THE GROSS-ZAGIER FORMULA AND THE BORCHERDS LIFT

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1. Introduction

The purpose of the present note is to report on recent joint work with T. Yang on Faltings heights of CM cycles and derivatives of L-functions [BY]. In this work we study the Faltings height pairing of arithmetic special divisors and CM cycles on Shimura varieties associated to orthogonal groups. We compute the archimedean contribution to the height pairing and derive a conjecture relating the total pairing to the central derivative of a Rankin L-function. We prove the conjecture in certain cases where the Shimura variety has dimension 0, 1, or 2. In particular, this leads to a new proof of the Gross-Zagier formula.

2. The Gross-Zagier formula

Let E be an elliptic curve over $\mathbb{Q}$, say, given by a Weierstrass equation

$$E : y^2 = x^3 + ax + b$$

with $a, b \in \mathbb{Q}$. The rational points of E, that is, in the solutions to the Weierstrass equation over $\mathbb{Q}$, form an abelian group $E(\mathbb{Q})$. By the Mordell-Weil theorem, this group is finitely generated. Therefore

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{tors},$$

where $r \in \mathbb{Z}_{\geq 0}$ is the rank of E and $E(\mathbb{Q})_{tors}$ is the finite subgroup of points of finite order in $E(\mathbb{Q})$. By a theorem of Mazur it is known which abelian groups can occur as $E(\mathbb{Q})_{tors}$. It is a list of fifteen groups, all of order ≤ 12. However, the rank r is still very mysterious. The Birch and Swinnerton-Dyer conjecture predicts that r can be computed by means of the Hasse-Weil L-function $L(E, s)$ of E. In view of the celebrated modularity theorem of Wiles et al. [Wi], [BCDT], the L-function has a holomorphic continuation to $\mathbb{C}$ and satisfies a functional equation relating the value at s to the value at $2 - s$. The Birch and Swinnerton-Dyer conjecture states that

$$r = \operatorname{ord}_{s=1} L(E, s).$$

Not much is known in general regarding the conjecture. By the work of Gross-Zagier and Kolyvagin we know that it is true when the analytic rank of E, the quantity $\operatorname{ord}_{s=1} L(E, s)$,

Date: April 23, 2009.

2000 *Mathematics Subject Classification.* 14G40, 11G18, 11F67.

I reported on this material at the Symposium on *Automorphic Forms, L-Functions and Shimura Varieties* at INHA University, Incheon, Korea (2008). I would like to thank the organizers for the invitation and for their hospitality during the conference.

is equal to 0 or 1. Here the contribution of Gross and Zagier is an explicit construction of a point of infinite order on elliptic curves with analytic rank 1. It can be deduced from the Gross-Zagier formula, which be briefly recall in a formulation which is convenient for the present note.

Assume that the L-function $L(E,s)$ of E has an odd functional equation so that the central critical value $L(E,1)$ vanishes. In this case the Birch and Swinnerton-Dyer conjecture predicts the existence of a rational point of infinite order on E. Let N be the conductor of E, and let $X_0(N)$ be the moduli space of cyclic isogenies of degree N of generalized elliptic curves. Let K be an imaginary quadratic field such that N is the norm of an integral ideal of K, and write D for the discriminant of K. We may consider the divisor $Z(D)$ on $X_0(N)$ given by elliptic curves with complex multiplication by the maximal order of K. By the theory of complex multiplication, this divisor is defined over K, and its degree h is given by the class number of K. Hence the divisor $y(D) = \operatorname{tr}_{K/\mathbb{Q}}(Z(D) - h\cdot(\infty))$ has degree zero and is defined over $\mathbb{Q}$. Using a modular parametrization $X_0(N) \to E$, we obtain a rational point $y^E(D)$ on E. The Gross-Zagier formula [GZ] states that the canonical height of $y^E(D)$ is given by the derivative of the L-function of E over K at $s=1$, more precisely

