Non-vanishing modulo ℓ of Fourier coefficients of half-integral weight modular forms

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1 Introduction

Let k be an integer and N a positive integer divisible by 4. If ℓ is a prime denote by v_ℓ a continuation of the usual ℓ-adic valuation on $\mathbb{Q}$ to a fixed algebraic closure. Let f be a modular form of weight $k+1/2$ with respect to $\Gamma_0(N)$ and Nebentypus character χ which has integral algebraic Fourier coefficients $a(n)$, and put $v_\ell(f) = \inf_n v_\ell(a(n))$. Suppose that f is a common eigenform of all Hecke operators $T(p^2)$ with corresponding eigenvalues λ_p.

In a recent paper, Ono and Skinner (under the additional assumption that f is "good") proved the following theorem [OnSk]: For all but finitely many primes ℓ there exist infinitely many square-free integers d for which $v_\ell(a(d)) = 0$. Their proof uses the theory of ℓ-adic Galois representations. Similar results were obtained by Jochnowitz [Jo] by developing a theory of half-integral weight modular forms modulo ℓ analogous to the integral weight theory due to Serre, Swinnerton-Dyer and Katz.

Results of this type can be viewed as mod ℓ versions of a well known theorem of Vignéras about the non-vanishing of Fourier coefficients of half-integral weight modular forms [Vi]. A new proof for this was given by the author [Br].

In the present paper we extend the method introduced in [Br] to the mod ℓ situation and thereby obtain a new approach to the above stated theorem and certain generalizations.

We shall use an application of the q-expansion principle of arithmetic algebraic geometry (Lemma 1) and exploit the properties of various well known operators defined on modular forms to infer our first result (Theorem 1). Roughly speaking it states that if for a given prime p and a given $\varepsilon \in \{\pm 1\}$ all Fourier coefficients $a(n)$ with $\left(\frac{n}{p}\right) = \varepsilon$ vanish modulo ℓ, then the Hecke eigenvalue λ_p satisfies a certain congruence modulo ℓ.

Under the (obviously necessary) assumption that f is not a linear combination of elementary theta series of weight $1/2$ or $3/2$, one can deduce several non-vanishing theorems. For instance in Theorem 4 we shall show that there exists a finite set $A_N(f)$ of primes which has an explicit description in terms of the eigenvalues λ_p with the property: For every prime ℓ with $(\ell, N) = 1$, $v_\ell(f) = 0$ and $\ell \notin A_N(f)$ there are infinitely many square-free d such that

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$v_\ell(a(d)) = 0$. Note that we do not need the notion of a "good" modular form. Theorem 2 and Theorem 3 contain certain refinements.

In the last section we will briefly indicate some applications. By the works of Waldspurger [Wa], Kohnen and Zagier [KoZa, Koh2] the results above have interesting consequences for the study of critical values of twisted L-series attached to newforms of weight $2k$ (Theorem 5). Moreover, one can consider the Cohen-Eisenstein series of level 4 and weight $k + 1/2$ to find indivisibility results for special values $L(1-k, \chi_D)$ of Dirichlet L-series (Theorem 6). Finally, we shall give a generalization of results due to Horie [Ho2] on the existence of certain infinite families of imaginary quadratic fields (Theorem 7).

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2 Notation

Let $GL_2^+(\mathbb{R})$ denote the group of all real 2×2-matrices with positive determinant. $GL_2^+(\mathbb{R})$ acts on the upper complex half plane $\mathbb{H}$ by Moebius transformations. As in [Sh2] we denote by G the set of ordered pairs $(\alpha, \phi(z))$, where $\alpha \in GL_2^+(\mathbb{R})$ with last row $(c\ d)$, and ϕ is a holomorphic function on $\mathbb{H}$ with the property $\phi^2(z) = t \det \alpha^{-1/2}(cz+d)$ (where $|t| = 1$). A group structure is defined on G by the multiplication law $(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z))$.

For an integer k, the group G acts on the set of holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f|_{k+\frac{1}{2}}\xi = f\,|\,\xi = \phi(z)^{-2k-1} f(\alpha z), \quad \xi = (\alpha, \phi(z)) \in G.$$

As usual, this operation can be extended to the group algebra $\mathbb{C}[G]$ by

$$f|_{k+\frac{1}{2}} \sum_\nu c_\nu \xi_\nu := \sum_\nu c_\nu f|_{k+\frac{1}{2}}\xi_\nu \quad \text{for} \quad \sum_\nu c_\nu \xi_\nu \in \mathbb{C}[G].$$

Write χ_0 for the trivial character, and if M is a non-zero integer, let χ_M denote the quadratic character corresponding to $\mathbb{Q}(\sqrt{M})$.

Let k be a non-negative integer, N a positive integer divisible by 4, and χ a Dirichlet character modulo N. We denote by $M_{k+\frac{1}{2}}(N, \chi)$ the space of modular forms of weight $k+\frac{1}{2}$ with respect to $\Gamma_0(N)$ with Nebentypus character χ (in the sense of [Sh2]). The subspace of cusp forms is denoted by $S_{k+\frac{1}{2}}(N, \chi)$. (Spaces of modular forms of integral weight occurring in the applications section will be denoted in an analogous way.)

For instance the standard theta function $\theta(z) = \sum_{n=-\infty}^{\infty} e(n^2 z)$ (where $e(z) := \exp(2\pi i z)$) is a modular form of weight $1/2$ with respect to $\Gamma_0(4)$. Moreover, for a primitive Dirichlet character ψ modulo r and a positive integer m we have the Shimura theta function

$$\theta_{\psi,m}(z) = \sum_{n=-\infty}^{\infty} \psi(n) n^\nu e(n^2 m z), \tag{1}$$

where ν is taken to be 0 resp. 1, if ψ is an even resp. odd character. It is shown in [Sh2] §2 that

$$\theta_{\psi,m} \in M_{\frac{1}{2}}(4r^2 m, \chi_m \psi) \qquad \text{if} \quad \nu = 0,$$

$$\theta_{\psi,m} \in S_{\frac{3}{2}}(4r^2m, \chi_{-m}\psi) \qquad \text{if} \quad \nu = 1.$$

Serre and Stark proved that every modular form in $M_{\frac{1}{2}}(N,\chi)$ is a linear combination of suitable theta series with even character of the above type [SeSt].

