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Borcherds Products on O$(2, l)$ and Chern Classes of Heegner Divisors

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To my parents

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Introduction

Let (V, q) be a quadratic space of signature $(2, l)$ and $L \subset V$ an even lattice. To simplify this exposition let us further assume that L is unimodular. This implies that l is even. Around 1994 R. Borcherds discovered a lifting from nearly holomorphic modular forms of weight $1 - l/2$ for $\mathrm{SL}_2(\mathbb{Z})$ to meromorphic modular forms for the orthogonal group of L, which have infinite product expansions analogous to the Dedekind eta function (see [Bo1]).

Here, by a *nearly holomorphic* modular form for $\mathrm{SL}_2(\mathbb{Z})$ we mean a function on the complex upper half plane $\mathbb{H} = \{\tau \in \mathbb{C};\ \Im(\tau) > 0\}$ with the usual transformation behavior, which is holomorphic on $\mathbb{H}$, but may have a pole at the cusp ∞. Such a modular form f has a Fourier expansion

$$f(\tau) = \sum_{n \gg -\infty} c(n) q^n,$$

where $q = e^{2\pi i \tau}$ as usual[1]. The Fourier polynomial

$$\sum_{0 > n \gg -\infty} c(n) q^n$$

is called the principal part of f.

To describe the image of f under the Borcherds lifting we first recall some facts on the Hermitean symmetric space associated with the orthogonal group $\mathrm{O}(V) \cong \mathrm{O}(2, l)$ of V. We extend the bilinear form $(\cdot, \cdot)$ attached to the quadratic form $q(\cdot)$ to a $\mathbb{C}$-bilinear form on the complexification $V(\mathbb{C})$ of V. The subset

$$\mathcal{K} = \big\{ [W] \in P(V(\mathbb{C}));\quad (W, W) = 0, \quad (W, \overline{W}) > 0 \big\}$$

of the projective space $P(V(\mathbb{C}))$ of $V(\mathbb{C})$ has 2 components. We chose one of them and denote it by $\mathcal{K}^+$. It is easily verified that the connected component $\mathrm{O}^+(V)$ of the identity of $\mathrm{O}(V)$ acts transitively on $\mathcal{K}^+$. The stabilizer H of a fixed point is a maximal compact subgroup and thereby $\mathcal{K}^+ \cong \mathrm{O}^+(V)/H$.

Primitive isotropic vectors in L correspond to rational zero-dimensional boundary components of $\mathcal{K}^+$ in the sense of Baily-Borel. Let $z \in L$ be a

[1] Confusion with the quadratic form q on V will not be possible.

primitive isotropic vector, and $z' \in L$ isotropic with $(z, z') = 1$. Then the lattice $K = L \cap z^\perp \cap z'^\perp$ is unimodular and has signature $(1, l-1)$. The assignment $Z \mapsto [Z + z' - q(Z)z]$ defines a biholomorphic map from the set of $Z \in K \otimes \mathbb{C}$ with positive imaginary part Y (i.e. $q(Y) > 0$) to $\mathcal{K}$. One connected component $\mathbb{H}_l$ of this set is mapped isomorphically to $\mathcal{K}^+$. The domain $\mathbb{H}_l$ is a tube domain realization of $\mathrm{O}^+(V)/H$ in $\mathbb{C}^l$ and can be viewed as a generalized upper half plane. The linear action of $\mathrm{O}^+(V)$ on $\mathcal{K}^+$ induces an action on $\mathbb{H}_l$ by fractional linear transformations.