$$\langle y^E(D), y^E(D)\rangle_{NT} = C\sqrt{|D|}L'(E,1)L(E,\chi_D,1).$$

Here C is an explicit non-zero constant which is independent of K, and $L(E,\chi_D,s)$ denotes the quadratic twist of $L(E,s)$ by the quadratic Dirichlet character χ_D corresponding to $K/\mathbb{Q}$. It is always possible to choose K such that $L(E,\chi_D,1)$ is non-vanishing. So, in this case, $y^E(D)$ has infinite order if and only if $L'(E,1)\neq 0$. In particular, if E has analytic rank 1, we see that $r\geq 1$. The inequality $r\leq 1$ follows from Kolyvagin's work. Besides its importance in the context of the Birch and Swinnerton-Dyer conjecture, the Gross-Zagier formula has a striking application to the Gauss class number problem. It can be used to find elliptic curves over $\mathbb{Q}$ whose analytic rank is at least 3, leading to effective lower bounds for class numbers by a result of Goldfeld.

The work of Gross and Zagier triggered a lot of further research on height pairings of algebraic cycles on Shimura varieties. For instance, Gross and Keating computed the intersection numbers of three Hecke correspondences on the product of two copies of the modular curve $X(1)$ over $\mathbb{Z}$ [GK]. Zhang considered heights of Heegner type cycles on Kuga-Sato fiber varieties over modular curves in [Zh1], and the heights of Heegner points on compact Shimura curves over totally real fields in [Zh2]. Kudla, Rapoport and Yang investigated Arakelov intersection numbers of special cycles on Shimura varieties of orthogonal type and related them to derivatives of Siegel Eisenstein series and modular L-functions, see e.g. [Ku1], [Ku4], [KRY2]. In most of this work, the connection between a height pairing and the derivative of an automorphic L-function comes up in a rather indirect way.

In our joint work with T. Yang [BY], we consider a different approach to obtain identities between certain height pairings on Shimura varieties of orthogonal type and derivatives of automorphic L-functions. It is based on the Borcherds lift [Bo1] and its generalization in [Br], [BF]. We propose a conjecture for the Faltings height pairing of arithmetic special divisors and CM cycles. We compute the archimedean contribution to the height pairing. Using this result we prove the conjecture in certain low dimensional cases.

3. Shimura varieties associated to orthogonal groups

Let (V,Q) be a quadratic space over $\mathbb{Q}$ of signature $(n,2)$, and let $H = \mathrm{GSpin}(V)$. We realize the hermitian symmetric space corresponding to $H(\mathbb{R})$ as the Grassmannian

$$\mathbb{D} = \{z \subset V(\mathbb{R});\quad \dim(z) = 2 \text{ and } Q\mid_z < 0\}$$

of oriented negative definite two-dimensional subspaces of $V(\mathbb{R})$. For a compact open subgroup $K \subset H(\mathbb{A}_f)$ we consider the Shimura variety

$$X_K = H(\mathbb{Q})\backslash\big(\mathbb{D} \times H(\mathbb{A}_f)/K\big).$$

It is a quasi-projective variety of dimension n, which is defined over $\mathbb{Q}$, see [Ku2]. Note that for small n there are exceptional isomorphisms relating H to other classical groups. For instance $\mathrm{GSpin}(1,2) \cong \mathrm{GL}_2(\mathbb{R})$, so in the $n = 1$ case we are essentially looking at modular curves. Hilbert modular surfaces can be viewed as a particular $n = 2$ case and Siegel modular threefolds as a $n = 3$ case.

Let $L \subset V$ be an even lattice, and write L' for the dual of L. The discriminant group L'/L is finite. Throughout we assume that $K \subset H(\mathbb{A}_f)$ stabilizes $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and that K acts trivially on L'/L. This is no loss of generality, since we can always fulfil this assumption by choosing K sufficiently small.