Let $S^*_{\frac{3}{2}}(N,\chi)$ denote the orthogonal complement (with respect to the Petersson inner product) of the subspace of $S_{\frac{3}{2}}(N,\chi)$, which is spanned by theta series $\theta_{\psi,m}$ with odd character ψ. According to [Ci] or [St], $S^*_{\frac{3}{2}}(N,\chi)$ maps to $S_2(N/2,\chi^2)$ under the Shimura lifting. For $k \geq 2$ we put for notational convenience $S^*_{k+\frac{1}{2}}(N,\chi) := S_{k+\frac{1}{2}}(N,\chi)$. Thus by [Sh2, Ni] and the results cited above, for every integer $k \geq 1$ we have the Shimura lifting which maps $S^*_{k+\frac{1}{2}}(N,\chi)$ to $S_{2k}(N/2,\chi^2)$.

For each rational prime ℓ let v_ℓ denote the ℓ-adic valuation on $\mathbb{Q}$ (recall the convention $v_\ell(0) = \infty$). Fix algebraic closures $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_\ell$. Thereby a continuation of v_ℓ to $\bar{\mathbb{Q}}$ is defined which we also denote by v_ℓ. For two elements a, b of the valuation ring $\mathcal{O}_\ell = \{z \in \bar{\mathbb{Q}};\ v_\ell(z) \geq 0\}$ we write $a \equiv b \pmod{\ell}$, if $a - b$ is contained in the maximal ideal of $\mathcal{O}_\ell$. Furthermore, for an integer n we write $\ell \| n$ if $\ell | n$ and ℓ does not divide n/ℓ.

We define a continuation of v_ℓ to the algebra $\bar{\mathbb{Q}}[[q]]$ of formal power series by

$$v_\ell(f) = \inf\{v_\ell(a(n))\,;\, n \geq 0\} \quad \text{for} \quad f = \sum_{n=0}^{\infty} a(n)q^n \in \bar{\mathbb{Q}}[[q]].$$

Obviously, if $c \in \bar{\mathbb{Q}}$ and $f, g \in \bar{\mathbb{Q}}[[q]]$, we have $v_\ell(cf) = v_\ell(c) + v_\ell(f)$ and $v_\ell(fg) \geq v_\ell(f) + v_\ell(g)$.

3 Operators

In the following section we briefly recall the properties of some operators defined on $M_{k+\frac{1}{2}}(N,\chi)$. Let $f = \sum_{n=0}^{\infty} a(n)e(nz) \in M_{k+\frac{1}{2}}(N,\chi)$ and put

$$W_N = \left(\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, N^{1/4}(-iz)^{1/2} \right) \in G.$$

Then the Fricke involution is given by $f \mapsto f|_{k+\frac{1}{2}} W_N$. It takes $M_{k+\frac{1}{2}}(N,\chi)$ to $M_{k+\frac{1}{2}}(N, \left(\frac{N}{\cdot}\right)\bar{\chi})$. Let m be a positive integer and

$$V_m = m^{-k/2-1/4} \left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}, m^{-1/4} \right) \in \mathbb{C}[G].$$

The shift $f \mapsto f|_{k+\frac{1}{2}} V_m$ induces a map from $M_{k+\frac{1}{2}}(N,\chi)$ to $M_{k+\frac{1}{2}}(Nm, \left(\frac{m}{\cdot}\right)\chi)$, and $f|_{k+\frac{1}{2}} V_m = \sum_{n=0}^{\infty} a(n)e(nmz)$.

If r is a real number, let $\xi(r) = (\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right), 1)$. For a primitive Dirichlet character ψ mod m we introduce the twist-operator. We put

$$f_\psi = f|_{k+\frac{1}{2}} \left(\frac{1}{W(\bar{\psi})} \sum_{u \bmod m} \bar{\psi}(u)\xi(u/m) \right),$$

where $W(\psi)$ denotes the Gauss sum of ψ. Then f_ψ has the Fourier expansion $f = \sum_{n=0}^\infty \psi(n)a(n)e(nz)$. The twist with ψ gives a map to $M_{k+\frac{1}{2}}(Nm^2, \chi\psi^2)$. Furthermore, if we define

$$B_m = \frac{1}{m} \sum_{u \bmod m} \xi(u/m) \in \mathbb{C}[G]$$

we have a projection operator $f \mapsto f|_{k+\frac{1}{2}} B_m = \sum_{n=0}^\infty a(mn)e(mnz)$. The latter is an element of $M_{k+\frac{1}{2}}(Nm^2, \chi)$. This can be proved by an obvious modification of [Sh2] Lemma 3.6.

For a prime p one has the Hecke operator $T(p^2)$ as defined in [Sh2] which takes $M_{k+\frac{1}{2}}(N, \chi)$ to itself. Let $f \,|\, T(p^2) = \sum_{n=0}^\infty c(n)e(nz)$. Then the action of $T(p^2)$ on the Fourier coefficients is given by

$$c(n) = a(p^2 n) + \chi^*(p) \left(\frac{n}{p}\right) p^{k-1} a(n) + \chi^*(p^2) p^{2k-1} a(n/p^2)$$

where $\chi^* = \left(\frac{-1}{\cdot}\right)^k \chi$. (Recall the convention $a(x) = 0$ for a number theoretic function a, if $x \notin \mathbb{N}_0$.)

All operators introduced above are linear and take $S^*_{k+\frac{1}{2}}(N, \chi)$ to a suitable $S^*_{k+\frac{1}{2}}(N', \chi')$. Moreover, they satisfy the following *commutation relations*:

For a positive integer m one immediately verifies

$$W_{Nm} = m^{k/2+1/4} W_N V_m. \tag{2}$$

Now let p be a prime not dividing N and φ the primitive Dirichlet character defined by $\varphi(x) = \left(\frac{x}{p}\right)$. For brevity define $g = f \,|\, W_N$. Then the identity

$$f_\varphi \,|\, W_{Np^2} = \chi^*(p) \left(p^{1/2} g \,|\, B_p - p^{-1/2} g \right) \tag{3}$$

holds (see [Sh2] §5). Note that by Mellin transform (3) is equivalent to the functional equation of the twisted Dirichlet series $\sum_{n \geq 1} \left(\frac{n}{p}\right) a(n) n^{-s}$. From (2) and (3) one immediately infers

$$f \,|\, B_p \,|\, W_{Np^2} = \chi^*(p) p^{-1/2} g_\varphi + p^{k-1/2} g \,|\, V_{p^2}. \tag{4}$$

Moreover the identity

$$pf \,|\, B_{p^2} \,|\, W_{Np^4} = \chi(p^2) \left(pg \,|\, B_{p^2} - g \,|\, B_p \right) + \chi^*(p) p^k g_\varphi \,|\, V_{p^2} + p^{2k} g \,|\, V_{p^4} \tag{5}$$

will be needed. As (5) is not standard, we briefly indicate how it can be proved: If we define for the moment

$$C_{p^2} = \frac{1}{p^2} \sum_{\substack{u \bmod p^2 \\ (u,p)=1}} \xi(u/p^2) \in \mathbb{C}[G],$$

then $B_{p^2} = C_{p^2} + \frac{1}{p} B_p$. In the usual way (as e.g. in [Sh2] Prop. 5.1) one easily shows

$$f \,|\, C_{p^2} \,|\, W_{Np^4} = \chi(p^2) g \,|\, C_{p^2}.$$

Now using (2) and (4) the stated equality can be deduced.

