Eisenstein series attached to lattices and modular forms on orthogonal groups

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1 Introduction

In the present paper we study certain vector valued Eisenstein series on the metaplectic cover $\mathrm{Mp}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$.

Let L be an even lattice of signature (b^+, b^-), equipped with a quadratic form q, and write L' for its dual. Recall that the Weil representation ρ_L^* attached to the quadratic module $(L'/L, q)$ is a unitary representation of the integral metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$ (cf. [No], [Bo2]). Let $k \in \frac{1}{2}\mathbb{Z}$ and $k > 2$. We consider the space $M_{k,L}$ of $\mathbb{C}[L'/L]$-valued holomorphic modular forms of weight k with respect to $\mathrm{Mp}_2(\mathbb{Z})$ and ρ_L^*. It is easily seen that $M_{k,L} = \{0\}$, if $2k$ is not congruent to $b^- - b^+$ modulo 2. Thus, if the rank $m = b^+ + b^-$ of L is even, any non-zero modular form in $M_{*,L}$ has integral weight. If m is odd, then any non-zero modular form in $M_{*,L}$ has half-integral weight.

We define Eisenstein series in the space $M_{k,L}$ in the usual way. The main purpose of the present paper is to compute their Fourier expansion explicitly. For simplicity we restrict ourselves to the case $2k - b^- + b^+ \equiv 0 \pmod 4$ and to one particular Eisenstein series $E(\tau)$ in $M_{k,L}$. This is sufficient for our later applications. The more general case can be treated similarly.

By a standard computation the (γ, n)-th Fourier coefficient $q(\gamma, n)$ of $E(\tau)$ can be expressed in terms of a rather complicated infinite series ($\gamma \in L'/L$ and $n \in \mathbb{Z} - q(\gamma)$). This was done in chapter 1.2.3 of [Br1]. The resulting formula was sufficient for the purposes of [Br1] but is not very satisfying. For instance, it does not even show that the $q(\gamma, n)$ are algebraic (or better rational) numbers. In section 4 we use Shintani's formula for the coefficients of the Weil representation [Sh] and results of Siegel on representation numbers of quadratic forms modulo prime powers [Si], to compute the series for $q(\gamma, n)$

more explicitly. We obtain a finite formula involving special values of Dirichlet L-series and finitely many representation numbers modulo prime powers attached to the lattice L (Theorem 4.6, Theorem 4.8). One pleasant property of the above class of Eisenstein series is that it includes both, the classical Eisenstein series of integral weight for $\mathrm{SL}_2(\mathbb{Z})$, and the half-integral weight Cohen-Eisenstein series for $\Gamma_0(4)$, as easy special cases.

In section 5 we consider the case that L has signature $(2, l)$ with $l \geq 3$. Let $\Gamma(L)$ be the subgroup of the orthogonal group of L that acts identically on the discriminant group L'/L. The Eisenstein series $E(\tau)$ of weight $1 + l/2$ attached to L can be used to prove some results on automorphic forms on the orthogonal group $\mathrm{O}(2, l)$. This is the main application of Theorem 4.8 and motivated the present paper. Recall that for any $\beta \in L'/L$ and any $t \in \mathbb{Z} + q(\beta)$ with $t < 0$ there is a $\Gamma(L)$-invariant divisor $H(\beta, t)$ on the Hermitean symmetric space $\mathcal{H} = \mathrm{O}^0(2, l)/K$ attached to $\mathrm{O}^0(2, l)$. Here K denotes a maximal compact subgroup (see section 5 for precise definitions). Following Borcherds we call $H(\beta, t)$ *Heegner divisor* of discriminant (β, t). These divisors generalize the usual Heegner points on the upper complex half plane.

In the context of Borcherds' theory of automorphic products (cf. [Bo1, Bo2, Bo3]) it was shown in [Br1] that the coefficients $q(\gamma, n)$ of $E(\tau)$ encode the weights of automorphic forms for $\Gamma(L)$, whose divisors are linear combinations of Heegner divisors. Hence, if we know the weight of such an automorphic form, we obtain some information on its divisor by means of Theorem 4.8.

In particular, if there are no cusp forms in $M_{k,L}$, then the $q(\gamma, n)$ determine completely the positions of the Heegner divisors $H(\gamma, -n)$ in the second cohomology of $\mathcal{H}/\Gamma(L)$. This generalizes van der Geer's result on Siegel modular threefolds [Ge].

We shall show that the $q(\gamma, n)$ are non-positive rational numbers. Moreover, $q(\gamma, n)$ is negative, if and only if $H(\gamma, -n)$ is non-trivial. As a consequence the main result (Theorem 5.2) of section 5 can be deduced. It roughly states: Let F be a holomorphic modular form of weight r for $\Gamma(L)$, whose divisor (F) is a linear combination of Heegner divisors. Let $D = \frac{1}{2} \sum_{\beta,t} c(\beta, t) H(\beta, t)$ be another linear combination of Heegner divisors with non-negative integral coefficients $c(\beta, t)$ such that $D \leq (F)$. Then the corresponding sum $-\frac{1}{4} \sum_{\beta,t} c(\beta, t) q(\beta, -t)$ of the coefficients of $E(\tau)$ is $\leq r$. It equals r, if and only if $D = (F)$.

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2 Notation

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers. As usual, we denote by $\mathbb{H} = \{\tau \in \mathbb{C};\ \Im(\tau) > 0\}$ the complex upper half plane. Throughout we will use τ as a standard variable on $\mathbb{H}$ and write x for its real part and y for its imaginary part, respectively. For $z \in \mathbb{C}$ we put $e(z) = e^{2\pi i z}$, and denote by $\sqrt{z} = z^{1/2}$ the principal branch of the square root, so that $\arg(\sqrt{z}) \in (-\pi/2, \pi/2]$. For any integer k we put $z^{k/2} = (z^{1/2})^k$. Moreover, if x is a real number, we let $[x] = \max\{n \in \mathbb{Z};\ n \leq x\}$. If x is non-zero we write $\mathrm{sgn}(x) = x/|x|$.

Let D be a discriminant, i.e. a non-zero integer congruent to 0 or 1 modulo 4. Then we write χ_D for the Dirichlet character modulo $|D|$, which is given by the Kronecker symbol: $\chi_D(a) = \left(\frac{D}{a}\right)$. The corresponding Dirichlet series is denoted by $L(s, \chi_D)$.