It is an important feature of such Shimura varieties that they come with natural families of algebraic cycles in all codimensions, see e.g. [Ku2]. These special cycles arise from embeddings of rational quadratic subspaces $V' \subset V$ of signature $(n',2)$ with $0 \leq n' \leq n$. It is an interesting problem to consider height pairings of arithmetic versions of special cycles in complementary codimension, see [Ku4]. In the present paper we study this problem for special divisors (where $n' = n - 1$) and special 0-cycles (where $n' = 0$). The latter are also called CM cycles since they are associated to CM number fields.

We define CM cycles on X_K following [Scho]. Let $U \subset V$ be a negative definite two-dimensional *rational* subspace of V. It determines a two point subset $\{z_U^\pm\} \subset \mathbb{D}$ given by $U(\mathbb{R})$ with the two possible choices of orientation. Let $V_+ \subset V$ be the orthogonal complement of U. Then V_+ is a positive definite subspace of dimension n, and we have the rational splitting

$$V = V_+ \oplus U.$$

Let $T = \mathrm{GSpin}(U)$, which we view as a subgroup of H acting trivially on V_+, and put $K_T = K \cap T(\mathbb{A}_f)$. We obtain the CM cycle

$$Z(U) = T(\mathbb{Q})\backslash\big(\{z_U^\pm\} \times T(\mathbb{A}_f)/K_T\big) \longrightarrow X_K.$$

The cycle $Z(U)$ is defined over $\mathbb{Q}$. It can be viewed as a generalization of CM points on modular curves.

Next we define special divisors on X_K (cf. [Bo1], [Br], [Ku3]). We follow the description in [Ku3, pp. 304]. Let $x \in V(\mathbb{Q})$ be a vector of positive norm. We write V_x for the orthogonal complement of x in V and H_x for the stabilizer of x in H. So $H_x \cong \mathrm{GSpin}(V_x)$.

The sub-Grassmannian

$$\mathbb{D}_x = \{z \in \mathbb{D};\ z \perp x\}$$

defines an analytic divisor on $\mathbb{D}$. For $h \in H(\mathbb{A}_f)$ we consider the natural map

$$H_x(\mathbb{Q})\backslash\mathbb{D}_x \times H_x(\mathbb{A}_f)/(H_x(\mathbb{A}_f) \cap hKh^{-1}) \longrightarrow X_K, \quad (z, h_1) \mapsto (z, h_1 h).$$

Its image defines a divisor $Z(x, h)$ on X_K, which is rational over $\mathbb{Q}$. For $m \in \mathbb{Q}_{>0}$ let

$$\Omega_m = \{x \in V;\ Q(x) = m\}$$

be the corresponding quadric in V. If $\Omega_m(\mathbb{Q})$ is non-empty, then by Witt's theorem, we have $\Omega_m(\mathbb{Q}) = H(\mathbb{Q})x_0$ and $\Omega_m(\mathbb{A}_f) = H(\mathbb{A}_f)x_0$ for a fixed element $x_0 \in \Omega_m(\mathbb{Q})$. For $\mu \in L'/L$ we may write

$$(\mu + \hat{L}) \cap \Omega_m(\mathbb{A}_f) = \coprod_j K\xi_j^{-1}x_0$$

as a finite disjoint union, where $\xi_j \in H(\mathbb{A}_f)$. This follows from the fact that $\mu + \hat{L}$ is compact and $\Omega_m(\mathbb{A}_f)$ is a closed subset of $V(\mathbb{A}_f)$. We define a composite special divisor by putting

$$Z(m, \mu) = \sum_j Z(x_0, \xi_j).$$

The definition is independent of the choice of x_0 and the representatives ξ_j. These divisors generalize Heegner divisors on modular curves.

4. Harmonic weak Maass forms and automorphic Green functions

For the divisors $Z(m, \mu)$ we obtain Arakelov Green functions by means of a regularized theta lift of harmonic weak Maass forms. We now describe this construction.