As above let p be a prime not dividing N. Then an easy calculation shows

$$f\,|\,W_N\,|\,T(p^2) \quad = \quad \bar{\chi}(p^2) f\,|\,T(p^2)\,|\,W_N. \tag{6}$$

If m is a positive integer prime to p and ψ is a quadratic character modulo m, one has

$$\begin{aligned} f\,|\,B_m\,|\,T(p^2) &= f\,|\,T(p^2)\,|\,B_m, \\ f_\psi\,|\,T(p^2) &= (f\,|\,T(p^2))_\psi. \end{aligned} \tag{7}$$

4 Results

Recall that N always denotes a positive integer divisible by 4 and χ a Dirichlet character modulo N. Let k be an integer, $k \geq 1$, and put $\chi^* := \left(\frac{-1}{\cdot}\right)^k \chi$. For a modular form f of integral weight k and level N, let us also denote the usual Fricke-involution by W_N. Thus $f|_k W_N = f|_k \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right) = N^{-k/2} z^{-k} f(-\frac{1}{Nz})$.

We need the following fundamental lemma:

Lemma 1. *Let f be a modular form of weight $k+1/2$ with respect to $\Gamma_1(N)$, whose Fourier coefficients $a(n)$ are algebraic integers. Let ℓ be a prime not dividing N. Then $f\,|\,W_N$ has algebraic Fourier coefficients and $v_\ell(f\,|\,W_N) = v_\ell(f)$.*

Let us first give a proof of an analogous statement for a modular form f of integral weight. By [Sh1] Th. 3.52, the Fourier coefficients $a(n)$ of f are contained in a finite extension K of $\mathbb{Q}$. Since $\mathcal{O}_{K,\ell} = \{z \in K\,;\, v_\ell(z) \geq 0\}$ is a discrete valuation ring, we may multiply f with a suitable power of a local parameter of the maximal ideal of $\mathcal{O}_{K,\ell}$, in order to normalize $v_\ell(f) = 0$. Then it suffices to show that $f\,|\,W_N$ has algebraic Fourier coefficients and $v_\ell(f\,|\,W_N) \geq 0$, i.e. that the coefficients of the expansion at the cusp 0 are all contained in $\mathcal{O}_\ell$. This follows immediately from the well known q-expansion principle, which is a deeper fact (see [DeRa] Th. VII 3.9 or [Katz] §1.6). Note that the assumption $(\ell, N) = 1$ is needed here. We will reduce Lemma 1 to this result, by multiplying with θ.

Proof of Lemma 1. Let m be a positive integer and $q = e^{2\pi i z}$. Then $\theta|_{\frac{1}{2}} V_m$ is invertible in $\mathbb{Z}[[q]]$, since its constant term equals 1, and $(\theta|_{\frac{1}{2}} V_m)^{-1}$ also has the constant term 1. Therefore $v_\ell(\theta|_{\frac{1}{2}} V_m) = v_\ell((\theta|_{\frac{1}{2}} V_m)^{-1}) = 0$ and

$$v_\ell(f) \geq v_\ell((\theta\,|\,V_m)^{-1}) + v_\ell(\theta\,|\,V_m \cdot f) = v_\ell(\theta\,|\,V_m \cdot f) \geq v_\ell(\theta\,|\,V_m) + v_\ell(f) = v_\ell(f),$$

which shows

$$v_\ell(f) = v_\ell(\theta\,|\,V_m \cdot f). \tag{8}$$

Since θf is a modular form of weight $k+1$ with respect to $\Gamma_1(N)$, by the earlier result $(\theta f)|_{k+1} W_N$ has algebraic Fourier coefficients and $v_\ell(\theta f) = v_\ell((\theta f)|_{k+1} W_N)$. If we write $N = 4M$, we have

$$\theta|_{\frac{1}{2}} W_N = M^{1/4} \theta|_{\frac{1}{2}} W_4|_{\frac{1}{2}} V_M = M^{1/4} \theta|_{\frac{1}{2}} V_M.$$

Thus $f|_{k+\frac{1}{2}}W_N$ has algebraic Fourier coefficients, and with (8) and $(\ell, M) = 1$ we get

$$v_\ell(f) = v_\ell(\theta f) = v_\ell((\theta f)|_{k+1}W_N) = v_\ell(\theta|_{\frac{1}{2}}V_M \cdot f|_{k+\frac{1}{2}}W_N) = v_\ell(f|_{k+\frac{1}{2}}W_N).$$

This proves the lemma. □

For the rest of this section (unless otherwise specified) let $f \in M_{k+\frac{1}{2}}(N,\chi)$ be a modular form with integral algebraic Fourier coefficients $a(n)$ ($n \in \mathbb{N}_0$) and define $g = \sum_{n=0}^\infty b(n)e(nz) = f \,|\, W_N$. By Lemma 1 the $b(n)$ are algebraic.

Lemma 2. *Let p be a prime not dividing N and ℓ a prime with $v_\ell(f) = 0$ and $(\ell, Np(p-1)) = 1$. Then there is an $n \in \mathbb{N}$ which is prime to p with $v_\ell(a(n)) = 0$.*

Proof. Since $v_\ell(f) = 0$ by Lemma 1 we find $v_\ell(g) = 0$. Using the assumption $(\ell, p(p-1)) = 1$ we see $v_\ell\left(p^{1/2}g \,|\, B_p - p^{-1/2}g\right) = 0$. If we put $\varphi = \left(\frac{\cdot}{p}\right)$, then according to (3) we have

$$f_\varphi = \chi^*(p)\left(p^{1/2}g \,|\, B_p - p^{-1/2}g\right) \,|\, W_{Np^2}.$$

By Lemma 1 we conclude $v_\ell(f_\varphi) = 0$ and this implies the assertion. □

Theorem 1. *Assume that N is a square. Let p be a prime not dividing N and $\varphi = \left(\frac{\cdot}{p}\right)$. Further, let ℓ be a prime with $(\ell, Np(p-1)) = 1$ and $v_\ell(f) = 0$. Suppose that f is an eigenform of $T(p^2)$ with corresponding eigenvalue λ_p.*

1. Let $\varepsilon \in \{\pm 1\}$ and

$$h := f \,|\, (1 - B_p) - \varepsilon f_\varphi = 2 \sum_{\substack{n \geq 0 \\ \varphi(n) = -\varepsilon}} a(n)e(nz).$$

Assume that $\nu := v_\ell(h) > 0$, then $v_\ell\left(\lambda_p - \varepsilon\chi^*(p)(p^k + p^{k-1})\right) \geq \nu$.