Let $n \in \mathbb{N}$ and χ be a Dirichlet character. We define the twisted divisor sum $\sigma_s(n, \chi)$ by

$$\sigma_s(n, \chi) = \sum_{d|n} \chi(d) d^s,$$

where the sum runs through all positive divisors of n. For $x \in \mathbb{R} - \mathbb{N}$ we understand $\sigma_s(x, \chi) = 0$. As usual, if $\chi = \chi_1$ is the trivial character modulo 1, we briefly write $\sigma_s(n)$ instead of $\sigma_s(n, \chi_1)$.

For any prime p we denote by v_p the (additive) p-adic valuation on $\mathbb{Q}$.

3 Modular forms and the Weil representation

In this section we briefly recall from [Br1] and [Bo2] some facts about the Weil representation and certain vector valued modular forms.

Let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$, realized as the group of pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, and ϕ is a holomorphic square root of $\tau \mapsto c\tau + d$. The assignment

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau + d}\right) \tag{3.1}$$

defines a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{Mp}_2(\mathbb{R})$.

We denote by $\mathrm{Mp}_2(\mathbb{Z})$ the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by the two elements

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \quad \text{and} \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

One has the relations $S^2 = (ST)^3 = Z$, where $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$ is the standard generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$. We put $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$ and write Γ_∞ resp. $\widetilde{\Gamma}_\infty$ for the subgroup of Γ_1 resp. $\mathrm{Mp}_2(\mathbb{Z})$ generated by $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ resp. T.

Let L be an even lattice, i.e. a free $\mathbb{Z}$-module of finite rank, equipped with a symmetric $\mathbb{Z}$-valued bilinear form $(\cdot, \cdot)$ such that the associated quadratic form $q(x) = \frac{1}{2}(x, x)$ takes its values in $\mathbb{Z}$. We assume that L is non-degenerated and denote its signature by (b^+, b^-) and its rank by $m = b^+ + b^-$. We write L' for the dual lattice of L. The modulo 1 reduction of $q(\cdot)$ is a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on the (finite) discriminant group L'/L, whose associated bilinear form is the modulo 1 reduction of the bilinear form $(\cdot, \cdot)$ on L'.

Recall that there is a particular unitary representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$. If we denote the standard basis of $\mathbb{C}[L'/L]$ by $(\mathfrak{e}_\gamma)_{\gamma \in L'/L}$ then ρ_L can be defined by the action of the generators $S, T \in \mathrm{Mp}_2(\mathbb{Z})$ as follows:

$$\rho_L(T)\mathfrak{e}_\gamma = e(q(\gamma))\mathfrak{e}_\gamma \tag{3.2}$$

$$\rho_L(S)\mathfrak{e}_\gamma = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} e(-(\gamma, \delta))\mathfrak{e}_\delta \tag{3.3}$$

(cp. [Bo2]). We denote by ρ_L^* the dual representation of ρ_L.

The representation ρ_L is essentially the Weil representation associated to the quadratic module $(L'/L, q)$ (see [No]). It factors through a finite quotient of $Mp_2(\mathbb{Z})$. Observe that $\rho_L(Z)\mathfrak{e}_\gamma = i^{b^- - b^+}\mathfrak{e}_{-\gamma}$.

Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on $\mathbb{C}[L'/L]$, which is linear in the first variable and anti-linear in the second. For $\beta, \gamma \in L'/L$ and $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ we define the coefficient $\rho_{\beta\gamma}(M, \phi)$ of the representation ρ_L by

$$\rho_{\beta\gamma}(M, \phi) = \langle \rho_L(M, \phi)\mathfrak{e}_\gamma, \mathfrak{e}_\beta \rangle.$$

The following result due to Shintani will be of fundamental importance to us (cf. [Sh], Prop. 1.6).

Proposition 3.1 (Shintani). *Let $\beta, \gamma \in L'/L$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then the coefficient $\rho_{\beta\gamma}(\widetilde{M})$ is given by*

$$\sqrt{i}^{(b^- - b^+)(1 - \mathrm{sgn}(d))} \delta_{\beta, a\gamma} e(abq(\beta)), \tag{3.4}$$

if $c = 0$, and by

$$\frac{\sqrt{i}^{(b^- - b^+)\,\mathrm{sgn}(c)}}{|c|^{(b^- + b^+)/2}\sqrt{|L'/L|}} \sum_{r \in L/cL} e\left(\frac{a(\beta + r, \beta + r) - 2(\gamma, \beta + r) + d(\gamma, \gamma)}{2c}\right), \tag{3.5}$$

if $c \neq 0$. Here, $\delta_{*,*}$ denotes the Kronecker-delta.

Let $k \in \frac{1}{2}\mathbb{Z}$ and f be a $\mathbb{C}[L'/L]$-valued function on $\mathbb{H}$. We define the Petersson slash operator by

$$\left(f \mid_k^* (M, \phi)\right)(\tau) = \phi(\tau)^{-2k} \rho_L^*(M, \phi)^{-1} f(M\tau) \tag{3.6}$$

for $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.

Any holomorphic function $f : \mathbb{H} \to \mathbb{C}[L'/L]$, which is invariant under the $|_k^*$-operation of $T \in \mathrm{Mp}_2(\mathbb{Z})$, has a Fourier expansion

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} - q(\gamma)} c(\gamma, n)\mathfrak{e}_\gamma(n\tau), \tag{3.7}$$

where $\mathfrak{e}_\gamma(\tau) := \mathfrak{e}_\gamma e(\tau)$.

Let $k \in \frac{1}{2}\mathbb{Z}$. We call a holomorphic function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ a *modular form* of weight k with respect to ρ_L^* and $\mathrm{Mp}_2(\mathbb{Z})$ if

i) $f \mid_k^* (M,\phi) = f$ for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and

ii) f is holomorphic in ∞.

Here, the second condition means that all coefficients $c(\gamma, n)$ with $n < 0$ vanish in the Fourier expansion (3.7) of f. The $\mathbb{C}$-vector space of modular forms of weight k with respect to ρ_L^* and $\mathrm{Mp}_2(\mathbb{Z})$ is denoted by $M_{k,L}$. It is easily seen that $M_{k,L}$ is finite dimensional. The transformation behavior under Z^2 implies that $M_{k,L} = \{0\}$, if $2k \not\equiv b^- - b^+ \pmod 2$.