The orthogonal group $\mathrm{O}(L)$ of the lattice L is an arithmetic subgroup of $\mathrm{O}(V)$, and the subgroup $\Gamma(L) = \mathrm{O}(L) \cap \mathrm{O}^+(V)$ acts on $\mathbb{H}_l$. We are interested in the geometry of the quotient $\mathcal{X}_L = \mathbb{H}_l/\Gamma(L)$. By the theory of Baily-Borel it is a quasi-projective algebraic variety over $\mathbb{C}$. There are certain special divisors on $\mathcal{X}_L$, which arise from embedded quotients of type $\mathrm{O}(2, l-1)$. Let m be a negative integer. The orthogonal complement of a vector $\lambda \in L$ with $q(\lambda) = m$ is a quadratic space of signature $(2, l-1)$. The sum

$$H(m) = \sum_{\substack{\lambda \in L/\{\pm 1\} \\ q(\lambda) = m}} \lambda^\perp \subset \mathcal{K}^+ \subset P(V(\mathbb{C}))$$

defines a $\Gamma(L)$-invariant divisor on $\mathbb{H}_l$. It is the inverse image of an algebraic divisor on $\mathcal{X}_L$ (also denoted by $H(m)$). Following Borcherds we call it the *Heegner divisor* of discriminant m.

Let f be a nearly holomorphic modular form of weight $1 - l/2$ with Fourier coefficients $c(n)$ as before, and assume that $c(n) \in \mathbb{Z}$ for $n < 0$. Then the Borcherds lifting of f is given by the infinite product

$$\Psi(Z) = e\big((\varrho_f(W), Z)\big) \prod_{\substack{\lambda \in K \\ (\lambda, W) > 0}} \big(1 - e((\lambda, Z))\big)^{c(q(\lambda))}$$

for $Z \in \mathbb{H}_l$ (as usual $e(\tau) := e^{2\pi i \tau}$). Here $\varrho_f(W) \in K \otimes \mathbb{Q}$ is a so-called Weyl vector, and $(\lambda, W) > 0$ under the product means a certain positivity condition.

Theorem 0.1 (Borcherds). *(See Theorem 13.3 in [Bo2] or Theorem 3.22 in this text.) The product $\Psi(Z)$ converges, if the imaginary part of Z is sufficiently large. It can be continued to a meromorphic function on $\mathbb{H}_l$ with the following properties:*

1. *The function Ψ is a meromorphic modular form for the group $\Gamma(L)$ with some multiplier system.*
2. *The weight of Ψ is given by the constant term of f. It equals $c(0)/2$.*
3. *The divisor of Ψ is determined by the principal part of f. It is given by the linear combination of Heegner divisors*

$$\sum_{m<0} c(m) H(m).$$

Borcherds' first proof of this result in [Bo1] was rather indirect. There he writes down the infinite product, shows that it can be meromorphically continued with the right poles and zeros using the Hardy-Ramanujan-Rademacher asymptotics, and proves that it is automorphic by looking at the transformation behavior under suitable generators of $\Gamma(L)$. Later the physicists Harvey and Moore discovered that Borcherds' lifting can be understood in the framework of the theta correspondence, if one regularizes the wildly divergent integrals in the right way [HM, Kon]. This idea was used by Borcherds in [Bo2] to develop a more conceptual account to his lifting and to extend it in many ways. We briefly indicate the idea of the latter approach:

The groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{O}(2,l)$ form a dual reductive pair in the sense of Howe [Ho]. Thus it is possible to lift automorphic forms on one group to automorphic forms on the other by integrating against a certain kernel function, the Siegel theta function $\Theta_L(\tau,Z)$ of the lattice L. This theta function transforms as a non-holomorphic modular form in the variable $\tau\in\mathbb{H}$ and is $\Gamma(L)$-invariant in the second variable $Z\in\mathbb{H}_l$. If f is a nearly holomorphic modular form of weight $1-l/2$ as above, then its theta lifting is formally given by

$$\Phi(Z)=\int\limits_{\mathcal{F}} f(\tau)\overline{\Theta_L(\tau,Z)}y\,\frac{dx\,dy}{y^2},$$

where $\mathcal{F}=\{\tau=x+iy\in\mathbb{H};\quad |x|\leq 1/2,\ |\tau|\geq 1\}$ denotes the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Unfortunately, since f grows exponentially as $\Im(\tau)\to\infty$, the integral diverges. However, it can be regularized by taking the constant term in the Laurent expansion at $s=0$ of the meromorphic continuation in s of

$$\lim_{u\to\infty}\int\limits_{\mathcal{F}_u} f(\tau)\overline{\Theta_L(\tau,Z)}y^{1+s}\,\frac{dx\,dy}{y^2},$$

which converges for $\Re(s)\gg 0$. Here $\mathcal{F}_u=\{\tau=x+iy\in\mathcal{F};\quad y\leq u\}$ denotes the truncated fundamental domain. The above limit roughly means that we first integrate over x and afterwards over y.

