FALTINGS HEIGHTS OF CM CYCLES AND DERIVATIVES OF L-FUNCTIONS

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Abstract. We study the Faltings height pairing of arithmetic special divisors and CM cycles on Shimura varieties associated to orthogonal groups. We compute the archimedean contribution to the height pairing and derive a conjecture relating the total pairing to the central derivative of a Rankin L-function. We prove the conjecture in certain cases where the Shimura variety has dimension 0, 1, or 2. In particular, we obtain a new proof of the Gross-Zagier formula.

1. Introduction

Let E be an elliptic curve over $\mathbb{Q}$. Assume that its L-function $L(E,s)$ has an odd functional equation so that the central critical value $L(E,1)$ vanishes. In this case the Birch and Swinnerton-Dyer conjecture predicts the existence of a rational point of infinite order on E. It is natural to ask if is possible to construct such a point explicitly. The celebrated work of Gross and Zagier [GZ] provides such a construction when $L'(E,1) \neq 0$. We briefly recall their main result, the Gross-Zagier formula, in a formulation which is convenient for the present paper.

Let N be the conductor of E, and let $X_0(N)$ be the moduli space of cyclic isogenies of degree N of generalized elliptic curves. Let K be an imaginary quadratic field such that N is the norm of an integral ideal of K, and write D for the discriminant of K (we may assume $D < -4$ for simplicity in the introduction). Gross and Zagier consider the divisor $Z(D)$ on $X_0(N)$ given by elliptic curves with complex multiplication by the maximal order of K. By the theory of complex multiplication, this divisor is defined over K, and its degree h is given by the class number of K. Hence the divisor $y(D) = \mathrm{tr}_{K/\mathbb{Q}}(Z(D) - h \cdot (\infty))$ has degree zero and is defined over $\mathbb{Q}$. By means of the work of Wiles et al. [Wi], [BCDT], one obtains a rational point $y^E(D)$ on E using a modular parametrization $X_0(N) \to E$. The Gross-Zagier formula states that the canonical height of $y^E(D)$ is given by the derivative of the L-function of E over K at $s = 1$, more precisely

$$\langle y^E(D), y^E(D)\rangle_{NT} = C\sqrt{|D|}L'(E,1)L(E,\chi_D,1).$$

Here C is an explicit non-zero constant which is independent of K, and $L(E,\chi_D,s)$ denotes the quadratic twist of $L(E,s)$ by the quadratic Dirichlet character χ_D corresponding to $K/\mathbb{Q}$. It is always possible to choose K such that $L(E,\chi_D,1)$ is non-vanishing. So, in this case, $y^E(D)$ has infinite order if and only if $L'(E,1) \neq 0$.

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The work of Gross and Zagier triggered a lot of further research on height pairings of algebraic cycles on Shimura varieties. For instance, Gross and Keating computed the intersection numbers of three Hecke correspondences on the product of two copies of the modular curve $X(1)$ over $\mathbb{Z}$ [GK]. Zhang considered heights of Heegner type cycles on Kuga-Sato fiber varieties over modular curves in [Zh1], and the heights of Heegner points on compact Shimura curves over totally real fields in [Zh2]. Kudla, Rapoport and Yang investigated Arakelov intersection numbers of special cycles on Shimura varieties of orthogonal type and related them to derivatives of Siegel Eisenstein series and modular L-functions, see e.g. [Ku2], [Ku5], [KRY2]. In most of this work, the connection between a height pairing and the derivative of an automorphic L-function comes up in a rather indirect way.

In the present paper we consider a different approach to obtain identities between certain height pairings on Shimura varieties of orthogonal type and derivatives of automorphic L-functions. It is based on the Borcherds lift [Bo1] and its generalization in [Br2], [BF]. We propose a conjecture for the Faltings height pairing of arithmetic special divisors and CM cycles. We compute the archimedean contribution to the height pairing. Using this result we prove the conjecture in certain low dimensional cases. We now describe the content of this paper in more detail.

Let (V, Q) be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$, and let $H = \mathrm{GSpin}(V)$. We realize the hermitian symmetric space corresponding to $H(\mathbb{R})$ as the Grassmannian $\mathbb{D}$ of oriented negative definite two-dimensional subspaces of $V(\mathbb{R})$. For a compact open subgroup $K \subset H(\mathbb{A}_f)$ we consider the Shimura variety

$$X_K = H(\mathbb{Q})\backslash\big(\mathbb{D} \times H(\mathbb{A}_f)/K\big).$$

It is a quasi-projective variety of dimension n, which is defined over $\mathbb{Q}$.

It is an important feature of such Shimura varieties that they come with natural families of algebraic cycles in all codimensions, see e.g. [Ku3]. These special cycles arise from embeddings of rational quadratic subspaces $V' \subset V$ of signature $(n', 2)$ with $0 \leq n' \leq n$. It is an interesting problem to consider height pairings of arithmetic versions of special cycles in complementary codimension, see [Ku5]. In the present paper we study this problem for special divisors (where $n' = n - 1$) and special 0-cycles (where $n' = 0$). The latter are also called CM cycles since they are associated to CM number fields.

We define CM cycles on X_K following [Scho]. Let $U \subset V$ be a negative definite two-dimensional *rational* subspace of V. It determines a two point subset $\{z_U^\pm\} \subset \mathbb{D}$ given by $U(\mathbb{R})$ with the two possible choices of orientation. Let $V_+ \subset V$ be the orthogonal complement of U. Then V_+ is a positive definite subspace of dimension n, and we have the rational splitting $V = V_+ \oplus U$. Let $T = \mathrm{GSpin}(U)$, which we view as a subgroup of H acting trivially on V_+, and put $K_T = K \cap T(\mathbb{A}_f)$. We obtain the CM cycle

$$Z(U) = T(\mathbb{Q})\backslash\big(\{z_U^\pm\} \times T(\mathbb{A}_f)/K_T\big) \longrightarrow X_K.$$

We aim to compute the Faltings height pairing of $Z(U)$ with arithmetic special divisors on X_K that are constructed by means of a regularized theta lift. We use a similar setup as in [Ku4]. Let $L \subset V$ be an even lattice, and write L' for the dual of L. The discriminant group L'/L is finite. We consider the space S_L of Schwartz functions on $V(\mathbb{A}_f)$ which are supported on $L' \otimes \hat{\mathbb{Z}}$ and which are constant on cosets of $\hat{L} = L \otimes \hat{\mathbb{Z}}$. The characteristic functions $\phi_\mu = \operatorname{char}(\mu+\hat{L})$ of the cosets $\mu \in L'/L$ form a basis of S_L. We write $\Gamma' = \mathrm{Mp}_2(\mathbb{Z})$ for the full inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in the two fold metaplectic covering of $\mathrm{SL}_2(\mathbb{R})$. Recall that there is a Weil representation ρ_L of Γ' on S_L, see (2.7).

Let $k \in \frac{1}{2}\mathbb{Z}$. We write $M^!_{k,\rho_L}$ for the space of S_L-valued weakly holomorphic modular forms of weight k for Γ' with representation ρ_L. Recall that weakly holomorphic modular forms are those meromorphic modular forms whose poles are supported at the cusps. The space of weakly holomorphic modular forms is contained in the space H_{k,ρ_L} of harmonic weak Maass forms of weight k for Γ' with representation ρ_L (see Section 3 for precise definitions). An element $f \in H_{k,\rho_L}$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c^+(n,\mu) q^n \phi_\mu + \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n<0}} c^-(n,\mu)\Gamma(1-k, 4\pi|n|v) q^n \phi_\mu,$$

where $\Gamma(a,t)$ denotes the incomplete Gamma function, and v is the imaginary part of $\tau \in \mathbb{H}$. Note that $c^\pm(n,\mu) = 0$ unless $n \in Q(\mu) + \mathbb{Z}$, and that there are only finitely many $n < 0$ for which $c^+(n,\mu)$ is non-zero. There is an antilinear differential operator $\xi : H_{k,\rho_L} \to S_{2-k,\bar{\rho}_L}$ to the space of cusp forms of weight $2-k$ with dual representation. It is surjective and its kernel is equal to $M^!_{k,\rho_L}$.

Assume that $K \subset H(\mathbb{A}_f)$ acts trivially on L'/L. Recall that for any $\mu \in L'/L$ and for any positive $m \in Q(\mu) + \mathbb{Z}$ there is a special divisor $Z(m,\mu)$ on X_K, see Section 4. An arithmetic divisor on X_K is a pair (x, g_x) consisting of a divisor x on X_K and a Green function g_x of logarithmic type for x. For the divisors $Z(m,\mu)$ we obtain such Green functions by means of the regularized theta lift of harmonic weak Maass forms. For $\tau \in \mathbb{H}$, $z \in \mathbb{D}$ and $h \in H(\mathbb{A}_f)$, let $\theta_L(\tau, z, h)$ be the Siegel theta function associated to the lattice L. Let $f \in H_{1-n/2,\bar{\rho}_L}$ be a harmonic weak Maass form of weight $1 - n/2$, and denote its Fourier expansion as above. We consider the regularized theta integral

$$\Phi(z,h,f) = \int_{\mathcal{F}}^{reg} \langle f(\tau), \theta_L(\tau,z,h)\rangle \, d\mu(\tau).$$

This theta lift was studied in [Br2], [BF], generalizing the Borcherds lift of weakly holomorphic modular forms [Bo1]. It turns out that $\Phi(z,h,f)$ is a logarithmic Green function for the divisor

$$Z(f) = \sum_{\mu \in L'/L} \sum_{m>0} c^+(-m,\mu) Z(m,\mu)$$

in the sense of Arakelov geometry (see [SABK]). It is harmonic when $c^+(0,0) = 0$. The pair $\hat{Z}(f) = (Z(f), \Phi(\cdot, f))$ defines an arithmetic divisor on X_K. We obtain a linear map

$$H_{1-n/2,\bar{\rho}_L} \longrightarrow \hat{\mathrm{Z}}^1(X_K)_{\mathbb{C}}, \quad f \mapsto \hat{Z}(f)$$

to the group of arithmetic divisors on X_K. Using the Borcherds lift [Bo1], we see that this map takes weakly holomorphic modular forms with vanishing constant term to arithmetic divisors which are rationally equivalent to zero.

Let $\mathcal{X} \to \operatorname{Spec}(\mathbb{Z})$ be a regular scheme which is projective and flat over $\mathbb{Z}$, of relative dimension n. An arithmetic divisor on $\mathcal{X}$ is a pair (x, g_x) of a divisor x on $\mathcal{X}$ and a logarithmic Green function g_x for the divisor $x(\mathbb{C})$ induced by x on the complex variety $\mathcal{X}(\mathbb{C})$, see [SABK]. Recall from [BGS] that there is a height pairing

$$\widehat{\mathrm{CH}}^1(\mathcal{X}) \times \mathrm{Z}^n(\mathcal{X}) \longrightarrow \mathbb{R}$$

between the first arithmetic Chow group of $\mathcal{X}$ and the group of codimension n cycles. When $\hat{x} = (x, g_x) \in \widehat{\mathrm{CH}}^1(\mathcal{X})$ and $y \in \mathrm{Z}^n(\mathcal{X})$ such that x and y intersect properly on the generic fiber, it is defined by

$$\langle \hat{x}, y\rangle_{Fal} = \langle x, y\rangle_{fin} + \langle \hat{x}, y\rangle_\infty,$$

where $\langle \hat{x}, y\rangle_\infty = \frac{1}{2} g_x(y(\mathbb{C}))$, and $\langle x, y\rangle_{fin}$ denotes the intersection pairing at the finite places. The quantity $\langle \hat{x}, y\rangle_{Fal}$ is called the Faltings height of y with respect to $\hat{x}$.

We now give a conjectural formula for the Faltings height pairing of arithmetic special divisors and CM cycles (see Section 5 for details). We are quite vague here and ignore various difficult technical problems regarding regular models. Assume that there is a regular scheme $\mathcal{X}_K \to \operatorname{Spec}\mathbb{Z}$, projective and flat over $\mathbb{Z}$, whose associated complex variety is a smooth compactification of X_K. Let $\mathcal{Z}(m,\mu)$ and $\mathcal{Z}(U)$ be suitable extensions to $\mathcal{X}_K$ of the cycles $Z(m,\mu)$ and $Z(U)$, respectively. Such extensions can be found in many cases using a moduli interpretation of $\mathcal{X}_K$, see e.g. [Ku5], [KRY2], or by taking flat closures as in [BBK]. For an $f \in H_{1-n/2,\bar\rho_L}$, we set $\mathcal{Z}(f) = \sum_\mu \sum_{m>0} c^+(-m,\mu)\mathcal{Z}(m,\mu)$. Then the pair

$$\hat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(\cdot, f))$$

defines an arithmetic divisor in $\widehat{\mathrm{CH}}^1(\mathcal{X}_K)_\mathbb{C}$. The pairing of this divisor with the CM cycle $\mathcal{Z}(U)$ should be given by the central derivative of a certain Rankin type L-function which we now describe.

Using the splitting $V = V_+ \oplus U$, we obtain definite lattices $N = L \cap U$ and $P = L \cap V_+$. Let

$$\theta_P(\tau) = \sum_{\mu\in P'/P} \sum_{m\geq 0} r(m,\mu) q^m \phi_\mu$$

be the Fourier expansion of the S_P-valued theta series associated to the positive definite lattice P. For a cusp form $g \in S_{1+n/2,\rho_L}$ with Fourier expansion $g = \sum_\mu \sum_{m>0} b(m,\mu) q^m \phi_\mu$, we consider the Rankin type L-function

$$L(g,U,s) = (4\pi)^{-(s+n)/2}\Gamma\left(\tfrac{s+n}{2}\right) \sum_{m>0} \sum_{\mu\in P'/P} r(m,\mu)\overline{b(m,\mu)} m^{-(s+n)/2}, \tag{1.1}$$

where g is considered as an $S_{P\oplus N}$-valued cusp form in a natural way (via Lemma 3.1). This L-function can be written as a Rankin-Selberg convolution against an incoherent Eisenstein series $E_N(\tau, s; 1)$ of weight 1 associated to the negative definite lattice N, see Section 4.1. Under mild assumptions on U, the completed L-function $L^*(g,U,s) := \Lambda(\chi_D, s+1)L(g,U,s)$ satisfies the functional equation

$$L^*(g,U,s) = -L^*(g,U,-s).$$

Consequently, it vanishes at $s = 0$, the center of symmetry, and it is of interest to describe the derivative $L'(g, U, 0)$.

Conjecture 1.1. *Let $f \in H_{1-n/2,\bar{\rho}_L}$, and assume that the constant term $c^+(0, 0)$ of f vanishes. Then*

$$\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(U)\rangle_{Fal} = \frac{\deg(Z(U))}{2} \cdot L'(\xi(f), U, 0). \tag{1.2}$$

In Section 4 we compute the archimedean contribution to the height pairing, see Theorem 4.7.

Theorem 1.2. *The archimedean height pairing $\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(U)\rangle_\infty$ is given by*

$$\frac{1}{2}\Phi(Z(U), f) = \frac{\deg(Z(U))}{2} \left(\mathrm{CT}\left(\langle f^+,\, \theta_P \otimes \mathcal{E}_N\rangle\right) + L'(\xi(f), U, 0)\right).$$

Here f^+ denotes the "holomorphic part" of the harmonic weak Maass form f and $\mathcal{E}_N(\tau)$ is the holomorphic part of the derivative $E'_N(\tau, 0; 1)$ of the Eisenstein series associated to N, see (2.26). Moreover, $\mathrm{CT}(\,\cdot\,)$ means the constant term of a holomorphic Fourier series. In the proof we combine the approach of Kudla and Schofer to evaluate regularized theta integrals on special cycles (see [Ku4], [Scho]) with results on harmonic weak Maass forms and automorphic Green functions obtained in [BF]. The basic idea is to view the evaluation of $\Phi(z, h, f)$ on $Z(U)$ as an integral over $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T$. Then the CM value $\Phi(Z(U), f)$ can be computed using a see-saw identity, the Siegel-Weil formula, and the properties of the Maass lowering and raising operators on Eisenstein series and harmonic weak Maass forms.

When f is actually weakly holomorphic then $\xi(f) = 0$ and Theorem 1.2 reduces to the main result of [Scho]. Moreover, the Borcherds lift of f gives rise to a relation which shows that the arithmetic divisor $\hat{\mathcal{Z}}(f)$ is rationally equivalent to zero. Hence the Faltings height in Conjecture 1.1 vanishes. Therefore the archimedean contribution to the height pairing must equal the negative of the contribution from the finite places. This leads to a general conjecture for the finite intersection pairing of $\mathcal{Z}(m, \mu)$ and $\mathcal{Z}(U)$ (see Conjecture 5.1) which motivates Conjecture 1.1:

Conjecture 1.3. *Let $\mu \in L'/L$, and let $m \in Q(\mu)+\mathbb{Z}$ be positive. Then $\langle \mathcal{Z}(m, \mu), \mathcal{Z}(U)\rangle_{fin}$ is equal to $-\frac{\deg(Z(U))}{2}$ times the (m, μ)-th Fourier coefficient of $\theta_P \otimes \mathcal{E}_N$.*

In view of Theorem 1.2, this conjecture is essentially equivalent to Conjecture 1.1. We discuss this in detail in Section 5, where we also give a slight generalization and derive some consequences.

In Section 6 we consider the case $n = 0$ where V is negative definite of dimension 2. Then we have $U = V$. The even Clifford algebra of V is an imaginary quadratic field $k = \mathbb{Q}(\sqrt{D})$, and $H = \mathrm{GSpin}(V) = k^*$. For simplicity we assume that the lattice L is isomorphic to a fractional ideal $\mathfrak{a} \subset k$ with the scaled norm $-\mathrm{N}(\cdot)/\mathrm{N}(\mathfrak{a})$ as the quadratic form. We take $K = \hat{\mathcal{O}}_k^*$, which acts on L'/L trivially. Then X_K is the union of two copies of the ideal class group $\mathrm{Cl}(k)$. An integral model over $\mathbb{Z}$ can be found by slightly varying the setup of [KRY1]. It is given as the moduli stack $\mathcal{C}$ over $\mathbb{Z}$ of elliptic curves with complex multiplication by the ring of integers of k. The special divisors can be defined on $\mathcal{C}$ by considering CM elliptic curves whose endomorphism ring is larger, and therefore equal to an order of a quaternion algebra. They are supported in finite characteristic.

In this case the lattice P is zero-dimensional and the L-function $L(\xi(f), U, s)$ vanishes identically. Therefore Conjecture 1.1 reduces to the statement that the arithmetic degree of the special divisor $\mathcal{Z}(f)$ on $\mathcal{C}$ should be given by the negative of the average of the regularized theta lift of f. We prove this identity using Theorem 1.2 and the results obtained in [KRY1], respectively their generalization in [KY1]. More precisely we show (see Theorem 6.5):

Theorem 1.4. *Let $f \in H_{1,\bar{\rho}_L}$ and assume that the constant term of f vanishes. Then*

$$\widehat{\deg}(\mathcal{Z}(f)) = -\frac{1}{2} \sum_{(z,h) \in X_K} \Phi(z, h, f).$$

In Section 7 we consider the case $n = 1$. We let V be the rational quadratic space of signature $(1, 2)$ given by the trace zero 2×2 matrices with the quadratic form $Q(x) = N \det(x)$, where N is a fixed positive integer. In this case $H \cong \mathrm{GL}_2$. We chose the lattice $L \subset V$ and the compact open subgroup $K \subset H(\mathbb{A}_f)$ such that X_K is isomorphic to the modular curve $\Gamma_0(N)\backslash\mathbb{H}$. The special divisors $Z(m, \mu)$ and the CM cycles $Z(U)$ are both supported on CM points and therefore closely related.

The space $S_{3/2,\rho_L}$ can be identified with the space of Jacobi cusp forms of weight 2 and index N. Recall that there is a Shimura lifting from this space to cusp forms of weight 2 for $\Gamma_0(N)$, see [GKZ]. Let G be a normalized newform of weight 2 for $\Gamma_0(N)$ whose Hecke L-function $L(G, s)$ satisfies an odd functional equation. There exists a newform $g \in S_{3/2,\rho_L}$ corresponding to G under the Shimura correspondence. It turns out that the L-function $L(g, U, s)$ is proportional to $L(G, s + 1)$, see Lemma 7.3.

We may choose $f \in H_{1/2,\bar{\rho}_L}$ with vanishing constant term such that $\xi(f) = \|g\|^{-2} g$ and such that the principal part of f has coefficients in the number field generated by the eigenvalues of G. Then $Z(f)$ defines an explicit point in the Jacobian of $X_0(N)$, which lies in the G isotypical component, see Theorem 7.6. In this case Conjecture 1.1 essentially reduces to the following Gross-Zagier type formula for the Neron-Tate height of $Z(f)$ (Theorem 7.7).

Theorem 1.5. *The Neron-Tate height of $Z(f)$ is given by*

$$\langle Z(f), Z(f)\rangle_{NT} = \frac{2\sqrt{N}}{\pi \|g\|^2} L'\big(G, 1\big).$$

The proof of this result which we give in Section 7.3 is quite different from the original proof of Gross and Zagier and uses *minimal* information on finite intersections between Heegner divisors. Instead, we derive it from Theorem 1.2, modularity of the generating series of Heegner divisors (Borcherds' approach to the Gross-Kohnen-Zagier theorem [Bo2]), and multiplicity one for the subspace of newforms in $S_{3/2,\rho_L}$ [SZ]. Another crucial ingredient is the non-vanishing result for coefficients of weight 2 Jacobi cusp forms by Bump, Friedberg, and Hoffstein [BFH]. Employing in addition the Waldspurger type formula for the coefficients of g [GKZ], we also obtain the Gross-Zagier formula as stated at the beginning.

We conclude Section 7 with a proof of Conjectures 1.1 and 1.3 in this case, which can be used to give another proof of Theorem 1.5. It relies on the computation of the finite intersection pairing of $\mathcal{Z}(f)$ and $\mathcal{Z}(U)$ by pulling back to $\mathcal{Z}(U)$ and employing the results for the $n=0$ case obtained in Section 6. Finally, in Section 8 we use the same idea to prove Conjecture 1.3 in certain special cases for $n=2$. Here we consider the case where the CM 0-cycle lies on the diagonal in a Hilbert modular surface. The normalization of the Hirzebruch-Zagier divisor given by the diagonal is the modular curve of level 1. We may pull back the divisor $\mathcal{Z}(f)$ to this modular curve and compute the intersection there using the results of Section 7 (see Theorem 8.1).

The paper is organized as follows. In Section 2 we collect important facts on theta series, Eisenstein series and the Siegel-Weil formula. In Section 3 we recall some results on vector valued modular forms and harmonic weak Maass forms. In Section 4 we define the regularized theta lift and compute the CM values of automorphic Green functions. Section 5 contains the conjectures on Faltings heights. In Section 6 we consider the case $n=0$, in Section 7 the case $n=1$, and in Section 8 the case $n=2$.