2. Let d^* be a square-free positive integer with $v_\ell(b(d^*m^2)) = 0$ for an $m \in \mathbb{N}$. Define

$$H := f \,|\, B_p - f \,|\, B_{p^2} = \sum_{\substack{n \geq 0 \\ p || n}} a(n)e(nz)$$

and assume that $\mu := v_\ell(H) > 0$. Then d^* is prime to p and the congruence $v_\ell\left(\lambda_p - \left(\frac{d^*}{p}\right)\chi^*(p)(p^k + p^{k-1})\right) \geq \mu$ holds.

Note that the assumption that N is a square means no restriction, because f can always be considered as a modular form of level N^2 without violating the conditions on p and ℓ. Moreover, note that according to Lemma 1 $v_\ell(f) = 0$ implies $v_\ell(g) = 0$. Thus by Lemma 2 there always exists a d^* as required in the second part.

Proof. 1. Since $h \in M_{k+\frac{1}{2}}(Np^2, \chi)$ and ℓ is prime to Np by Lemma 1 we have

$$v_\ell(h) = v_\ell(h \,|\, W_{Np^2}) = \nu. \tag{9}$$

We consider the twist of g with φ and use (3):

$$\begin{aligned} g_\varphi \mid W_{Np^2} &= \bar{\chi}^*(p)\left(p^{1/2} f \mid B_p - p^{-1/2} f\right) \\ &= \bar{\chi}^*(p)\left(p^{1/2}(f - \varepsilon f_\varphi - h) - p^{-1/2} f\right) \\ &= \bar{\chi}^*(p)\left(p^{1/2} - p^{-1/2}\right) f - \varepsilon \bar{\chi}^*(p) p^{1/2} f_\varphi - \bar{\chi}^*(p) p^{1/2} h \end{aligned}$$

Applying W_{Np^2} for a second time yields:

$$\begin{aligned} g_\varphi &= \bar{\chi}^*(p)\left(p^{1/2} - p^{-1/2}\right) p^{k+1/2} f \mid W_N V_{p^2} \\ &\quad - \varepsilon\left(pg \mid B_p - g\right) - \bar{\chi}^*(p) p^{1/2} h \mid W_{Np^2} \\ &= \bar{\chi}^*(p)\left(p^{k+1} - p^k\right) g \mid V_{p^2} + \varepsilon g - \varepsilon p g \mid B_p - \bar{\chi}^*(p) p^{1/2} h \mid W_{Np^2} \end{aligned}$$

Thus we get an identity of power series

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{n}{p}\right) b(n) e(nz) &= \bar{\chi}^*(p)\left(p^{k+1} - p^k\right) \sum_{n=0}^{\infty} b(n/p^2) e(nz) \\ &\quad - \varepsilon(p-1) \sum_{n=0}^{\infty} b(n) e(nz) + \varepsilon p \sum_{\substack{n=0 \\ (n,p)=1}}^{\infty} b(n) e(nz) \\ &\quad - \bar{\chi}^*(p) p^{1/2} h \mid W_{Np^2}. \end{aligned} \tag{10}$$

By (9) we have $v_\ell(\bar{\chi}^*(p) p^{1/2} h \mid W_{Np^2}) = \nu$. Hence, comparing Fourier coefficients in (10) we obtain the following congruences:

(i) If p does not divide n, then $v_\ell\left(\left(\frac{n}{p}\right) b(n) - \varepsilon b(n)\right) \geq \nu$.

(ii) If $p|n$ and p^2 does not divide n, we get $v_\ell\left(\varepsilon(p-1) b(n)\right) \geq \nu$. Since ℓ is prime to $p-1$, we obtain $v_\ell(b(n)) \geq \nu$.

(iii) If $p^2|n$, we see after division by $p-1$ that

$$v_\ell\left(b(n) - \varepsilon \bar{\chi}^*(p) p^k b(n/p^2)\right) \geq \nu. \tag{11}$$

Now put $g \mid T(p^2) = \sum_{n=0}^{\infty} c(n) e(nz)$. Since p is prime to N and $f \mid T(p^2) = \lambda_p f$, by (6) one has

$$g \mid T(p^2) = f \mid W_N \mid T(p^2) = \bar{\chi}(p^2) f \mid T(p^2) \mid W_N = \bar{\chi}(p^2) \lambda_p g,$$

and therefore

$$c(n) = \bar{\chi}(p^2) \lambda_p b(n) \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

On the other hand we know

$$c(n) = b(p^2 n) + \bar{\chi}^*(p)\left(\frac{n}{p}\right) p^{k-1} b(n) + \bar{\chi}^*(p^2) p^{2k-1} b(n/p^2).$$

If we consider this equation modulo ℓ and use (i)–(iii) above, we obtain for *every* non-negative integer n by an easy calculation

$$v_\ell\left(c(n) - \varepsilon \bar{\chi}^*(p)\left(p^k + p^{k-1}\right) b(n)\right) \geq \nu. \tag{13}$$

Comparing (12) and (13) gives
$$v_\ell\left(\bar\chi(p^2)\lambda_p b(n) - \varepsilon\bar\chi^*(p)\left(p^k + p^{k-1}\right) b(n)\right) \geq \nu.$$
Since $v_\ell(g) = 0$, there exists an n with $v_\ell(b(n)) = 0$. For this n we may divide by $b(n)$ and get
$$v_\ell\left(\lambda_p - \varepsilon\chi^*(p)\left(p^k + p^{k-1}\right)\right) \geq \nu.$$
This proves the first part of the lemma.