The integral representation can be used to compute the singularities of $\Phi(Z)$ and its Fourier expansion for $q(Y)\gg 0$. It turns out that $\Phi(Z)=\log|\Psi(Z)|$. In that way the above properties of Ψ can be deduced.

To illustrate this result, let us describe a famous example. Let K be the even unimodular lattice $II_{1,25}$ and $L=K\oplus II_{1,1}$ be the orthogonal sum of K with a hyperbolic plane. Then $l=26$, and by the above theorem there is a lifting from nearly holomorphic modular forms of weight -12 for $\mathrm{SL}_2(\mathbb{Z})$ to meromorphic modular forms for the group $\Gamma(L)$ on $\mathbb{H}_{26}$. If $\Delta(\tau)=q\prod_{n=1}^\infty(1-q^n)^{24}$ denotes the classical Delta-function, then we can consider the Borcherds lifting of

$$1/\Delta(\tau)=\sum_n p_{24}(n+1)q^n=q^{-1}+24+324q+3200q^2+25650q^3+\dots.$$

We find that

$$\Psi(Z) = e\big((\varrho, Z)\big) \prod_{\substack{\lambda \in K \\ \lambda > 0}} \big(1 - e((\lambda, Z))\big)^{p_{24}(1+q(\lambda))}$$

is a holomorphic modular form of weight 12 for the group $\Gamma(L)$. Its divisor is equal to $H(-1)$. Here we are in the special situation that 12 is the singular weight for $\mathrm{O}(2, 26)$. Hence all Fourier coefficients $a(\lambda)$ of Ψ must vanish unless $q(\lambda) = 0$. The latter coefficients can be easily determined. One finds that

$$\Psi(Z) = \sum_{w \in W} \det(w) \Delta\big((w(\varrho), Z)\big),$$

where W is the reflection group of the lattice $II_{1,25}$. If we compare the two equalities, we get the Weyl denominator formula for the fake monster Lie algebra (see [Bo1]).

In the present book we are mainly interested in geometric aspects of the Borcherds lifting. The fact that it gives us explicit relations between Heegner divisors in the divisor class group $\mathrm{Cl}(\mathcal{X}_L)$ of $\mathcal{X}_L$ makes it a very useful tool for the study of the geometry of these divisors. In this context it is convenient to work with a slightly modified divisor class group $\widetilde{\mathrm{Cl}}(\mathcal{X}_L)$, which is defined as the quotient of the group of divisors on $\mathcal{X}_L$ modulo the subgroup of divisors coming from meromorphic modular forms for $\Gamma(L)$ of rational weight r with some multiplier system.

By Theorem 0.1 we know that if $\sum_{m<0} c(m) q^m$ is a Fourier polynomial, which is the principal part of a nearly holomorphic modular form of weight $1 - l/2$, then the corresponding linear combination $\sum_{m<0} c(m) H(m)$ of Heegner divisors is 0 in $\widetilde{\mathrm{Cl}}(\mathcal{X}_L)$. It is natural to ask, which Fourier polynomials occur as principal parts of nearly holomorphic modular forms.