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2. Theta series and Eisenstein series

Here we fix the basic setup taken from [Ku1], [Ku4]. We present some facts on theta series, Eisenstein series, and the Siegel-Weil formula.

Let (V,Q) be a quadratic space over $\mathbb{Q}$ of signature $(n,2)$. Let $H=\mathrm{GSpin}(V)$, and $G=\mathrm{SL}_2$, viewed as an algebraic groups over $\mathbb{Q}$. Recall that there is an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow H \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. We write $G'_{\mathbb{A}}$ for the twofold metaplectic cover of $G(\mathbb{A})$. We frequently identify $G'_{\mathbb{R}}$, the full inverse image in $G'_{\mathbb{A}}$ of $G(\mathbb{R})$, with the group of pairs

$$(g,\phi(\tau))$$

where $g=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{R})$ and $\phi(\tau)$ is a holomorphic function on the upper complex half plane $\mathbb{H}$ such that $\phi(\tau)^2=c\tau+d$. The multiplication is given by $(g_1,\phi_1(\tau))(g_2,\phi_2(\tau))=(g_1g_2,\phi_1(g_2\tau)\phi_2(\tau))$.

Let K' be the full inverse image in $G'_{\mathbb{A}}$ of $K=\mathrm{SL}_2(\hat{\mathbb{Z}})\subset G(\mathbb{A}_f)$. Let K'_∞ be the full inverse image in $G'_{\mathbb{R}}$ of $K_\infty=\mathrm{SO}(2,\mathbb{R})\subset G(\mathbb{R})$. We write $G'_{\mathbb{Q}}$ for the image in $G'_{\mathbb{A}}$ of $G(\mathbb{Q})$ under the canonical splitting. We have $G'_{\mathbb{A}}=G'_{\mathbb{Q}}G'_{\mathbb{R}}K'$ and

$$\Gamma := \mathrm{SL}_2(\mathbb{Z}) \cong G'_{\mathbb{Q}}\cap G'_{\mathbb{R}}K'.$$

We write $\Gamma' = \mathrm{Mp}_2(\mathbb{Z})$ for the full inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in $G'_\mathbb{R}$. Then for every $\gamma' \in \Gamma'$ there are unique elements $\gamma \in \Gamma$ and $\gamma'' \in K'$ such that

$$\gamma = \gamma'\gamma''. \tag{2.1}$$

The assignment $\gamma' \mapsto \gamma''$ defines a homomorphism $\Gamma' \to K'$. Let ψ be the standard non-trivial additive character of $\mathbb{A}/\mathbb{Q}$. The groups $G'_\mathbb{A}$ and $H(\mathbb{A})$ act on the space $S(V(\mathbb{A}))$ of Schwartz-Bruhat functions of $V(\mathbb{A})$ via the Weil representation $\omega = \omega_\psi$.

For $\varphi \in S(V(\mathbb{A}))$ we have the usual theta function

$$\vartheta(g, h; \varphi) = \sum_{x \in V(\mathbb{Q})} (\omega(g, h)\varphi)(x),$$

where $g \in G'_\mathbb{A}$ and $h \in H(\mathbb{A})$. It is left invariant under $G'_\mathbb{Q}$ by Poisson summation, and it is trivially left invariant under $H(\mathbb{Q})$.

Here we consider the following specific Schwartz functions. We realize the hermitean symmetric space corresponding to $H(\mathbb{R})$ as the Grassmannian

$$\mathbb{D} = \{z \subset V(\mathbb{R}); \quad \dim(z) = 2 \text{ and } Q\mid_z < 0\}$$

of oriented negative definite 2-dimensional subspaces of $V(\mathbb{R})$. For any $z \in \mathbb{D}$, we may consider the corresponding majorant

$$(x, x)_z = (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z),$$

which is a positive definite quadratic form on the vector space $V(\mathbb{R})$. The Gaussian

$$\varphi_\infty(x, z) = \exp(-\pi(x, x)_z)$$

belongs to $S(V(\mathbb{R}))$. It has the invariance property $\varphi_\infty(hx, hz) = \varphi_\infty(x, z)$ for any $h \in H(\mathbb{R})$. Moreover, it has weight $n/2-1$ under the action of the maximal compact subgroup $K'_\infty \subset G'_\mathbb{R}$. Let $\varphi_f \in S(V(\mathbb{A}_f))$. We obtain a theta function on $G'_\mathbb{A} \times H(\mathbb{A})$ by putting

$$\theta(g, h; \varphi_f) = \vartheta\big(g, h; \varphi_\infty(\cdot, z_0) \otimes \varphi_f(\cdot)\big), \tag{2.2}$$

where $z_0 \in \mathbb{D}$ denotes a fixed base point. This theta function can be written as a theta function on $\mathbb{H} \times \mathbb{D}$ in the usual way. For $z \in \mathbb{D}$ we chose a $h_z \in H(\mathbb{R})$ such that $h_z z_0 = z$. Notice that $\omega(h_z)\varphi_\infty(\cdot, z_0) = \varphi_\infty(\cdot, z)$. Moreover, choosing i as a base point for $\mathbb{H}$, we put

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}$$

for $\tau = u + iv \in \mathbb{H}$ and write $g'_\tau = (g_\tau, 1) \in G'_\mathbb{R}$. So we have $g'_\tau i = \tau$. We then obtain the theta function

$$\begin{aligned} \theta(\tau, z, h_f; \varphi_f) &= v^{-n/4+1/2}\vartheta\big(g'_\tau, (h_z, h_f); \varphi_\infty(\cdot, z_0) \otimes \varphi_f(\cdot)\big) \\ &= v^{-n/4+1/2} \sum_{x \in V(\mathbb{Q})} \omega(g'_\tau)\big(\varphi_\infty(\cdot, z) \otimes \omega(h_f)\varphi_f\big)(x) \end{aligned} \tag{2.3}$$

for $h_f \in H(\mathbb{A}_f)$. Using the fact that

$$v^{-n/4+1/2}\omega(g'_\tau)\big(\varphi_\infty(\cdot, z)\big)(x) = v\, e\big(Q(x_{z^\perp})\tau + Q(x_z)\bar{\tau}\big),$$

we find more explicitly

$$\theta(\tau, z, h_f; \varphi_f) = v \sum_{x\in V(\mathbb{Q})} e\big(Q(x_{z^\perp})\tau + Q(x_z)\bar{\tau}\big) \otimes \varphi_f(h_f^{-1}x). \tag{2.4}$$

By means of the argument of [Ku4, Lemma 1.1], we obtain the following transformation formula for $\theta(\tau, z, h_f; \varphi_f)$ under Γ'. Let $\gamma' = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \phi) \in \Gamma'$, and write $\gamma = \gamma'\gamma''$ as in (2.1). Then we have

$$\theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau, z, h_f; \varphi_f\right) = \phi(\tau)^{n-2}\theta\left(\tau, z, h_f; \omega_f(\gamma'')^{-1}\varphi_f\right). \tag{2.5}$$

If we view $\theta(\tau, z, h_f; \cdot)$ as a function on $\mathbb{H}$ with values in the dual space $S(V(\mathbb{A}_f))^\vee$ of $S(V(\mathbb{A}_f))$, we see that $\theta(\tau, z, h_f; \cdot)$ transforms as a (non-holomorphic) modular form of weight $n/2 - 1$ with representation $\omega_f^\vee$.

Let $L \subset V$ be an even lattice and write L' for the dual lattice. The discriminant group L'/L is finite. We consider the subspace S_L of Schwartz functions in $S(V(\mathbb{A}_f))$ which are supported on $L' \otimes \hat{\mathbb{Z}}$ and which are constant on cosets of $\hat{L} = L \otimes \hat{\mathbb{Z}}$. For any $\mu \in L'/L$, the characteristic function

$$\phi_\mu = \mathrm{char}(\mu + \hat{L})$$

belongs to S_L, and we have

$$S_L = \bigoplus_{\mu\in L'/L} \mathbb{C}\phi_\mu \subset S(V(\mathbb{A}_f)).$$

In particular, the dimension of S_L is equal to $|L'/L|$. The space S_L is stable under the action of K' via the Weil representation (see [Ku4]).

We define a S_L-valued theta function by putting

$$\theta_L(\tau, z, h_f) = \sum_{\mu\in L'/L} \theta(\tau, z, h_f; \phi_\mu)\phi_\mu. \tag{2.6}$$

If we identify S_L with the group ring $\mathbb{C}[L'/L]$ by mapping ϕ_μ to the standard basis element $\mathfrak{e}_\mu$ of $\mathbb{C}[L'/L]$, then $\theta_L(\tau, z, 1)$ is exactly the Siegel theta function $\Theta_L(\tau, z)$ considered by Borcherds in [Bo1] §4 for the polynomial $p = 1$. (Under this identification of S_L with $\mathbb{C}[L'/L]$ the L^2 scalar product on S_L corresponds to the standard scalar product on $\mathbb{C}[L'/L]$. The convolution product corresponds to the usual product in $\mathbb{C}[L'/L]$.) Let $\gamma' = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \phi) \in \Gamma'$. We write $\gamma = \gamma'\gamma''$ as in (2.1) and put

$$\rho_L(\gamma') = \bar{\omega}_f(\gamma''). \tag{2.7}$$

Then ρ_L defines a representation of Γ' on S_L. The transformation formula (2.5) implies that

$$\theta_L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau, z, h_f\right) = \phi(\tau)^{n-2}\rho_L(\gamma')\theta_L(\tau, z, h_f). \tag{2.8}$$

Let $T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1 \right)$, and $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau} \right)$ denote the standard generators of Γ'. Recall that the action of ρ_L is given by

$$\rho_L(T)(\phi_\mu) = e(\mu^2/2)\phi_\mu, \tag{2.9}$$

$$\rho_L(S)(\phi_\mu) = \frac{e((2-n)/8)}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e(-(\mu,\nu))\phi_\nu, \tag{2.10}$$

see e.g. [Bo1], [Ku4], [Br2].

2.1. **The Siegel-Weil formula.** Here we briefly recall the Siegel-Weil formula in our setting (see [We], [KR1], [KR2], [Ku1]). We assume that n is even, which is sufficient for our purposes. In this case the dimension of V is even so that the Weil representation factors through $G = \mathrm{SL}_2$.

For $a \in \mathbb{G}_m$ we put $m(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, and for $b \in \mathbb{G}_a$ we put $n(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. Let $P = MN \subset G$ be the parabolic subgroup of upper triangular matrices, where

$$M = \{m(a);\ a \in \mathbb{G}_m\}, \tag{2.11}$$

$$N = \{n(b);\ b \in \mathbb{G}_a\}. \tag{2.12}$$

Let χ_V denote the quadratic character of $\mathbb{A}^\times/\mathbb{Q}^\times$ associated to V given by

$$\chi_V(x) = (x, (-1)^{\dim V/2} \det(V))_{\mathbb{A}}.$$

Here $\det(V)$ denotes the Gram determinant of V and $(\cdot,\cdot)$ is the Hilbert symbol on $\mathbb{A}^*$. For $s \in \mathbb{C}$ we denote by $I(s, \chi_V)$ the principal series representation of $G(\mathbb{A})$ induced by $\chi_V|\cdot|^s$. It consists of all smooth functions $\Phi(g,s)$ on $\mathbb{G}(\mathbb{A})$ satisfying

$$\Phi(n(b)m(a)g, s) = \chi_V(a)|a|^{s+1}\Phi(g,s)$$

for all $b \in \mathbb{A}$, $a \in \mathbb{A}^\times$, and the action of $G(\mathbb{A})$ is given by right translations. There is a $G(\mathbb{A})$-intertwining map

$$\lambda : S(V(\mathbb{A})) \longrightarrow I(s_0, \chi_V), \quad \lambda(\varphi)(g) = (\omega(g)\varphi)(0), \tag{2.13}$$

where $s_0 = \dim(V)/2 - 1$. A section $\Phi(s) \in I(s, \chi_V)$ is called standard, if its restriction to $K_\infty K$ is independent of s. Using the Iwasawa decomposition $G(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K_\infty K$, we see that the function $\lambda(\varphi) \in I(s_0, \chi_V)$ has a unique extension to a standard section $\lambda(\varphi, s) \in I(s, \chi_V)$ such that $\lambda(\varphi, s_0) = \lambda(\varphi)$.

We give an example at the archimedean place. For $\ell \in \mathbb{Z}$, let χ_ℓ be the character of K_∞ defined by

$$\chi_\ell(k_\theta) = e^{i\ell\theta},$$

where $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in K_\infty$. Let $\Phi_\infty^\ell(s) \in I_\infty(s, \chi_V)$ be the unique standard section such that

$$\Phi_\infty^\ell(k_\theta, s) = \chi_\ell(k_\theta) = e^{i\ell\theta}.$$

In terms of the Iwasawa decomposition we have

$$\Phi_\infty^\ell(n(b)m(a)k_\theta, s) = \chi_V(a)|a|^{s+1}e^{i\ell\theta}. \tag{2.14}$$

Then it is easily seen that for the Gaussian we have

$$\lambda_\infty(\varphi_\infty(\cdot, z)) = \Phi_\infty^{n/2-1}(s_0). \tag{2.15}$$

For any standard section $\Phi(s)$, the Eisenstein series

$$E(g, s; \Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma g, s)$$

converges for $\Re(s) > 1$ and defines an automorphic form on $G(\mathbb{A})$. It has a meromorphic continuation in s to the whole complex plane and satisfies a functional equation relating $E(g, s; \Phi)$ and $E(g, -s; M(s)\Phi)$. In our special case, the Siegel Weil formula says the following (see [We], [Ku4, Theorem 4.1]).

Theorem 2.1. *Let V be a rational quadratic space of signature $(n, 2)$ as above. Assume that V is anisotropic or that $\dim(V) - r_0 > 2$, where r_0 is the Witt index of V. Then $E(g, s; \lambda(\varphi))$ is holomorphic at s_0 and*

$$\frac{\alpha}{2} \int_{\mathrm{SO}(V)(\mathbb{Q})\backslash \mathrm{SO}(V)(\mathbb{A})} \vartheta(g, h; \varphi)\, dh = E(g, s_0; \lambda(\varphi)).$$

Here dh is the Tamagawa measure on $\mathrm{SO}(V)(\mathbb{A})$, and $\alpha = 2$ if $n = 0$, and $\alpha = 1$ if $n > 0$.

Note that the theta integral on the right hand side converges absolutely by Weil's convergence criterion.

2.2. **Quadratic spaces of signature** $(0, 2)$**.** Here, as in [Scho], we are interested in the special case, where V is a definite space of signature $(0, 2)$. Then (V, Q) is isometric to $(k, -c\,\mathrm{N}(\cdot))$ for an imaginary quadratic field k with the negative of the norm form scaled by a constant $c \in \mathbb{Q}_{>0}$. The group $H(\mathbb{Q})$ can be identified with the multiplicative group k^* of k, and $\mathrm{SO}(V)$ is the group of norm 1 elements in k. The homomorphism $H \to \mathrm{SO}(V)$ is given by $h \mapsto h\bar{h}^{-1}$, and $\mathrm{SO}(V)$ acts on k by multiplication. Moreover, the Grassmannian $\mathbb{D}$ consists of the two points z_V^+ and z_V^- given by $V(\mathbb{R})$ with positive and negative orientation, respectively. We want to compute the integral of the theta function $\theta_L(\tau, z_V, h_f)$ in (2.6), where $z_V \in \mathbb{D}$. To this end, for $\ell \in \mathbb{Z}$, we define a S_L-valued Eisenstein series of weight ℓ by putting

$$E_L(\tau, s; \ell) = v^{-\ell/2} \sum_{\mu \in L'/L} E(g_\tau, s; \Phi_\infty^\ell \otimes \lambda_f(\phi_\mu))\phi_\mu. \tag{2.16}$$

We normalize the measure on $\mathrm{SO}(V)(\mathbb{R}) \cong \mathrm{SO}(2, \mathbb{R})$ such that $\mathrm{vol}(\mathrm{SO}(V)(\mathbb{R})) = 1$. This determines the normalization of the measure dh_f on $\mathrm{SO}(V)(\mathbb{A}_f)$. Note that in this normalization we have $\mathrm{vol}(\mathrm{SO}(V)(\mathbb{Q})\backslash \mathrm{SO}(V)(\mathbb{A}_f)) = 2$. By Theorem 2.1 and (2.15) we obtain:

Proposition 2.2. *We have*

$$\int_{\mathrm{SO}(V)(\mathbb{Q})\backslash \mathrm{SO}(V)(\mathbb{A}_f)} \theta_L(\tau, z_V, h_f)\, dh = E_L(\tau, 0; -1).$$

Following [Ku1, §IV.2], we write down this Eisenstein series more classically. It is easily seen that $P(\mathbb{Q})\backslash G(\mathbb{Q}) = \Gamma_\infty\backslash\Gamma$, where $\Gamma_\infty = P(\mathbb{Q}) \cap \Gamma$. Hence we have

$$E(g_\tau, s; \Phi) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Phi(\gamma g_\tau, s).$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we consider the Iwasawa decomposition $\gamma g_\tau = nm(\alpha)k_\theta$, where $\alpha \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}$. A calculation shows that

$$\begin{aligned} \alpha &= v^{1/2}|c\tau + d|^{-1}, \\ e^{i\theta} &= \frac{c\bar\tau + d}{|c\tau + d|}. \end{aligned}$$

Inserting this into (2.14), we see that

$$\Phi^\ell_\infty(\gamma g_\tau, s) = v^{s/2+1/2}(c\tau + d)^{-\ell}|c\tau + d|^{\ell-s-1}.$$

Therefore we obtain

$$\begin{aligned} E(g_\tau, s, \Phi^\ell_\infty \otimes \lambda_f(\phi_\mu)) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (c\tau + d)^{-\ell} \frac{v^{s/2+1/2}}{|c\tau + d|^{s+1-\ell}} \cdot \lambda_f(\phi_\mu)(\gamma) \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (c\tau + d)^{-\ell} \frac{v^{s/2+1/2}}{|c\tau + d|^{s+1-\ell}} \cdot \langle \phi_\mu, (\omega_f^{-1}(\gamma))\phi_0\rangle. \end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ denotes the L^2 scalar product on S_L. Consequently, for the vector valued Eisenstein series we find

$$E_L(\tau, s; \ell) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left[\Im(\tau)^{(s+1-\ell)/2}\phi_0\right] |_{\ell,\rho_L} \gamma, \tag{2.17}$$

where $|_{\ell,\rho_L}$ is the usual Petersson slash operator of weight ℓ for the representation ρ_L. In particular we see that this Eisenstein series coincides up to a shift in the argument with the Eisenstein series considered in [BK, §3].

Recall that the Maass lowering and raising operators in weight ℓ are defined by

$$L_\ell = -2iv^2 \frac{\partial}{\partial\bar\tau}, \tag{2.18}$$

$$R_\ell = 2i\frac{\partial}{\partial\tau} + \ell v^{-1}. \tag{2.19}$$

They lower (respectively raise) the weight of an automorphic form by 2. It is easily seen that

$$\begin{aligned} L_\ell E_L(\tau, s; \ell) &= \frac{1}{2}(s + 1 - \ell)E_L(\tau, s; \ell - 2), \\ R_\ell E_L(\tau, s; \ell) &= \frac{1}{2}(s + 1 + \ell)E_L(\tau, s; \ell + 2) \end{aligned}$$

(see also [Ku4, Lemma 2.7]). In particular, we see that

$$L_1 E_L(\tau, s; 1) = \frac{s}{2}E_L(\tau, s; -1). \tag{2.20}$$

Since $E_L(\tau, s; -1)$ is holomorphic at $s = 0$ by Theorem 2.1, we find that $E_L(\tau, s; 1)$ vanishes at $s = 0$, the center of symmetry. This corresponds to the fact that $E_L(\tau, s; 1)$ is an incoherent Eisenstein series, see [Ku2], because it is constructed at all finite places from the data corresponding to the quadratic space (V, Q), but at the archimedean place one takes $(V, -Q)$. In particular $E_L(\tau, s; 1)$ satisfies an odd functional equation under $s \mapsto -s$, which explains the vanishing at $s = 0$ (see also Proposition 2.5). The identity (2.20) implies that

$$L_1 E_L'(\tau, 0; 1) = \frac{1}{2} E_L(\tau, 0; -1), \tag{2.21}$$

where $E_L'(\tau, s; 1)$ denotes the derivative of $E_L(\tau, s; 1)$ with respect to s. This identity can be written in terms of differential forms as follows.

Lemma 2.3. *We have*

$$-2\bar{\partial}\left(E_L'(\tau, 0; 1)\, d\tau\right) = E_L(\tau, 0; -1)\, d\mu(\tau).$$

As in [Scho] we write the Fourier expansion of the Eisenstein series in the form

$$E_L(\tau, s; 1) = \sum_{\mu \in L'/L} \sum_{m \in \mathbb{Q}} A_\mu(s, m, v) q^m \phi_\mu, \tag{2.22}$$

where $q = e^{2\pi i \tau}$ as usual. The coefficients $A_\mu(s, m, v)$ are computed in [KY2], [KRY1], [Scho], and [BK]. The formulas we will need later are summarized in Theorem 2.6 below. Notice that $A_\mu(s, m, v) = 0$ unless $m \in Q(\mu) + \mathbb{Z}$. Since the Eisenstein series vanishes at $s = 0$, the coefficients have a Laurent expansion of the form

$$A_\mu(s, m, v) = b_\mu(m, v)s + O(s^2) \tag{2.23}$$

at $s = 0$, and we have

$$E_L'(\tau, 0; 1) = \sum_{\mu \in L'/L} \sum_{m \in \mathbb{Q}} b_\mu(m, v) q^m \phi_\mu. \tag{2.24}$$

For the evaluation of an automorphic Green function at a CM cycle, the following quantities play a key role:

$$\kappa(m, \mu) := \begin{cases} \lim_{v \to \infty} b_\mu(m, v), & \text{if } m \neq 0 \text{ or } \mu \neq 0, \\ \lim_{v \to \infty} b_0(0, v) - \log(v), & \text{if } m = 0 \text{ and } \mu = 0. \end{cases} \tag{2.25}$$

According to [Scho, Proposition 2.20 and Lemma 2.21], (see also [Ku4, Theorem 2.12]), the limits exist. If $m > 0$, then $b_\mu(m, v)$ is actually independent of v and equal to $\kappa(m, \mu)$. We also have $\kappa(m, \mu) = 0$ for $m < 0$ or $m = 0$, $\mu \neq 0$. Using the quantities $\kappa(m, \mu)$ we define a holomorphic S_L valued function on $\mathbb{H}$ by

$$\mathcal{E}_L(\tau) = \sum_{\mu \in L'/L} \sum_{m \in \mathbb{Q}} \kappa(m, \mu) q^m \phi_\mu. \tag{2.26}$$

This function is clearly periodic, but it is not invariant under the $|_{1,\rho_L}$-action of $S \in \Gamma'$.