We consider the space S_L of Schwartz functions on $V(\mathbb{A}_f)$ which are supported on $\hat{L}'$ and which are constant on cosets of $\hat{L}$. The characteristic functions

$$\phi_\mu = \mathrm{char}(\mu + \hat{L})$$

of the cosets $\mu \in L'/L$ form a basis of S_L. We write $\Gamma' = \mathrm{Mp}_2(\mathbb{Z})$ for the full inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in the two fold metaplectic covering of $\mathrm{SL}_2(\mathbb{R})$. Recall that there is a Weil representation ρ_L of Γ' on S_L, see e.g. [Bo1], [BY].

Let $k \in \frac{1}{2}\mathbb{Z}$. We write $M^!_{k,\rho_L}$ for the space of S_L-valued weakly holomorphic modular forms of weight k for Γ' with representation ρ_L. Recall that weakly holomorphic modular forms are those meromorphic modular forms whose poles are supported at the cusps. The space of weakly holomorphic modular forms is contained in the space H_{k,ρ_L} of harmonic weak Maass forms of weight k for Γ' with representation ρ_L. Recall that harmonic weak Maass forms are real analytic modular forms which are annihilated by the weight k Laplacian and which may have poles at the cusps. An element $f \in H_{k,\rho_L}$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c^+(n,\mu) q^n \phi_\mu + \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n<0}} c^-(n,\mu) \Gamma(1-k, 4\pi |n| v) q^n \phi_\mu,$$

where $\Gamma(a,t)$ denotes the incomplete Gamma function, and v is the imaginary part of $\tau \in \mathbb{H}$. Note that $c^{\pm}(n,\mu) = 0$ unless $n \in Q(\mu) + \mathbb{Z}$, and that there are only finitely many $n < 0$ for which $c^+(n,\mu)$ is non-zero. There is an antilinear differential operator $\xi : H_{k,\rho_L} \to S_{2-k,\bar{\rho}_L}$ to the space of cusp forms of weight $2-k$ with dual representation. It is surjective and gives rise to an exact sequence

$$0 \longrightarrow M^!_{k,\rho_L} \longrightarrow H_{k,\rho_L} \xrightarrow{\ \xi\ } S_{2-k,\bar{\rho}_L} \longrightarrow 0\ .$$

For $\tau \in \mathbb{H}$, $z \in \mathbb{D}$ and $h \in H(\mathbb{A}_f)$, let $\theta_L(\tau, z, h)$ be the Siegel theta function associated to the lattice L. Let $f \in H_{1-n/2,\bar{\rho}_L}$ be a harmonic weak Maass form of weight $1 - n/2$, and denote its Fourier expansion as above. We consider the regularized theta integral

$$\Phi(z,h,f) = \int_{\mathcal{F}}^{reg} \langle f(\tau), \theta_L(\tau, z, h)\rangle \, d\mu(\tau).$$

This theta lift was studied in [Br], [BF], generalizing the Borcherds lift of weakly holomorphic modular forms [Bo1]. The following theorem is proved in [Br], [BF].

Theorem 4.1. *The function $\Phi(z,h,f)$ is a logarithmic Green function for the divisor*

$$Z(f) = \sum_{\mu \in L'/L} \sum_{m>0} c^+(-m,\mu) Z(m,\mu)$$

in the sense of Arakelov geometry.

This means that $\Phi(z,h,f)$ is smooth on $X_K \backslash Z(f)$ with a logarithmic singularity along the divisor $-2Z(f)$. The $(1,1)$-form $dd^c\Phi(z,h,f)$ can be continued to a smooth form on all of X_K, and we have the Green current equation

$$dd^c[\Phi(z,h,f)] + \delta_{Z(f)} = [dd^c \Phi(z,h,f)],$$

where δ_Z denotes the Dirac current of a divisor Z (cf. [SABK]). Note that the Green function $\Phi(z,h,f)$ is harmonic when $c^+(0,0) = 0$.