In the above example it is clear that every Fourier polynomial occurs as the principal part of a nearly holomorphic modular form of weight -12. To see this one can for instance apply the Hecke operator $T(n)$ in weight -12 to $1/\Delta$, to get a nearly holomorphic modular form with principal part q^{-n}. More generally, this argument shows that every Heegner divisor is trivial in $\widetilde{\mathrm{Cl}}(\mathcal{X}_L)$, if there exists a nearly holomorphic modular form of weight $1 - l/2$ with principal part q^{-1}. However, the situation gets more complicated, if there are non-zero holomorphic cusp forms of complementary weight $1 + l/2$. If $g = \sum_n a(n) q^n$ is such a cusp form and $f = \sum_n c(n) q^n$ any nearly holomorphic modular form of weight $1 - l/2$, then

$$f(\tau) g(\tau) d\tau$$

is a meromorphic differential form on the Riemann surface $\overline{\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})} = P^1(\mathbb{C})$. It has just one pole, which lies at the cusp ∞. The residue is given by

$$\mathcal{C}[fg] = \sum_{n<0} c(n)a(-n),$$

where $\mathcal{C}[h]$ means the constant term of a Laurent series $h \in \mathbb{C}((q))$. By the residue theorem this quantity has to vanish. We get a necessary condition for the existence of nearly holomorphic modular forms. In particular we see that there is no such form with principal part q^{-1}, if there are non-zero cusp forms of weight $1 + l/2$. As an application of Serre duality on Riemann surfaces, Borcherds proved (more generally for vector valued modular forms) that the above residue condition is also sufficient (see [Bo3] Theorem 3.1):

Proposition 0.2. *There exists a nearly holomorphic modular form f for $\mathrm{SL}_2(\mathbb{Z})$ of weight $k \leq 2$ with prescribed principal part $\sum_{n<0} c(n)q^n$, if and only if*

$$\sum_{n<0} c(n)a(-n) = 0$$

for all cusp forms $g = \sum_n a(n)q^n$ of weight $2 - k$ for $\mathrm{SL}_2(\mathbb{Z})$.

In the easy special case of scalar valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$ this can be proved directly by induction: If $k \geq -12$ and $k \neq -10$, there are no non-zero cusp forms of weight $2 - k$. The quotient of a modular form of weight $k+12$ with Fourier expansion $1+O(q)$ and Δ is a nearly holomorphic modular form of weight k with principal part q^{-1}. For instance the above Hecke operator argument shows that any principal part can be realized.

Now let $k = -10$ or $k < -12$. Suppose that the Fourier polynomial $p = \sum_{n<0} c(n)q^n$ satisfies the condition of the proposition. Then the principal part of the Laurent series $p\Delta$ clearly satisfies the condition in weight $k + 12$. By induction assumption there is a nearly holomorphic modular form $\tilde{f}$ of weight $k + 12$ with the same principal part as $p\Delta$. We now show that the constant terms of $p\Delta$ and $\tilde{f}$ also agree. If E is a modular form of weight $2 - k - 12$ with $E = 1 + O(q)$, then $\mathcal{C}[p\Delta - \tilde{f}] = \mathcal{C}[p\Delta E] - \mathcal{C}[\tilde{f}E]$. But the first quantity on the right hand side vanishes by assumption and the second by the residue theorem. Hence $\tilde{f} = p\Delta + O(q)$, and $f = \tilde{f}/\Delta$ is a nearly holomorphic modular form of weight k with principal part p.

The proposition in particular tells us that whenever there are cusp forms of weight $1+l/2$, then there are many linear combinations of Heegner divisors, which are not the divisor of a Borcherds product. Here the natural question arises, whether all relations between Heegner divisors in $\widetilde{\mathrm{Cl}}(\mathcal{X}_L)$ are given by Borcherds products. In other words we ask:

Question 0.3. Let F be any meromorphic modular form for $\Gamma(L)$ (with some multiplier system), whose divisor is a linear combination of Heegner divisors $H(m)$. Is f then a Borcherds product?

The present work was motivated by this question. Before we describe our approach to the problem, we rephrase it in a more algebraic way. Let

$\kappa = 1 + l/2$ and denote by S_κ the space of cusp forms of weight κ for $\mathrm{SL}_2(\mathbb{Z})$. We write $\mathcal{A}_\kappa(\mathbb{Z})$ for the $\mathbb{Z}$-submodule of the dual space S_κ^* of S_κ generated by the functionals

$$a_r : g \mapsto b(r), \qquad \text{for } g = \sum_{n \geq 1} b(n) q^n \in S_\kappa.$$

The fact that S_κ has a basis of modular forms with coefficients in $\mathbb{Z}$ implies that $\mathcal{A}_\kappa(\mathbb{Z}) \otimes \mathbb{C} = S_\kappa^*$. Combining the Borcherds lifting with the proposition, we get the following result.