Remark 2.4. Another way of interpreting (2.21) is that $E_L'(\tau,0;1)$ is a harmonic weak Maass form (see Section 3.1) of weight 1 which is mapped to $v^{-1}\overline{E_L(\tau,0;-1)}$ under ξ. The function $\mathcal{E}_L(\tau)$ is simply the holomorphic part of $E_L'(\tau,0;1)$.

Now we assume that $(L,Q)=(\mathfrak{a},-\frac{\mathrm{N}}{\mathrm{N}(\mathfrak{a})})$ where $\mathfrak{a}$ is a fractional ideal of an imaginary quadratic field $k=\mathbb{Q}(\sqrt{D})$ with fundamental discriminant $D\equiv 1 \pmod 4$. We denote by $\mathcal{O}_k$ the ring of integers in k, and write ∂ for the different of k. In this case, $V=k$, the dual lattice is given by $L'=\partial^{-1}\mathfrak{a}$, and

$$L'/L=\partial^{-1}\mathfrak{a}/\mathfrak{a}\cong\partial^{-1}/\mathcal{O}_k\cong\mathbb{Z}/D\mathbb{Z}.$$

Proposition 2.5. *Let the notation be as above. Let χ_D be the quadratic Dirichlet character associated to $k/\mathbb{Q}$, and let*

$$\Lambda(\chi_D,s)=|D|^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s+1)L(\chi_D,s),\quad \Gamma_{\mathbb{R}}(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$$

be its complete L-function. Let

$$E_L^*(\tau,s)=\Lambda(\chi_D,s+1)E_L(\tau,s),$$

then

$$E_L^*(\tau,s)=-E_L^*(\tau,-s).$$

Proof. It is equivalent to prove the equation for each $E(\tau,s,\mu)=E(\tau,s,\Phi_\infty^1\otimes\lambda_f(\phi_\mu))$. By Langlands' general theory of Eisenstein series, one has

$$E(g,s,\Phi)=E(g,-s,M(s)\Phi).$$

Here $M(s)=\prod_{p\le\infty}M_p(s):I(s,\chi_D)\to I(-s,\chi_D)$ is the usual intertwining operator given by (when $\Re(s)\gg 0$)

$$M_p(s)\Phi_p(g,s)=\int_{\mathbb{Q}_p}\Phi_p(wn(b)g,s)db \tag{2.27}$$

for $\Phi_p\in I_p(s,\chi_D)$, where $I_p(s,\chi_D)$ is the local principal series.

When $p=\infty$, it is well-known that

$$M_\infty(s)\Phi_\infty^1(g,s)=C_\infty(s)\Phi_\infty^1(g,-s) \tag{2.28}$$

with

$$C_\infty(s)=M_\infty(s)\Phi_\infty^1(1,s). \tag{2.29}$$

It is also known (see for example [KRY1, Proposition 2.6]) that

$$C_\infty(s)=-\gamma_\infty(V)\frac{\Gamma_{\mathbb{R}}(s+1)}{\Gamma_{\mathbb{R}}(s+2)} \tag{2.30}$$

where $\gamma_\infty(V)=-\gamma_\infty(-V)=i$ is the local Weil index associated to the local Weil representation ω_V of $\mathrm{SL}_2(\mathbb{R})$ on the Schwartz space $S(V\otimes\mathbb{R})$ with respect to the dual pair $(\mathrm{O}(V),\mathrm{SL}_2)$. The fact we need here is that $\gamma_\infty(V)=-\gamma_\infty(-V)$, not the explicit formula, and Φ_∞^1 coming from the dual pair $(\mathrm{O}(-V),\mathrm{SL}_2)$.

When $p \nmid D\infty$, L is unimodular, and it is well-known (see [KRY1, Section 2]) that

$$M_p(s)\Phi_\mu(g,s) = C_p(s)\Phi_\mu(g,-s) \tag{2.31}$$

with

$$C_p(s) = \frac{L_p(\chi_D,s)}{L_p(\chi_D,s+1)}. \tag{2.32}$$

When $p|D$ and $\mu = 0$, the intertwining operator is also computed in [KRY1, Sections 2, 3]. We do it here in general. Let

$$K_0(p) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p); \quad c \equiv 0 \pmod p\}.$$

Every $g \in K_0(p)$ can be written as product

$$g = n_-(pc)n(b)m(a), \quad n_-(c) = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right), \quad n(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), \quad m(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$$

with $a \in \mathbb{Z}_p^*$, $b, c \in \mathbb{Z}_p$. Let $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, then

$$\mathrm{SL}_2(\mathbb{Z}_p) = K_0(p) \cup B(\mathbb{Z}_p)wN(\mathbb{Z}_p),$$

where

$$B(\mathbb{Z}_p) = \{m(a);\ a \in \mathbb{Z}_p^*\}, \quad N(\mathbb{Z}_p) = \{n(b);\ b \in \mathbb{Z}_p\}.$$

Notice also that $\mathrm{SL}_2(\mathbb{Q}_p) = B(\mathbb{Q}_p)\,\mathrm{SL}_2(\mathbb{Z}_p)$. So $\Phi_p \in I(s,\chi_D)$ is determined by its value in $K_p = \mathrm{SL}_2(\mathbb{Z}_p)$. Recall locally that

$$\Phi_\mu(g,s) = |a(g)|^s(\omega_V(g)\phi_\mu)(0),$$

where $\phi_\mu = \mathrm{char}(\mu + L_p)$ is the p-part of ϕ_μ and $|a(g)| = |a|$ if $g = n(b)m(a)k$ with $k \in \mathrm{SL}_2(\mathbb{Z}_p)$. One can check that

$$\begin{aligned} \Phi_\mu(gn(b)) &= \psi(bQ(\mu))\Phi_\mu(g), \quad b \in \mathbb{Z}_p, \\ \Phi_\mu(gm(a)) &= \chi_D(a)\Phi_{a^{-1}\mu}(g), \quad a \in \mathbb{Z}_p^*, \\ \Phi_\mu(gn_-(pc)) &= \Phi_\mu(g), \quad c \in \mathbb{Z}_p, \\ \Phi_\mu(gw) &= \gamma_p(V)\,\mathrm{vol}(L_p)\sum_{\lambda \in L_p'/L_p} \psi_p(-(\mu,\lambda))\Phi_\lambda(g), \end{aligned} \tag{2.33}$$

since $\omega_V(g)\phi_\mu$ satisfies similar equations. Here $\psi = \prod_p \psi_p$ is the 'canonical' additive character of $\mathbb{Q}_\mathbb{A}$, and $\mathrm{vol}(L_p) = [L_p' : L_p]^{-\frac{1}{2}}$ is the measure of L_p with respect to the self-dual Haar measure on L_p (with respect to ψ_p). By definition, it is easy to see that $\tilde{\Phi}_\mu(g,-s) = M(s)\Phi_\mu(g,s) \in I_p(-s,\chi_D)$ satisfies the same equations. So both Φ_μ and $\tilde{\Phi}_\mu$ are determined by their values at $g = 1$. A simple computation gives

$$\tilde{\Phi}_\mu(1) = \gamma_p(V)\,\mathrm{vol}(L_p)\delta_{\mu,0} = \gamma_p(V)\,\mathrm{vol}(L_p)\Phi_\mu(1).$$

Since both functions satisfy the same set of equations and are determined by their values at $g = 1$, one has

$$M_p(s)\Phi_\mu(g,s) = \tilde{\Phi}_\mu(g,-s) = \gamma_p(V)\,\mathrm{vol}(L_p)\Phi_\mu(g,-s). \tag{2.34}$$

Combining (2.28)–(2.34) together with

$$\prod_{p|D} \mathrm{vol}(L_p) = D^{-\frac{1}{2}}, \qquad \prod_{p\le\infty} \gamma_p(V) = 1,$$

one sees that

$$E(g, s, \Phi^1_\infty \otimes \lambda_f(\phi_\mu)) = \frac{\Lambda(\chi_D, s)}{\Lambda(\chi_D, s+1)} E(g, -s, \Phi^1_\infty \otimes \lambda_f(\phi_\mu)).$$

This proves the proposition since $\Lambda(\chi_D, s) = \Lambda(\chi_D, 1-s)$. □

We end this section with a theorem of Schofer [Scho, Theorem 4.1], which will be used later.

Theorem 2.6. *Let the notation be as above, and let h_k be the class number of $k = \mathbb{Q}(\sqrt{D})$. Write $\chi = (D, \cdot)_\mathbb{A} = \prod_p \chi_p$ as a product of local quadratic characters. Let $\mu \in L'/L$ and $m > 0$ such that $m \in Q(\mu) + \mathbb{Z}$. Then*

$$\begin{aligned} -\Lambda(\chi_D, 1)\kappa(m, \mu) = \eta_0(m, \mu) &\sum_{p \text{ inert}} (\mathrm{ord}_p(m) + 1)\rho(m|D|/p) \log p \\ &+ \rho(m|D|) \sum_{p|D} \eta_p(m, \mu)(\mathrm{ord}_p(m) + 1) \log p \end{aligned}$$

where

$$\eta_p(m, \mu) = (1 - \chi_p(-m\,\mathrm{N}(\mathfrak{a}))) \prod_{\substack{q|D, q\ne p \\ \mu_q = 0}} (1 + \chi_q(-m\,\mathrm{N}(\mathfrak{a}))),$$

$$\eta_0(m, \mu) = \prod_{\substack{q|D \\ \mu_q = 0}} (1 + \chi_q(-m\,\mathrm{N}(\mathfrak{a}))).$$

Here we take $\eta_0(m, \mu) = 1$ and $\eta_p(m, \mu) = 0$ if $\mu_q \ne 0$ for all $q|D$. Finally,

$$\rho(n) = \#\{\mathfrak{b} \subset \mathcal{O}_k;\ \mathrm{N}(\mathfrak{b}) = n\}.$$

We also have

$$\kappa(0, 0) = \log|D| - 2\frac{\Lambda'(\chi_D, 0)}{\Lambda(\chi_D, 0)}.$$

Proof. The formula for $\kappa(0, 0)$ follows from [Scho, Lemma 2.21]. The other formula is [Scho, Theorem 4.1] when $D < -3$. Looking into his proof, the formula is true in general for $D \equiv 1 \pmod 4$ if we replace h_k by $\Lambda(\chi_D, 1)$. Notice that

$$\Lambda(\chi_D, 1) = \frac{\sqrt{|D|}}{\pi} L(\chi_D, 1) = \frac{2}{w_k} h_k. \tag{2.35}$$

Here w_k is the number of roots of unity in k. □

3. Vector valued modular forms

Let (V, Q) be a quadratic space as in Section 2, and let $L \subset V$ be an even lattice. In this section we make no restriction on the signature (b^+, b^-) of V. We consider the subspace S_L of Schwartz functions in $S(V(\mathbb{A}_f))$ which are supported on $\hat{L}' = L' \otimes \hat{\mathbb{Z}}$ and which are constant on cosets of $\hat{L}$. For any $\mu \in L'/L$, we write ϕ_μ for the characteristic function of $\mu + \hat{L}$.

We use τ as a standard variable on $\mathbb{H}$ and write u for its real and v for its imaginary part. If $f : \mathbb{H} \to S_L$ is a function, we write $f = \sum_{\mu \in L'/L} f_\mu \phi_\mu$ for its decomposition in components with respect to the standard basis (ϕ_μ) of S_L. Let $k \in \frac{1}{2}\mathbb{Z}$, and assume for simplicity that $k \equiv \frac{b^+ - b^-}{2} \pmod 2$. Let $\rho = \rho_L$ be the Weil representation of $\Gamma' = \mathrm{Mp}_2(\mathbb{Z})$ on S_L, see (2.7). We denote by $A_{k,\rho}$ the space of S_L-valued C^∞ modular forms of weight k for Γ' with representation ρ. The subspaces of weakly holomorphic modular forms (resp. holomorphic modular forms, cusp forms) are denoted by $M^!_{k,\rho}$ (resp. $M_{k,\rho}$, $S_{k,\rho}$). We need a few facts about such vector valued modular forms.

If $f \in A_{k,\rho}$ and $g \in A_{-k,\bar{\rho}}$, then the scalar valued function

$$\langle f(\tau), g(\tau) \rangle = \sum_{\mu \in L'/L} f_\mu(\tau) g_\mu(\tau) \tag{3.1}$$

is invariant under Γ'. For $f, g \in A_{k,\rho}$, we define the Petersson scalar product by

$$(f, g)_{Pet} = \int_{\mathcal{F}} \langle f, \bar{g} \rangle v^k \, d\mu(\tau), \tag{3.2}$$

provided the integral converges. Here $d\mu(\tau) = \frac{du\,dv}{v^2}$ is the invariant measure on $\mathbb{H}$, and $\mathcal{F} = \{\tau \in \mathbb{H};\ |u| \leq 1/2 \text{ and } |\tau| \geq 1\}$ denotes the standard fundamental domain for the action of Γ on $\mathbb{H}$.

Let K and L be even lattices. Then the Weil representation $\rho_{K \oplus L}$ is isomorphic to the tensor product of ρ_K and ρ_L. Moreover, if $f = \sum_{\mu \in K'/K} f_\mu \phi_\mu \in A_{k,\rho_K}$ and $g = \sum_{\nu \in L'/L} g_\nu \phi_\nu \in A_{l,\rho_L}$, then

$$f \otimes g = \sum_{\mu,\nu} f_\mu g_\nu \phi_{\mu+\nu} \in A_{k+l,\rho_{K \oplus L}}.$$

Let $M \subset L$ be a sublattice of finite index, then a vector valued modular form $f \in A_{k,\rho_L}$ can be naturally viewed as a vector valued modular form in A_{k,ρ_M}. Indeed, we have the inclusions $M \subset L \subset L' \subset M'$ and therefore

$$L/M \subset L'/M \subset M'/M.$$

We have the natural map $L'/M \to L'/L$, $\mu \mapsto \bar{\mu}$.

Lemma 3.1. *There are two natural maps*

$$\mathrm{res}_{L/M} : A_{k,\rho_L} \to A_{k,\rho_M}, \quad f \mapsto f_M$$

and

$$\mathrm{tr}_{L/M} : A_{k,\rho_M} \to A_{k,\rho_L}, \quad g \mapsto g^L$$

such that for any $f \in A_{k,\rho_L}$ *and* $g \in A_{k,\rho_M}$

$$\langle f, \bar{g}^L \rangle = \langle f_M, \bar{g} \rangle.$$

They are given as follows. For $\mu \in M'/M$ *and* $f \in A_{k,\rho_L}$*,*

$$(f_M)_\mu = \begin{cases} f_{\bar{\mu}}, & \text{if } \mu \in L'/M, \\ 0, & \text{if } \mu \notin L'/M. \end{cases}$$

For any $\bar{\mu} \in L'/L$*, and* $g \in A_{k,\rho_M}$*, let* μ *be a fixed preimage of* $\bar{\mu}$ *in* L'/M*. Then*

$$(g^L)_{\bar{\mu}} = \sum_{\alpha \in L/M} g_{\alpha+\mu}.$$

Proof. See [Sche, Proposition 6.9] for the map $\mathrm{res}_{L/M}$. The assertion for $\mathrm{tr}_{L/M}$ can be proved analogously. □

Remark 3.2. The following fact about the trace map and theta functions, which is easy to check, will be used in Section 4:

$$\theta_L = (\theta_M)^L. \tag{3.3}$$

3.1. **Harmonic weak Maass forms.** Now assume that $k \leq 1$. A twice continuously differentiable function $f : \mathbb{H} \to S_L$ is called a *harmonic weak Maass form* (of weight k with respect to Γ' and ρ_L) if it satisfies:

(i) $f \mid_{k,\rho_L} \gamma' = f$ for all $\gamma' \in \Gamma'$;
(ii) there is a S_L-valued Fourier polynomial

$$P_f(\tau) = \sum_{\mu \in L'/L} \sum_{n \leq 0} c^+(n, \mu) q^n \phi_\mu$$

such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$;
(iii) $\Delta_k f = 0$, where

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)$$

is the usual weight k hyperbolic Laplace operator (see [BF]).

The Fourier polynomial P_f is called the *principal part* of f. We denote the vector space of these harmonic weak Maass forms by H_{k,ρ_L} (it was called H^+_{k,ρ_L} in [BF]). Any weakly holomorphic modular form is a harmonic weak Maass form. The Fourier expansion of any $f \in H_{k,\rho_L}$ gives a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c^+(n, \mu) q^n \phi_\mu, \tag{3.4a}$$

$$f^-(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c^-(n, \mu) W(2\pi n v) q^n \phi_\mu, \tag{3.4b}$$

and $W(a) = W_k(a) := \int_{-2a}^{\infty} e^{-t} t^{-k}\, dt = \Gamma(1-k, 2|a|)$ for $a < 0$. We refer to f^+ as the *holomorphic part* and to f^- as the *non-holomorphic part* of f.

Recall that there is an antilinear differential operator $\xi = \xi_k : H_{k,\rho_L} \to S_{2-k,\bar\rho_L}$, defined by

$$f(\tau) \mapsto \xi(f)(\tau) := v^{k-2}\overline{L_k f(\tau)}. \tag{3.5}$$

Here L_k is the Maass lowering operator defined in (2.18). The kernel of ξ is equal to $M^!_{k,\rho_L}$. By [BF, Corollary 3.8], the sequence

$$0 \longrightarrow M^!_{k,\rho_L} \longrightarrow H_{k,\rho_L} \xrightarrow{\xi} S_{2-k,\bar\rho_L} \longrightarrow 0 \tag{3.6}$$

is exact.

There is a bilinear pairing between the spaces $M_{2-k,\bar\rho_L}$ and H_{k,ρ_L} defined by the Petersson scalar product

$$\{g, f\} = \big(g,\, \xi(f)\big)_{Pet} \tag{3.7}$$

for $g \in M_{2-k,\bar\rho_L}$ and $f \in H_{k,\rho_L}$. If g has the Fourier expansion $g = \sum_{\mu,n} b(n,\mu) q^n \phi_\mu$, and we denote the Fourier expansion of f as in (3.4), then by [BF, Proposition 3.5] we have

$$\{g, f\} = \sum_{\mu\in L'/L} \sum_{n\le 0} c^+(n,\mu) b(-n,\mu). \tag{3.8}$$

Hence $\{g, f\}$ only depends on the principal part of f. The exactness of (3.6) implies that the induced pairing between $S_{2-k,\bar\rho_L}$ and $H_{k,\rho_L}/M^!_{k,\rho_L}$ is non-degenerate.

Lemma 3.3. *Let $f \in H_{k,\rho_L}$ and assume that P_f is constant. Then $f \in M_{k,\rho_L}$.*

Proof. It follows from the assumption and (3.8) that

$$\big(\xi(f),\, \xi(f)\big)_{Pet} = \{\xi(f), f\} = 0.$$

Hence $\xi(f) = 0$ and f is weakly holomorphic. Since P_f is constant we find that $f \in M_{k,\rho_L}$. □

Lemma 3.4. *Let $\mu \in L'/L$, and let $m \in \mathbb{Q}_{>0}$ such that $m \equiv -Q(\mu) \pmod 1$. There exists a harmonic weak Maass form $f_{m,\mu} \in H_{k,\rho_L}$ whose Fourier expansion starts as*

$$f_{m,\mu}(\tau) = \frac{1}{2}(q^{-m}\phi_\mu + q^{-m}\phi_{-\mu}) + O(1), \quad v \to \infty.$$

Proof. This is an immediate consequence of [BF, Proposition 3.11]. □

4. Regularized theta integrals

Let (V, Q) be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$. We use the setup of Section 2. In particular, $L \subset V$ is an even lattice. Let $K \subset H(\mathbb{A}_f)$ be a compact open subgroup acting trivially on S_L. We consider the Shimura variety

$$X_K = H(\mathbb{Q})\backslash\big(\mathbb{D} \times H(\mathbb{A}_f)/K\big). \tag{4.1}$$

It is a quasi-projective variety of dimension n defined over $\mathbb{Q}$.

On X_K we consider the following special divisors (cf. [Bo1], [Br2], [Ku4]). We follow the description in [Ku4, pp. 304]. Let $x \in V(\mathbb{Q})$ be a vector of positive norm. We write

V_x for the orthogonal complement of x in V and H_x for the stabilizer of x in H. So $H_x \cong \mathrm{GSpin}(V_x)$. The sub-Grassmannian

$$\mathbb{D}_x = \{z \in \mathbb{D};\ z \perp x\} \tag{4.2}$$

defines an analytic divisor of $\mathbb{D}$. For $h \in H(\mathbb{A}_f)$ we consider the natural map

$$H_x(\mathbb{Q})\backslash \mathbb{D}_x \times H_x(\mathbb{A}_f)/(H_x(\mathbb{A}_f) \cap hKh^{-1}) \longrightarrow X_K, \quad (z, h_1) \mapsto (z, h_1 h). \tag{4.3}$$

Its image defines a divisor $Z(x,h)$ on X_K, which is rational over $\mathbb{Q}$. For $m \in \mathbb{Q}_{>0}$ let

$$\Omega_m = \{x \in V;\ Q(x) = m\} \tag{4.4}$$

be the corresponding quadric in V. If $\Omega_m(\mathbb{Q})$ is non-empty, then by Witt's theorem, we have $\Omega_m(\mathbb{Q}) = H(\mathbb{Q})x_0$ and $\Omega_m(\mathbb{A}_f) = H(\mathbb{A}_f)x_0$ for a fixed element $x_0 \in \Omega_m(\mathbb{Q})$. For a Schwartz function $\varphi \in S_L$, we may write

$$\operatorname{supp}(\varphi) \cap \Omega_m(\mathbb{A}_f) = \coprod_j K\xi_j^{-1} x_0 \tag{4.5}$$

as a finite disjoint union, where $\xi_j \in H(\mathbb{A}_f)$. This follows from the fact that $\operatorname{supp}(\varphi)$ is compact and $\Omega_m(\mathbb{A}_f)$ is a closed subset of $V(\mathbb{A}_f)$. We define a weighted special divisor by putting

$$Z(m, \varphi) = \sum_j \varphi(\xi_j^{-1} x_0) Z(x_0, \xi_j). \tag{4.6}$$

This definition is independent of the choice of x_0 and the representatives ξ_j. For $\mu \in L'/L$ we briefly write $Z(m,\mu) := Z(m, \phi_\mu)$. The following lemma is a special case of [Ku3, Proposition 5.4].