The pair $\hat{Z}(f) = (Z(f), \Phi(\cdot, f))$ defines an arithmetic divisor on X_K. We obtain a linear map

$$H_{1-n/2,\bar{\rho}_L} \longrightarrow \hat{\mathrm{Z}}^1(X_K)_{\mathbb{C}}, \quad f \mapsto \hat{Z}(f)$$

to the group of arithmetic divisors on X_K. It gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^!_{1-n/2,\bar{\rho}_L,0} & \longrightarrow & H_{1-n/2,\bar{\rho}_L,0} & \xrightarrow{\ \xi\ } & S_{1+n/2,\rho_L} & \longrightarrow & 0\ . \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widehat{\mathrm{Rat}}(X_K)_{\mathbb{C}} & \longrightarrow & \widehat{\mathrm{Z}}^1(X_K)_{\mathbb{C}} & \longrightarrow & \widehat{\mathrm{CH}}^1(X_K)_{\mathbb{C}} & \longrightarrow & 0
\end{array}$$

Here $H_{1-n/2,\bar\rho_L,0}$ denotes the subspace of $H_{1-n/2,\bar\rho_L}$ consisting of those weak Maass forms with vanishing constant term $c^+(0,0)$, and $M^!_{1-n/2,\bar\rho_L,0}$ denotes the corresponding space of weakly holomorphic modular forms. Moreover, $\widehat{\mathrm{Rat}}(X_K)$ is the group of arithmetic divisors given by $(\mathrm{div}(r), -\log|r|^2)$ for a rational function r on X_K, and $\widehat{\mathrm{CH}}^1(X_K)$ is the first arithmetic Chow group. The left vertical arrow in the above diagram is the Borcherds lift of weakly holomorphic modular forms, see [Bo1].

5. CM values of Green functions

We aim to compute the Faltings height pairing of the arithmetic special divisor $\hat Z(f)$ and the CM cycle $Z(U)$. The pairing is a sum of an archimedean and a non-archimedean contribution. We begin by computing the archimedean part. It is given by the evaluation

$$\frac{1}{2}\Phi(Z(U),f) = \frac{1}{2}\sum_{(z,h)\in Z(U)} \Phi(z,h,f) \tag{5.1}$$

of the Green function of $\hat Z(f)$ at the cycle $Z(U)$. We now describe the quantities that enter in the formula for (5.1).

By means of the splitting $V = V_+ \oplus U$, we obtain definite lattices $N = L\cap U$ and $P = L\cap V_+$. Let

$$\theta_P(\tau) = \sum_{\lambda\in P'} q^{Q(\lambda)}\phi_\lambda = \sum_{\mu\in P'/P}\sum_{m\geq 0} r(m,\mu)q^m\phi_\mu$$

be the theta series in $M_{n/2,\rho_P}$ associated to the positive definite lattice P. The Fourier coefficients $r(m,\mu)$ are the representation numbers of m by the coset $\mu+P$. For to the negative definite 2-dimensional lattice N there is a similar theta series. The corresponding genus theta series is related to an incoherent Eisenstein series $E_N(\tau,s;1)$ of weight 1 via the Siegel Weil formula. The central derivative

$$\mathcal{E}_N(\tau) = \frac{d}{ds}E_N(\tau,s;1)\mid_{s=0}$$

is a harmonic weak Maass form in H_{1,ρ_N}.

For a cusp form $g\in S_{1+n/2,\rho_L}$ with Fourier expansion $g=\sum_\mu\sum_{m>0} b(m,\mu)q^m\phi_\mu$, we consider the Rankin type L-function

$$L(g,U,s) = (4\pi)^{-(s+n)/2}\Gamma\left(\tfrac{s+n}{2}\right)\sum_{m>0}\sum_{\mu\in P'/P} r(m,\mu)\overline{b(m,\mu)}m^{-(s+n)/2}. \tag{5.2}$$

This L-function can be written as a Rankin-Selberg convolution against the Eisenstein series $E_N(\tau,s;1)$. Under mild assumptions on U, the completed L-function $L^*(g,U,s) := \Lambda(\chi_D,s+1)L(g,U,s)$ satisfies the functional equation

$$L^*(g,U,s) = -L^*(g,U,-s).$$

Consequently, it vanishes at $s=0$, the center of symmetry, and it is of interest to describe the derivative $L'(g,U,0)$.