4 Eisenstein series

We now construct Eisenstein series $E_\beta(\tau)$ for the space $M_{k,L}$ and determine their Fourier coefficients $q_\beta(\gamma, n)$. Throughout we assume that $k \in \frac{1}{2}\mathbb{Z}$ and $k > 2$. For simplicity we only consider the case $2k - b^- + b^+ \equiv 0 \pmod 4$, the case $2k - b^- + b^+ \equiv 2 \pmod 4$ can be treated similarly.

Let $\beta \in L'/L$ with $q(\beta) \in \mathbb{Z}$. Then the vector $\mathfrak{e}_\beta \in \mathbb{C}[L'/L]$, considered as a constant function $\mathbb{H} \to \mathbb{C}[L'/L]$, is invariant under the $|_k^*$-action of $T, Z^2 \in \mathrm{Mp}_2(\mathbb{Z})$. The Eisenstein series

$$E_\beta(\tau) = \frac{1}{2} \sum_{(M,\phi) \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \mathfrak{e}_\beta \mid_k^* (M,\phi) \tag{4.1}$$

of weight k converges normally on $\mathbb{H}$ and therefore defines a $\mathrm{Mp}_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$.

The following proposition can be proved in the standard way (see [Br1] chapter 1.2.3). We omit the proof.

Proposition 4.1. *The Eisenstein series E_β has the Fourier expansion*

$$E_\beta(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} - q(\gamma) \\ n \geq 0}} q_\beta(\gamma, n) \mathfrak{e}_\gamma(n\tau)$$

with

$$q_\beta(\gamma, n) = \begin{cases} \delta_{\beta,\gamma} + \delta_{-\beta,\gamma}, & \text{if } n = 0, \\ \dfrac{(2\pi)^k n^{k-1}}{\Gamma(k)} \displaystyle\sum_{c \in \mathbb{Z} - \{0\}} |c|^{1-k} H_c^*(\beta, 0, \gamma, n), & \text{if } n > 0. \end{cases} \tag{4.2}$$

Here, $H_c^*(\beta, m, \gamma, n)$ denotes the generalized Kloosterman sum

$$H_c^*(\beta, m, \gamma, n) = \frac{e^{-\pi i \operatorname{sgn}(c) k/2}}{|c|} \sum_{\substack{d(c)^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty}} \rho_{\beta\gamma} \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} e\left(\frac{ma + nd}{c}\right) \tag{4.3}$$

($\beta, \gamma \in L'/L$ and $m \in \mathbb{Z} - q(\beta)$, $n \in \mathbb{Z} - q(\gamma)$). In particular E_β is an element of $M_{k,L}$.

The sum in (4.3) runs over all primitive residues d modulo c and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a representative for the double coset in $\Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty$ with lower row $(c\ d')$ and $d' \equiv d \pmod{c}$. Observe that the expression $\rho_{\beta\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}\, e\left(\frac{ma+nd}{c}\right)$ does not depend on the choice of the coset representative.

The coefficients $\rho_{\beta\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}$ are universally bounded, since ρ_L factors through a finite group. Hence there is a constant $C > 0$ such that $H_c^*(\beta, m, \gamma, n) < C$ for all $\gamma \in L'/L$, $n \in \mathbb{Z} - q(\gamma)$, and $c \in \mathbb{Z} - \{0\}$. This implies that the series (4.2) converges absolutely.

We will mainly be interested in the Eisenstein series $E_0(\tau)$ which we simply denote by $E(\tau)$. In the same way we write $q(\gamma, n)$ for the Fourier coefficients $q_0(\gamma, n)$ of $E(\tau)$.

The rest of this section is devoted to finding a more explicit formula for the coefficients $q(\gamma, n)$ of $E(\tau)$. Note that the coefficients of the more general Eisenstein series $E_\beta(\tau)$ can be computed analogously.

Proposition 4.2. *The generalized Kloosterman sum $H_c^*(0, 0, \gamma, n)$ equals*

$$\frac{(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}} |c|^{-1+m/2} \sum_{a|c} a^{1-m} \mu(|c|/a) N_{\gamma,n}(a).$$

Here μ denotes the Moebius function and

$$N_{\gamma,n}(a) = \#\{r \in L/aL; \quad q(r-\gamma) + n \equiv 0 \pmod{a}\}. \tag{4.4}$$

Notice that the left hand side of the congruence in (4.4) is always integral, because $n \in \mathbb{Z} - q(\gamma)$.

Proof. If we insert the formula for the coefficients of the representation ρ_L (Proposition 3.1) into the definition of $H_c^*(0, 0, \gamma, n)$, we obtain

$$\begin{aligned} H_c^*(0,0,\gamma,n) &= \frac{e^{-\pi i \operatorname{sgn}(c)(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}|c|^{1+m/2}} \sum_{\substack{d\ (c)^* \\ ad \equiv 1\ (c)}} \sum_{r \in L/cL} e\left(\frac{aq(r) - (\gamma, r) + dq(\gamma) + nd}{c}\right) \\ &= \frac{(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}|c|^{1+m/2}} \sum_{r \in L/cL} \sum_{d\ (c)^*} e\left(\frac{d(q(r-\gamma)+n)}{c}\right), \end{aligned}$$

where the sums $\sum_{d\ (c)^*}$ run through all primitive residues d modulo c. We use the evaluation of the Ramanujan sum

$$\sum_{d\ (c)^*} e\left(\frac{dn}{c}\right) = \sum_{a|(c,n)} \mu(|c|/a) a$$

by means of the Moebius function ([Ap] Chapter 8.3). We get

$$H_c^*(0,0,\gamma,n) = \frac{(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}|c|^{1+m/2}} \sum_{a|c} \mu(|c|/a) a \sum_{\substack{r \in L/cL \\ q(r-\gamma)+n \equiv 0\ (a)}} 1.$$

The condition $q(r-\gamma)+n \equiv 0 \pmod a$ in the inner sum depends only on r modulo aL. Thus

$$H_c^*(0,0,\gamma,n) = \frac{(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}|c|^{1+m/2}} \sum_{a|c} \mu(|c|/a) a (|c|/a)^m \sum_{\substack{r \in L/aL \\ q(r-\gamma)+n \equiv 0\ (a)}} 1$$
$$= \frac{(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}} |c|^{-1+m/2} \sum_{a|c} \mu(|c|/a) a^{1-m} N_{\gamma,n}(a).$$