Theorem 0.4. *The assignment $a_r \mapsto H(-r)$ defines a homomorphism*

$$\eta : \mathcal{A}_\kappa(\mathbb{Z}) \longrightarrow \widetilde{\mathrm{Cl}}(\mathcal{X}_L).$$

The above Question 0.3 is equivalent to the question whether η is injective.

Our approach to this problem consists in the following steps.

1. Extend the regularized theta lifting to Maass wave forms with singularities at the cusps to find for *every* Heegner divisor an interesting analytic object, which can be used in the study of the next two steps.
2. Compose the map η with the Chern class map to the second cohomology of the space $\mathcal{X}_L$.
3. Find an automorphic description of the resulting map from S_κ to the cohomology in terms of square integrable harmonic differential forms. Use its properties to obtain a criterion for the injectivity of η.

The first step is carried out in chapters 1 to 3, the second and third in chapters 4 and 5. We now describe the content of this book in more detail. We stick to our simplifying assumption that L be unimodular, a much more general case is considered in the body of the text. Moreover, we assume from now on that $l > 2$.

We want to extend the regularized theta lifting to Maass wave forms of (negative) weight $k = 1 - l/2$ for $\mathrm{SL}_2(\mathbb{Z})$ with singularities at the cusps. Recall that such a Maass wave form is a function f on the upper half plane, which transforms under $\mathrm{SL}_2(\mathbb{Z})$ like a modular form of weight k, but which is only an eigenfunction of the hyperbolic Laplacian Δ_k instead of being holomorphic. Moreover, we require that f grows slower than $O(e^{2\pi M y})$ as $y \to \infty$ for some positive integer M.

It can be shown that the space of these Maass wave forms with fixed eigenvalue is spanned by certain Poincaré series, which we now describe. Let m be a negative integer, and $M_{\nu,\mu}(y)$ be the M-Whittaker function. If we put $\mathcal{M}_s(y) = y^{-k/2} M_{-k/2,\, s-1/2}(y)$ for $s \in \mathbb{C}$, then $\mathcal{M}_s(4\pi|m|y)e(mx)$ is an eigenfunction of Δ_k with eigenvalue $s(1-s) + (k^2 - 2k)/4$. We consider the Poincaré series of weight k and index m

$$F_m(\tau,s)=\frac{1}{2\Gamma(2s)}\sum_{M\in\Gamma_\infty\setminus\operatorname{SL}_2(\mathbb{Z})}[\mathcal{M}_s(4\pi|m|y)e(mx)]\mid_k M,$$

where $\Gamma_\infty = \{\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right),\ n\in\mathbb{Z}\}$, and $|_k$ denotes the usual Petersson slash operator in weight k. It is easily seen that the series converges normally for $\tau\in\mathbb{H}$ and $\Re(s)>1$. It is an eigenfunction of Δ_k, since the action of Δ_k and the slash operator commute. The asymptotic behavior of the M-Whittaker function implies that $F_m(\tau,s)$ increases exponentially as $e^{2\pi|m|y}$ for $y\to\infty$. We determine the Fourier expansion of $F_m(\tau,s)$ explicitly in Theorem 1.9.

For $s=1-k/2$ the functions $F_m(\tau,s)$ are particularly interesting, because $F_m(\tau,1-k/2)$ is annihilated by the Laplacian Δ_k and thereby close to being holomorphic on $\mathbb{H}$. We prove that any nearly holomorphic modular form of weight k is a linear combination of the $F_m(\tau,1-k/2)$.

The point is that (in any weight k) any Fourier polynomial can be realized as the principal part of a linear combination of these Poincaré series. So they can be viewed as a generalization of nearly holomorphic modular forms.