Theorem 5.1. *Let $f \in H_{1-n/2,\bar{\rho}_L}$, and assume that the constant term $c^+(0,0)$ of f vanishes. We have*

$$\Phi(Z(U), f) = \deg(Z(U)) \cdot \left(\mathrm{CT}\left(\langle f^+,\, \theta_P \otimes \mathcal{E}_N^+\rangle\right) + L'(\xi(f), U, 0)\right). \tag{5.3}$$

Here f^+ and $\mathcal{E}_N^+$ denote the holomorphic parts of the harmonic weak Maass forms f and $\mathcal{E}_N$. Moreover, $\mathrm{CT}(S)$ *denotes the constant term of a q-series S.*

The first summand on the right hand side is an explicit (rational) linear combination of the coefficients $\kappa(m,\mu)$ of $\mathcal{E}_N^+$. Each of these coefficients is a rational linear combination of $\log(p)$ for primes p, which can be computed explicitly.

The theorem can be proved by combining the approach of Kudla and Schofer to evaluate regularized theta integrals on special cycles (see [Ku3], [Scho]) with results on harmonic weak Maass forms and automorphic Green functions obtained in [BF]. The basic idea is to view the evaluation of $\Phi(z,h,f)$ on $Z(U)$ as an integral over $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T$. Then the CM value $\Phi(Z(U), f)$ can be computed using the see-saw dual pair

$$\begin{tikzcd}
\mathrm{SL}_2 \times \mathrm{SL}_2 \arrow[d, dash] \arrow[dr, dash] & \mathrm{SO}(V) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{SL}_2 & \mathrm{SO}(V^+) \times \mathrm{SO}(U),
\end{tikzcd}$$

the Siegel-Weil formula (see [We], [KR1], [KR2]), and the properties of the Maass lowering and raising operators on Eisenstein series and harmonic weak Maass forms.

When f is actually weakly holomorphic then $\xi(f) = 0$. So the second summand on the right hand side of (5.3) vanishes. Moreover, $\Phi(z,h,f) = -2\log|\Psi(z,h,f)|^2$ where $\Psi(z,h,f)$ is a rational function on X_K, namely the Borcherds lift of f, see [Bo1]. Hence Theorem 5.1 says that

$$\log|\Psi(Z(U), f)| = -\frac{\deg(Z(U))}{4}\,\mathrm{CT}\left(\langle f^+,\, \theta_P \otimes \mathcal{E}_N^+\rangle\right).$$

One obtains an explicit formula for the prime factorization of $\Psi(Z(U), f)$, see [Scho]. It generalizes the formula of Gross and Zagier on singular moduli, that is, CM values of the j-function.

6. Faltings heights

In this section we give a conjectural interpretation of the central derivative of the L-function $L(g, U, s)$ occuring in Theorem 5.1 as a height pairing. We are quite vague here and ignore various difficult technical problems regarding regular models.

Let $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ be a regular scheme which is projective and flat over $\mathbb{Z}$, of relative dimension n. An arithmetic divisor on $\mathcal{X}$ is a pair (x, g_x) of a divisor x on $\mathcal{X}$ and a logarithmic Green function g_x for the divisor $x(\mathbb{C})$ induced by x on the complex variety $\mathcal{X}(\mathbb{C})$, see [SABK]. Recall from [BGS] that there is a height pairing

$$\widehat{\mathrm{CH}}^1(\mathcal{X}) \times \mathrm{Z}^n(\mathcal{X}) \longrightarrow \mathbb{R}$$