Lemma 4.1. *Assume that $H(\mathbb{A}_f) = H(\mathbb{Q})K$ and put $\Gamma_K = H(\mathbb{Q}) \cap K$. Then*

$$Z(m, \varphi) = \sum_{x \in \Gamma_K \backslash \Omega_m(\mathbb{Q})} \varphi(x) \operatorname{pr}(\mathbb{D}_x, 1),$$

where $\operatorname{pr} : \mathbb{D} \times H(\mathbb{A}_f) \to X_K$ *denotes the natural projection.* □

Let $f \in H_{1-n/2, \bar\rho_L}$ be a harmonic weak Maass form of weight $1 - n/2$ with representation $\bar\rho_L$ for Γ', and denote its Fourier expansion as in (3.4). We consider the regularized theta integral

$$\Phi(z, h, f) = \int_{\mathcal{F}}^{reg} \langle f(\tau), \theta_L(\tau, z, h)\rangle \, d\mu(\tau) \tag{4.7}$$

for $z \in \mathbb{D}$ and $h \in H(\mathbb{A}_f)$. The integral is regularized as in [Bo1], [BF], that is, $\Phi(z,h,f)$ is defined as the constant term in the Laurent expansion at $s = 0$ of the function

$$\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle f(\tau), \theta_L(\tau, z, h)\rangle \, v^{-s} d\mu(\tau). \tag{4.8}$$

Here $\mathcal{F}_T = \{\tau \in \mathbb{H};\ |u| \le 1/2,\ |\tau| \ge 1, \text{ and } v \le T\}$ denotes the truncated fundamental domain. The following theorem summarizes some properties of the function $\Phi(z,h,f)$ in the setup of the present paper (see [Br2], [BF]).

Theorem 4.2. *The function $\Phi(z,h,f)$ is smooth on $X_K\backslash Z(f)$, where*

$$Z(f)=\sum_{\mu\in L'/L}\sum_{m>0}c^+(-m,\mu)Z(m,\mu). \tag{4.9}$$

It has a logarithmic singularity along the divisor $-2Z(f)$. The $(1,1)$-form $dd^c\Phi(z,h,f)$ can be continued to a smooth form on all of X_K. We have the Green current equation

$$dd^c[\Phi(z,h,f)]+\delta_{Z(f)}=[dd^c\Phi(z,h,f)], \tag{4.10}$$

where δ_Z denotes the Dirac current of a divisor Z. Moreover, if Δ_z denotes the invariant Laplace operator on $\mathbb{D}$, normalized as in [Br2]*, we have*

$$\Delta_z\Phi(z,h,f)=\frac{n}{4}\cdot c^+(0,0). \tag{4.11}$$

In particular, the theorem implies that $\Phi(z,h,f)$ is a Green function for the divisor $Z(f)$ in the sense of Arakelov geometry in the normalization of [SABK]. (If the constant term $c^+(0,0)$ of f does not vanish, one actually has to work with the generalization of Arakelov geometry given in [BKK], see also [BBK].) Moreover, we see that $\Phi(z,h,f)$ is harmonic when $c^+(0,0)=0$. Therefore, it is called the *automorphic Green function* associated with $Z(f)$. Observe that the divisor $Z(f)$ has coefficients in the field of definition of the principal part of f.

In the special case when f is weakly holomorphic, $\Phi(z,h,f)$ is essentially equal to the logarithm of the Petersson metric of a Borcherds product $\Psi(z,h,f)$ on X_K. Note that $\Phi(z,h,f)$ has a finite value for *every* $z\in\mathbb{D}$, even on $Z(f)$, where it is not smooth, see [Scho]. Similar Green functions are investigated from the point of view of spherical functions on real Lie groups in [OT]. The following theorem gives a characterization of $\Phi(z,h,f)$. Although it is not needed in the rest of the paper, we include it here to provide some background.

Theorem 4.3. *Assume that the Witt rank of V over $\mathbb{Q}$ is smaller than n. Let G be a smooth real valued function on $X_K\backslash Z(f)$ with the properties:*

(i) *G has a logarithmic singularity along $-2Z(f)$,*
(ii) *$\Delta_z G=$ constant,*
(iii) *$G\in L^{1+\varepsilon}(X_K,d\mu(z))$ for some $\varepsilon>0$.*

Then $G(z,h)$ differs from $\Phi(z,h,f)$ by a constant.

Here $d\mu(z)$ is the measure on X_K induced from the Haar measure on the group $H(\mathbb{A})$. If the Witt rank of V is equal to n, one can obtain a similar characterization by also requiring growth conditions at the boundary of X_K. The constant could be fixed, for instance, by adding a condition on the value of the integral $\int_{X_K}G\,d\mu(z)$.

Idea of the proof. First, we notice that $\Phi(z,h,f)$ satisfies the properties (i)–(iii). The first two are contained in Theorem 4.2. The third can be proved using the Fourier expansion of $\Phi(z,h,f)$ (see [Br2]) and the 'curve lemma' as in [Br3, Theorem 2].

Hence the difference $G(z,h)-\Phi(z,h,f)$ is a smooth subharmonic function on the complete Riemann manifold X_K which is contained in $L^{1+\varepsilon}(X_K,d\mu(z))$. By a result of Yau, such a function must be constant (see e.g. [Br2, Corollary 4.22]). □

4.1. **CM values of automorphic Green functions.** We define CM cycles on X_K as follows. Let $U \subset V$ be a negative definite 2-dimensional rational subspace of V. It determines a two point subset $\{z_U^\pm\} \subset \mathbb{D}$ given by $U(\mathbb{R})$ with the two possible choices of orientation. Let $V_+ \subset V$ be the orthogonal complement of U over $\mathbb{Q}$. Then V_+ is a positive definite subspace of dimension n, and we have the rational splitting

$$V = V_+ \oplus U. \tag{4.12}$$

Let $T = \mathrm{GSpin}(U)$, which we view as a subgroup of H acting trivially on V_+, and put $K_T = K \cap T(\mathbb{A}_f)$. We obtain the CM cycle

$$Z(U) = T(\mathbb{Q})\backslash\big(\{z_U^\pm\} \times T(\mathbb{A}_f)/K_T\big) \longrightarrow X_K. \tag{4.13}$$

Here each point in the cycle is counted with multiplicity $\frac{2}{w_{K,T}}$, where $w_{K,T} = \#(T(\mathbb{Q})\cap K_T)$.

It is our goal to compute the value of $\Phi(z, f)$ on $Z(U)$. In the special case when f is weakly holomorphic this was done by Schofer [Scho], whose argument we will extend here. The related problem of computing the integrals of logarithms of Petersson norms of Borcherds products is considered in [Ku4], [BK].

We fix the Tamagawa Haar measure on $\mathrm{SO}(U)(\mathbb{A})$ so that $\mathrm{vol}(\mathrm{SO}(U)(\mathbb{R})) = 1$, and $\mathrm{vol}(\mathrm{SO}(U)(\mathbb{Q})\backslash \mathrm{SO}(U)(\mathbb{A}_f)) = 2$. We also fix the usual Haar measure on $\mathbb{A}_f^*$ so that $\mathrm{vol}(\mathbb{Z}_p^*) = 1$. So $\mathrm{vol}(\hat{\mathbb{Z}}^*) = 1$, and $\mathrm{vol}(\mathbb{Q}^*\backslash\mathbb{A}_f^*) = 1/2$. We then use the exact sequence

$$1 \to \mathbb{A}_f^* \to T(\mathbb{A}_f) \to \mathrm{SO}(U)(\mathbb{A}_f) \to 1$$

to define the Haar measure on $T(\mathbb{A}_f)$. The following is a special case of [Scho, Lemma 2.13].

Lemma 4.4. *With the notation as above, one has*

$$\begin{aligned}\Phi(Z(U), f) &= \frac{2}{w_{K,T}} \sum_{z\in\mathrm{supp}(Z(U))} \Phi(z, f) \\ &= \frac{\deg Z(U)}{2} \int_{h\in\mathrm{SO}(U)(\mathbb{Q})\backslash \mathrm{SO}(U)(\mathbb{A}_f)} \Phi(z_U^+, h, f)\, dh\end{aligned} \tag{4.14}$$

and

$$\deg Z(U) = \frac{4}{\mathrm{vol}(K_T)}. \tag{4.15}$$

Proof. Taking $B(h) = \Phi(z_U^\pm, h, f)$ and $B(h) = 1$ in [Scho, Lemma 2.13], one gets (4.14) and (4.15), respectively. □

Using the splitting (4.12), we obtain definite lattices

$$N = L \cap U, \quad P = L \cap V_+.$$

Then $N \oplus P \subset L$ is a sublattice of finite index. Since $\theta_{P\oplus N} = \theta_P \otimes \theta_N$, and $\theta_L = (\theta_{P\oplus N})^L$ by (3.3), Lemma 3.1 implies that

$$\langle f, \theta_L\rangle = \langle f_{P\oplus N}, \theta_P \otimes \theta_N\rangle.$$

So we may assume in the following calculation $L = P \oplus N$ if we replace f by $f_{P\oplus N}$.

For $z = z_U^\pm$ and $h \in T(\mathbb{A}_f)$, the Siegel theta function $\theta_L(\tau, z, h)$ splits up as a product

$$\theta_L(\tau, z_U^\pm, h) = \theta_P(\tau) \otimes \theta_N(\tau, z_U^\pm, h). \tag{4.16}$$

Here $\theta_P(\tau) = \theta_P(\tau, 1)$ is the holomorphic S_P-valued theta function of weight $n/2$ associated to the positive definite lattice P.

For the computation of the CM value $\Phi(Z(U), f)$ it is convenient to write the regularized theta integral as a limit of truncated integrals by means of the following lemma. If $S(q) = \sum_{n\in\mathbb{Z}} a_n q^n$ is a Laurent series in q (or a holomorphic Fourier series in τ), we write

$$\mathrm{CT}(S) = a_0 \tag{4.17}$$

for the constant term in the q-expansion.

Lemma 4.5. *If we define*

$$A_0 = \mathrm{CT}(\langle f^+(\tau),\, \theta_P(\tau) \otimes \phi_{0+N}\rangle), \tag{4.18}$$

we have

$$\Phi(z_U^\pm, h, f) = \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle f(\tau),\, \theta_P(\tau) \otimes \theta_N(\tau, z_U^\pm, h)\rangle \, d\mu(\tau) - A_0 \log(T) \right].$$

Proof. We use the splitting $f = f^+ + f^-$ of f into its holomorphic and non-holomorphic part. If we insert it into the definition (4.7), we obtain

$$\Phi(z_U^\pm, h, f) = \int_{\mathcal{F}}^{reg} \langle f^+(\tau), \theta_L(\tau, z_U^\pm, h)\rangle \, d\mu(\tau) + \int_{\mathcal{F}} \langle f^-(\tau), \theta_L(\tau, z_U^\pm, h)\rangle \, d\mu(\tau). \tag{4.19}$$

Since f^- is rapidly decreasing as $v \to \infty$, the second integral on the right hand side converges absolutely. For the first integral on the right hand side we insert the factorization (4.16) of $\theta_L(\tau, z_U^\pm, h)$ and argue as in [Scho, Proposition 2.19] or [Ku4, Proposition 2.5]. We find that it is equal to

$$\lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle f^+(\tau),\, \theta_P(\tau) \otimes \theta_N(\tau, z_U^\pm, h)\rangle \, d\mu(\tau) - A_0 \log(T) \right].$$

Adding the two contributions, we obtain the assertion. □

Lemma 4.6. *We have*

$$\Phi(Z(U), f) = \frac{\deg(Z(U))}{2} \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle f(\tau),\, \theta_P(\tau) \otimes E_N(\tau, 0; -1)\rangle \, d\mu(\tau) - 2A_0 \log(T) \right],$$

where $E_N(\tau, 0; -1)$ denotes the Eisenstein series defined in (2.16).

Proof. We insert the formula of Lemma 4.5 into the Definition (4.14). The evaluation of $\Phi(z, h, f)$ at the CM cycle is a finite sum, which may be interchanged with the limit. Consequently, the lemma follows from Lemma 4.4 and the Siegel-Weil formula, see Proposition 2.2. □

For any $g \in S_{1+n/2,\rho_L}$ we define an L-function by means of the convolution integral

$$L(g, U, s) = \big(\theta_P(\tau) \otimes E_N(\tau, s; 1),\, g(\tau)\big)_{Pet}. \tag{4.20}$$

The meromorphic continuation of the Eisenstein series $E_N(\tau, s; 1)$ leads to the meromorphic continuation of $L(g, U, s)$ to the whole complex plane. At $s = 0$, the center of symmetry, $L(g, U, s)$ vanishes because the Eisenstein series $E_N(\tau, s; 1)$ is incoherent. Proposition 2.5 gives the following simple functional equation for

$$L^*(g, U, s) := \Lambda(\chi_D, s + 1)L(g, U, s)$$

when $N \cong (\mathfrak{a}, -\frac{\mathrm{N}}{\mathrm{N}(\mathfrak{a})})$ for a fractional ideal $\mathfrak{a}$ of $k = \mathbb{Q}(\sqrt{D})$:

$$L^*(g, U, s) = -L^*(g, U, -s). \tag{4.21}$$

Let

$$g(\tau) = \sum_{\mu \in L'/L} \sum_{m>0} b(m, \mu) q^m \phi_\mu, \tag{4.22}$$

$$\theta_P(\tau) = \sum_{\mu \in P'/P} \sum_{m \geq 0} r(m, \mu) q^m \phi_\mu \tag{4.23}$$

be the Fourier expansion of g and θ_P, respectively. Using the usual unfolding argument, we obtain the Dirichlet series expansion

$$L(g, U, s) = (4\pi)^{-(s+n)/2}\Gamma\left(\tfrac{s+n}{2}\right) \sum_{m>0} \sum_{\mu \in P'/P} r(m, \mu)\overline{b(m, \mu)} m^{-(s+n)/2}. \tag{4.24}$$

Theorem 4.7. *The value of the automorphic Green function $\Phi(z, h, f)$ at the CM cycle $Z(U)$ is given by*

$$\Phi(Z(U), f) = \deg(Z(U)) \cdot \big(\mathrm{CT}\left(\langle f^+(\tau),\, \theta_P(\tau) \otimes \mathcal{E}_N(\tau)\rangle\right) + L'(\xi(f), U, 0)\big)\,.$$

Here $\mathcal{E}_N(\tau)$ denotes the function defined in (2.26)*, and $L'(\xi(f), U, s)$ the derivative with respect to s of the L-series* (4.20)*.*

Proof. In view of Lemma 4.6 we have

$$\Phi(Z(U), f) = \frac{\deg(Z(U))}{2} \lim_{T\to\infty} \left[I_T(f) - 2A_0 \log(T)\right], \tag{4.25}$$

where

$$I_T(f) := \int_{\mathcal{F}_T} \langle f(\tau),\, \theta_P(\tau) \otimes E_N(\tau, 0; -1)\rangle \, d\mu(\tau).$$

We compute $I_T(f)$ combining the ideas of [Scho] and [BF]. According to Lemma 2.3, we have

$$\begin{aligned} I_T(f) &= -2 \int_{\mathcal{F}_T} \langle f(\tau),\, \theta_P(\tau) \otimes \bar\partial(E'_N(\tau, 0; 1)\, d\tau)\rangle \\ &= -2 \int_{\mathcal{F}_T} d\langle f(\tau),\, \theta_P(\tau) \otimes E'_N(\tau, 0; 1)\, d\tau\rangle + 2 \int_{\mathcal{F}_T} \langle(\bar\partial f),\, \theta_P(\tau) \otimes E'_N(\tau, 0; 1)\, d\tau\rangle. \end{aligned}$$

Using Stokes' theorem and the definition of the Maass lowering operator, we get

$$\begin{aligned} I_T(f) &= -2\int_{\partial\mathcal{F}_T} \langle f(\tau),\, \theta_P(\tau)\otimes E_N'(\tau,0;1)\,d\tau\rangle \\ &\quad + 2\int_{\mathcal{F}_T} \langle L_{1-n/2}f,\, \theta_P(\tau)\otimes E_N'(\tau,0;1)\rangle\,d\mu(\tau) \\ &= 2\int_{\tau=iT}^{iT+1} \langle f(\tau),\, \theta_P(\tau)\otimes E_N'(\tau,0;1)\rangle\,d\tau \\ &\quad + 2\int_{\mathcal{F}_T} \langle\overline{\xi(f)},\, \theta_P(\tau)\otimes E_N'(\tau,0;1)\rangle\,v^{1+n/2}d\mu(\tau). \end{aligned}$$

If we insert this formula into (4.25), we obtain

$$\begin{aligned} \Phi(Z(U),f) = \deg(Z(U)) \lim_{T\to\infty}\left[\int_{\tau=iT}^{iT+1} \langle f(\tau),\, \theta_P(\tau)\otimes E_N'(\tau,0;1)\rangle\,d\tau - A_0\log(T)\right] \\ + \deg(Z(U))\int_{\mathcal{F}} \langle\overline{\xi(f)},\, \theta_P(\tau)\otimes E_N'(\tau,0;1)\rangle\,v^{1+n/2}d\mu(\tau). \end{aligned}$$

The second summand on the right hand side leads to $L'(\xi(f),U,0)$ via the integral representation (4.20). For the first summand, we may replace f by its holomorphic part f^+, since f^- is rapidly decreasing as $v\to\infty$. Inserting the Fourier expansion of $E_N'(\tau,0;1)$ and the definition of A_0, we get

$$\begin{aligned} &\lim_{T\to\infty}\left[\int_{\tau=iT}^{iT+1} \langle f(\tau),\, \theta_P(\tau)\otimes E_N'(\tau,0;1)\rangle\,d\tau - A_0\log(T)\right] \\ &= \lim_{T\to\infty}\int_{\tau=iT}^{iT+1} \left\langle f^+(\tau),\, \theta_P(\tau)\otimes \sum_{\mu\in N'/N}\sum_{m\in\mathbb{Q}} \big(b_\mu(m,v) - \delta_{\mu,0}\delta_{m,0}\log(v)\big)q^m\phi_\mu \right\rangle d\tau. \end{aligned}$$

Here $\delta_{*,*}$ denotes the Kronecker delta. The limit is equal to

$$\mathrm{CT}\left(\langle f^+(\tau),\, \theta_P(\tau)\otimes\mathcal{E}_N(\tau)\rangle\right).$$

This conclude the proof of the theorem. □

In the special case when f is weakly holomorphic we have $\xi(f)=0$. Hence the L-function term vanishes and the above formula reduces to [Scho, Theorem 1.1].

Remark 4.8. If the principal part P_f is constant, then

$$\Phi(Z(U),f) = \deg(Z(U))c^+(0,0)\kappa(0,0).$$

Proof. In view of Lemma 3.3 the assumption implies that $f\in M_{1-n/2,\bar\rho_L}$. Hence $\xi(f)=0$ and the assertion follows. □

5. Faltings heights of CM cycles

Let $\mathcal{X} \to \operatorname{Spec}(\mathbb{Z})$ be an arithmetic variety, that is, a regular scheme which is projective and flat over $\mathbb{Z}$, of relative dimension n. Let $\mathrm{Z}^d(\mathcal{X})$ denote the group of codimension d cycles on $\mathcal{X}$. Recall that an arithmetic divisor on $\mathcal{X}$ is a pair $\hat{x} = (x, g_x)$ of a divisor x on $\mathcal{X}$ and a Green function g_x for the divisor $x(\mathbb{C})$ induced by x on the complex variety $\mathcal{X}(\mathbb{C})$. So g_x is a smooth real function on $\mathcal{X}(\mathbb{C}) \setminus x(\mathbb{C})$ with a logarithmic singularity on $x(\mathbb{C})$ satisfying the current equation

$$dd^c[g_x] + \delta_{x(\mathbb{C})} = [\omega_x]$$

with a smooth $(1,1)$-form ω_x on $\mathcal{X}(\mathbb{C})$. We write $\widehat{\mathrm{CH}}^1(\mathcal{X})$ for the first arithmetic Chow group of $\mathcal{X}$, that is, the free abelian group generated by the arithmetic divisors on $\mathcal{X}$ modulo rational equivalence, see [SABK]. Moreover, if $F \subset \mathbb{C}$ is a subfield we put

$$\widehat{\mathrm{CH}}^1(\mathcal{X})_F = \widehat{\mathrm{CH}}^1(\mathcal{X}) \otimes_{\mathbb{Z}} F.$$

Recall from [BGS, Section 2.3] that there is a height pairing

$$\widehat{\mathrm{CH}}^1(\mathcal{X}) \times \mathrm{Z}^n(\mathcal{X}) \longrightarrow \mathbb{R}.$$

When $\hat{x} = (x, g_x) \in \widehat{\mathrm{CH}}^1(\mathcal{X})$ and $y \in \mathrm{Z}^n(\mathcal{X})$ such that x and y intersect properly, it is defined by

$$\langle \hat{x}, y\rangle_{Fal} = \langle x, y\rangle_{fin} + \langle \hat{x}, y\rangle_\infty,$$

where

$$\langle \hat{x}, y\rangle_\infty = \frac{1}{2} g_x(y(\mathbb{C})),$$

and $\langle x, y\rangle_{fin}$ denotes the intersection pairing at the finite places. When x and y do not intersect properly, one defines the pairing by replacing $\hat{x}$ by a suitable arithmetic divisor which is rationally equivalent. The quantity $\langle \hat{x}, y\rangle_{Fal}$ is called the Faltings height of y with respect to $\hat{x}$ (see also [BKK, §6.3]). When $y \in \mathrm{Z}^n(\mathcal{X})$ is a horizontal cycle, the condition that x and y intersect properly means that on the generic fiber $y(\mathbb{C})$ is disjoint from $x(\mathbb{C})$.

