^{2,0}_{\text{squ}}(X_K, \mathbb{C}) \oplus H^{1,1}_{\text{squ}}(X_K, \mathbb{C}) \oplus H^{0,2}_{\text{squ}}(X_K, \mathbb{C}).$$

Here the classes of the Eisenstein cohomology $H^2_{\text{Eis}}(X_K, \mathbb{C})$ are given by Eisenstein series, and $H^{p,q}_{\text{squ}}(X_K, \mathbb{C})$ consists of all cohomology classes which can be represented by a square integrable differential form of type (p, q).

It will turn out that the Chern classes of the Hirzebruch-Zagier divisors $T(m)$ lie in the subspace

$$H^{1,1}_{\text{univ}}(X_K, \mathbb{C}) \oplus H^{1,1}_{\text{sym}}(X_K, \mathbb{C}) \subset H^{1,1}_{\text{squ}}(X_K, \mathbb{C}),$$

where

$$H^{1,1}_{\text{univ}}(X_K, \mathbb{C}) = \mathbb{C}\frac{dz_1 \wedge d\bar{z}_1}{y_1^2} \oplus \mathbb{C}\frac{dz_2 \wedge d\bar{z}_2}{y_2^2}$$

is given by the universal classes, and $H^{1,1}_{\text{sym}}(X_K,\mathbb{C})$ can be described by Hilbert cusp forms as follows: Let Γ^1_K be the subgroup

$$\Gamma^1_K = \left\{ \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & -b' \\ -c' & d' \end{smallmatrix}\right) \right); \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_K \right\}$$

of $\mathrm{Sl}_2(\mathbb{R}) \times \mathrm{Sl}_2(\mathbb{R})$ and denote by $S_2(\Gamma^1_K)$ the space of holomorphic cusp forms of weight 2 with respect to Γ^1_K. Then $H^{1,1}_{\text{sym}}(X_K,\mathbb{C})$ is the image of the (well defined) map

$$\alpha : S_2(\Gamma^1_K) \to H^{1,1}_{\text{squ}}(X_K,\mathbb{C}), \quad g(z_1,z_2) \mapsto g(z_1,-\bar{z}_2)dz_1 \wedge d\bar{z}_2 + g(z_2,-\bar{z}_1)dz_2 \wedge d\bar{z}_1. \quad (41)$$

In the same way as Zagier (cf. [Za1] Appendix 1, [Za2] §6) we define for $s=\sigma+it \in \mathbb{C}$ with $\sigma>0$ and $(z_1,z_2) \in \mathbb{H}\times\mathbb{H}$:

$$\omega^1_m(z_1,z_2,s) = \sum_{\substack{a,b\in\mathbb{Z}\\ \lambda\in\mathfrak{d}^{-1}\\ ab-\mathrm{N}(\lambda)=m/D}} \frac{1}{(-az_1z_2+\lambda z_1-\lambda' z_2+b)^2} \frac{(y_1y_2)^s}{|-az_1z_2+\lambda z_1-\lambda' z_2+b|^{2s}}. \quad (42)$$

Since this series converges normally for $\sigma>0$, the function $\omega^1_m(z_1,z_2,s)$ transforms like a modular form of weight 2 under Γ^1_K. It can be seen that $\omega^1_m(z_1,z_2,s)$ has a holomorphic continuation in s to $\{s\in\mathbb{C};\ \sigma>-1/4\}$, and that

$$\omega^1_m(z_1,z_2) := \omega^1_m(z_1,z_2,0) \quad (43)$$

belongs to $S_2(\Gamma^1_K)$. The Fourier expansion of $\omega^1_m(z_1,z_2)$ has the form

$$\omega_m(z_1,z_2) = 8\pi^2 \sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu>0\\ \nu'<0}} \Bigg\{ - \sum_{\substack{r\in\mathbb{N}\\ r|\nu\sqrt{D}\\ \mathrm{N}(\nu/r)=-m/D}} r + 2\pi\sqrt{\frac{|\nu\nu'|}{m}} \sum_{a\geq 1} \frac{1}{a} G_a(m,\nu) J_1\left(\frac{4\pi}{a}\sqrt{\frac{m|\nu\nu'|}{D}}\right) \Bigg\} e(\nu z_1 - \nu' z_2). \quad (44)$$

Theorem 7. *The Chern class of the Hirzebruch-Zagier divisor $T(m)$ is given by*

$$c(T(m)) = -\frac{q_0(m)}{16}\left(\frac{dz_1\wedge d\bar{z}_1}{y_1^2} + \frac{dz_2\wedge d\bar{z}_2}{y_2^2}\right) + \frac{m}{4D}\alpha(\omega^1_m(z_1,z_2)),$$

where $q_0(m)$ denotes the m-th Fourier coefficient of the Eisenstein series $E(z)$ and α the map in (41).