2. The proof of the second part is based on the same idea. Since $H \in M_{k+\frac{1}{2}}(Np^4, \chi)$ and ℓ is prime to Np, by Lemma 1 we have
$$v_\ell(H) = v_\ell(H \,|\, W_{Np^4}) = \mu. \tag{14}$$

As above we consider the twist of g with φ:
$$\begin{aligned}
g_\varphi \,|\, W_{Np^2} &= \bar\chi^*(p)\left(p^{1/2} f \,|\, B_p - p^{-1/2} f\right)\\
&= \bar\chi^*(p)\left(p^{1/2} f \,|\, B_{p^2} - p^{-1/2} f\right) + \bar\chi^*(p) p^{1/2} H\\
p^{1/2} g_\varphi \,|\, W_{Np^2} \,|\, W_{Np^4} &= \bar\chi^*(p) p f \,|\, B_{p^2} \,|\, W_{Np^4} - \bar\chi^*(p) f \,|\, W_{Np^4}\\
&\quad + \bar\chi^*(p) p\, H \,|\, W_{Np^4}
\end{aligned}$$

Using (2) and (5) we obtain
$$\begin{aligned}
p^k(p-1) g_\varphi \,|\, V_{p^2} &= \chi^*(p)\left(pg \,|\, B_{p^2} - g \,|\, B_p\right)\\
&\quad - \bar\chi^*(p) p^{2k}(p-1) g \,|\, V_{p^4} + \bar\chi^*(p) p\, H \,|\, W_{Np^4},
\end{aligned} \tag{15}$$
where $v_\ell(\bar\chi^*(p) pH \,|\, W_{Np^4}) = \mu$ according to (14). Carefully comparing Fourier coefficients and using $(\ell, (p-1)) = 1$ we find for every square-free $d \in \mathbb{N}$ and every positive integer m:

(i) $v_\ell\left(b(dm^2p^2) - \left(\frac{d}{p}\right)\bar\chi^*(p) p^k b(dm^2)\right) \geq \mu.$

(ii) $v_\ell\left(b(dm^2)\right) \geq \mu$, if $p|d$.

The latter congruence implies $(p, d^*) = 1$. Again, denote the Fourier coefficients of $g \,|\, T(p^2)$ by $c(n)$. In the same way as in the first part we find
$$v_\ell\left(c(dm^2) - \left(\frac{d}{p}\right)\bar\chi^*(p)\left(p^k + p^{k-1}\right) b(dm^2)\right) \geq \mu$$
and therefore
$$v_\ell\left(\lambda_p b(dm^2) - \left(\frac{d}{p}\right)\chi^*(p)\left(p^k + p^{k-1}\right) b(dm^2)\right) \geq \mu \tag{16}$$
for every square-free d and every positive integer m.

Applying (16) for d^* and an $m \in \mathbb{N}$ with $v_\ell(b(d^*m^2)) = 0$ we infer
$$v_\ell\left(\lambda_p - \left(\frac{d^*}{p}\right)\chi^*(p)\left(p^k + p^{k-1}\right)\right) \geq \mu.$$
This concludes the proof of the second part. □

Corollary 1. *Let p, φ, f, λ_p, ℓ be defined as in Theorem 1. For $\varepsilon \in \{\pm 1\}$ define $w_\varepsilon = v_\ell\left(\lambda_p - \varepsilon\chi^*(p)(p^k + p^{k-1})\right)$. 1. There is an integer m_ε with $\varphi(m_\varepsilon) = -\varepsilon$ and $v_\ell(a(m_\varepsilon)) \le w_\varepsilon$. 2. Let d^* be a square-free positive integer with $(p, d^*) = 1$ and $v_\ell(b(d^* m^2)) = 0$ for an $m \in \mathbb{N}$. Then there is an integer m_0 with $p || m_0$ and $v_\ell(a(m_0)) \le w_{\varphi(d^*)}$.*

Proof. 1. Suppose that there is no such m_ε. Then using the notation of Theorem 1 we have $v_\ell(h) > w_\varepsilon$. (Note that the Fourier coefficients of f are contained in a finite extension of $\mathbb{Q}$.) Hence, we find $v_\ell\left(\lambda_p - \varepsilon\chi^*(p)(p^k + p^{k-1})\right) > w_\varepsilon$, a contradiction. 2. The second statement can be proved in a completely analogous way. □

Lemma 3. *Let p, φ, f, λ_p be defined as in Theorem 1 and $\varepsilon \in \{\pm 1\}$. As in Corollary 1 put $w_\varepsilon = v_\ell\left(\lambda_p - \varepsilon\chi^*(p)(p^k + p^{k-1})\right)$. Further, let m^* be a positive integer with $\varphi(m^*) = \varepsilon$ and assume that ℓ is a prime not dividing $Np(p-1)$ such that $v_\ell(a(m^*)) = 0$. Then there exist infinitely many square-free integers d, such that $\varphi(d) = -\varepsilon$ and $v_\ell(a(dm_d^2)) \le w_\varepsilon$ for an integer m_d.*

Proof. If $w_\varepsilon = \infty$ we have nothing to prove, so let us assume $w_\varepsilon < \infty$. We are going to show by induction on t that there are integers $m_1, \ldots, m_t$ and distinct square-free integers $d_1, \ldots, d_t$, with $\varphi(d_j) = -\varepsilon$ and $v_\ell(a(d_j m_j^2)) \le w_\varepsilon$ for $j = 1, \ldots, t$.

For $t = 1$ the assertion follows from Corollary 1.

Now let $t > 1$. According to the above argument, there is a square-free integer d_1 and an integer m_1, such that

$$v_\ell(a(d_1 m_1^2)) \le w_\varepsilon, \quad \text{and} \quad \left(\frac{d_1}{p}\right) = -\varepsilon.$$

Thus, in particular d_1 is different from the square-free part of m^*. We can choose a prime q with

$$(q, \ell) = 1, \quad (q, p) = 1, \quad \left(\frac{m^*}{q}\right) = +\varepsilon, \quad \left(\frac{d_1}{q}\right) = -\varepsilon.$$

(In fact, there exist infinitely many such primes.) Put $\psi = \left(\frac{\cdot}{q}\right)$ and

$$f_1 = \frac{1}{2}\left(f \,|\, (1 - B_q) + \varepsilon f_\psi\right) \in M_{k+\frac{1}{2}}(Nq^2, \chi).$$

If we denote the Fourier coefficients of f_1 by $a_1(n)$, we have

$$f_1 = \sum_{n=0}^{\infty} a_1(n) e(nz) \quad \text{with} \quad a_1(n) = \begin{cases} a(n) & \text{if } \psi(n) = \varepsilon, \\ 0 & \text{if } \psi(n) \ne \varepsilon. \end{cases} \tag{17}$$

By (17) we find in particular $a_1(m^*) = a(m^*)$, and therefore $v_\ell(f_1) = 0$. Further, we have $a_1(d_1 m^2) = 0$ for all $m \in \mathbb{N}$.

Since by (7)

$$f_1 \,|\, T(p^2) = \frac{1}{2}\left(f \,|\, (1 - B_q) \,|\, T(p^2) + \varepsilon f_\psi \,|\, T(p^2)\right) = \lambda_p f_1,$$

we may apply the induction assumption on f_1. Thus there exist $t-1$ distinct square-free integers $d_2, \ldots, d_t$ and integers $m_2, \ldots, m_t$, such that $\varphi(d_j) = -\varepsilon$

and $v_\ell(a_1(d_j m_j^2)) \le w_\varepsilon$. Hence, in particular $a_1(d_j m_j^2) \neq 0$ and thereby $d_1 \neq d_j$ for $j = 2, \dots, t$.