between the first arithmetic Chow group of $\mathcal{X}$ and the group of codimension n cycles. When $\hat{x} = (x, g_x) \in \widehat{\mathrm{CH}}^1(\mathcal{X})$ and $y \in \mathrm{Z}^n(\mathcal{X})$ such that x and y intersect properly, it is defined by

$$\langle \hat{x}, y\rangle_{Fal} = \langle x, y\rangle_{fin} + \langle \hat{x}, y\rangle_{\infty},$$

where $\langle \hat{x}, y\rangle_{\infty} = \frac{1}{2} g_x(y(\mathbb{C}))$, and $\langle x, y\rangle_{fin}$ denotes the intersection pairing at the finite places. The quantity $\langle \hat{x}, y\rangle_{Fal}$ is called the Faltings height of y with respect to $\hat{x}$.

Assume that there is a regular scheme $\mathcal{X}_K \to \operatorname{Spec}\mathbb{Z}$, projective and flat over $\mathbb{Z}$, whose associated complex variety is a smooth compactification of X_K. Let $\mathcal{Z}(m,\mu)$ and $\mathcal{Z}(U)$ be suitable extensions to $\mathcal{X}_K$ of the cycles $Z(m,\mu)$ and $Z(U)$, respectively. Such extensions can be found in many cases using a moduli interpretation of $\mathcal{X}_K$, see e.g. [Ku4], [KRY2], or by taking flat closures as in [BBK]. For an $f \in H_{1-n/2,\bar{\rho}_L}$, we set $\mathcal{Z}(f) = \sum_\mu \sum_{m>0} c^+(-m,\mu)\mathcal{Z}(m,\mu)$. Then the pair

$$\hat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(\cdot, f))$$

defines an arithmetic divisor in $\widehat{\mathrm{CH}}^1(\mathcal{X}_K)_{\mathbb{C}}$.

Conjecture 6.1. *Let $f \in H_{1-n/2,\bar{\rho}_L}$, and assume that the constant term $c^+(0,0)$ of f vanishes. Then*

$$\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(U)\rangle_{Fal} = \frac{\deg(Z(U))}{2} L'(\xi(f), U, 0).$$

When f is actually weakly holomorphic then $\xi(f) = 0$ and the right hand side of the equality in Conjecture 6.1 vanishes. On the other hand, the Borcherds lift of f gives rise to a relation in the arithmetic Chow group which shows that the arithmetic divisor $\hat{\mathcal{Z}}(f)$ is rationally equivalent to zero. Hence the Faltings height on the left hand side vanishes as well. Moreover, the archimedean contribution to the height pairing must equal the negative of the contribution from the finite places. This leads to a general conjecture for the finite intersection pairing of $\mathcal{Z}(m,\mu)$ and $\mathcal{Z}(U)$ which motivates Conjecture 6.1:

Conjecture 6.2. *Let $\mu \in L'/L$, and let $m \in Q(\mu)+\mathbb{Z}$ be positive. Then $\langle \mathcal{Z}(m,\mu), \mathcal{Z}(U)\rangle_{fin}$ is equal to $-\frac{\deg(Z(U))}{2}$ times the (m,μ)-th Fourier coefficient of $\theta_P \otimes \mathcal{E}_N^+$.*

In view of Theorem 5.1, this conjecture is essentially equivalent to Conjecture 6.1.

7. The $n = 0$ case

Here we discuss Conjecture 6.1 in the case $n = 0$ where V is negative definite of dimension 2. Then we have $U = V$. The even Clifford algebra of V is an imaginary quadratic field $k = \mathbb{Q}(\sqrt{D})$, and $H = \operatorname{GSpin}(V) = k^*$. For simplicity we assume that the lattice L is isomorphic to a fractional ideal $\mathfrak{a} \subset k$ with the scaled norm $-\mathrm{N}(\cdot)/\mathrm{N}(\mathfrak{a})$ as the quadratic form. We take $K = \hat{\mathcal{O}}_k^*$, which acts on L'/L trivially. Then X_K is the union of two copies of the ideal class group $\mathrm{Cl}(k)$. An integral model over $\mathbb{Z}$ can be found by slightly varying the setup of [KRY1]. It is given as the moduli stack $\mathcal{C}$ over $\mathbb{Z}$ of elliptic curves with complex multiplication by the ring of integers of k. The special divisors can be defined on $\mathcal{C}$ by considering CM elliptic curves whose endomorphism ring is larger, and therefore equal to an order of a quaternion algebra. They are supported in finite characteristic.