Theorem 4.7 and the examples of the next sections lead to the following conjectures. We are quite vague here and ignore various difficult technical problems regarding integral models. Assume that there is a regular scheme $\mathcal{X}_K \to \operatorname{Spec}\mathbb{Z}$, projective and flat over $\mathbb{Z}$, whose associated complex variety is a smooth compactification X_K^c of X_K. Let $\mathcal{Z}(m,\mu)$ and $\mathcal{Z}(U)$ be suitable extensions to $\mathcal{X}_K$ of the cycles $Z(m,\mu)$ and $Z(U)$, respectively. Such extensions can be found in many cases using a moduli interpretation of $\mathcal{X}_K$, see e.g. [Ku5], [KRY2]. (When $n > 0$ one can often also take the flat closures in $\mathcal{X}_K$ of $Z(m,\mu)$ and $Z(U)$, respectively.) For an $f \in H_{1-n/2,\bar{\rho}_L}$, the function $\Phi(\cdot, f)$ is a Green function for the divisor $Z(f)$. Set $\mathcal{Z}(f) = \sum_\mu \sum_{m>0} c^+(-m,\mu)\mathcal{Z}(m,\mu)$. Then the pair

$$\hat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(\cdot, f))$$

defines an arithmetic divisor in $\widehat{\mathrm{CH}}^1(\mathcal{X}_K)_{\mathbb{C}}$. (When X_K is non-compact, one has to add suitable components to the divisor $\mathcal{Z}(f)$ which are supported at the boundary, see Section 7. Moreover, if the constant term $c^+(0,0)$ of f does not vanish, one actually has to work with the generalized arithmetic Chow groups defined in [BKK].) Theorem 4.7 provides a formula for the quantity

$$\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(U) \rangle_\infty = \frac{1}{2}\Phi(Z(U), f). \tag{5.1}$$

Now suppose that f is weakly holomorphic with coefficients $c^+(m,\mu) \in \mathbb{Z}$ for all $m \le 0$ and constant term $c^+(0,0) = 0$. Then $\hat{\mathcal{Z}}(f)$ should be rationally equivalent to a torsion element, the relation being given by the Borcherds lift of f. Assuming this, we would have

$$0 \overset{??}{=} \langle \hat{\mathcal{Z}}(f), \mathcal{Z}(U) \rangle_{Fal} = \langle \mathcal{Z}(f), \mathcal{Z}(U) \rangle_{fin} + \frac{1}{2}\Phi(Z(U), f).$$

Theorem 4.7 then implies that

$$\langle \mathcal{Z}(f), \mathcal{Z}(U) \rangle_{fin} \overset{??}{=} -\frac{\deg(Z(U))}{2} \operatorname{CT}\left(\langle f^+(\tau),\, \theta_P(\tau) \otimes \mathcal{E}_N(\tau) \rangle \right). \tag{5.2}$$

Expanding both sides would suggest the following conjecture on the arithmetic intersection.

Conjecture 5.1. *Let $\mu \in L'/L$, and let $m \in Q(\mu)+\mathbb{Z}$ be positive. Then $\langle \mathcal{Z}(m,\mu), \mathcal{Z}(U) \rangle_{fin}$ is equal to $-\frac{\deg Z(U)}{2}$ times the (m,μ)-th Fourier coefficient of $\theta_P \otimes \mathcal{E}_N$, that is,*

$$\langle \mathcal{Z}(m,\mu), \mathcal{Z}(U) \rangle_{fin} = -\frac{\deg(Z(U))}{2} \sum_{\substack{\mu_1 \in P'/P \\ \mu_2 \in N'/N \\ \mu_1+\mu_2 \equiv \mu\ (L)}} \sum_{\substack{m_i \in \mathbb{Q}_{\ge 0} \\ m_1+m_2=m}} r(m_1,\mu_1)\kappa(m_2,\mu_2).$$

Here $r(m,\mu)$ is the (m,μ)-coefficient of θ_P, and $\kappa(m,\mu)$ is the (m,μ)-th coefficient of $\mathcal{E}_N$.

This conjecture and Theorem 4.7 would imply the following conjecture.

Conjecture 5.2. *For any $f \in H_{1-n/2,\bar{\rho}_L}$, one has*

$$\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(U) \rangle_{Fal} = \frac{\deg(Z(U))}{2} \left(c^+(0,0)\kappa(0,0) + L'(\xi(f), U, 0) \right). \tag{5.3}$$

In view of Lemma 3.4, for $\mu \in L'/L$ and $m \in Q(\mu) + \mathbb{Z}$ positive, there is an $f_{m,\mu} \in H_{1-n/2,\bar{\rho}_L}$ such that $\mathcal{Z}(f_{m,\mu}) = \mathcal{Z}(m,\mu)$. Evaluating Conjecture 5.2 for $\hat{\mathcal{Z}}(f_{m,\mu})$ and using Theorem 4.7, we see that the two conjectures are equivalent.

Conjecture 5.2 has also the following consequence. Let $Q_- \subset P(V(\mathbb{C}))$ be defined by

$$Q_- = \{ w \in V(\mathbb{C});\ (w,w) = 0,\ (w,\bar{w}) < 0 \}/\mathbb{C}^*. \tag{5.4}$$

It is isomorphic to $\mathbb{D}$ via $w = v_1 - v_2 i$ maps to the oriented negative 2-plane z with oriented $\mathbb{R}$-basis $\{v_1, v_2\}$ (see e.g. [Bo1], [Br2], and [Ku4]). The restriction to Q_- of the tautological line bundle on $P(V(\mathbb{C}))$ induces a line bundle ω on X_K. In certain low dimensional cases it can be identified with the Hodge bundle. We define the Petersson metric on ω via

$$\|w\|^2_{Pet} := -\frac{a}{2}(w,\bar{w}). \tag{5.5}$$

Here $a := 2\pi e^{-\gamma}$ is a normalizing factor (which is inserted to make (5.6) hold without any extra constant), and $\gamma = -\Gamma'(1)$ is Euler's constant. Rational sections of ω^k can be identified with meromorphic modular forms $\Psi(z,h)$ of weight k and level K for $\mathrm{SO}(V)$.

The Petersson metric coincides with the usual Petersson metric for a modular form (due to our choice of the normalizing factor a). Let us assume that it has an integral model which we still denote by ω. Using the Petersson metric, we get a metrized line bundle $\hat{\omega} = (\omega, \|\cdot\|_{Pet})$. An integral modular form Ψ of weight k can be viewed as a section of ω, and we have

$$k\hat{c}_1(\hat{\omega}) = (\operatorname{div}\Psi, -\log\|\Psi\|^2_{Pet}) \in \widehat{\operatorname{CH}}^1(\mathcal{X}_K). \tag{5.6}$$

Now let f be a weakly holomorphic modular form for Γ' with $c^+(0,0) \neq 0$ and $c^+(m,\mu) \in \mathbb{Z}$ for $m \leq 0$. Let $\Psi(z,h,f)$ be its Borcherds lifting which is a meromorphic modular form of weight $c^+(0,0)/2$ for $\operatorname{SO}(V)$ of level K, see [Bo1]. Then we have

$$\Phi(z,h,f) = -2\log\|\Psi(z,h,f)\|^2_{Pet},$$

see [Bo1, Theorem 13.3]. Consequently,

$$c^+(0,0)\hat{c}_1(\hat{\omega}) = \hat{\mathcal{Z}}(f).$$

So Conjecture 5.2 leads to the following conjecture.

Conjecture 5.3. *One has*

$$\frac{1}{\deg Z(U)}\langle\hat{\omega}, \mathcal{Z}(U)\rangle_{Fal} = \frac{1}{2}\kappa(0,0).$$

It is interesting that the right hand side depends only on K_T. When $K_T \cong \hat{\mathcal{O}}_D^*$ for a fundamental discriminant $D < 0$,

$$\kappa(0,0) = \log|D| - 2\frac{\Lambda'(\chi_D,0)}{\Lambda(\chi_D,0)} = 4h_{Fal}(E)$$

is four times the Faltings height of an elliptic curve with complex multiplication by $\mathcal{O}_D$. This follows from the Chowla-Selberg formula as reformulated by Colmez [Co]. When $n = 1$ and $X_K = Y_0(N)$, the sections of ω^k actually correspond to weight $2k$ modular forms in the usual sense, and $\mathcal{Z}(U)$ is the moduli stack of CM elliptic curves with CM by $\mathcal{O}_D$ (see Section 7). In this case, the conjecture is simply the the celebrated Chowla-Selberg formula just mentioned. When $n = 2$, and X_K is a Hilbert modular surface, sections of ω^k correspond to weight k Hilbert modular forms, and the left hand side of the conjecture is the Faltings height of a CM abelian surface of the CM type (K,Φ) where $K = \mathbb{Q}(\sqrt{\Delta},\sqrt{D})$ is a biquadratic CM quartic field with real quadratic subfield $F = \mathbb{Q}(\sqrt{\Delta})$, and $\Phi = \operatorname{Gal}(K/k_D)$ as a CM type of K. In this case, the conjecture is a special case of Colmez' conjecture and follows from the Chowla-Selberg formula (see for example [Ya1, Proposition 3.3]).

Remark 5.4. In view of Conjectures 5.2 and 5.3, we can consider the degree 0 divisor

$$\hat{\mathcal{Z}}(f)^0 = \hat{\mathcal{Z}}(f) - c^+(0,0)\hat{\omega}.$$

Then Conjecture 5.2 (assuming Conjecture 5.3) becomes

$$\langle\hat{\mathcal{Z}}(f)^0, \mathcal{Z}(U)\rangle_{Fal} = \frac{\deg(Z(U))}{2}L'(\xi(f),U,0). \tag{5.7}$$

6. The $n = 0$ case

Here we consider the case $n = 0$ where V is negative definite of dimension 2. Then $U = V$ and the even Clifford algebra $C^0(V)$ of V is an imaginary quadratic field $k = \mathbb{Q}(\sqrt{D})$. For simplicity we assume that $(L, Q) \cong (\mathfrak{a}, -\frac{\mathrm{N}}{\mathrm{N}(\mathfrak{a})})$ for a fractional ideal $\mathfrak{a} \subset k$ as in the end of Section 2. So $L' = \partial^{-1}\mathfrak{a}$. In this case $H = T = \mathrm{GSpin}(V) = k^*$. We take

$$K = K_T = \hat{\mathcal{O}}_k^*, \tag{6.1}$$

which acts on L'/L trivially. So

$$X_K = Z(U) = k^*\backslash\{z_U^\pm\} \times k_f^*/\hat{\mathcal{O}}_k^* = \{z_U^\pm\} \times \mathrm{Cl}(k)$$

is the union of two copies of the ideal class group $\mathrm{Cl}(k)$ (a finite collection of points). It has the following integral model over $\mathbb{Z}$.

Let $\mathcal{C}$ be the moduli stack over $\mathbb{Z}$ representing the moduli problem which assigns to every scheme S over $\mathbb{Z}$ the set $\mathcal{C}(S)$ of the CM elliptic curves (E, ι) where E is an elliptic curve over S and $\iota : \mathcal{O}_k \hookrightarrow \mathrm{End}_S(E) =: \mathcal{O}_E$ is an $\mathcal{O}_k$-action on E such that the main involution on $\mathcal{O}_E$ gives the complex conjugation on k. Indeed, let $\mathcal{C}^+$ be the moduli stack over $\mathcal{O}_k$ defined in [KRY1], representing the moduli problem which assigns to every scheme S over $\mathcal{O}_k$ the set $\mathcal{C}^+(S)$ of CM elliptic curves (E, ι) over S such that the CM action $\iota : \mathcal{O}_k \hookrightarrow \mathcal{O}_E$ gives rise to the structure map $\mathcal{O}_k \to \mathcal{O}_S$ on the lie algebra $\mathrm{Lie}(E)$. Then $\mathcal{C}$ is the restriction of coefficients of $\mathcal{C}^+$ in the sense of Grothendieck, i.e., it is $\mathcal{C}^+$ but viewed as a stack over $\mathbb{Z}$: $\mathcal{C} = (\mathcal{C}^+ \to \mathrm{Spec}(\mathcal{O}_k) \to \mathrm{Spec}(\mathbb{Z}))$.

Lemma 6.1. *One has a bijective map between $\mathcal{C}(\mathbb{C})$ and X_K.*

Proof. It is well-known that every elliptic curve with CM by $\mathcal{O}_k$ over $\mathbb{C}$ is isomorphic to $E_\mathfrak{a} = \mathbb{C}/\mathfrak{a}$ for some fractional ideal $\mathfrak{a}$ of k, and that the isomorphism class of $E_\mathfrak{a}$ depends only on the ideal class of $\mathfrak{a}$. On the other hand, $E_\mathfrak{a}$ has two $\mathcal{O}_k$-actions induced by

$$\iota_+(r)z = rz, \quad \iota_-(r)z = \bar{r}z,$$

respectively. So $(z_U^\pm, [\mathfrak{a}]) \mapsto (E_\mathfrak{a}, \iota_\pm)$ gives a bijection between X_K and $\mathcal{C}(\mathbb{C})$. ☐

For $(E, \iota) \in \mathcal{C}(S)$, let

$$V(E, \iota) = \{x \in \mathcal{O}_E;\ \iota(\alpha)x = x\iota(\bar{\alpha}) \text{ for all } \alpha \in \mathcal{O}_k, \text{ and } \mathrm{tr}\, x = 0\} \tag{6.2}$$

be the space of 'special endomorphisms' with the definite quadratic form $\mathrm{N}(x) := \deg x = -x^2$, see [KRY1], [KY1]. When $S = \mathrm{Spec}(F)$ for an algebraically closed field F, then $V(E, \iota)$ is empty if $F = \mathbb{C}$ or $F = \bar{\mathbb{F}}_p$ for a prime p which is split in k. When p is non-split in k, then $\mathcal{O}_E$ is a maximal order of the unique quaternion algebra $\mathbb{B}$ which is ramified exactly at p and ∞. In this case $V(E, \iota)$ is a positive definite lattice of rank 2 and $\mathrm{N}(x)$ is the reduced norm of x.

For $\mu \in L'/L = \partial^{-1}\mathfrak{a}/\mathfrak{a}$ and $m \in \mathbb{Q}_{>0}$, consider the moduli problem which assigns to every scheme S (over $\mathbb{Z}$) the set $\mathcal{Z}(S)$ of triples $(E, \iota, \boldsymbol{\beta})$ where

(i) $(E, \iota) \in \mathcal{C}(S)$, and

(ii) $\boldsymbol{\beta} \in V(E,\iota)\partial^{-1}\mathfrak{a}$ such that

$$\mathrm{N}\,\boldsymbol{\beta} = m\,\mathrm{N}\,\mathfrak{a}, \quad \mu + \boldsymbol{\beta} \in \mathcal{O}_E\mathfrak{a}.$$

It is empty unless $m \in Q(\mu) + \mathbb{Z}$.

Lemma 6.2. *Let the notation be as above, and assume that $m \in Q(\mu) + \mathbb{Z}$. Then the above moduli problem is represented by an algebraic stack $\mathcal{Z}(m, \mathfrak{a}, \mu)$ of dimension 0. Furthermore, the forgetful map $(E, \iota, \boldsymbol{\beta}) \mapsto (E, \iota)$ is a finite étale map from $\mathcal{Z}(m, \mathfrak{a}, \mu)$ into $\mathcal{C}$.*

We will view $\mathcal{Z}(m, \mathfrak{a}, \mu)$ as a cycle in $\mathcal{C}$ by identifying it with its direct image under the forgetful map. It is supported at finitely many primes which are non-split in k.

Proof. Consider the similar moduli problem which assigns to each scheme S over $\mathcal{O}_k$ the set $\mathcal{Z}^+(S)$ of the triples $(E, \iota, \boldsymbol{\beta})$ where $(E, \iota) \in \mathcal{C}^+(S)$ and $\boldsymbol{\beta}$ satisfies the same conditions as above. Choose a $\lambda \in \mathfrak{a}^{-1}/\partial\mathfrak{a}^{-1}$ such that the multiplication by λ gives an isomorphism

$$\partial^{-1}\mathfrak{a}/\mathfrak{a} \cong \partial^{-1}/\mathcal{O}_k, \quad x \mapsto \lambda x.$$

Then $\mathcal{Z}^+(S)$ consists of the triples $(E, \iota, \boldsymbol{\beta})$ where $(E, \iota) \in \mathcal{C}^+(S)$,

$$\boldsymbol{\beta} \in V(E,\iota)(\partial\mathfrak{a}^{-1})^{-1}, \quad \mathrm{N}(\partial\mathfrak{a}^{-1})\,\mathrm{N}\,\boldsymbol{\beta} = m|D|,$$

and

$$\lambda\mu + \bar{\lambda}\boldsymbol{\beta} \in \mathcal{O}_E.$$

It is proved in [KY1] that this moduli problem is represented by a DM-stack $\mathcal{Z}^+(m, \mathfrak{a}, \mu)$ (denoted there by $\mathcal{Z}(m|D|, \partial\mathfrak{a}^{-1}, \bar{\lambda}, \lambda\mu)$). Let $\mathcal{Z}(m, \mathfrak{a}, \mu)$ be the restriction of coefficients of $\mathcal{Z}^+(m, \mathfrak{a}, \mu)$, then $\mathcal{Z}(m, \mathfrak{a}, \mu)$ represents the moduli problem $S \mapsto \mathcal{Z}(S)$. The forgetful map is clearly a finite étale map. □

Following [KRY2, Section 2], we define the arithmetic degree of a 0-dimensional DM-stack $\mathcal{Z}$ as

$$\widehat{\deg}(\mathcal{Z}) = \sum_p \sum_{x \in \mathcal{Z}(\bar{\mathbb{F}}_p)} \frac{1}{\#\mathrm{Aut}(x)} i_p(\mathcal{Z}, x) \log p. \tag{6.3}$$

Here $i_p(\mathcal{Z}, x)$ is defined as follows. Let $\tilde{\mathcal{O}}_{\mathcal{Z},x}$ be the strictly Henselian local ring of $\mathcal{Z}$ at x, then

$$i_p(x) = \mathrm{Length}(\tilde{\mathcal{O}}_{\mathcal{Z},x})$$

is the length of the local Artin ring $\tilde{\mathcal{O}}_{\mathcal{Z},x}$. It is well-known that

$$\widehat{\deg}(\mathcal{Z}) = \widehat{\deg}(\mathrm{cRes}_{\mathcal{O}_K/\mathbb{Z}}\,\mathcal{Z}) \tag{6.4}$$

for a DM-stack $\mathcal{Z}$ over the ring of integers $\mathcal{O}_K$ of some number field K, where $\mathrm{cRes}_{\mathcal{O}_K/\mathbb{Z}}\,\mathcal{Z}$ is the restriction of coefficients of $\mathcal{Z}$. In particular, one has

$$\widehat{\deg}(\mathcal{Z}(m, \mathfrak{a}, \mu)) = \widehat{\deg}(\mathcal{Z}^+(m, \mathfrak{a}, \mu)). \tag{6.5}$$

It is also well-known that

$$\widehat{\deg}(\mathcal{Z}) = \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}(\mathcal{Z} \otimes_{\mathbb{Z}} \mathcal{O}_K) \tag{6.6}$$

for a DM-stack $\mathcal{Z}$ over $\mathbb{Z}$.

Lemma 6.3. *Let $w_k = \#\mathcal{O}_k^*$. We have*

$$\frac{1}{4}\deg Z(U) = \frac{1}{\operatorname{vol}(K_T)} = \frac{h_k}{w_k} = \frac{\sqrt{|D|}}{2\pi} L(\chi_D, 1).$$

Proof. Recall that $T = k^*$ and $K = \hat{\mathcal{O}}_k^*$ in our case. Hence $w_{K,T} = w_k$. Moreover, recall our Haar measure choice just before Lemma 4.4. Since

$$1 \to \mathbb{Q}^* \backslash \mathbb{Q}_f^* \to k^* \backslash k_f^* \to k^1 \backslash k_f^1 \to 1$$

is exact, and $\operatorname{vol}(\mathbb{Q}^* \backslash \mathbb{Q}_f^*) = 1/2$, we see that

$$\operatorname{vol}(k^* \backslash k_f^*) = \operatorname{vol}(\mathbb{Q}^* \backslash \mathbb{Q}_f^*) \operatorname{vol}(k^1 \backslash k_f^1) = 1.$$

On the other hand, we have

$$\int_{k^* \backslash k_f^*} d^*x = \int_{k^* \backslash k_f^* / \hat{\mathcal{O}}_k^*} \int_{\mathcal{O}_k^* \backslash \hat{\mathcal{O}}_k^*} d^*x = \frac{h_k}{w_k} \operatorname{vol}(\hat{\mathcal{O}}_k^*).$$

Hence $\operatorname{vol}(K_T) = \operatorname{vol}(\hat{\mathcal{O}}_k^*) = \frac{w_k}{h_k}$, and the assertion follows from (2.35). □

Conjecture 5.1 is just the following theorem in this special case, which is a reformulation of [KY1, Corollary 7.9].

Theorem 6.4. *Let the notation be as above and assume that D is odd. Then*

$$\widehat{\deg}(\mathcal{Z}(m, \mathfrak{a}, \mu)) = -\frac{\deg Z(U)}{2} \kappa(m, \mu).$$

Here $\deg Z(U)$ is given explicitly by Lemma 6.3.

Sketch of the proof. For the convenience of the reader, we derive the theorem from [KY1, Corollary 7.9], which is restated as formula (6.9) below. For a prime p which is inert or ramified in k, let $\mathbb{B}$ be the unique quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞. Choose a prime $p_0 \nmid 2pD$ (depending on p) satisfying

$$\operatorname{inv}_l \mathbb{B} = \begin{cases} (D, -p_0 p)_l, & \text{if } p \text{ is inert in } k, \\ (D, -p_0)_l, & \text{if } p \text{ is ramified in } k \end{cases} \tag{6.7}$$

for every prime l. Here $\operatorname{inv}_l \mathbb{B} = \pm 1$ depends on whether $\mathbb{B}$ is a matrix algebra or a division algebra.