□

Proposition 4.3. *Let $\gamma \in L'$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$. The coefficient $q(\gamma,n)$ equals the value at $s = k$ of the analytic continuation in s of*

$$\frac{2^{k+1}\pi^k n^{k-1}(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}\Gamma(k)\zeta(s-m/2)} L_{\gamma,n}(s).$$

Here $\zeta(s)$ denotes the Riemann zeta function and $L_{\gamma,n}(s)$ the L-series

$$L_{\gamma,n}(s) = \sum_{a \geq 1} N_{\gamma,n}(a) a^{1-m/2-s}. \tag{4.5}$$

Proof. We consider the L-series

$$\widetilde{L}_{\gamma,n}(s) = \sum_{c \in \mathbb{Z}-\{0\}} |c|^{1-s} H_c^*(0,0,\gamma,n). \tag{4.6}$$

It converges normally for $\Re(s) > 2$. According to Proposition 4.2 one has

$$\widetilde{L}_{\gamma,n}(s) = \frac{2(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}} \sum_{c \geq 1} c^{m/2-s} \sum_{a|c} \mu(c/a) a^{1-m} N_{\gamma,n}(a).$$

Substituting $d = c/a$ in the above sums we find

$$\widetilde{L}_{\gamma,n}(s) = \frac{2(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}} \sum_{a \geq 1} N_{\gamma,n}(a) a^{1-m/2-s} \sum_{d \geq 1} \mu(d) d^{m/2-s}$$
$$= \frac{2(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}\zeta(s-m/2)} L_{\gamma,n}(s). \tag{4.7}$$

If we insert this into the formula for $q(\gamma,n)$ given in Proposition 4.1, we obtain the assertion.

□

Note that the L-series $L_{\gamma,n}(s)$ only converges for $\Re(s) > 1 + m/2$. Using the equality of (4.6) and (4.7), and the properties of the Riemann zeta function, we see that it has a meromorphic continuation to $\Re(s) > 2$ and a simple pole at $s = 1 + m/2$.

Let S denote the Gram matrix of the lattice L with respect to a fixed basis. Then $|L'/L| = |\det(S)|$. We use the common abbreviation $S[x] = x^t Sx$, whenever the matrix product makes sense. We may obviously write

$$N_{\gamma,n}(a) = \#\left\{r \in (\mathbb{Z}/a\mathbb{Z})^m; \quad \tfrac{1}{2}S[r-\gamma] + n \equiv 0 \pmod{a}\right\},$$

where we have identified $\gamma \in L'$ with its coordinate vector.

Let $d_\gamma = \min\{b \in \mathbb{N}; \quad b\gamma \in L\}$ be the level of γ and put

$$\tilde{\gamma} = d_\gamma \gamma, \tag{4.8}$$
$$\tilde{n} = d_\gamma^2 n. \tag{4.9}$$

Then $\tilde{n}$ is integral, and d_γ divides $\det(S)$ and $2\tilde{n}$. If a is coprime to $\det(S)$, then d_γ is invertible modulo a. Hence

$$N_{\gamma,n}(a) = \#\left\{r \in (\mathbb{Z}/a\mathbb{Z})^m; \quad \tfrac{1}{2}S[r] \equiv -\tilde{n} \pmod{a}\right\} \tag{4.10}$$

in this case. For general a we have

$$N_{\gamma,n}(a) = \#\left\{r \in (\mathbb{Z}/d_\gamma a\mathbb{Z})^m; \quad \tfrac{1}{2}S[r] + \tilde{n} \equiv 0 \ (d_\gamma^2 a), \quad r \equiv \tilde{\gamma} \ (d_\gamma)\right\}. \tag{4.11}$$

It is easily seen that $N_{\gamma,n}(a)$ is multiplicative:

$$N_{\gamma,n}(a_1 a_2) = N_{\gamma,n}(a_1) N_{\gamma,n}(a_2)$$

for coprime a_1 and a_2. This implies that $L_{\gamma,n}(s)$ has an Euler product expansion

$$L_{\gamma,n}(s) = \prod_p \left(\sum_{\nu \geq 0} N_{\gamma,n}(p^\nu) p^{\nu(1-m/2-s)} \right), \tag{4.12}$$

where the product extends over all primes p.

Lemma 4.4. *Let p be a prime. Put*

$$w_p = 1 + 2v_p(2nd_\gamma). \tag{4.13}$$

Then the equality

$$N_{\gamma,n}(p^{\alpha+1}) = p^{m-1} N_{\gamma,n}(p^\alpha)$$

holds for any $\alpha \geq w_p$.

Proof. This can be proved in the same way as Hilfssatz 13 in [Si]. □

Note that $2nd_\gamma$ is always integral and thereby $w_p \geq 1$.

Using the above Lemma, the Euler product (4.12) can be simplified:

$$\begin{aligned} L_{\gamma,n}(s) &= \prod_p \left(\sum_{\nu=0}^{w_p-1} N_{\gamma,n}(p^\nu) p^{\nu(1-m/2-s)} + N_{\gamma,n}(p^{w_p}) p^{w_p(1-m/2-s)} \sum_{\nu\geq 0} p^{\nu(m/2-s)} \right) \\ &= \zeta(s-m/2) \prod_p L_{\gamma,n}(s,p), \end{aligned} \tag{4.14}$$

where $L_{\gamma,n}(s,p)$ denotes the local Euler factor

$$L_{\gamma,n}(s,p) = (1-p^{m/2-s}) \sum_{\nu=0}^{w_p-1} N_{\gamma,n}(p^\nu) p^{\nu(1-m/2-s)} + N_{\gamma,n}(p^{w_p}) p^{w_p(1-m/2-s)}. \tag{4.15}$$

The following Theorem is crucial for the further computation of $L_{\gamma,n}(s)$.