In this case the lattice P is zero-dimensional and the L-function $L(\xi(f), U, s)$ vanishes identically. Therefore Conjecture 6.1 reduces to the statement that the arithmetic degree of the special divisor $\mathcal{Z}(f)$ on $\mathcal{C}$ should be given by the negative of the average of the regularized theta lift of f. This identity is proved in [BY] using Theorem 5.1 and the results obtained in [KRY1], respectively their generalization in [KY]. More precisely the following theorem is proved:

Theorem 7.1. *Let $f \in H_{1,\bar{\rho}_L}$ and assume that the constant term of f vanishes. Then*

$$\widehat{\deg}(\mathcal{Z}(f)) = -\frac{1}{2} \sum_{(z,h)\in X_K} \Phi(z,h,f).$$

8. The $n = 1$ case

Finally, we consider the case $n = 1$. We let V be the rational quadratic space of signature $(1, 2)$ given by the trace zero 2×2 matrices with the quadratic form $Q(x) = N \det(x)$, where N is a fixed positive integer. In this case $H \cong \mathrm{GL}_2$. We chose the lattice $L \subset V$ and the compact open subgroup $K \subset H(\mathbb{A}_f)$ such that X_K is isomorphic to the modular curve $\Gamma_0(N)\backslash\mathbb{H}$. The special divisors $Z(m, \mu)$ and the CM cycles $Z(U)$ are both supported on CM points and therefore closely related.

The space $S_{3/2,\rho_L}$ can be identified with the space of Jacobi cusp forms of weight 2 and index N. Recall that there is a Shimura lifting from this space to cusp forms of weight 2 for $\Gamma_0(N)$, see [GKZ]. Let G be a normalized newform of weight 2 for $\Gamma_0(N)$ whose Hecke L-function $L(G, s)$ satisfies an odd functional equation. There exists a newform $g \in S_{3/2,\rho_L}$ corresponding to G under the Shimura correspondence. It turns out that the L-function $L(g, U, s)$ is proportional to $L(G, s + 1)$.

We may choose $f \in H_{1/2,\bar{\rho}_L}$ with vanishing constant term such that $\xi(f) = \|g\|^{-2} g$ and such that the principal part of f has coefficients in the number field generated by the eigenvalues of G. Then $Z(f)$ defines an explicit point in the Jacobian of $X_0(N)$, which lies in the G isotypical component. In this case Conjecture 6.1 essentially reduces to the following Gross-Zagier type formula for the Neron-Tate height of $Z(f)$.

Theorem 8.1. *The Neron-Tate height of $Z(f)$ is given by*

$$\langle Z(f), Z(f)\rangle_{NT} = \frac{2\sqrt{N}}{\pi \|g\|^2} L'\big(G, 1\big).$$

The proof of this result which we give in [BY] is quite different from the original proof of Gross and Zagier and uses *minimal* information on finite intersections between special divisors. Instead, we derive it from Theorem 5.1, modularity of the generating series of special divisors (Borcherds' approach to the Gross-Kohnen-Zagier theorem [Bo2], [BrO]), and multiplicity one for the subspace of newforms in $S_{3/2,\rho_L}$ [SZ]. Another crucial ingredient is the non-vanishing result for coefficients of weight 2 Jacobi cusp forms by Bump, Friedberg, and Hoffstein [BFH]. Employing in addition the Waldspurger type formula for the coefficients of g [GKZ], we also obtain the Gross-Zagier formula as stated at the beginning.

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