In particular, $p_0 = \mathfrak{p}_0 \bar{\mathfrak{p}}_0$ is split in k. For an ideal $\mathfrak{b}$ of k, let $[\mathfrak{b}]$ be the ideal class of $\mathbb{B}$ and $[[\mathfrak{b}]]$ be its associated genus, i.e., the set of (fractional) ideals $\alpha \mathfrak{c}^2 \mathfrak{b}$. Moreover, let

$$\rho(n, [[\mathfrak{b}]]) = \#\{\mathfrak{c} \subset \mathcal{O}_k;\ \mathfrak{a} \in [[\mathfrak{b}]],\quad \mathrm{N}\,\mathfrak{c} = n\}. \tag{6.8}$$

Notice that it is equal to

$$\rho(n) = \#\{\mathfrak{c} \subset \mathcal{O}_k;\ \mathrm{N}\,\mathfrak{c} = n\}$$

if it is non-zero. In [KY1], Kudla and the second author proved the following formula:

$$\widehat{\deg}(\mathcal{Z}(m,\mathfrak{a},\mu)) = 2^{o(\mu)} \Bigg[\sum_{p \text{ inert}} (\operatorname{ord}_p m + 1)\rho(m|D|/p, [[\mathfrak{p}_0 \partial \bar{\mathfrak{a}}]]) \log p$$

$$+ \sum_{p|D,\ \mu_p=0} (\operatorname{ord}_p m + 1)\rho(m|D|/p, [[\mathfrak{p}_0 \mathfrak{p}^{-1} \partial \bar{\mathfrak{a}}]]) \Bigg]. \tag{6.9}$$

Here we decompose

$$\partial^{-1}\mathfrak{a}/\mathfrak{a} = \oplus_{p|D} (\partial^{-1}\mathfrak{a}/\mathfrak{a}) \otimes \mathbb{Z}_p, \quad \mu = (\mu_p)_{p|D},$$

and $o(\mu) = \#\{p|D;\ \mu_p = 0\}$. Comparing this with Theorem 2.6, one sees that it suffices to verify that for positive $m \in Q(\mu) + \mathbb{Z} = -\frac{\mu\bar{\mu}}{\mathrm{N}\,\mathfrak{a}} + \mathbb{Z}$ we have

$$2^{o(\mu)} \rho(m|D|/p, [[\mathfrak{p}_0 \partial \bar{\mathfrak{a}}]]) = \eta_0(m,\mu)\rho(m|D|/p) \tag{6.10}$$

when p is inert in k, and

$$2^{o(\mu)} \rho(m|D|/p, [[\mathfrak{p}_0 \partial^{-1} \partial \bar{\mathfrak{a}}]]) = \eta_p(m,\mu)\rho(m|D|) \tag{6.11}$$

when p is ramified in k and $\mu_p = 0$. For $p|D$, let ξ_p be the genus character of $\mathrm{Cl}(k)/\mathrm{Cl}(k)^2$ given by

$$\xi_p([\mathfrak{b}]) = \chi_p(\mathrm{N}\,\mathfrak{b}) = (D, \mathrm{N}\,\mathfrak{b})_p.$$

Then $\mathfrak{c} \in [[\mathfrak{b}]]$ if and only if $\xi_p(\mathrm{N}\,\mathfrak{c}) = \xi_p(\mathrm{N}\,\mathfrak{b})$ for all $p|D$. So just as

$$\rho(n) = \prod_{l<\infty} \rho_l(n),$$

one has

$$\rho(n, [[\mathfrak{b}]]) = \prod_{l \nmid D\infty} \rho_l(n) \prod_{l|D} \rho_l(n, [[\mathfrak{b}]]),$$

where $\rho_l(n, [[\mathfrak{b}]]) = 1$ or 0 depending on whether there is an integral ideal $\mathfrak{c}$ such that $\mathrm{N}\,\mathfrak{c} = n$ and $\xi_l(n) = \xi_l(\mathrm{N}\,\mathfrak{b})$, and

$$\rho_l(n) = \begin{cases} 1, & \text{if } l|D, \\ \frac{1+(-1)^{\operatorname{ord}_l n}}{2}, & \text{if } \chi_p(l) = -1, \\ \operatorname{ord}_l n + 1, & \text{if } \chi_p(l) = 1. \end{cases}$$

To see (6.10), we may assume that there is an integral ideal $\mathfrak{c}$ with $\mathrm{N}\,\mathfrak{c} = m|D|/p$ (otherwise both sides are zero). For any $l|D$, one has by (6.7)

$$\begin{aligned} \xi_l(\frac{m|D|}{p} \mathrm{N}(\mathfrak{p}_0 \partial \bar{\mathfrak{a}})) &= (D, mpp_0 \,\mathrm{N}\,\mathfrak{a})_l \\ &= (D, -m \,\mathrm{N}\,\mathfrak{a})_l. \end{aligned} \tag{6.12}$$

When $\mu_l \neq 0$, $\mu\bar{\mu} \notin \mathbb{Z}_l$, and $-m\,\mathrm{N}\,\mathfrak{a} \in \mu\bar{\mu} + \mathbb{Z}_l$. So

$$-m\,\mathrm{N}\,\mathfrak{a}|D| \in \mu\bar{\mu}|D| + \mathbb{Z}_l|D|$$

and $\mu\bar{\mu}|D| \in \mathbb{Z}_l^*$ (note that $l \neq 2$, since D is odd). We may assume that $\mathfrak{a}$ is prime to ∂, so $\mathrm{N}\,\mathfrak{a}$ does not interfere here. Hence

$$\xi_l(\frac{m|D|}{p}\,\mathrm{N}(\mathfrak{p}_0\partial\bar{\mathfrak{a}})) = (D, -m\,\mathrm{N}\,\mathfrak{a}|D|)_l = (D, \mu\bar{\mu}|D|)_l = 1.$$

That is $\rho_l(m|D|/p, [[\mathfrak{p}_0\partial\bar{\mathfrak{a}}]]) = 1$ when $\mu_l \neq 0$. When $\mu_l = 0$, (6.12) implies that

$$\rho_l(m|D|/p, [[\mathfrak{p}_0\partial\bar{\mathfrak{a}}]]) = \frac{1}{2}(1 + \chi_l(-m\,\mathrm{N}\,\mathfrak{a})).$$

This proves (6.10). The verification of (6.11) is the same plus the fact $\rho(m|D|) = \rho(m|D|/p)$ for $p|\gcd(m|D|, |D|)$. □

Notice that the L-function $L(\xi(f), U, s)$ vanishes identically, since it is given as the Petersson scalar product of a cusp form and an Eisenstein series. The lattice N is equal to L. So Conjecture 5.2 is simply the following theorem in our special case.

Theorem 6.5. *Let $f \in H_{1,\bar{\rho}_L}$ and assume that the constant term $c^+(0,0)$ of f vanishes. Then*

$$\widehat{\deg}(\mathcal{Z}(f)) = -\frac{1}{2}\Phi(Z(U), f).$$

Proof. Since

$$\mathcal{Z}(f) = \sum_{\mu\in L'/L}\sum_{m>0} c^+(-m,\mu)\mathcal{Z}(m,\mathfrak{a},\mu),$$

one has by Theorem 6.4 that

$$\widehat{\deg}(\mathcal{Z}(f)) = -\frac{\deg Z(U)}{2}\sum_{\mu\in L'/L}\sum_{m>0} c^+(-m,\mu)\kappa(m,\mu).$$

On the other hand, Theorem 4.7 asserts in this case

$$\Phi(Z(U), f) = \deg Z(U)\,\mathrm{CT}\left(\langle f^+(\tau), \mathcal{E}_L(\tau)\rangle\right) = \deg Z(U)\sum_{\mu\in L'/L}\sum_{m>0} c^+(-m,\mu)\kappa(m,\mu).$$

Comparing the two equalities, we obtain the assertion. □

7. Height pairings on modular curves

Throughout this section we assume that (V, Q) has signature $(1, 2)$. Then X_K is a modular or a Shimura curve defined over $\mathbb{Q}$. The special divisors $Z(m,\mu)$ and the CM cycles are both divisors on X_K (both supported on CM points). Moreover, the Faltings height pairing is closely related to the Neron-Tate height pairing. Here we compute the heights of special divisors employing Theorem 4.7, modularity of the generating series of special divisors, and multiplicity one for newforms in $S_{3/2,\rho_L}$. Another crucial ingredient is the non-vanishing result for coefficients of weight 2 Jacobi cusp forms by Bump, Friedberg, and Hoffstein [BFH]. This leads to a proof of the Gross-Zagier formula which uses minimal information on the intersections of special divisors at the finite places. Moreover, we also prove Conjectures 5.1 and 5.2 by pulling back special divisors to the moduli space $\mathcal{C}$ defined in Section 6.

7.1. **The modular curve** $X_0(N)$**.** In this example we chose L such that $X_K = Y_0(N)$. Then the compactification of X_K by the cusps is isomorphic to the modular curve $X_0(N)$. The basic setup is the same as in [BrO, Section 2.4] with the difference that the quadratic form is replaced by its negative (which is slightly more convenient for the present paper).

Let N be a positive integer. We consider the rational quadratic space

$$V := \{x \in \mathrm{Mat}_2(\mathbb{Q});\ \mathrm{tr}(x) = 0\} \tag{7.1}$$

with the quadratic form $Q(x) := N\det(x)$. The corresponding bilinear form is given by $(x, y) = -N\,\mathrm{tr}(xy)$ for $x, y \in V$. The signature of V is $(1, 2)$. The group $\mathrm{GL}_2(\mathbb{Q})$ acts on V by conjugation

$$\gamma.x = \gamma x\gamma^{-1}, \qquad \gamma \in \mathrm{GL}_2(\mathbb{Q}),$$

leaving the quadratic form invariant. This induces an isomorphism $H = \mathrm{GSpin}(V) \cong \mathrm{GL}_2$. The domain $\mathbb{D}$ can be identified with $\mathbb{H} \cup \bar{\mathbb{H}}$ via

$$z = x + iy \mapsto \mathbb{R}\Re\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} + \mathbb{R}\Im\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \in \mathbb{D}. \tag{7.2}$$

Under this identification, the action of $H(\mathbb{R})$ on $\mathbb{D}$ becomes the usual linear fractional action.

Let L be the lattice

$$L = \left\{\begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix};\quad a, b, c \in \mathbb{Z}\right\}. \tag{7.3}$$

The dual lattice is given by

$$L' = \left\{\begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix};\quad a, b, c \in \mathbb{Z}\right\}. \tag{7.4}$$

We frequently identify $\mathbb{Z}/2N\mathbb{Z}$ with L'/L via $r \mapsto \mu_r = \mathrm{diag}(r/2N, -r/2N)$. Here the quadratic form on L'/L is identified with the quadratic form $x \mapsto -x^2$ on $\mathbb{Z}/4N\mathbb{Z}$. The level of L is $4N$. For $m \in \mathbb{Q}$ and $\mu \in L'/L$, we define

$$L_{m,\mu} := \{x \in \mu + L;\ Q(x) = m\}.$$

Notice that $L_{m,\mu}$ is empty unless $Q(\mu) \equiv m \pmod 1$.

Let $K_p \subset H(\mathbb{Q}_p)$ be the compact open subgroup

$$K_p = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p);\ c \in N\mathbb{Z}_p\right\},$$

and let $K = \prod_p K_p \subset H(\mathbb{A}_f)$. Then K takes the lattice L to itself and acts trivially on the discriminant group L'/L. Since $H(\mathbb{A}_f) = H(\mathbb{Q})K$, it is easily seen that

$$\alpha : \Gamma_0(N)\backslash\mathbb{H} \to X_K = H(\mathbb{Q})\backslash\mathbb{D} \times H(\mathbb{A}_f)/K, \quad \Gamma_0(N)z \mapsto H(\mathbb{Q})(z, 1)K$$

is an isomorphism.

Let $m \in \mathbb{Q}_{>0}$ and let $\mu \in L'/L$ such that $Q(\mu) \equiv m \pmod 1$. Then $D := -4Nm \in \mathbb{Z}$ is a negative discriminant. If $r \in \mathbb{Z}$ with $\mu = \mu_r \pmod L$, then $D \equiv r^2 \pmod{4N}$, and

$$x = \begin{pmatrix} \frac{r}{2N} & \frac{1}{N} \\ \frac{D-r^2}{4N} & -\frac{r}{2N} \end{pmatrix} \in L_{m,\mu}. \tag{7.5}$$

Conversely, for a pair of integers $D < 0$ and r with $D \equiv r^2 \pmod{4N}$, let $m = -D/4N$ and $\mu = \mu_r$. Then $m \in Q(\mu) + \mathbb{Z}$ is positive. We will use this correspondence in this section freely without mentioning it. Moreover, it is easy to check from Lemma 4.1 that

$$Z(m, \mu) = P_{D,r} + P_{D,-r} \tag{7.6}$$

where $P_{D,r}$ is the Heegner divisor defined in [GKZ].

For a positive norm vector x as in (7.5) we put

$$V_+ = \mathbb{Q}x, \qquad U = V \cap x^\perp, \tag{7.7}$$

$$\mathcal{P} = L \cap V_+, \qquad \mathcal{N} = L \cap U. \tag{7.8}$$

Then V_+ is a positive definite line and U is a 2-dimensional negative definite subspace in V. Here we use $\mathcal{N}$ instead of N as in the previous section to avoid confusion with the level N. An easy computation gives

$$\mathcal{N} = \mathbb{Z}\begin{pmatrix} 1 & 0 \\ -r & -1 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 & 1/N \\ \frac{r^2-D}{4N} & 0 \end{pmatrix}. \tag{7.9}$$

In particular, the determinant of $\mathcal{N}$ is $-D$. It is also easy to check that

$$\mathcal{P} = \mathbb{Z}\begin{pmatrix} r & 2 \\ \frac{D-r^2}{2} & -r \end{pmatrix} = \mathbb{Z}\frac{2N}{t}x, \quad \mathcal{P}' = \mathbb{Z}\frac{t}{D}x. \tag{7.10}$$

with $t = \gcd(r, 2N)$. We consider the ideal $\mathfrak{n} = [N, \frac{r+\sqrt{D}}{2}]$ of $\mathcal{O}_D = \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$. The norm of $\mathfrak{n}$ is equal to N. We define a quadratic form Q on $\mathfrak{n}$ via

$$Q(z) = -\frac{z\bar{z}}{N} = -\frac{\mathrm{N}(z)}{\mathrm{N}(\mathfrak{n})}. \tag{7.11}$$

Lemma 7.1. *Assume that D is the fundamental discriminant of $k = \mathbb{Q}(\sqrt{D})$. Then the following map gives an isomorphism of quadratic lattices:*

$$f : (\mathfrak{n}, Q) \to (\mathcal{N}, Q), \quad xN + y\frac{r+\sqrt{D}}{2} \mapsto \{x, y\} := \begin{pmatrix} x & -\frac{y}{N} \\ -rx-y\frac{r^2-D}{4N} & -x \end{pmatrix}.$$

Moreover, both are equivalent to the integral quadratic form $[-N, -r, -\frac{r^2-D}{4N}] = -Nx^2 - rxy - \frac{r^2-D}{4N}y^2$.

Proof. Clearly,

$$Q(xN + y\frac{r+\sqrt{D}}{2}) = -\frac{1}{N}\left((xN + \frac{yr}{2})^2 - \frac{D}{4}y^2\right) = -Nx^2 - rxy - \frac{r^2-D}{4}y^2,$$

so $(\mathfrak{n}, Q)$ is equivalent to $[-N, -r, -\frac{r^2-D}{4}]$. On the other hand, by definition we have

$$Q(\{x,y\}) = N \det \begin{pmatrix} x & -\frac{y}{N} \\ -rx-y\frac{r^2-D}{4N} & -x \end{pmatrix} = -Nx^2 - rxy - \frac{r^2-D}{4N}y^2.$$

By means of (7.9), one sees then that $\mathcal{N}$ is equivalent to $[-N, -r, -\frac{r^2-D}{4}]$, too. This proves the lemma. □

By Lemma 7.1, we see that $T = \mathrm{GSpin}(U) \cong k^*$ with $k = \mathbb{Q}(\sqrt{D})$. It is easily checked that $K_T \cong \hat{\mathcal{O}}_k^*$.

Proposition 7.2. *Assume that D is a fundamental discriminant coprime to N. Then*

$$Z(U) = Z(m, \mu).$$

Proof. We claim that under the assumption on D we have

$$\Omega_m(\mathbb{A}_f) \cap \mathrm{supp}(\phi_\mu) = Kx. \tag{7.12}$$

Then the assertion follows directly from the definitions of the cycles (4.6) and (4.13). To prove the claim, we have to show for all primes p that $\Omega_m(\mathbb{Q}_p) \cap \mathrm{supp}(\phi_\mu) = K_p x$. This is a direct computation which we omit. □

7.2. **The Shimura lifting and Hecke eigenforms.** Recall from [EZ, §5] that for the lattice L (defined in Section 7.1) the space of cusp forms $S_{3/2,\rho_L}$ is isomorphic to the space $J_{2,N}$ of Jacobi forms of weight 2 and index N. There is a Hecke theory and a newform theory for $J_{2,N}$ which give rise to the corresponding notions on $S_{3/2,\rho_L}$. Let $S_2^-(N)$ denote the space of cusp forms of weight 2 for $\Gamma_0(N)$ which are invariant under the Fricke involution. Note that the Hecke L-function of any $G \in S_2^-(N)$ satisfies a functional equation with root number -1 and therefore vanishes at the central critical point. According to [SZ], the subspace of newforms $J_{2,N}^{new}$ of $J_{2,N}$ is isomorphic to the subspace of newforms $S_2^{new,-}(N)$ of $S_2^-(N)$ as a module over the Hecke algebra. The isomorphism is given by the Shimura correspondence.

More precisely, let $m_0 \in \mathbb{Q}_{>0}$ and $\mu_0 \in L'/L$ such that $m_0 \equiv Q(\mu_0) \pmod 1$. Assume that $D_0 := -4Nm_0 \in \mathbb{Z}$ is a fundamental discriminant. Let $x \in L_{m_0,\mu_0}$ be as in (7.5) and let U be defined by (7.7). There is a linear map $\mathcal{S}_{m_0,\mu_0} : S_{3/2,\rho_L} \to S_2(N)$ defined by

$$g = \sum_\mu \sum_{m>0} b(m,\mu) q^m \phi_\mu \mapsto \mathcal{S}_{m_0,\mu_0}(g) = \sum_{n=1}^\infty \sum_{d|n} \left(\frac{D_0}{d}\right) b\left(m_0 \frac{n^2}{d^2}, \mu_0 \frac{n}{d}\right) q^n, \tag{7.13}$$

see [GKZ, Section II.3], or [Sk, Section 2]. If we denote the Fourier coefficients of $\mathcal{S}_{m_0,\mu_0}(g)$ by $B(n)$, then we may rewrite the formula for the image as the Dirichlet series identity

$$L\left(\mathcal{S}_{m_0,\mu_0}(g), s\right) = \sum_{n>0} B(n) n^{-s} = L(\chi_{D_0}, s) \cdot \sum_{n>0} b\left(m_0 n^2, \mu_0 n\right) n^{-s}. \tag{7.14}$$

The maps $\mathcal{S}_{m_0,\mu_0}$ are Hecke-equivariant and there is a linear combination of them which provides the above isomorphism of $S_{3/2,\rho_L}^{new}$ and $S_2^{new,-}(N)$. Notice that if $g \in S_{3/2,\rho_L}^{new}$ is a newform that corresponds to the normalized newform $G \in S_2^{new,-}(N)$ under the Shimura correspondence, then

$$L\left(\mathcal{S}_{m_0,\mu_0}(g), s\right) = b\left(m_0, \mu_0\right) \cdot L(G, s). \tag{7.15}$$

Lemma 7.3. *Let m_0, μ_0, D_0, U be as above. If $g \in S_{3/2,\rho_L}$, then*

$$L(g, U, s) = 2^{-s} \left(\pi m_0\right)^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(\chi_{D_0}, s+1)^{-1} L\big(\mathcal{S}_{m_0,\mu_0}(g), s+1\big).$$

In particular,

$$L'(g, U, 0) = \frac{4\sqrt{N}}{\pi \deg(Z(m_0, \mu_0))} b(m_0, \mu_0) L'(G, 1),$$

if $g \in S^{new}_{3/2,\rho_L}$ and $G \in S_2^{new,-}(N)$ are further related by (7.15).

Proof. In view of (4.24) we have

$$L(g, U, s) = (4\pi)^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) \sum_{\lambda \in \mathcal{P}'} b(Q(\lambda), \lambda) Q(\lambda)^{-(s+1)/2},$$

where we view g as a modular form with representation $\rho_{\mathcal{P}\oplus\mathcal{N}}$ via Lemma 3.1. Using the fact that $b(Q(\lambda), \lambda) = 0$ for $\lambda \in \mathcal{P}'$ unless $\lambda \in \mathcal{P}' \cap L' = \mathbb{Z}x$, the assertion follows by a straightforward computation. □

Let $G \in S_2^{new,-}(N)$ be a normalized newform of weight 2, and write F_G for the totally real number field generated by the eigenvalues of G. There is a newform $g \in S^{new}_{3/2,\rho_L}$ mapping to G under the Shimura correspondence. We normalize g such that all its coefficients $b(m, \mu)$ are contained in F_G.

Lemma 7.4. *There is a $f \in H_{1/2,\bar{\rho}_L}$ with Fourier coefficients $c^{\pm}(m, \mu)$ such that*

(i) $\xi(f) = \|g\|^{-2} g$,
(ii) *the coefficients of the principal part P_f lie in F_G,*
(iii) *the constant term $c^+(0, 0)$ vanishes.*

Proof. The existence of an $f \in H_{1/2,\bar{\rho}_L}$ satisfying (i) and (ii) follows from Lemma 7.3 in [BrO]. We may in addition attain (iii) by adding a suitable multiple of the theta series in $M_{1/2,\bar{\rho}_L}$ for the lattice $\mathbb{Z}$ with the quadratic form $x \mapsto Nx^2$. □

Lemma 7.5. *Let $g \in S^{new}_{3/2,\rho_L}$ be a newform with Fourier coefficients $b(m, \mu)$ as above. Let S be a finite set of primes including all those dividing N. There exist infinitely many fundamental discriminants $D < 0$ such that*

(i) *q splits in $\mathbb{Q}(\sqrt{D})$ for all primes $q \in S$,*
(ii) *$b(m, \mu) \neq 0$ for $m = -\frac{D}{4N}$ and any $\mu \in L'/L$ such that $m \equiv Q(\mu) \pmod{1}$.*

Proof. This is a consequence of the non-vanishing theorem for the central critical values of quadratic twists of Hecke L-functions proved in [BFH], together with the Waldspurger type formula for Jacobi forms, see [GKZ, Chapter II, Corollary 1] and [Sk]. □

An alternative proof could probably be given by employing the relationship between vector valued modular forms (respectively Jacobi forms) and scalar valued modular forms and using the non-vanishing result proved in [Br1].

7.3. **The Gross-Zagier Formula.** Let $\mathcal{Y}_0(N)$ (respectively $\mathcal{X}_0(N)$) be the moduli stack over $\mathbb{Z}$ of cyclic isogenies of degree N of elliptic curves (respectively generalized elliptic curves) $\pi : E \to E'$ such that $\ker \pi$ meets every irreducible component of each geometric fiber as in [KM]. Then $\mathcal{X}_0(N)(\mathbb{C}) = X_0(N)$. The stack $\mathcal{X}_0(N)$ is a proper flat curve over $\mathbb{Z}$. It is smooth over $\mathbb{Z}[1/N]$ and regular except at closed supersingular points $\underline{x}$ in characteristic p dividing N where $\mathrm{Aut}(\underline{x}) \neq \{\pm 1\}$ (see [GZ, Chapter 3, Proposition 1.4]).

Let $\mathcal{Z}(m, \mu)$ be the DM-stack representing the moduli problem which assigns to a base scheme S over $\mathbb{Z}$ the set of pairs $(\pi : E \to E', \iota)$ where

(i) $\pi : E \to E'$ is a cyclic isogeny of two elliptic curves E and E' over S of degree N,
(ii) $\iota : \mathcal{O}_D \hookrightarrow \mathrm{End}(\pi) = \{\alpha \in \mathrm{End}(E);\ \pi\alpha\pi^{-1} \in \mathrm{End}(E')\}$ is an $\mathcal{O}_D$ action on π such that $\iota(\mathfrak{n}) \ker \pi = 0$.

Here $\mathfrak{n} = [N, \frac{r+\sqrt{D}}{2}]$ is one ideal of $k = \mathbb{Q}(\sqrt{D})$ above N and $\mu_r = \mu$ (recall that $D = -4Nm$ and $\mu_r = \mathrm{diag}(\frac{r}{2N}, -\frac{r}{2N})$). Moreover, $\mathcal{O}_D$ denotes the order of discriminant D in k.