Theorem 4.5 (Siegel). *Let p be a prime not dividing $2\det(S)$ and $\alpha \in \mathbb{Z}$ with $\alpha > v_p(n)$.*
i) Suppose that m is even. Put $D = (-1)^{m/2}\det(S)$. Then

$$p^{\alpha(1-m)} N_{\gamma,n}(p^\alpha) = \left(1 - \chi_D(p) p^{-m/2}\right) \left(1 + \chi_D(p) p^{1-m/2} + \cdots + \chi_D(p^{v_p(n)}) p^{v_p(n)(1-m/2)}\right).$$

ii) Suppose that m is odd. Write $n = n_0 f^2$ (where $n_0 \in \mathbb{Q}$ and $f \in \mathbb{N}$) such that $(f, 2\det S) = 1$ and $v_\ell(n_0) \in \{0,1\}$ for all primes ℓ with $(\ell, 2\det S) = 1$. Let $\tilde{n}_0 = n_0 d_\gamma^2$ and $\mathcal{D} = 2(-1)^{(m+1)/2}\tilde{n}_0\det(S)$. If $m \geq 3$, then

$$p^{\alpha(1-m)} N_{\gamma,n}(p^\alpha) = \frac{1-p^{1-m}}{1-\chi_{\mathcal{D}}(p)p^{(1-m)/2}} \left(\sigma_{2-m}(p^{v_p(f)}) - \chi_{\mathcal{D}}(p) p^{(1-m)/2} \sigma_{2-m}(p^{v_p(f)-1})\right).$$

If $m = 1$, we have

$$N_{\gamma,n}(p^\alpha) = \left(\chi_{\mathcal{D}}(p) + \chi_{\mathcal{D}}(p)^2\right) p^{v_p(f)}.$$

It is well known that $(-1)^{m/2}\det(S) \equiv 0, 1 \pmod 4$, if m is even, and that $\det(S) \equiv 0 \pmod 2$, if m is odd. Thus D and $\mathcal{D}$ are discriminants.

Proof. Since p is coprime to $\det(S)$, the number $N_{\gamma,n}(p^\alpha)$ is given by (4.10). So the assertion is just a reformulation of Hilfssatz 16 in [Si]. (The formula for $m = 1$ has to be extracted directly from the proof.) □

We may now state a formula for $q(\gamma, n)$ which is accessible for computer computation.

Theorem 4.6. *Let $\gamma \in L'$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$. The coefficient $q(\gamma, n)$ equals*

$$\frac{2^{k+1}\pi^k n^{k-1}(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}\Gamma(k)}$$

times

$$\begin{cases} \dfrac{1}{L(k,\chi_D)} \displaystyle\prod_{p\mid 2\tilde{n}\det(S)} \dfrac{L_{\gamma,n}(k,p)}{1-\chi_D(p)p^{-k}}, & \textit{if } m \textit{ is even,} \\ \dfrac{L(k-1/2,\chi_{\mathcal{D}})}{\zeta(2k-1)} \displaystyle\prod_{p\mid 2\tilde{n}\det(S)} \dfrac{1-\chi_{\mathcal{D}}(p)p^{1/2-k}}{1-p^{1-2k}} L_{\gamma,n}(k,p), & \textit{if } m \textit{ is odd.} \end{cases}$$

Here $L_{\gamma,n}(k,p)$ is given by (4.15) and D, $\mathcal{D}$ are defined as in Theorem 4.5.

Proof. By Proposition 4.3 and (4.14) we know that

$$q(\gamma,n) = \frac{2^{k+1}\pi^k n^{k-1}(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}\Gamma(k)} \prod_p L_{\gamma,n}(k,p). \tag{4.16}$$

Let p be a prime with $(p, 2\tilde{n}\det S) = 1$. According to Theorem 4.5 we have

$$p^{1-m}N_{\gamma,n}(p) = \begin{cases} 1-\chi_D(p)p^{-m/2}, & \text{if } 2 \mid m, \\ 1+\chi_{\mathcal{D}}(p)p^{(1-m)/2}, & \text{if } 2 \nmid m. \end{cases}$$

Noting that $w_p = 1$ we find

$$L_{\gamma,n}(s,p) = \begin{cases} 1-\chi_D(p)p^{-s}, & \text{if } 2 \mid m, \\ \dfrac{1-p^{1-2s}}{1-\chi_{\mathcal{D}}(p)p^{1/2-s}}, & \text{if } 2 \nmid m. \end{cases}$$

If we insert this into (4.16), we obtain the assertion. □

Corollary 4.7. *The coefficients of $E(\tau)$ are rational numbers.*

Proof. This can be deduced using the functional equation of the Dirichlet series $L(s,\chi_D)$ (resp. $L(s,\chi_{\mathcal{D}})$ and $\zeta(s)$) and the fact that the values at negative integers can be expressed in terms of Bernoulli polynomials [Za]. □

Example 1. Let L be a hyperbolic plane, i.e. the lattice $\mathbb{Z}^2$ with the quadratic form $q((a,b)) = ab$. This is obviously a unimodular lattice of signature $(1,1)$. Let k be an even integer. In this case the space $M_{k,L}$ is simply the space of elliptic modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$. The function $E(\tau)$ is the classical Eisenstein series of weight k for $\mathrm{SL}_2(\mathbb{Z})$, normalized such that its constant term equals 2. According to Theorem 4.6, for any positive integer n the n-th Fourier coefficient $q(n) = q(0,n)$ is given by

$$\frac{2^{k+1}\pi^k n^{k-1}(-1)^{k/2}}{\Gamma(k)\zeta(k)} \prod_{p\mid 2n} \frac{L_{0,n}(k,p)}{1-p^{-k}}. \tag{4.17}$$

We leave it to the reader to verify that

$$N_{0,n}(p^\alpha) = \begin{cases} (v_p(n)+1)(1-1/p)p^\alpha, & \text{if } v_p(n) < \alpha, \\ (\alpha+1)p^\alpha - \alpha p^{\alpha-1}, & \text{if } v_p(n) \geq \alpha, \end{cases}$$

for any prime p. A straightforward computation yields $L_{0,n}(s,p) = (1-p^{-s})\sigma_{1-s}(p^{v_p(n)})$. If we insert this into (4.17), we find

$$q(n) = \frac{2^{k+1}\pi^k n^{k-1}(-1)^{k/2}}{\Gamma(k)\zeta(k)}\sigma_{1-k}(n).$$

Using $\zeta(k) = -\frac{(2\pi i)^k}{2k!}B_k$, with the k-th Bernoulli number B_k, we get

$$q(n) = -\frac{4k}{B_k}\sigma_{k-1}(n),$$

in accordance with the classical result.