The Fourier coefficients of the non-holomorphic part of $F_m(\tau,1-k/2)$ can be identified with the coefficients of holomorphic Poincaré series of weight $\kappa=1+l/2$ in S_κ (Proposition 1.16). As a first application of this crucial observation we derive another proof of Proposition 0.2, which works more generally for vector valued modular forms. We show that a finite linear combination

$$\sum_{m<0}c(m)F_m(\tau,1-k/2)$$

is nearly holomorphic, if and only if the functional $\sum_m c(m)a_{-m}$ is equal to zero in S_κ^*. Similar results in a slightly different setting are well known, see for instance [He], [Ni].

In view of the above discussion it is natural to investigate the theta lifting of $F_m(\tau,1-k/2)$ to the orthogonal group $\mathrm{O}^+(V)$. Unfortunately, in the same way as for the Borcherds lifting of nearly holomorphic modular forms, the theta integral diverges because of the exponential growth of $F_m(\tau,1-k/2)$ as $y\to\infty$. In section 2.2 we show that it can be regularized as follows. For $s\in\mathbb{C}$ with real part greater than $1-k/2$ we define a function $\Phi_m(Z,s)$ on the generalized upper half plane $\mathbb{H}_l$ by

$$\Phi_m(Z,s)=\lim_{u\to\infty}\int_{\mathcal{F}_u}F_m(\tau,s)\overline{\Theta_L(\tau,v)}y\,\frac{dx\,dy}{y^2}.$$

We show that $\Phi_m(Z,s)$ is holomorphic in s and can be continued to a holomorphic function on $\{s\in\mathbb{C};\ \Re(s)>1,\ s\neq 1-k/2\}$ with a simple pole at $s=1-k/2$. We define the regularized theta integral $\Phi_m(Z)$ to be the constant term of the Laurent expansion in s of the function $\Phi_m(Z,s)$ at $s=1-k/2$. Note that this regularization is different from the regularization due to Harvey, Moore and Borcherds described before. Nevertheless, up to an additive constant it leads to the same lifting. (Hence the present work can be viewed as an answer to Problem 16.13 in [Bo2].) We compute the singularities of $\Phi_m(Z)$. We find that it is a real analytic function on $\mathbb{H}_l - H(m)$ with a logarithmic singularity along $H(m)$.

In section 2.3 the theta integral $\Phi_m(Z,s)$ is evaluated by unfolding it against the Poincaré series $F_m(\tau,s)$. Since the integral is not absolutely convergent, we have to make sure that the argument can still be justified. We obtain a formula for $\Phi_m(Z,s)$ as a Poincaré series for the orthogonal group $\Gamma(L)$ involving the Gauss hypergeometric function (Theorem 2.14). Note that in [Br1, Br2] we started with such a Poincaré series, calculated its Fourier expansion, and thereby found a connection to certain elliptic modular forms in a rather indirect way.

We then determine the Fourier expansion of the function $\Phi_m(Z,s)$ using the same method as Borcherds in [Bo2]. By applying a partial Fourier transform and the Poisson summation formula, the theta series Θ_L can be rewritten as a Poincaré series involving similar theta series Θ_K attached to the Lorentzian sub-lattice K. Thus the theta integral can also be unfolded against the kernel Θ_L (Theorem 2.15).

In chapter 3 we investigate the Fourier expansion of $\Phi_m(Z)$ in more detail. We find that it can be written as the sum of two real valued functions $\psi_m(Z)$ and $\xi_m(Z)$. Here $\xi_m(Z)$ is real analytic on the whole $\mathbb{H}_l$, and $-\frac{1}{4}\psi_m(Z)$ is the logarithm of the absolute value of a holomorphic function $\Psi_m(Z)$ on $\mathbb{H}_l$ whose divisor equals $H(m)$. As in [Br1] this splitting is fundamental for the results of the present work. In the Fourier expansion of $\xi_m(Z)$ a certain special function $\mathcal{V}_\kappa(A,B)$ occurs. It can be regarded as a generalized Whittaker function, which might be of independent interest.