The forgetful map $(\pi : E \to E', \iota) \mapsto (\pi : E \to E')$ is a finite étale map from $\mathcal{Z}(m, \mu)$ into $\mathcal{Y}_0(N)$, which is generically 2 to 1, and its direct image is the flat closure of $Z(m, \mu)$ in $\mathcal{X}_0(N)$. It does not intersect with the boundary $\mathcal{X}_0(N) \backslash \mathcal{Y}_0(N)$, and lies in the regular locus of $\mathcal{X}_0(N)$ (see [Co, Lemma 2.2 and Remark 2.3]). In particular, we may use intersection theory for these divisors and for cuspidal divisors on $\mathcal{X}_0(N)$ even though $\mathcal{X}_0(N)$ is not regular.

Let $f \in H_{1/2,\bar{\rho}_L}$, and denote the Fourier expansion of f as in (3.4). Assume that the principal part of f has coefficients in $\mathbb{R}$ and that $c^+(0, 0) = 0$. There is a divisor $C(f)$ on $X_0(N)$ supported at the cusps such that $\Phi(z, h, f)$ is a Green function for the divisor

$$Z^c(f) = Z(f) + C(f)$$

of degree 0 on $X_0(N)$. Here the cuspidal contribution is given by the Weyl vector term in the Fourier expansion of $\Phi(z, h, f)$ at a cusp, see for instance [BrO, Theorem 5.3]. This term has logarithmic growth at the cusp with multiplicity determined by the Weyl vector. Let $\mathcal{Z}^c(f)$ be the flat closure of $Z^c(f)$ in $\mathcal{X}_0(N)$. We write $\hat{\mathcal{Z}}^c(f)$ for the arithmetic divisor given by the pair

$$\big(\mathcal{Z}^c(f), \Phi(\cdot, f)\big) \in \widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))_{\mathbb{R}}.$$

For $m \in \mathbb{Q}_{>0}$ and $\mu \in L'/L$ we define

$$y(m, \mu) = Z(m, \mu) - \frac{\deg Z(m, \mu)}{2}((\infty) + (0)). \tag{7.16}$$

This divisor has degree 0 and is invariant under the Fricke involution. Moreover, using the principal part of the weak Maass form f, we put

$$y(f) = \sum_{\mu \in L'/L} \sum_{m>0} c^+(-m, \mu) y(m, \mu). \tag{7.17}$$

We let $\mathcal{Y}(m,\mu)$ and $\mathcal{Y}(f)$ denote their flat closures in $\mathcal{X}_0(N)$. Note that for primes p not dividing the discriminant $D=-4Nm$, the divisor $\mathcal{Y}(m,\mu)$ has zero intersection with every fibral component of $\mathcal{X}_0(N)$ over $\mathbb{F}_p$, see e.g. [GKZ, Chapter IV.4, Proposition 1].

Let $J=J_0(N)$ be the Jacobian of $X_0(N)$, and let $J(F)$ denote its points over any number field F. They correspond to divisor classes of degree zero on $X_0(N)$ which are rational over F. Note that $y(f)$ is a divisor of degree 0 which differs from $Z^c(f)$ by a divisor of degree zero on $X_0(N)$ which is supported at the cusps. By the Manin-Drinfeld theorem, $Z^c(f)$ and $y(f)$ define the same point in the Mordell-Weil space $J(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{C}$.

We now fix some notation for the rest of this subsection. Let $G\in S_2^{new,-}(N)$ be a normalized newform defined over the number field F_G. Let $g\in S^{new}_{3/2,\rho_L}$ be a cusp form corresponding to G under the Shimura correspondence with coefficients $b(m,\mu)\in F_G$. Let $f\in H_{1/2,\bar\rho_L}$ be a harmonic weak Maass form associated to g as in Lemma 7.4.

We now consider the generating series

$$A(\tau)=\sum_{\mu\in L'/L}\sum_{m>0} y(m,\mu)q^m\phi_\mu. \tag{7.18}$$

By the Gross-Kohnen-Zagier theorem, $A(\tau)$ is a modular form with values in $J(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{C}$. Borcherds gave a different proof for this result using Borcherds products associated to weakly holomorphic modular forms in $M^!_{1/2,\bar\rho_L}$, see [Bo2].

We may look at the projection $A^G(\tau)$ of $A(\tau)$ to the G-isotypical component of $J(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{C}$. So the coefficients of $A^G(\tau)$ are the projections $y^G(m,\mu)$ of the special divisors $y(m,\mu)$ to the G-isotypical component. [BrO, Theorem 7.7] describes this generating series as follows.

Theorem 7.6. *Let f, g, and G be as above. We have the identity*

$$A^G(\tau)=g(\tau)\otimes y(f)\in S_{3/2,\rho_L}\otimes_{\mathbb{Z}}J(\mathbb{Q}).$$

In particular, the divisor $y(f)$ lies in the G-isotypical component of $J(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{C}$.

The proof is based on a comparison of the action of the Hecke algebra on the Jacobian and on harmonic weak Maass forms, and on multiplicity one for the space $S^{new}_{3/2,\rho_L}$.

Theorem 7.7. *Let G be a normalized cuspidal new form of weight 2, level N whose L-function has an odd functional equation. Let f and g be associated to G as above. Then the Neron-Tate height of $y(f)$ is given by*

$$\langle y(f),y(f)\rangle_{NT}=\frac{2\sqrt{N}}{\pi\|g\|^2}L'(G,1).$$

Proof. According to Theorem 7.6, we have $b(m,\mu)y(f)=y^G(m,\mu)$, and therefore

$$\begin{aligned}\langle y(f),y(f)\rangle_{NT}b(m,\mu)&=\langle y(f),y(m,\mu)\rangle_{NT}\\&=\langle Z^c(f),y(m,\mu)\rangle_{NT}\end{aligned}$$

for all (m,μ). Here we have also used the Manin-Drinfeld theorem. Set $d(m,\mu)=\deg Z(m,\mu)$. For two pairs (m_0,μ_0) and (m_1,μ_1) which we will specify appropriately later, we put

$$c=c(m_0,m_1,\mu_0,\mu_1)=d(m_1,\mu_1)b(m_0,\mu_0)-d(m_0,\mu_0)b(m_1,\mu_1).$$

We consider the degree zero divisor

$$\begin{aligned} Z &= d(m_1,\mu_1)y(m_0,\mu_0) - d(m_0,\mu_0)y(m_1,\mu_1) \\ &= d(m_1,\mu_1)Z(m_0,\mu_0) - d(m_0,\mu_0)Z(m_1,\mu_1). \end{aligned}$$

on $X_0(N)$. This divisor is supported outside the cusps. Let M be the least common multiple of the discriminants of the special divisors in the support of $Z(f)$. We assume that $D_i = -4Nm_i$ is coprime to MN. This implies that $Z^c(f)$ and Z are relatively prime. Moreover, it implies that for every prime p, the divisor $\mathcal{Z}^c(f)$ or the flat closure of Z has zero intersection with every fibral component of $\mathcal{X}_0(N)$ over $\mathbb{F}_p$. By means of [Gr, Section 3], we find

$$\begin{aligned} c\langle y(f), y(f)\rangle_{NT} &= \langle Z^c(f), d(m_1,\mu_1)Z(m_0,\mu_0) - d(m_0,\mu_0)Z(m_1,\mu_1)\rangle_{NT} \\ &= d(m_1,\mu_1)\langle \hat{\mathcal{Z}}^c(f), \mathcal{Z}(m_0,\mu_0)\rangle_{Fal} - d(m_0,\mu_0)\langle \hat{\mathcal{Z}}^c(f), \mathcal{Z}(m_1,\mu_1)\rangle_{Fal}. \end{aligned}$$

Notice that $\mathcal{C}(f)$, the flat closure in $\mathcal{X}_0(N)$ of the cuspidal part $C(f)$, lies in the cuspidal part of $\mathcal{X}_0(N)$ and thus does not intersect with $\mathcal{Z}(m,\mu)$. One has by Theorem 4.7, and Lemma 7.3:

$$\begin{aligned} (7.19)\quad & \langle \hat{\mathcal{Z}}^c(f), \mathcal{Z}(m_0,\mu_0)\rangle_{Fal} \\ &= \frac{1}{2}\Phi(Z(m_0,\mu_0), f) + \langle \mathcal{Z}(f), \mathcal{Z}(m_0,\mu_0)\rangle_{fin} + \langle \mathcal{C}(f), \mathcal{Z}(m_0,\mu_0)\rangle_{fin} \\ &= \frac{d(m_0,\mu_0)}{2} L'(\xi(f), U_0, 0) + \frac{d(m_0,\mu_0)}{2}\,\mathrm{CT}\langle f^+, \theta_{\mathcal{P}_0}\otimes \mathcal{E}_{\mathcal{N}_0}\rangle + \langle \mathcal{Z}(f), \mathcal{Z}(m_0,\mu_0)\rangle_{fin} \\ &= \frac{2\sqrt{N}}{\pi\|g\|^2} b(m_0,\mu_0) L'(G,1) + \frac{d(m_0,\mu_0)}{2}\,\mathrm{CT}\langle f^+, \theta_{\mathcal{P}_0}\otimes \mathcal{E}_{\mathcal{N}_0}\rangle + \langle \mathcal{Z}(f), \mathcal{Z}(m_0,\mu_0)\rangle_{fin}. \end{aligned}$$

Here the subscript 0 in $\mathcal{P}_0$, $\mathcal{N}_0$, U_0, and T_0 indicates its relation to D_0. So we see that

$$c\langle y(f), y(f)\rangle_{NT} = c\frac{2\sqrt{N}}{\pi\|g\|^2} L'(G,1) + \sum_{p \text{ prime}} \alpha_p \log p$$

with coefficients $\alpha_p \in F_G$.

We claim that we can choose D_0 and D_1 such that $c \neq 0$. In fact, according to Lemma 7.5, we may fix a pair (m_0,μ_0) such that D_0 is coprime to MN and such that $b(m_0,\mu_0) \neq 0$. We let (m_1,μ_1) run through the pairs such that D_1 is a square modulo $4N$ and coprime to MN. By Siegel's lower bound for the class numbers we have for any $\varepsilon > 0$ that

$$d(m_1,\mu_1) \gg_\varepsilon m_1^{1/2-\varepsilon}, \quad m_1 \to \infty.$$

On the other hand, by Iwaniec's bound for the coefficients of half integral weight modular forms as refined by Duke [Iw], [Du], we have

$$b(m_1,\mu_1) \ll_\varepsilon m_1^{1/2-1/28+\varepsilon}, \quad m_1 \to \infty.$$

This implies that $c \neq 0$ for m_1 sufficiently large. Hence we find hat

$$(7.20)\qquad \langle y(f), y(f)\rangle_{NT} = \frac{2\sqrt{N}}{\pi\|g\|^2} L'(G,1) + \sum_p \beta_p \log p$$

for some coefficients $\beta_p \in F_G$ independent of all choices that we made above.

Now we prove that $\beta_p = 0$ for every p. Let p be any fixed prime. According to Lemma 7.5, we may fix a pair (m_0, μ_0) such that D_0 is coprime to MN, p splits in $\mathbb{Q}(\sqrt{D_0})$, and such that $b(m_0, \mu_0) \neq 0$. We let (m_1, μ_1) run through the pairs such that D_1 is a square modulo $4N$ coprime to MN, and such that p splits in $\mathbb{Q}(\sqrt{D_1})$. As above, we have $c \neq 0$ when m_1 is sufficiently large. We write

$$\mathrm{CT}\langle f^+, \theta_{\mathcal{P}_i} \otimes \mathcal{E}_{\mathcal{N}_i}\rangle = \sum_{q \text{ prime}} a_{i,q} \log q$$

by Theorem 2.6, and

$$\langle \mathcal{Z}(f), \mathcal{Z}(m_i, \mu_i)\rangle_{fin} = \sum_{q \text{ prime}} b_{i,q} \log q$$

by definition. In view of (7.19) and (7.20), one has

$$\beta_p = \frac{d(m_1,\mu_1)d(m_0,\mu_0)}{2c} a_{0,p} - \frac{d(m_0,\mu_0)d(m_1,\mu_1)}{2c} a_{1,p} + \frac{d(m_1,\mu_1)}{c} b_{0,p} - \frac{d(m_0,\mu_0)}{c} b_{1,p}.$$

By Theorem 2.6, one sees immediately that $a_{i,p} = 0$ since p is split in $k_i = \mathbb{Q}(\sqrt{D_i})$. On the other hand, if $x = (\pi : E \to E', \iota) \in \mathcal{Z}(m_0, \mu_0)(\bar{\mathbb{F}}_p)$, then E and E' are ordinary since p is split in k_0. This means that ι is an isomorphism. So there is no action of $\mathcal{O}_D$ on E if D/D_0 is not a square. This implies

$$\langle \mathcal{Z}(m, \mu), \mathcal{Z}(m_0, \mu_0)\rangle_p = 0$$

if $m/m_0 = D/D_0$ is not a square. Consequently,

$$\langle \mathcal{Z}(f), \mathcal{Z}(m_0, \mu_0)\rangle_p = 0,$$

that is, $b_{0,p} = 0$. For the same reason, $b_{1,p} = 0$ and thus $\beta_p = 0$. This proves the theorem. □

Corollary 7.8. *(Gross-Zagier formula* [GZ, Theorem I.6.3]*) For any any $\mu \in L'/L$ and any positive $m \in Q(\mu) + \mathbb{Z}$ we have*

$$\langle y^G(m, \mu), y^G(m, \mu)\rangle_{NT} = \frac{\sqrt{|D|}}{4\pi^2 \|G\|^2} L(G, \chi_D, 1) L'\big(G, 1\big).$$

Here $D = -4Nm$ and $\|G\|$ denotes the Petersson norm of G.

Proof. This follows from Theorem 7.7 using the fact that $y^G(m, \mu) = b(m, \mu) y(f)$ and the Waldspurger type formula

$$b(m, \mu)^2 = \frac{\|g\|^2}{8\pi\sqrt{N}\|G\|^2} \sqrt{|D|} L(G, \chi_D, 1),$$

see [GKZ, Chapter II, Corollary 1], and [Sk]. Here we have also used the fact that the Petersson norm $\|g\|$ is equal to $2N^{1/4}\|\phi\|$, where ϕ is the Jacobi form of weight 2 corresponding to g and $\|\phi\|$ is its Petersson norm, see [EZ, Theorem 5.3]. (Notice that a factor of 2 is missing in [EZ] which is due to the fact that the element $(-1, 0)$ of the Jacobi group acts as $(\tau, z) \mapsto (\tau, -z)$ on $\mathbb{H} \times \mathbb{C}$.) □

7.4. **Pull-back of special divisors.** We continue to use the notation of Section 7.1. Given two cycles $\mathcal{Z}(m_i, \mu_i)$ in $\mathcal{Y}_0(N)$, let $D_i = -4Nm_i$ and $r_i \in \mathbb{Z}/2N\mathbb{Z}$ with $\mu_i = \mu_{r_i}$ as before. We assume that D_0 is prime to $2N$ and is fundamental, and that D_0D_1 is not a square so that $\mathcal{Z}(m_0, \mu_0)$ and $\mathcal{Z}(m_1, \mu_1)$ intersect properly. In this setting, Conjecture 5.1 is just the following theorem.

Theorem 7.9. *Under the above assumptions on D_0 and D_1, the finite intersection pairing $\langle \mathcal{Z}(m_1, \mu_1), \mathcal{Z}(m_0, \mu_0)\rangle_{fin}$ is equal to the (m_1, μ_1)-th coefficient of $\frac{-2}{\mathrm{vol}(K_T)}\theta_{\mathcal{P}_0}(\tau) \otimes \mathcal{E}_{\mathcal{N}_0}(\tau)$. That is,*

$$\langle \mathcal{Z}(m_1, \mu_1), \mathcal{Z}(m_0, \mu_0)\rangle_{fin} = -\frac{\deg Z(m_0, \mu_0)}{2} \sum_{\substack{l\in\mathbb{Q} \\ lx_0\in\mathcal{P}_0' \\ l^2m_0\leq m_1}} \sum_{\substack{\nu\in\mathcal{N}_0'/\mathcal{N}_0 \\ \nu+lx_0\equiv\mu_1\ (L)}} \kappa(m_1 - l^2m_0, \nu).$$

In this subsection, we prove the result by pulling the intersection back to the CM stack $\mathcal{C}$ studied in Section 6 and using Theorem 6.4. Let $\mathfrak{n}_i = [N, \frac{r_i+\sqrt{D_i}}{2}]$. Let $\mathcal{C}$ be the moduli stack of CM elliptic curves associated to the quadratic field $k_0 = \mathbb{Q}(\sqrt{D_0})$ defined in Section 6. For a CM elliptic curve $(E, \iota) \in \mathcal{C}(S)$, let $E_{\mathfrak{n}_0} = E/E[\mathfrak{n}_0]$ and let $\pi : E \to E_{\mathfrak{n}_0}$ be the natural map. Write

$$\mathcal{O}_{E,\mathfrak{n}_0} = \mathrm{End}_S(\pi) = \{\alpha \in \mathcal{O}_E;\ \pi\alpha\pi^{-1} \in \mathrm{End}_S(E_{\mathfrak{n}_0})\}.$$

The starting point is

Lemma 7.10. *There is a natural isomorphism of stacks*

$$j : \mathcal{C} \to \mathcal{Z}(m_0, \mu_0), \quad j(E, \iota) = (\pi : E \to E_{\mathfrak{n}_0}, \iota).$$

Proof. Since $\iota(\mathfrak{n}_0)\ker\pi = \iota(\mathfrak{n}_0)E[\mathfrak{n}_0] = 0$, and $\iota(\mathcal{O}_{D_0}) \subset \mathcal{O}_{E,\mathfrak{n}_0}$, one has for $(E, \iota) \in \mathcal{C}(S)$ that $j(E, \iota) \in \mathcal{Z}(m_0, \mu_0)(S)$. The map j is obviously a bijection. It is also easy to check that $\mathrm{Aut}_S(E, \iota) = \mathrm{Aut}_S(j(E, \iota))$. So j is an isomorphism. □

Combining this map with the natural map from $\mathcal{Z}(m_0, \mu_0)$ to $\mathcal{X}_0(N)$, we obtain a natural map from $\mathcal{C}$ to $\mathcal{X}_0(N)$, still denoted by j. Its direct image is the cycle $\mathcal{Z}(m_0, \mu_0)$. So

$$\langle \mathcal{Z}(m_1, \mu_1), \mathcal{Z}(m_0, \mu_0)\rangle_{fin} = \widehat{\deg}(j^*\mathcal{Z}(m_1, \mu_1)).$$

Looking at the fiber product diagram

$$\begin{array}{ccc} j^*\mathcal{Z}(m_1, \mu_1) = \mathcal{Z}(m_1, \mu_1) \times_{\mathcal{X}_0(N)} \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{Z}(m_1, \mu_1) & \longrightarrow & \mathcal{X}_0(N) \end{array},$$

one sees that $j^*\mathcal{Z}(m_1, \mu_1)(S)$ consists of triples (E, ι, ϕ) where $(E, \iota) \in \mathcal{C}(S)$, and

$$\phi :\ \mathcal{O}_{D_1} \hookrightarrow \mathcal{O}_{E,\mathfrak{n}_0}$$

such that

$$\phi(\mathfrak{n}_1)E[\mathfrak{n}_0] = 0.$$

Proposition 7.11. *One has*

$$\langle \mathcal{Z}(m_1,\mu_1), \mathcal{Z}(m_0,\mu_0)\rangle_{fin} = -\frac{\deg Z(m_0,\mu_0)}{2} \sum_{\substack{n\equiv r_0r_1 \pmod{2N} \\ n^2\le D_0D_1}} \kappa(\frac{D_0D_1-n^2}{4N|D_0|}, \frac{\tilde{2}n}{\sqrt{D_0}}).$$

Here $\tilde{2}\in \mathbb{Z}/D_0\mathbb{Z}$ is determined by the condition $2\cdot\tilde{2}\equiv 1 \pmod{D_0}$.