Example 2. Let L be the 1-dimensional lattice $\mathbb{Z}$ equipped with the quadratic form $q(a) = a^2$. Then $L'/L \cong \mathbb{Z}/2\mathbb{Z}$. Let k be half-integral such that $k+1/2$ is even. The space $M_{k,L}$ is isomorphic to the space of Jacobi forms of weight $k+1/2$ and index 1 ([EZ] Theorem 5.1) and thereby isomorphic to the Kohnen space M_k^+ of modular forms of weight k for the group $\Gamma_0(4)$ whose n-th Fourier coefficient equals zero unless $-n \equiv 0,1 \pmod 4$ ([Ko], [EZ] Theorem 5.4). In this case $E(\tau)$ essentially equals the Cohen-Eisenstein series of weight k (cf. [Co]). Let $\gamma \in L'/L$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$. Moreover, let Δ be the unique fundamental discriminant such that $-4n = \Delta f^2$ with $f \in \mathbb{N}$. By Theorem 4.6 we have

$$q(\gamma,n) = \frac{2^{k+1/2}\pi^k n^{k-1}(-1)^{(2k+1)/4}}{\Gamma(k)} \frac{L(k-1/2,\chi_\Delta)}{\zeta(2k-1)} \prod_{p|2\tilde{n}} \frac{1-\chi_\Delta(p)p^{1/2-k}}{1-p^{1-2k}} L_{\gamma,n}(k,p). \quad (4.18)$$

To compute the finite Euler product we note that

$$N_{\gamma,n}(p^\alpha) = \begin{cases} \left(\chi_\Delta(p) + \chi_\Delta(p)^2\right) p^{v_p(f)}, & \text{if } v_p(n) < \alpha, \\ p^{[\alpha/2]}, & \text{if } v_p(n) \geq \alpha, \end{cases}$$

for any odd prime p. In fact, the case $v_p(n) \geq \alpha$ is easy, and the case $v_p(n) < \alpha$ follows from Theorem 4.5. With some extra work it can be seen that this formula still holds for $p = 2$. (That is why we have worked with Δ instead of $\mathcal{D}$.) It can be deduced that

$$L_{\gamma,n}(s,p) = \frac{1-p^{1-2s}}{1-\chi_\Delta(p)p^{1/2-s}} \left(\sigma_{2-2s}(p^{v_p(f)}) - \chi_\Delta(p)p^{1/2-s}\sigma_{2-2s}(p^{v_p(f)-1})\right).$$

Inserting this into (4.18), we obtain

$$q(\gamma,n) = \frac{2^{k+1/2}\pi^k n^{k-1}(-1)^{(2k+1)/4}}{\Gamma(k)} \frac{L(k-1/2,\chi_\Delta)}{\zeta(2k-1)} \sum_{d|f} \mu(d)\chi_\Delta(d)d^{1/2-k}\sigma_{2-2k}(f/d).$$

Throughout the rest of this paper we suppose that $k = m/2$. (For later applications we will only need this case.) Then the condition $2k - b^- + b^+ \equiv 0 \pmod 4$ is equivalent to requiring that b^+ is even. The condition $k > 2$ implies $m \geq 5$.

Under this assumption the formula of Theorem 4.6 can be considerably simplified.

Theorem 4.8. *Let $\gamma \in L'$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$. The coefficient $q(\gamma, n)$ of the Eisenstein series $E(\tau)$ of weight $k = m/2$ is equal to*

$$\frac{2^{k+1}\pi^k n^{k-1}(-1)^{b^+/2}}{\sqrt{|L'/L|}\Gamma(k)}$$

times

$$\begin{cases} \dfrac{\sigma_{1-k}(\tilde{n}, \chi_{4D})}{L(k, \chi_{4D})} \displaystyle\prod_{p \mid 2\det(S)} p^{w_p(1-2k)} N_{\gamma,n}(p^{w_p}), & \text{if } 2 \mid m, \\ \dfrac{L(k-1/2, \chi_{\mathcal{D}})}{\zeta(2k-1)} \displaystyle\sum_{d \mid f} \mu(d)\chi_{\mathcal{D}}(d) d^{1/2-k} \sigma_{2-2k}(f/d) \prod_{p \mid 2\det(S)} \frac{p^{w_p(1-2k)} N_{\gamma,n}(p^{w_p})}{1-p^{1-2k}}, & \text{if } 2 \nmid m. \end{cases}$$

Here $N_{\gamma,n}(p^{w_p})$ is given by (4.4), (4.13); and D, $\mathcal{D}$, f are defined as in Theorem 4.5. Moreover, $\sigma_{1-k}(\tilde{n}, \chi_{4D})$ denotes the twisted divisor sum (see section 2).

Proof. Let p be a prime. Since $k = m/2$ the local Euler factor $L(k, p)$ (4.15) is equal to

$$p^{w_p(1-2k)} N_{\gamma,n}(p^{w_p}).$$

If we put this into the formula given in Theorem 4.6, we find that $q(\gamma, n)$ is equal to

$$\frac{2^{k+1}\pi^k n^{k-1}(-1)^{b^+/2}}{\sqrt{|L'/L|}\Gamma(k)}$$

times

$$\begin{cases} \dfrac{1}{L(k, \chi_D)} \displaystyle\prod_{p \mid 2\tilde{n}\det(S)} \frac{p^{w_p(1-2k)} N_{\gamma,n}(p^{w_p})}{1-\chi_D(p)p^{-k}}, & \text{if } 2 \mid m, \\ \dfrac{L(k-1/2, \chi_{\mathcal{D}})}{\zeta(2k-1)} \displaystyle\prod_{p \mid 2\tilde{n}\det(S)} \frac{1-\chi_{\mathcal{D}}(p)p^{1/2-k}}{1-p^{1-2k}} p^{w_p(1-2k)} N_{\gamma,n}(p^{w_p}), & \text{if } 2 \nmid m. \end{cases} \tag{4.19}$$