Now $d_1, \dots, d_t$ satisfy the desired properties (for f):

$$\varphi(d_j) = -\varepsilon, \quad v_\ell(a(d_j m_j^2)) \le w_\varepsilon$$

for $j = 1, \dots, t$. □

In order to get non-trivial statements by Corollary 1 and Lemma 3 we now have to show that in the relevant situations $\lambda_p \neq \varepsilon \chi^*(p)(p^k + p^{k-1})$.

Lemma 4. *Let p be a prime and λ_p an eigenvalue of the Hecke operator $T(p^2)$ on $S^*_{k+\frac{1}{2}}(N, \chi)$. Then the estimate $|\lambda_p| < p^k + p^{k-1}$ holds.*

Proof. Via Shimura lifting (see [Sh2], [Ni], [Ci], [St]) λ_p also is an eigenvalue of the Hecke operator $T(p)$ on $S_{2k}(N/2, \chi^2)$. The assertion now follows for instance from a simple estimate due to Kohnen [Koh1] (which applies to our situation by an obvious modification). □

Remark. The numbers $\chi^*(p)(p^k + p^{k-1})$ occur as eigenvalues of the restriction of $T(p^2)$ on the space spanned by Shimura theta series (as defined in (1)).

Theorem 2. *Let ℓ be a prime not dividing N and suppose that f is a common eigenform of all $T(p^2)$ with corresponding eigenvalues λ_p. Assume that there is an $m^* \in \mathbb{N}$ with $v_\ell(a(m^*)) = 0$. Put*

$$w(f; \ell, m^*) = \min_p v_\ell \left(\lambda_p - \left(\frac{m^*}{p} \right) \chi^*(p)(p^k + p^{k-1}) \right),$$

where the minimum is taken over all primes p with $(p, N\ell m^*) = 1$ and $p \not\equiv 1 \pmod{\ell}$. Then there are infinitely many square-free integers d with $v_\ell(a(d)) \le w(f; \ell, m^*)$.

Proof. We may choose a prime p with $(p, N\ell m^*) = 1$, $p \not\equiv 1 \pmod{\ell}$ and

$$v_\ell \left(\lambda_p - \left(\frac{m^*}{p} \right) \chi^*(p)(p^k + p^{k-1}) \right) = w(f; \ell, m^*).$$

Then by Lemma 3 it suffices to prove the following

Lemma 5. *Assume that f is a Hecke eigenform. Let ℓ be a prime and d a positive square-free integer. Then for every $m \in \mathbb{N}$ the relation $v_\ell(a(dm^2)) \ge v_\ell(a(d))$ holds.*

Proof by induction on m. For $m = 1$ we have nothing to prove. Let $m > 1$ and choose a prime q dividing m and define m_1 by $m_1 q = m$. Since $f \,|\, T(q^2) = \lambda_q f$, one has

$$a(dm^2) = \left(\lambda_q - \chi^*(q) \left(\frac{dm_1^2}{q} \right) q^{k-1} \right) a(dm_1^2) - \chi(q^2) q^{2k-1} a(dm_1^2/q^2).$$

Considering $v_\ell(\lambda_q) \ge 0$ (λ_q is an algebraic integer) and $k \ge 1$ we obtain

$$v_\ell(a(dm^2)) \ge \min\{v_\ell(a(dm_1^2)), v_\ell(a(dm_1^2/q^2))\}.$$

Hence the assertion follows from the induction assumption. □

Before we state the next result, let us first introduce some notation. For a prime p and an algebraic integer λ we define

$$\begin{aligned} A(p,\lambda) &= \left\{q \in \mathbb{N} \text{ prime; } v_q(\lambda - \varepsilon\chi(p)(p^k + p^{k-1})) > 0 \text{ for an } \varepsilon = \pm 1\right\}, \\ A'(p,\lambda) &= A(p,\lambda) \cup \{q \in \mathbb{N} \text{ prime}\,;\; q|(p-1)\} \cup \{p\}. \end{aligned}$$

If λ_p is an eigenvalue of the Hecke operator $T(p^2)$ on $S^*_{k+\frac{1}{2}}(N,\chi)$ then $\#A(p,\lambda_p) < \infty$ according to Lemma 4.

Theorem 3. *Let $p_1, \ldots, p_r$ be distinct primes not dividing N, and put $\varphi_j = \left(\frac{\cdot}{p_j}\right)$ for $j = 1, \ldots, r$. Let $\varepsilon_1, \ldots, \varepsilon_r \in \{-1, 0, +1\}$. Suppose that f is a common eigenform of all $T(q^2)$ with corresponding eigenvalues λ_q. Let ℓ be a prime not dividing N with $v_\ell(f) = 0$ and $\ell \notin \bigcup_{j=1}^r A'(p_j, \lambda_{p_j})$. Then there exist infinitely many square-free integers d, such that*

$$\varphi_j(d) = \varepsilon_j \quad \textit{for every } j = 1, \ldots, r \textit{ and } \quad v_\ell(a(d)) = 0.$$

Proof. By Lemma 5 it suffices to prove the existence of infinitely many square-free integers d with $\varphi_j(d) = \varepsilon_j$ for every $j = 1, \ldots, r$ and $v_\ell(a(dm_d^2)) = 0$ for a suitable integer m_d. This will be done by induction on r.

For $r = 1$ and $\varepsilon_1 \in \{\pm 1\}$ the assertion follows from Corollary 1 and Lemma 3. If $r = 1$ and $\varepsilon_1 = 0$, in view of Lemma 1 and 2 there is an $n \in \mathbb{N}$ with $(n, p_1) = 1$ and $v_\ell(b(n)) = 0$. Hence, by Corollary 1 and the assumption on ℓ there exists an $m \in \mathbb{N}$ with $p_1 || m$ and $v_\ell(a(m)) = 0$. Moreover there is an $m^* \in \mathbb{N}$ with $(m^*, p_1) = 1$ and $v_\ell(a(m^*)) = 0$. The assertion can be deduced by an inductive argument similar to the proof of Lemma 3. We leave it to the reader to carry out the details.