Proof. By Theorem 4 and (38) the function $e^{\xi_m(z_1,z_2)/2}$ defines a Hermitean metric on $\mathcal{L}(T(m))$. Thus the Chern class of $T(m)$ equals

$$c(T(m)) = \frac{1}{2}\,\partial\bar{\partial}\xi_m(z_1,z_2).$$

The Fourier expansion of ξ_m can be written in the form (cp. Lemma 4 and Theorem 2)

$$\xi_m(z_1,z_2) = \frac{q_0(m)}{2}\log(y_1y_2) - \sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu>0\\ \mathrm{N}(\nu)=-m/D}} \sum_{n\geq 1} \frac{1}{n}\left(e(n\nu z_1 + n\nu'\bar z_2) + e(-n\nu\bar z_1 - n\nu' z_2)\right)$$

$$+ \frac{2\pi}{D} \sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu>0\\ \nu'<0}} \sqrt{\frac{m}{|\nu\nu'|}} \sum_{a\geq 1} \frac{1}{a} G_a(m,\nu) J_1\left(\frac{4\pi}{aD}\sqrt{mD|\nu\nu'|}\right)\left(e(\nu z_1 + \nu'\bar z_2) + e(-\nu\bar z_1 - \nu' z_2)\right).$$

Now the claim can be deduced by a straight-forward calculation. □

5.1 The Doi-Naganuma lifting

Let $\mathrm{Cl}_{\mathrm{H}}(X_K)$ resp. $\widetilde{\mathrm{Cl}}_{\mathrm{H}}(X_K)$ be the subgroup of $\mathrm{Cl}(X_K)$ resp. $\widetilde{\mathrm{Cl}}(X_K)$ generated by the $T(m)$. The composition of the Chern class map

$$c : \mathrm{Cl}_{\mathrm{H}}(X_K) \longrightarrow H^{1,1}_{\mathrm{univ}}(X_K,\mathbb{C}) \oplus H^{1,1}_{\mathrm{sym}}(X_K,\mathbb{C})$$

with the canonical projection on $H^{1,1}_{\mathrm{sym}}(X_K,\mathbb{C})$ induces a homomorphism

$$\tilde c : \widetilde{\mathrm{Cl}}_{\mathrm{H}}(X_K) \longrightarrow H^{1,1}_{\mathrm{sym}}(X_K,\mathbb{C}). \tag{45}$$

If we combine β, $\tilde c$ and α^{-1} (cf. (40), (45), (41)) we obtain a homomorphism

$$A(D,\chi_D) \longrightarrow S_2(\Gamma_K^1), \tag{46}$$

which is characterized by $a_r \mapsto \frac{r}{4D}\omega_r^1(z_1,z_2)$ (cf. Theorem 6, Theorem 7). We may tensor by $\mathbb{C}$ and identify $S_2(D,\chi_D) \to S_2^*(D,\chi_D)$, $f \mapsto \langle\cdot, f\rangle$ and get a map

$$S_2(D,\chi_D) \longrightarrow S_2(\Gamma_K^1) \quad \text{with} \quad G_r^D \mapsto \frac{1}{16\pi D}\omega_r^1(z_1,z_2). \tag{47}$$

In the following we show that it essentially equals the Doi-Naganuma map [DN, Na, Za1, As].

For simplicity we assume from now on that $\mathcal{O}$ contains a unit $\varepsilon_0 > 0 > \varepsilon_0'$ of negative norm. In this case the spaces $S_2(\Gamma_K^1)$ and $S_2(\Gamma_K)$ are identical via the isomorphism

$$S_2(\Gamma_K^1) \longrightarrow S_2(\Gamma_K), \quad f(z_1,z_2) \mapsto f(\varepsilon_0 z_1, -\varepsilon_0' z_2). \tag{48}$$

Let ω_m denote the image of ω_m^1 under (48), i.e. $\omega_m(z_1,z_2) = \omega_m^1(\varepsilon_0 z_1, -\varepsilon_0' z_2)$. (Notice that it equals the function ω_m defined in [Za1] Appendix 1.) The composition of (47) and (48) yields a homomorphism

$$\jmath : S_2(D,\chi_D) \longrightarrow S_2(\Gamma_K) \quad \text{with} \quad G_r^D \mapsto \frac{1}{16\pi D}\omega_r(z_1,z_2). \tag{49}$$

The latter property characterizes $\jmath$ uniquely, since the Poincaré series G_r^D generate the space $S_2(D,\chi_D)$.

But it follows from the description of the Doi-Naganuma map

$$\iota : S_2(D,\chi_D) \longrightarrow S_2(\Gamma_K), \qquad f \mapsto -\frac{1}{2\pi}\langle f(\tau), \Omega(-\bar{z}_1, -\bar{z}_2, \tau)\rangle_\tau$$

by means of its holomorphic kernel function

$$\Omega(z_1,z_2,\tau) = \sum_{n\geq 1} n\omega_n(z_1,z_2)e(n\tau)$$

([Za1] Theorem 4) that ι satisfies $\iota(G_r^D) = -\frac{1}{8\pi^2}\omega_r$. Indeed, we find that $\jmath = -\frac{\pi}{2D}\iota$. (Here ι is normalized such that normalized eigenforms in $S_2(D,\chi_D)$ are mapped to normalized Hilbert eigenforms.) Let us now summarize the above discussion.