If m is even, then according to Theorem 4.5 the finite Euler product over $p \mid 2\tilde{n}\det(S)$ in (4.19) is given by

$$\begin{aligned} &\prod_{\substack{p \mid \tilde{n} \\ p \nmid 2\det(S)}} \sigma_{1-k}(p^{v_p(n)}, \chi_{4D}) \prod_{p \mid 2\det(S)} \frac{p^{w_p(1-2k)} N_{\gamma,n}(p^{w_p})}{1-\chi_D(p)p^{-k}} \\ &= \sigma_{1-k}(\tilde{n}, \chi_{4D}) \frac{1}{1-\chi_D(2)2^{-k}} \prod_{p \mid 2\det(S)} p^{w_p(1-2k)} N_{\gamma,n}(p^{w_p}). \end{aligned} \tag{4.20}$$

If m is odd, then the finite Euler product over $p \mid 2\tilde{n}\det(S)$ in (4.19) is equal to

$$\prod_{\substack{p|\tilde{n} \\ p\nmid 2\det(S)}} \left(\sigma_{2-2k}(p^{v_p(f)}) - \chi_{\mathcal{D}}(p)p^{1/2-k}\sigma_{2-2k}(p^{v_p(f)-1})\right) \prod_{p|2\det(S)} \frac{p^{w_p(1-2k)}N_{\gamma,n}(p^{w_p})}{1-p^{1-2k}}$$

$$= \sum_{d|f} \mu(d)\chi_{\mathcal{D}}(d)d^{1/2-k}\sigma_{2-2k}(f/d) \prod_{p|2\det(S)} \frac{p^{w_p(1-2k)}N_{\gamma,n}(p^{w_p})}{1-p^{1-2k}}. \tag{4.21}$$

Inserting (4.20) resp. (4.21) into (4.19), we obtain the assertion. □

We have written a C++ program to evaluate the above formula. The source code and the binary (compiled for x86-Linux) can be downloaded from the first author's home-page.

In [Bu] for some lattices L the Eisenstein series $E(\tau)$ is expressed in terms of elementary theta functions.

5 Modular forms on $\mathrm{O}(2,l)$

Throughout this section we assume that L has signature $(2,l)$ with $l \geq 3$. Moreover, we suppose that L splits two hyperbolic planes over $\mathbb{Q}$. (This is always true if $l \geq 5$.) We put $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and denote by

$$\mathrm{O}(V) = \{g \in \mathrm{SL}(V); \quad q(ga) = q(a) \text{ for all } a \in V\}$$

the (special) orthogonal group of V. If $\mathrm{O}^0(V)$ denotes the connected component of the identity and K a maximal compact subgroup, then $\mathrm{O}^0(V)/K$ is a Hermitean symmetric space. The Hermitean structure can be described explicitly as follows.

We extend the bilinear form $(\cdot,\cdot)$ on V to a $\mathbb{C}$-bilinear form on the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V. Let $P(V_{\mathbb{C}})$ denote the associated projective space and write $W \mapsto [W]$ for the canonical projection $V_{\mathbb{C}} \to P(V_{\mathbb{C}})$. Consider the subset

$$\mathcal{K} = \left\{[W] \in P(V_{\mathbb{C}}); \quad (W,W) = 0,\ (W,\overline{W}) > 0\right\}$$

of $P(V_{\mathbb{C}})$. It is easily seen that $\mathcal{K}$ is a complex manifold of dimension l that consists of 2 connected components. The action of the orthogonal group $\mathrm{O}(V)$ on V induces an action on $\mathcal{K}$. The connected component of the identity preserves the components of $\mathcal{K}$, whereas $\mathrm{O}(V) - \mathrm{O}^0(V)$ interchanges them. We choose one fixed component of $\mathcal{K}$ and denote it by $\mathcal{H}$. Then $\mathrm{O}^0(V)$ acts transitively on $\mathcal{H}$ and the stabilizer K of a fixed point is a maximal compact subgroup. Thus $\mathrm{O}^0(V)/K = \mathcal{H}$.

Let $\mathrm{O}(L) = \left\{g \in \mathrm{O}^0(V); \quad gL = L\right\}$ be the orthogonal group of L. We denote by $\Gamma(L)$ the subgroup of finite index of $\mathrm{O}(L)$ consisting of all elements which act trivially on the discriminant group L'/L. According to Baily-Borel the quotient $\mathcal{H}/\Gamma(L)$ is a quasi-projective algebraic variety.

Let X be a normal irreducible complex space. By a divisor D on X we mean a formal linear combination $D = \sum n_Y Y$ $(n_Y \in \mathbb{Z})$ of irreducible closed analytic subsets Y of codimension 1 such that the support $\bigcup_{n_Y \neq 0} Y$ is a closed analytic subset of everywhere pure codimension 1. For two divisors $D = \sum n_Y Y$ and $D' = \sum n'_Y Y$ on X we write $D \leq D'$, if $n_Y \leq n'_Y$ for all irreducible closed analytic subsets Y of codimension 1.

Recall that for any vector $\lambda \in L'$ of negative norm there is a divisor $\lambda^\perp$ on $\mathcal{H}$ which is given by the orthogonal complement of λ in $\mathcal{H}$. Let $\beta \in L'/L$ and $t \in \mathbb{Z} + q(\beta)$ with $t < 0$. Then

$$H(\beta, t) = \sum_{\substack{\lambda \in L' \\ q(\lambda) = t \\ \lambda + L = \beta}} \lambda^\perp \tag{5.1}$$

is a $\Gamma(L)$-invariant divisor on $\mathcal{H}$. It is the inverse image under the canonical projection of an algebraic divisor on $\mathcal{H}/\Gamma(L)$ (which will also be denoted by $H(\beta,t)$). The multiplicities of all irreducible components equal 2, if $2\beta = 0$, and 1, if $2\beta \neq 0$ in L'/L. Following Borcherds we call this divisor *Heegner divisor* of discriminant (β, t). Note that $H(\beta,t) = H(-\beta,t)$.

We now define automorphic forms for the group $\Gamma(L)$. Denote by

$$\widetilde{\mathcal{H}} = \{W \in V_\mathbb{C} - \{0\}; \quad [W] \in \mathcal{H}\} \subset V_\mathbb{C} \tag{5.2}$$

the cone over $\mathcal{H}$. Let $r \in \mathbb{Q}$ and χ be a character of $\Gamma(L)$. A meromorphic function G on $\widetilde{\mathcal{H}}$ is called *automorphic form* of weight r and character χ with respect to $\Gamma(L)$, if

i) G is homogeneous of degree $-r$, i.e. $G(cW) = c^{-r} G(W)$ for any $c \in \mathbb{C} - \{0\}$;

ii) G is invariant under Γ, i.e. $G(\sigma W) = \chi(\sigma) G(W)$ for any $\sigma \in \Gamma(L)$.