Now let $r > 1$. Define

$$f_1 = \begin{cases} \frac{1}{2}\left(f \,|\, (1 - B_{p_1}) + \varepsilon_1 f_{\varphi_1}\right) \in M_{k+\frac{1}{2}}(Np_1^2, \chi) & \text{if } \varepsilon_1 \in \{\pm 1\}, \\ f \,|\, B_{p_1} - f \,|\, B_{p_1^2} \in M_{k+\frac{1}{2}}(Np_1^4, \chi) & \text{if } \varepsilon_1 = 0. \end{cases}$$

In the same way as in (17) f_1 has the Fourier expansion

$$f_1 = \sum_{\substack{n \geq 0 \\ \varphi_1(n) = \varepsilon_1}} a(n) e(nz) \quad \text{if } \varepsilon_1 \in \{\pm 1\}, \text{ resp. } \quad \sum_{\substack{n \geq 0 \\ p_1 || n}} a(n) e(nz) \quad \text{if } \varepsilon_1 = 0,$$

and according to Corollary 1 we have $v_\ell(f_1) = 0$ (if $\varepsilon_1 = 0$, one has to argue as indicated above). Moreover, by (7) we see $f_1 \,|\, T(p_j^2) = \lambda_{p_j} f_1$ for $j = 2, \ldots, r$. Applying the induction assumption on f_1 concludes the proof. $\square$

Finally we state a non-vanishing result comparable to Theorem 2 but not depending on the existence of a particular $m^* \in \mathbb{N}$ with $v_\ell(a(m^*)) = 0$.

For an eigenform f of all $T(q^2)$ with corresponding eigenvalues λ_q (q prime, $(q, N) = 1$) put

$$A_N(f) = \bigcap_{\substack{q \text{ prime} \\ (q,N)=1}} A'(q, \lambda_q). \tag{18}$$

Theorem 4. *Suppose that $f \in M_{k+\frac{1}{2}}(N, \chi)$ is a common eigenform of all $T(q^2)$ with corresponding eigenvalues λ_q (q prime). Then for every prime ℓ with $(\ell, N) = 1$, $v_\ell(f) = 0$ and $\ell \notin A_N(f)$ there exist infinitely many square-free integers d, such that $v_\ell(a(d)) = 0$.*

Proof. Since $\ell \notin A_N(f)$, there is a prime p with $(p,N)=1$ and $\ell \notin A'(p,\lambda_p)$. Thus the statement follows from Theorem 3. $\square$

5 Applications

1. *Critical values of twisted L-series attached to newforms of weight $2k$.* Let M be an odd square-free integer and $M^+_{k+\frac{1}{2}}(M)$ resp. $S^+_{k+\frac{1}{2}}(M)$ the subspace of $M_{k+\frac{1}{2}}(4M,\chi_0)$ resp. $S^*_{k+\frac{1}{2}}(4M,\chi_0)$ consisting of forms having a Fourier expansion $\sum_{n=0}^\infty c(n)e(nz)$ with $c(n)=0$ unless $(-1)^k n \equiv 0,1 \pmod 4$ (cf. [Koh2]).

Let $F=\sum_{n=1}^\infty a(n)e(nz)$ be a normalized newform in $S_{2k}(M,\chi_0)$ and denote by λ_p the corresponding eigenvalue of $T(p)$ (p prime) and by w_q the eigenvalue of the Atkin-Lehner involution at q (q prime, $q|M$). If D is an integer, then denote by $L(s;F,\chi_D)$ the twisted L-series $\sum_{n=1}^\infty \chi_D(n)a(n)n^{-s}$ attached to F.

According to [Koh2] the spaces $S_{2k}(M,\chi_0)$ and $S^+_{k+\frac{1}{2}}(M)$ are isomorphic as modules over the Hecke algebra. Via this isomorphism, there is an eigenform $g=\sum_{n=1}^\infty c(n)e(nz)$ in $S^+_{k+\frac{1}{2}}(M)$, uniquely determined up to multiplication with a non-zero complex number, such that

$$L(k;F,\chi_D)|D|^{k-1/2} = \sigma_0(M)^{-1}\frac{\pi^k}{(k-1)!}\frac{\langle F,F\rangle}{\langle g,g\rangle}|c(|D|)|^2 \tag{19}$$

for any fundamental discriminant D, for which $(-1)^k D>0$ and $\chi_D(q)=w_q$ for each prime q dividing M.

If on the contrary D is a fundamental discriminant with $\chi_D(q)\neq w_q$ for some prime divisor q of M, then $c(|D|)=0$. We may now deduce

Theorem 5. *For a normalized newform F in $S_{2k}(M,\chi_0)$, there exists a non-zero complex number Ω (which is an algebraic multiple of a period of F) with the property: For any odd prime ℓ with $(\ell,M)=1$ and $\ell\notin A_{4M}(F)$, there are infinitely many fundamental discriminants D with $(-1)^kD>0$ and*

$$v_\ell\left(\frac{L(k;F,\chi_D)|D|^{k-1/2}}{\Omega}\right)=0.$$

2. *Indivisibility of special values of Dirichlet L-series.* Let k be an even positive integer and let ℓ be a prime such that $\ell-1$ does not divide $2k$. Denote by $H_{k+\frac{1}{2}}$ the Cohen-Eisenstein series of weight $k+\frac{1}{2}$ (cf. [Co]), normalized such that

$$H_{k+\frac{1}{2}}=\sum_{n=0}^\infty c(n)e(nz)=\zeta(1-2k)+\zeta(1-k)e(z)+\dots.$$

Cohen showed that $H_{k+\frac{1}{2}}$ is an eigenform in $M^+_{k+\frac{1}{2}}(1)$ with corresponding eigenvalues $\sigma_{2k-1}(p)$ (p prime). Furthermore, if $D>0$ is a fundamental discriminant then

$$c(D)=L(1-k,\chi_D),$$

where $L(s,\chi_D)$ denotes the Dirichlet L-series attached to χ_D. It is well known that $L(1-k,\chi_D)=-B_{k,\chi_D}/k$ with the k-th generalized Bernoulli number

B_{k,χ_D}. Hence, by the Staudt-Clausen theorem and results of Carlitz on generalized Bernoulli numbers [Ca], the $c(n)$ are ℓ-integral.

According to Dirichlet's theorem on primes in arithmetic progressions we may choose a prime $p \geq 3$ such that the reduction of p modulo ℓ generates the multiplicative group $(\mathbb{Z}/\ell\mathbb{Z})^*$. Then $p \not\equiv 0, 1 \pmod{\ell}$ and it can be easily checked that ℓ does not divide $\sigma_{2k-1}(p) - p^k - p^{k-1}$. By Theorem 2 (applied with $m^* = 1$) we find

Theorem 6. *Let ℓ be a prime such that $\ell - 1$ does not divide $2k$ and $v_\ell(\zeta(1-k)) = 0$. Then there are infinitely many fundamental discriminants $D > 0$ with $v_\ell(L(1-k, \chi_D)) = 0$.*

Obviously, an analogous result could be deduced for odd $k \geq 3$. Theorem 3 could be applied to find a similar statement where the discriminants D satisfy prescribed local conditions at given primes.