Theorem 8. *Suppose that $\mathcal{O}$ contains a unit $\varepsilon_0 > 0 > \varepsilon_0'$ of negative norm. Denote by β the map (40), by $\tilde{c}$ the Chern class map (45), and by ι the Doi-Naganuma lifting. Identify $S_2(\Gamma_K)$ with $H^{1,1}_{\text{sym}}(X_K,\mathbb{C})$ via*

$$f(z_1,z_2) \mapsto -\frac{\pi}{2D}\alpha(f(\varepsilon_0 z_1, -\varepsilon_0' z_2)),$$

and consider the map $S_2(D,\chi_D) \to \widetilde{\text{Cl}}_{\text{H}}(X_K)\otimes_{\mathbb{Z}}\mathbb{C}$ which is given by $G_r^D \mapsto \frac{1}{4\pi r}T(r) = \beta(\langle\cdot, G_r^D\rangle)$. Then the following diagram commutes:

$$\begin{array}{ccc} S_2(D,\chi_D) & \longrightarrow & \widetilde{\text{Cl}}_{\text{H}}(X_K)\otimes_{\mathbb{Z}}\mathbb{C} \\ \iota\downarrow & & \downarrow\tilde{c} \\ S_2(\Gamma_K) & \longrightarrow & H^{1,1}_{\text{sym}}(X_K,\mathbb{C}). \end{array} \qquad (50)$$

5.2 Automorphic forms whose zeros and poles lie on Hirzebruch-Zagier divisors

Regarding Theorem 8 we may use the properties of the Doi-Naganuma lifting to study the map β. In this way we now show that every automorphic form with respect to Γ_K, whose zeros and poles lie on Hirzebruch-Zagier divisors can be written as a finite product of the functions $\Psi_m(z_1,z_2)$. Note that the $\Psi_m(z_1,z_2)$ itself are not necessarily automorphic. In this sense they could be viewed as "generalized Borcherds products".

By a standard lemma of the theory of newforms, $S_2(D,\chi_D)$ is the direct sum of the subspaces

$$S^+ = \{f \in S_2(D,\chi_D);\quad \chi_D(n) = -1 \Rightarrow a_n(f) = 0\},$$
$$S^- = \{f \in S_2(D,\chi_D);\quad \chi_D(n) = +1 \Rightarrow a_n(f) = 0\}.$$

It is well known that the Doi-Naganuma lifting ι is injective on S^+ and zero on S^-. This follows from the description of ι on Hecke eigenforms by means of the attached L-functions [Na, As, Ge].

It can be easily seen that there is a corresponding decomposition $A(D,\chi_D) = A^+ \oplus A^-$, where A^+ resp. A^- denotes the $\mathbb{Z}$-submodule of $A(D,\chi_D)$, which is generated by the a_r with $\chi_D(r) = +1$ resp. $\chi_D(r) = -1$. By Theorem 8 we find that β (40) is injective on A^+ and zero on A^-.

Theorem 9. *Assume that $\mathcal{O}$ contains a unit of negative norm. Let f be an automorphic form of weight k with respect to Γ_K with an arbitrary character. Suppose that the divisor of f in $D(X_K)$ has the form*

$$(f) = \sum_{j=1}^{N} c_j T(j) \qquad (c_j \in \mathbb{Z}).$$

Then up to a constant multiple f equals

$$\Psi(z_1, z_2) = \prod_{j=1}^{N} \Psi_j(z_1, z_2)^{c_j},$$

with the functions Ψ_j defined in (34). In particular f has a Borcherds product expansion, and $k = -\frac{1}{2}\sum_{j=1}^{N} c_j q_0(j)$ is given by the coefficients of the Eisenstein series $E(z)$.

Proof. The assumption implies that

$$\beta\left(\sum_{j=1}^{N} c_j a_j\right) = 0 \in \widetilde{\mathrm{Cl}}_{\mathrm{H}}(X_K).$$

Therefore $\sum_{j=1}^{N} c_j a_j$ lies in A^-. Thus there is an $N' \in \mathbb{N}$ and integers b_j $(j = 1, \dots, N')$ with $b_j = 0$ for $\chi_D(j) \neq -1$ such that $\sum_{j=1}^{N} c_j a_j + \sum_{j=1}^{N'} b_j a_j = 0 \in A(D, \chi_D)$. By Theorem 5 we may infer that

$$\Psi(z_1, z_2) = \prod_{j=1}^{N} \Psi_j(z_1, z_2)^{c_j}$$

is an automorphic form of weight $-\frac{1}{2}\sum_{j=1}^{N} c_j q_0(j)$ with $(\Psi) = (f)$. (Note that $\Psi_j(z_1, z_2)$ is trivially constant for $j \in \mathbb{N}$ with $\chi_D(j) = -1$.) Thus $\Psi(z_1, z_2)/f$ is an automorphic form without zeros and poles and thereby constant. □

Remark. To prove Theorem 8 and Theorem 9 without the restriction that $\mathcal{O}$ contains a unit of negative norm, one would have to compare the map (47) with a modified Doi-Naganuma lifting, given by the kernel function $\Omega^1(z_1, z_2, \tau) = \sum_{n \geq 1} n\omega_n^1(z_1, z_2)e(n\tau)$.

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