If G is in addition holomorphic on $\widetilde{\mathcal{H}}$, it is called *modular form.* (Since $l \geq 3$, then the Koecher principle ensures that G is also holomorphic on the Satake boundary.)

Let $E(\tau)$ be the Eisenstein series of weight $k = 1 + l/2$ with constant term $2\mathfrak{e}_0$ in $M_{k,L}$ (as in Theorem 4.8) and write $q(\gamma, n)$ for its Fourier coefficients ($\gamma \in L'/L$ and $n \in \mathbb{Z} - q(\gamma)$). The significance of $E(\tau)$ lies in the following theorem which was proved in [Br1] Theorem 13.15 and Corollary 13.15 (see also [Br2] Theorem 9).

Theorem 5.1. *Let F be an automorphic form of weight r with some character for the group $\Gamma(L)$. Suppose that its divisor (F) is a linear combination of Heegner divisors*

$$(F) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{t \in \mathbb{Z} + q(\beta) \\ t < 0}} c(\beta, t) H(\beta, t), \tag{5.3}$$

where the $c(\beta,t)$ are integral coefficients with $c(\beta,t) = c(-\beta,t)$. Then r satisfies

$$r = -\frac{1}{4} \sum_{\beta \in L'/L} \sum_{\substack{t \in \mathbb{Z} + q(\beta) \\ t < 0}} c(\beta, t) q(\beta, -t).$$

Using Theorem 4.8 of the present paper, the $q(\gamma, n)$ can be computed explicitly. By Theorem 5.1 we obtain some information on the existence of automorphic forms for $\Gamma(L)$ with prescribed zeros and poles along Heegner divisors.

Theorem 5.2. *Let F be a holomorphic modular form of weight r with some character for the group $\Gamma(L)$, whose divisor (F) is a linear combination of Heegner divisors. Let*

$$D = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{t \in \mathbb{Z}+q(\beta) \\ t<0}} c(\beta, t) H(\beta, t)$$

be a linear combination with non-negative integral coefficients $c(\beta, t)$ (satisfying $c(\beta, t) = c(-\beta, t)$) such that $D \leq (F)$. Then

$$-\frac{1}{4} \sum_{\beta \in L'/L} \sum_{\substack{t \in \mathbb{Z}+q(\beta) \\ t<0}} c(\beta, t) q(\beta, -t) \leq r. \tag{5.4}$$

The equality sign in (5.4) holds, if and only if $D = (F)$.

This theorem is an immediate consequence of Theorem 5.1 and the following proposition. (Observe that $H(\beta, t)$ equals 0 in the divisor group, if there is no $\lambda \in L'$ such that $\lambda \equiv \beta \pmod{L}$ and $q(\lambda) = t$.)

Proposition 5.3. *The Fourier coefficients $q(\gamma, n)$ ($\gamma \in L'$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$) of $E(\tau)$ are non-positive rational numbers. Furthermore, $q(\gamma, n) < 0$, if and only if there exists a $\lambda \in L'$ such that $\lambda \equiv \gamma \pmod{L}$ and $q(\lambda) = -n$.*

Proof. We consider the formula for $q(\gamma, n)$ of Theorem 4.8. Obviously the first factor is negative. Moreover, it is easily seen that $\sigma_{1-k}(\tilde{n}, \chi_{4D})$ and

$$\sum_{d|f} \mu(d) \chi_{\mathcal{D}}(d) d^{1/2-k} \sigma_{2-2k}(f/d)$$

are positive.

The following argument shows that $L(u, \chi_D) > 0$ for any $u \in \mathbb{R}$ with $u > 1$ and any discriminant D. The Euler product expansion implies that we may assume that D is a fundamental discriminant, i.e. the discriminant of a quadratic field K over $\mathbb{Q}$. It is well known that the L-series $L_K(s)$ attached to K is equal to $\zeta(s)L(s, \chi_D)$ (cf. [Za] §11). The values $L_K(u)$ and $\zeta(u)$ are positive by definition.

We find that $q(\gamma, n)$ is the product of a negative (rational) number with

$$\prod_{p | 2\det(S)} p^{w_p(1-2k)} N_{\gamma,n}(p^{w_p}).$$

Hence $q(\gamma, n) \leq 0$. If there is a $\lambda \in L'$ such that $\lambda \equiv \gamma \pmod{L}$ and $q(\lambda) = -n$, then

$$N_{\gamma,n}(a) = N_{\lambda,-q(\lambda)}(a) = \#\{r \in L/aL; \quad q(r - \lambda) - q(\lambda) \equiv 0 \pmod{a}\}$$

($a \in \mathbb{N}$). Since $r = 0$ is a solution of the congruence, we have $N_{\gamma,n}(a) \geq 1$ and thereby $q(\gamma, n) < 0$.

Now suppose that there is no $\lambda \in L'$ such that $\lambda \equiv \gamma \pmod{L}$ and $q(\lambda) = -n$. Assume that $q(\gamma, n) < 0$. Then $N_{\gamma,n}(p^{w_p}) \geq 1$ for any prime p dividing $2\det(S)$. Hence, by Lemma 4.4 the equation $q(r - \gamma) + n = 0$ has a solution r over $\mathbb{Z}_p$ for any prime p. Since q is indefinite of rank ≥ 5 we may infer that there exists a global solution $r \in L$ of the latter equation (cf. [Wa] Theorem 63 and 72). But then $-r + \gamma \in L'$ satisfies $-r + \gamma \equiv \gamma \pmod{L}$ and $q(-r + \gamma) = -n$ contradicting our assumption. (The latter statement can also be proved in a rather indirect way: If there is no $\lambda \in L'$ such that $\lambda \equiv \gamma \pmod{L}$ and $q(\lambda) = -n$, then $H(\gamma, -n) = 0$ in the divisor group of $\mathcal{H}/\Gamma(L)$. Thus any constant non-zero function F on $\widetilde{\mathcal{H}}$ is a modular form of weight 0 with divisor $(F) = H(\gamma, -n)$. By Theorem 5.1 we obtain $q(\gamma, n) = 0$.) □

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