I am indebted to W. Kohnen for pointing out the following application of Theorem 6. By the work of Mazur and Wiles [MaWi], if F is a real quadratic field and $\mathcal{O}_F$ its ring of integers then the order of the corresponding K_2-group is given by

$$\#K_2\mathcal{O}_F = 2^\nu w_2(F)|\zeta_F(-1)|,$$

where ν is an integer, and ζ_F denotes the zeta function of F. If F has discriminant D, then $\zeta_F(-1) = \zeta(-1)L(-1, \chi_D)$. Moreover, $w_2(F)$ denotes the largest positive integer n such that the Galois group $Gal(F(\zeta_n)/F)$ has exponent 2 (here ζ_n is a primitive n-th root of unity). By elementary Galois theory it can be seen that $Gal(F(\zeta_n)/F)$ is equal to the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$ or to a subgroup of index 2. Hence, for every prime $\ell \geq 7$ we find $v_\ell(w_2(F)) = 0$. Applying Theorem 6 for $k = 2$ we infer the following

Corollary 2. *Let ℓ be a prime ≥ 7. Then there exist infinitely many real quadratic fields F such that $K_2\mathcal{O}_F$ contains no element of order ℓ.*

3. *Indivisibility of class numbers of imaginary quadratic fields.* For a fundamental discriminant D let as usual $h(D)$ denote the class number of $\mathbb{Q}(\sqrt{D})$. In the following we shall give a generalization of results due to Horie on the existence of certain infinite families of imaginary quadratic fields (Th. 1, Th. 2 in [Ho2]). For related results also see [Ho1] and [HoOn].

Theorem 7. *Let $p_1, \dots, p_r$ be distinct odd primes and $\varepsilon_1, \dots, \varepsilon_r \in \{-1, 0, +1\}$. Let ℓ be a prime ≥ 5 such that ℓ does not divide $p_j(p_j-1)(p_j+1)$ for $j = 1, \dots, r$. Then there are infinitely many fundamental discriminants $D < 0$ for which $h(D)$ is not divisible by ℓ and $\left(\frac{D}{p_j}\right) = \varepsilon_j$ for $j = 1, \dots, r$.*

Proof. Consider the theta series $\theta^3(z) = \sum_{n \geq 0} r(n)e(nz) \in M_{3/2}(4, \chi_0)$. By the work of Gauss it is known that

$$r(n) = \begin{cases} 12H(4n) & \text{if } n \equiv 1, 2 \pmod{4}, \\ 24H(n) & \text{if } n \equiv 3 \pmod{8}, \\ r(n/4) & \text{if } n \equiv 0 \pmod{4}, \\ 0 & \text{if } n \equiv 7 \pmod{8}, \end{cases} \tag{20}$$

where $H(n)$ denotes the Hurwitz-Kronecker class number for $n \equiv 0, 3 \pmod 4$. If $-n = Dm^2$ with a fundamental discriminant $D < 0$, then

$$H(n) = \frac{h(D)}{w(D)} \sum_{t|m} \mu(t) \left(\frac{D}{t}\right) \sigma_1(m/t),$$

where $w(D)$ is half the number of units in $\mathbb{Q}(\sqrt{D})$.

Since $M_{3/2}(4, \chi_0)$ is one dimensional, θ^3 is an eigenform of all $T(p^2)$ and $\theta^3 \,|\, W_4 = \theta^3$. In view of (20) it suffices to show that there are infinitely many square-free $d \in \mathbb{N}$ with $\left(\frac{d}{p_j}\right) = \varepsilon_j$ $(j = 1, \dots, r)$ and $v_\ell(r(d)) = 0$.

Unfortunately, in this particular case Theorem 3 does not apply because $\theta^3 \,|\, T(p^2) = \sigma_1(p)\theta^3$ for every odd prime p. However, one can use Lemma 1 and Lemma 3 in a direct way to prove the assertion.

For a modular form f, a prime p and an $\varepsilon \in \{-1, 0, +1\}$ put $\varphi = \left(\frac{\cdot}{p}\right)$ and

$$f \,|\, \mathcal{B}_{p,\varepsilon} = \begin{cases} \frac{1}{2}\left(f \,|\, (1 - B_p) + \varepsilon f_\varphi\right) & \text{if } \varepsilon \in \{\pm 1\}, \\ f \,|\, B_p - f \,|\, B_{p^2} & \text{if } \varepsilon = 0. \end{cases}$$

For $0 \le t \le r$ define

$$M_t = p_1 \cdot \ldots \cdot p_t, \qquad f_t = \theta^3 \,|\, \mathcal{B}_{p_1,\varepsilon_1} \,|\, \ldots \,|\, \mathcal{B}_{p_t,\varepsilon_t} \in M_{3/2}(4M_t^4, \chi_0),$$

and denote the constant term of $f_t \,|\, W_{4M_t^4}$ by $b_t(0)$.

We claim that $v_\ell(b_t(0)) = 0$ for all $0 \le t \le r$. For $t = 0$ this is obvious and for $t > 0$ using (3), (4) and (5) the constant term $b_t(0)$ can be easily computed:

$$b_t(0) = \begin{cases} \frac{p_t}{2}\left(p_t + \varepsilon_t \left(\frac{-1}{p_t}\right)\right)(p_t - 1) b_{t-1}(0) & \text{if } \varepsilon_t \in \{\pm 1\}, \\ (1 - p_t^{-1})(p_t^2 - 1) b_{t-1}(0) & \text{if } \varepsilon_t = 0. \end{cases}$$

Hence, the claim follows by induction.

Now $v_\ell(b_r(0)) = 0$ implies $v_\ell(f_r \,|\, W_{4M_r^4}) \le 0$. Hence, by Lemma 1 and the definition of f_r we find $v_\ell(f_r) = 0$. If we write $f_r = \sum_{n \ge 0} a(n) e(nz)$, then for every integer m and every square-free d we have $a(dm^2) = r(dm^2)$, whenever $\left(\frac{d}{p_j}\right) = \varepsilon_j$ for all $j = 1, \dots, r$ and $(m, M_r) = 1$; and otherwise $a(dm^2) = 0$. Furthermore, f_r is an eigenform of all primes p with $(p, M_r) = 1$.

Let $m^* \in \mathbb{N}$ with $v_\ell(a(m^*)) = 0$ and choose a prime p with $\left(\frac{-m^*}{p}\right) = -1$, $p \not\equiv \pm 1 \pmod \ell$ and $(p, 4M_r\ell) = 1$. Then by Lemma 3 there exist infinitely many square-free d such that $\left(\frac{d}{p}\right) = -\left(\frac{m^*}{p}\right)$ and $v_\ell(a(dm_d^2)) = 0$ for an $m_d \in \mathbb{N}$. Now the assertion follows by an argument similar to Lemma 5. $\square$

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