Proof. First we look at geometric points $(E,\iota,\phi)\in j^*\mathcal{Z}(m_1,\mu_1)(F)$, with $F=\mathbb{C}$ or $F=\bar{\mathbb{F}}_p$. Then $\mathcal{O}_{E,\mathfrak{n}_0}$ contains $\iota(\mathcal{O}_{k_0})$ and $\phi(\mathcal{O}_{k_1})$, and is thus at least of rank four over $\mathbb{Z}$. This implies $F=\mathbb{F}_p$ for p non-split in k_i, $i=0,1$, and E is supersingular. Assuming this, let $\mathbb{B}$ be the quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞, and let

$$\iota_0 : k_0 \hookrightarrow \mathbb{B}$$

be a fixed embedding. Choose a prime $p_0\nmid 2pD_0$ such that (as in Section 6)

$$\mathrm{inv}_l\mathbb{B} = \begin{cases} (D_0,-p_0p)_l & \text{if } p \text{ inert in } k_0,\\ (D_0,-p_0)_l & \text{if } p \text{ ramified in } k_0\end{cases}$$

for every prime l. In particular, $p_0=\mathfrak{p}_0\bar{\mathfrak{p}}_0$ is split in k_0. Let $\kappa_{\mathbb{B}}=-p_0p$ or $-p_0$ depending on whether p is inert or ramified in k_0, and let $\delta_{\mathbb{B}}\in\mathbb{B}^*$ such that $\delta^2=\kappa_{\mathbb{B}}$ and $\delta_{\mathbb{B}}\alpha=\bar{\alpha}\delta$ for $\alpha\in k_0$. Here we identify $\alpha\in k_0$ with $\iota_0(\alpha)\in\mathbb{B}$. Then $\mathcal{O}_E=\operatorname{End}E$ is a maximal order of $\mathbb{B}$. Write

$$\phi(\frac{r_1+\sqrt{D_1}}{2}) = \alpha+\boldsymbol{\beta}\in\mathcal{O}_{E,\mathfrak{n}_0} \tag{7.21}$$

with $\alpha\in k_0$ and $\boldsymbol{\beta}\in\delta_{\mathbb{B}}k_0$. The condition $\phi(\mathfrak{n}_1)E[\mathfrak{n}_0]=0$ is the same as

$$\phi(\frac{r_1+\sqrt{D_1}}{2})E[\mathfrak{n}_0]=0,$$

which is the same as

$$\alpha+\boldsymbol{\beta}\in\mathcal{O}_E\mathfrak{n}_0.$$

In particular, $\alpha\in\partial_0^{-1}\mathfrak{n}_0$. One sees from (7.21) that

$$\phi(\sqrt{D_1})=\alpha_1+2\boldsymbol{\beta}$$

with $\alpha_1=-r_1+2\alpha\in\partial_0^{-1}$ and $\operatorname{tr}\alpha_1=0$. We write

$$\alpha_1=\frac{n}{\sqrt{D_0}},\quad \alpha=\frac{1}{\sqrt{D_0}}(aN+b\frac{r_0+\sqrt{D_0}}{2}).$$

Then we see

$$n=-r_1\sqrt{D_0}+2aN+br_0+b\sqrt{D_0}$$

and thus $b=r_1$, and

$$n=2aN+r_0r_1\equiv r_0r_1 \pmod{2N}.$$

Moreover,

$$D_1=\alpha_1^2-4\,\mathrm{N}(\boldsymbol{\beta})=\frac{n^2}{D_0}-4\,\mathrm{N}(\boldsymbol{\beta}),$$

and so

$$\mathrm{N}(\boldsymbol{\beta}) = \frac{D_0 D_1 - n^2}{4|D_0|} = \frac{D_0 D_1 - n^2}{4N|D_0|} \mathrm{N}(\mathfrak{n}_0) \in \frac{1}{|D_0|}\mathbb{Z}_{>0}.$$

This implies that

$$(E, \iota, \boldsymbol{\beta}) \in \mathcal{Z}(\frac{D_0 D_1 - n^2}{4N|D_0|}, \mathfrak{n}_0, \frac{n + r_1\sqrt{D_0}}{2\sqrt{D_0}})(\bar{\mathbb{F}}_p). \tag{7.22}$$

Conversely, if $(E, \iota, \boldsymbol{\beta}) \in \mathcal{Z}(\frac{D_0 D_1 - n^2}{4N}, \mathfrak{n}_0, \frac{n+r_1\sqrt{D_0}}{2\sqrt{D_0}})(\bar{\mathbb{F}}_p)$ for some $n \equiv r_0 r_1 \pmod{2N}$, then

$$\boldsymbol{\beta} \in \delta_{\mathbb{B}} \partial_0^{-1}\mathfrak{n}_0, \quad \mathrm{N}(\boldsymbol{\beta}) = \frac{D_0 D_1 - n^2}{4N|D_0|} \mathrm{N}(\mathfrak{n}_0)$$

and

$$\alpha + \boldsymbol{\beta} \in \mathcal{O}_E \mathfrak{n}_0$$

with $\alpha = \frac{n+r_1\sqrt{D_0}}{2\sqrt{D_0}}$. If we write $n = r_0 r_1 + 2aN$, then

$$\alpha = \frac{n + r_1\sqrt{D_0}}{2\sqrt{D_0}} = \frac{1}{\sqrt{D_0}}(aN + r_1 \frac{r_0 + \sqrt{D_0}}{2}) \in \partial_0^{-1}\mathfrak{n}_0.$$

So $\phi(\frac{r_1+\sqrt{D_1}}{2}) = \alpha + \boldsymbol{\beta} \in \mathcal{O}_E \mathfrak{n}_0$ gives $(E, \iota, \phi) \in j^*\mathcal{Z}(m_1, \mu_1)(\bar{\mathbb{F}}_p)$. Hence we have proved the following lemma which might be of independent interest.

Lemma 7.12. *There is an isomorphism*

$$j^*\mathcal{Z}(m_1, \mu_1)(\bar{\mathbb{F}}_p) \cong \bigsqcup_{\substack{n \equiv r_0 r_1 \pmod{2N} \\ n^2 \le D_0 D_1}} \mathcal{Z}(\frac{D_0 D_1 - n^2}{4N|D_0|}, \mathfrak{n}_0, \frac{n + r_1\sqrt{D_0}}{2\sqrt{D_0}})(\bar{\mathbb{F}}_p), \tag{7.23}$$

given by $(E, \iota, \phi) \mapsto (E, \iota, \boldsymbol{\beta})$ *via the relation*

$$\phi(\frac{r_1 + \sqrt{D_1}}{2}) = \frac{n + r_1\sqrt{D_0}}{2\sqrt{D_0}} + \boldsymbol{\beta}.$$

□

Let $W = W(\bar{\mathbb{F}}_p)$ be the Witt ring of $\bar{\mathbb{F}}_p$. It is not hard to check that for any locally complete W-algebra R with residue field $\bar{\mathbb{F}}_p$, (E, ι, ϕ) lifts to an element in $j^*\mathcal{Z}(m_1, \mu_1)(R)$ if and only if $(E, \iota, \boldsymbol{\beta})$ lifts to an element in $\mathcal{Z}(\frac{D_0 D_1 - n^2}{4N}, \mathfrak{n}_0, \frac{n+2r_1\sqrt{D_0}}{2\sqrt{D_0}})(R)$. So we have by Theorem 6.4 that

$$\begin{aligned}
\langle \mathcal{Z}(m_1, \mu_1), \mathcal{Z}(m_0, \mu_0)\rangle &= \widehat{\deg}(j^*\mathcal{Z}(m_1, \mu_1)) \\
&= \sum_{\substack{n \equiv r_0 r_1 \pmod{2N} \\ n^2 \le D_0 D_1}} \widehat{\deg}\mathcal{Z}(\frac{D_0 D_1 - n^2}{4N|D_0|}, \mathfrak{n}_0, \frac{n + r_1\sqrt{D_0}}{2\sqrt{D_0}}) \\
&= -\frac{\deg Z(m_0, \mu_0)}{2} \sum_{\substack{n \equiv r_0 r_1 \pmod{2N} \\ n^2 \le D_0 D_1}} \kappa(\frac{D_0 D_1 - n^2}{4N|D_0|}, \frac{n + r_1\sqrt{D_0}}{2\sqrt{D_0}}).
\end{aligned}$$

Since $\frac{n+r_1\sqrt{D_0}}{2\sqrt{D_0}} \equiv \frac{\tilde{2}n}{\sqrt{D_0}} \pmod{\mathcal{O}_{D_0}}$, this concludes the proof of Proposition 7.11. □

Now Theorem 7.9 follows from the above proposition and the following lemma.

Lemma 7.13. *Let the notation be as above. Then one has*

$$\sum_{\substack{l\in\mathbb{Q}\\ lx_0\in\mathcal{P}_0'\\ l^2m_0\le m_1}} \sum_{\substack{\nu\in\mathcal{N}_0'/\mathcal{N}_0\\ \nu+lx_0\equiv\mu_1\ (L)}} \kappa_\nu(m_1-l^2m_0) = \sum_{\substack{n\in\mathbb{Z},\, n^2\le D_0D_1\\ n\equiv r_0r_1\ (2N)}} \kappa(\frac{D_0D_1-n^2}{4N|D_0|}, \frac{\tilde{2}n}{\sqrt{D_0}}). \tag{7.24}$$

Proof. It is clear that $lx_0\in\mathcal{P}_0'$ if and only if $l=\frac{n}{D}$ with $n\in\mathbb{Z}$. The inequality $l^2m_0\le m_1$ is the same as $n^2\le D_0D_1$. By Lemma 7.1, one sees that

$$\nu(a) = f(\frac{Na}{\sqrt{D_0}}) = \frac{a}{D_0}\begin{pmatrix} -r_0 & -2\\ \frac{D_0+r_0^2}{2} & r_0\end{pmatrix}, \quad a\in\mathbb{Z}/D_0\mathbb{Z} \tag{7.25}$$

gives a complete set of representatives of $\mathcal{N}_0'/\mathcal{N}_0$. Write

$$\nu(a)+lx_0 = \begin{pmatrix} \frac{ur_0}{2N} & \frac{u}{N}\\ aD_0+\frac{D_0-r_0^2}{4N}u & -\frac{ur_0}{2N}\end{pmatrix}$$

with $u=\frac{n-2Na}{D_0}$. So $\nu(a)+lx_0\in\mu_1+L$ if and only if

$$u=\frac{n-2Na}{D_0}\in\mathbb{Z}, \quad \text{and} \quad ur_0\equiv r_1 \pmod{2N}.$$

Since $(D_0,2N)=1$, and $D_0\equiv r_0^2 \pmod{4N}$, one sees that the above condition is equivalent to

$$n\equiv 2Na \pmod{D_0}, \quad n\equiv r_0r_1 \pmod{2N}.$$

So $\nu(a)+lx_0\in\mu_1+L$ if and only if $n\equiv r_0r_1 \pmod{2N}$ and $Na\equiv\tilde{2}n \pmod{D_0}$. In such a case, Lemma 7.1 implies

$$\kappa(t,\nu(a)) = \kappa(t,\frac{Na}{\sqrt{D_0}}) = \kappa(t,\frac{\tilde{2}n}{\sqrt{D_0}}).$$

Finally, one checks

$$m_1-l^2m_0 = -\frac{D_1}{4N}+\frac{n^2}{D_0^2}\frac{D_0}{4N} = \frac{D_0D_1-n^2}{4N|D_0|}.$$

Putting this together, one proves the proposition. □

Now Conjecture 5.2 becomes the following theorem in our setting.

Theorem 7.14. *Assume that D_0 is a fundamental discriminant coprime to $2N$. Let f be any element of $H_{1/2,\bar{\rho}_L}$. Then*

$$\langle\hat{\mathcal{Z}}^c(f),\mathcal{Z}(m_0,\mu_0)\rangle_{Fal} = \frac{\deg Z(m_0,\mu_0)}{2}\left(c^+(0,0)\kappa(0,0)+L'(\xi(f),U,0)\right). \tag{7.26}$$

Proof. We first assume that $\mathcal{Z}(f)$ and $\mathcal{Z}(m_0,\mu_0)$ intersect properly. According to Proposition 7.2, the assumption on D_0 implies that $Z(U)=Z(m_0,\mu_0)$. Hence Theorem 4.7 says that

$$\begin{aligned}\langle \hat{\mathcal{Z}}^c(f), \mathcal{Z}(m_0,\mu_0)\rangle_\infty &= \frac{1}{2}\Phi(Z(U),f)\\ &= \frac{\deg Z(m_0,\mu_0)}{2}\left(\mathrm{CT}\left(\langle f^+,\, \theta_{\mathcal{P}_0}(\tau)\otimes \mathcal{E}_{\mathcal{N}_0}(\tau)\rangle\right) + L'(\xi(f),U,0)\right).\end{aligned}$$

According to Theorem 7.9, we have

$$\begin{aligned}\langle \mathcal{Z}^c(f), \mathcal{Z}(m_0,\mu_0)\rangle_{fin} &= \sum_{m>0,\mu\in L'/L} c^+(-m,\mu)\langle \mathcal{Z}(m,\mu), \mathcal{Z}(m_0,\mu_0)\rangle_{fin}\\ &= -\frac{\deg Z(m_0,\mu_0)}{2}\,\mathrm{CT}\left(\langle f^+ - c^+(0,0)\phi_0,\, \theta_{\mathcal{P}_0}(\tau)\otimes \mathcal{E}_{\mathcal{N}_0}(\tau)\rangle\right).\end{aligned}$$

Adding the two identities together, we obtain the assertion in the case when $\mathcal{Z}(f)$ and $\mathcal{Z}(m_0,\mu_0)$ intersect properly. Finally, for general f, we notice that there always exists a weakly holomorphic modular form $f'\in M^!_{1/2,\bar\rho_L}$ with vanishing constant term such that $\mathcal{Z}(f+f')$ and $\mathcal{Z}(m_0,\mu_0)$ intersect properly. For f' both sides of the claimed identity (7.26) vanish. Hence, the general case follows from the linearity of (7.26) in f. □

Notice that Theorem 7.14 and Lemma 7.3 can be used to give another proof of the Gross-Zagier formula in Theorem 7.7.

8. The case $n=2$

In this section, we verify a very special case of Conjecture 5.1 when $n=2$. We plan to study the case $n=2$ systematically in a sequel to this paper.

Let $F=\mathbb{Q}(\sqrt{\Delta})$ be a real quadratic field with prime discriminant $\Delta\equiv 1 \pmod 4$. We denote by $\mathcal{O}_F$ the ring of integers in F, and write ∂_F for the different of F. Let V be the quadratic space

$$V=\{A\in M_2(F);\ A'=A^t\}=\{A=\left(\begin{smallmatrix} a & \lambda\\ \lambda' & b\end{smallmatrix}\right);\ a,b\in\mathbb{Q},\ \lambda\in F\} \tag{8.1}$$

with the quadratic form $Q(A)=\det A$, which has signature $(2,2)$. We consider the even lattice $L=V\cap M_2(\mathcal{O}_F)$. The dual lattice is

$$L'=\{A=\left(\begin{smallmatrix} a & \lambda\\ \lambda' & b\end{smallmatrix}\right);\ a,b\in\mathbb{Z},\ \lambda\in\partial_F^{-1}\}.$$

In this case,

$$H=\mathrm{GSpin}(V)=\{g\in\mathrm{GL}_2(F);\ \det g\in\mathbb{Q}^*\}$$

acts on V via

$$g.A=\frac{1}{\det g}gA\,{}^tg'.$$

Take

$$K=H(\hat{\mathbb{Z}})=\{g\in\mathrm{GL}_2(\hat{\mathcal{O}}_F);\ \det g\in\hat{\mathbb{Z}}\}.$$

The following identification is well-known

$$(\mathbb{H}^\pm)^2\to\mathbb{D},\quad z=(z_1,z_2)\mapsto U=\mathbb{R}\left(\begin{smallmatrix} 1 & x_1\\ x_2 & x_1x_2-y_1y_2\end{smallmatrix}\right)\oplus\mathbb{R}\left(\begin{smallmatrix} 0 & -y_1\\ -y_2 & -x_1y_2-x_2y_1\end{smallmatrix}\right). \tag{8.2}$$

Since $H(\mathbb{A}_f) = H(\mathbb{Q})^+K$, one can show that

$$X_K = H(\mathbb{Q})\backslash\mathbb{D} \times H(\mathbb{A}_f)/K \cong \mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^2,$$

which we will denote simply by X in this section.

8.1. **The CM cycle $Z(U)$.** There are many CM 0-cycles $Z(U)$. Here we choose a special one for simplicity. Let $k_D = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field with fundamental discriminant D and assume $(D, 2\Delta) = 1$. The oriented negative 2-plane associated to the CM point $z = (\frac{D+\sqrt{D}}{2}, \frac{D+\sqrt{D}}{2}) \in X$ via (8.2) is actually rational and is given by

$$U = \mathbb{Q}f_1 \oplus \mathbb{Q}f_2, \quad f_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & D \end{smallmatrix}\right), \quad f_2 = \begin{pmatrix} 2 & D \\ D & \frac{D^2+D}{2} \end{pmatrix}. \tag{8.3}$$

The lattice $N = U \cap L$ is isomorphic to

$$(N, Q) \cong (\mathcal{O}_D, -\mathrm{N}), \quad f_1 \mapsto 1, \quad f_2 \mapsto \sqrt{D}, \tag{8.4}$$

where $\mathcal{O}_D$ is the ring of integers in k_D. It is easy to check that

$$V_+ = U^\perp = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2,$$
$$P = V_+ \cap L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \cong (\mathfrak{d}, \frac{\mathrm{N}}{\Delta}),$$

where

$$e_1 = \begin{pmatrix} 0 & \sqrt{\Delta} \\ -\sqrt{\Delta} & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & \frac{D+\sqrt{\Delta}}{2} \\ \frac{D-\sqrt{\Delta}}{2} & \frac{D^2-D}{4} \end{pmatrix},$$

and $\mathfrak{d}$ is the ideal of $k_{D\Delta} = \mathbb{Q}(\sqrt{D\Delta})$ over Δ. Let $P_i = \mathbb{Z}e_i$ and

$$M = P_1^\perp \cap L = \{A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right);\ a, b, c \in \mathbb{Z}\} \cong \{A = \left(\begin{smallmatrix} b & a \\ c & -b \end{smallmatrix}\right);\ a, b, c \in \mathbb{Z}\}.$$

So the cycle $Z(e_1, 1)$ defined in (4.3) is naturally isomorphic to the modular curve $Y_0(1)$ defined in Section 7. The inclusion $N \subset M \subset L$ gives rise to natural morphisms

$$Z(U) \xrightarrow{j_0} Y_0(1) \xrightarrow{j_1} X. \tag{8.5}$$

In terms of coordinates in upper half planes, they are given by

$$j_0([z_U^\pm, h]) = \frac{b+\sqrt{D}}{2a}, \quad j_1(z) = (z, z),$$

if $[a, \frac{b+\sqrt{D}}{2}]$ is the ideal of k_D associated to h. The morphism j_0 is two-to-one, and j_1 is an injection. It is not hard to check for a non-square integer $m > 0$ that

$$j_1^* Z(m, \mu) = \sum_{\substack{\mu_1 \in P_1'/P_1,\, \mu_2 \in M'/M \\ \mu_1+\mu_2 \equiv \mu\ (L) \\ m_1+m_2=m, m_i \geq 0}} r_{P_1}(m_1, \mu_1) Z(m_2, \mu_2)_{Y_0(1)}. \tag{8.6}$$

Here we use the subscript P_1 to indicate the dependence of Fourier coefficients $r_{P_1}(m, \mu)$ on P_1, and the subscript $Y_0(1)$ to indicate the cycles in $Y_0(1)$. We remark that $Z(m, \mu)$ is basically the Hirzebruch-Zagier divisor $T_{m\Delta}$.

8.2. **Integral Model.** Let $\mathcal{X}$ be the Hilbert moduli stack assigning to a base scheme S over $\mathbb{Z}$ the set of the triples (A, ι, λ), where

(i) A is a abelian surface over S.
(ii) $\iota : \mathcal{O}_F \hookrightarrow \mathrm{End}_S(A)$ is real multiplication of $\mathcal{O}_F$ on A.
(iii) $\lambda : \partial_F^{-1} \to P(A) = \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)^{\mathrm{sym}}$ is a ∂_F^{-1}-polarization (in the sense of Deligne-Papas) satisfying the condition:

$$\partial_F^{-1} \otimes A \to A^\vee, \quad r \otimes a \mapsto \lambda(r)(a)$$

is an isomorphism.

(See [Go, Chapter 3] and [Vo, Section 3].) Then it is well-known that $\mathcal{X}(\mathbb{C}) = X$. Let $\mathcal{Z}(m, \mu)$ be the flat closure of $Z(m, \mu)$ in $\mathcal{X}$, and let $\mathcal{Z}(U)$ be the flat closure of $Z(U)$ in $\mathcal{X}$. Let $\mathcal{C}$ and $\mathcal{Y}_0(1)$ be as in Sections 6 and 7. Let $j_0 : \mathcal{C} \to \mathcal{Y}_0(1)$ be the map defined in Lemma 7.10 (with abuse of notation). The map j_1 extends integrally to a closed immersion j_1 from $\mathcal{Y}_0(1)$ to $\mathcal{X}$ defined in [Ya2, Lemma 2.2]. Let $j = j_1 \circ j_0$ be the map from $\mathcal{C}$ to $\mathcal{X}$. Then the direct image of $\mathcal{C}$ is $\mathcal{Z}(U)$, so j can be viewed as the integral extension of the map j defined in (8.5). Taking the flat closures on both sides of (8.6), one sees that (8.6) holds also integrally. So we have by Theorem 7.9 and Lemma 7.10 that

$$\begin{aligned}
\langle \mathcal{Z}(U), \mathcal{Z}(m, \mu) \rangle_{\mathcal{X}} &= \langle (j_0)_* \mathcal{C}, j_1^* \mathcal{Z}(m, \mu) \rangle_{\mathcal{Y}_0(1)} \\
&= \sum_{\substack{\mu_1 \in P_1'/P_1,\, \mu_2 \in M'/M \\ \mu_1 + \mu_2 \equiv \mu \pmod{L} \\ m_1 + m_2 = m,\, m_i \geq 0}} r_{P_1}(m_1, \mu_1) \langle \mathcal{Z}(-\frac{D}{4}, \frac{D}{2}), \mathcal{Z}(m_2, \mu_2) \rangle_{\mathcal{Y}_0(1)} \\
&= c \sum_{\substack{\mu_1 \in P_1'/P_1,\, \mu_2 \in M'/M \\ \mu_1 + \mu_2 \equiv \mu \pmod{L} \\ m_1 + m_2 = m,\, m_i \geq 0}} r_{P_1}(m_1, \mu_1) \sum_{\substack{\mu_3 \in P_2'/P_2,\, \mu_4 \in N'/N \\ \mu_3 + \mu_4 \equiv \mu \pmod{M} \\ m_3 + m_4 = m_2,\, m_i \geq 0}} r_{P_2}(m_3, \mu_3) \kappa_N(m_4, \mu_4) \\
&= c \sum_{\substack{\mu_1 \in (P_1 + P_2)'/(P_1 + P_2),\, \mu_2 \in N'/N \\ \mu_1 + \mu_2 \equiv \mu \pmod{L} \\ m_1 + m_2 = m,\, m_i \geq 0}} r_{P_1 + P_2}(m_1, \mu_1) \kappa_N(m_2, \mu_2).
\end{aligned}$$

Here $c = -\frac{2}{\mathrm{vol}(K_T)} = -\frac{2h_D}{w_D}$ as in Lemma 6.3. Since

$$P_1 \oplus P_2 \oplus N \subset P \oplus N \subset L \subset L' \subset P' \oplus N' \subset (P_1 \oplus P_2)' \oplus N',$$

it is easy to see that for $\mu_1 \in (P_1 \oplus P_2)'$ and $\mu_2 \in N'$, the condition $\mu_1 + \mu_2 \in L'$ implies that $\mu_1 \in P'$. So we have proved Conjecture 5.1 in this special case, which we state as a theorem.

Theorem 8.1. *Let $F = \mathbb{Q}(\sqrt{\Delta})$ be a real quadratic field with prime discriminant $\Delta \equiv 1 \pmod{4}$, and let $\mathcal{X}$ be the associated Hilbert modular surface. Let U be as above, and assume that $m > 0$ is not a square. Then*

$$\langle \mathcal{Z}(U), \mathcal{Z}(m, \mu) \rangle_{fin} = -\frac{2h_D}{w_D} \sum_{\substack{\mu_1 \in P'/P,\, \mu_2 \in N'/N \\ \mu_1 + \mu_2 \equiv \mu \pmod{L} \\ m_1 + m_2 = m,\, m_i \geq 0}} r_P(m_1, \mu_1) \kappa_N(m_2, \mu_2)$$

is $-\frac{2h_D}{w_D}$ *times the* (m,μ)*-th Fourier coefficient of* $\theta_P(\tau)\otimes\mathcal{E}_N(\tau)$.

As discussed in Section 5, this implies Conjecture 5.2. We also remark that the L-series $L(\xi(f),U,s)$ is the Rankin-Selberg L-function of a cusp form of weight 2, level Δ and *non-trivial* Nebentypus χ_Δ with a theta function of weight 1. This is new in the sense that it is associated to the Jacobian of $X_1(\Delta)$.

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