Observe that the function $\Psi_m(Z)$ is not necessarily automorphic; by construction we only know that

$$|\Psi_m(Z)|e^{-\xi_m(Z)/4}$$

is invariant under $\Gamma(L)$. However, taking suitable finite products of the Ψ_m, one can attain that the main parts of the ξ_m cancel out. Thereby Theorem 0.1 can be recovered from a cohomological point of view (see Theorem 3.22). If f is a nearly holomorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ as before, then the function

$$\prod_{m<0} \Psi_m(Z)^{c(m)}$$

equals the Borcherds lifting $\Psi(Z)$ of f up to a multiplicative constant of absolute value 1.

Chapter 4 is the technical heart of this book. We consider the $\mathrm{O}(2,l)$-invariant Laplace operator Ω acting on functions on the generalized upper half plane $\mathbb{H}_l$. We prove that $\Phi_m(Z,s)$ is an eigenfunction of Ω (Theorem 4.6). This gives a partial answer to Problem 16.6 in [Bo2]. More precisely we show

Theorem 0.5. *The function $\Phi_m(Z,s)$ satisfies*

$$\Omega\Phi_m(Z,s) = \frac{1}{2}(s-1/2-l/4)(s-1/2+l/4)\Phi_m(Z,s).$$

The proof relies on the following well known argument: Using the machinery of the Weil representation it can be shown that the Siegel theta function satisfies the differential equation

$$\Delta_k \Theta_L(\tau,Z)y^{l/2} = -2\Omega\Theta_L(\tau,Z)y^{l/2}$$

(cf. [Sn]). Thus the assertion formally follows from the self-adjointness of the Laplacian Δ_k and the fact that $F_m(\tau,s)$ is an eigenfunction of Δ_k. But since the theta integral $\Phi_m(Z,s)$ does not converge absolutely, we have to prove step by step that this argument can still be justified for the regularized integral. As a consequence we find for the regularized function $\Phi_m(Z)$ that

$$\Omega\Phi_m(Z) = c$$

with some explicit real constant c (Theorem 4.7).

Let F be a meromorphic modular form for $\Gamma(L)$ of weight r. Suppose that its divisor is a linear combination of Heegner divisors:

$$(F) = \sum_{m<0} c(m)H(m).$$

Then the function $G(Z) = \log(|F(Z)|q(Y)^{r/2})$ is $\Gamma(L)$-invariant and has logarithmic singularities along Heegner divisors. It satisfies the differential equation $\Omega G(Z) = -rl/8$, similarly as the functions $\Phi_m(Z)$. We are now ready to state the main result of section 4.3, which is a first step towards answering Question 0.3 (see Theorem 4.23).

Theorem 0.6. *Up to an additive constant the function G is equal to the regularized theta lifting*

$$\Phi(Z) = -\frac{1}{4}\sum_{m<0} c(m)\Phi_m(Z).$$

Hence, loosely speaking, any modular form, whose divisor is a linear combination of Heegner divisors, is given by the regularized theta lifting of a Maass wave form with singularity at ∞. One might expect that this Maass form actually has to be holomorphic on $\mathbb{H}$. However, this seems far from being obvious. We will come back to this problem in section 5.2 and Theorem 0.8.

In the proof of Theorem 0.6 we first show that the difference $G - \Phi$ is a square-integrable function on the quotient $\mathcal{X}_L$. (If $l = 3$, then it is only in $L^p(\mathcal{X}_L)$ for $p < 2$.) This requires some reduction theory for the group $\mathrm{O}(2, l)$. In a couple of technical lemmas we define Siegel domains in $\mathbb{H}_l$ and derive some of their important properties. Moreover, we use the properties of the splitting $\Phi_m = \psi_m + \xi_m$ and an application of the Koecher boundedness principle. (Thus our proof in general breaks down for $l \leq 2$. In fact, this is not the only place where it breaks down.)

In the second step of the proof we exploit the fact that $G-\Phi$ is a smooth solution of the differential equation $\Omega(G-\Phi) = \text{constant}$. By means of results of Andreotti-Vesentini and Yau on (sub-) harmonic functions on complete Riemann manifolds that satisfy certain integrability conditions, we may infer that $G - \Phi$ is constant.

As a corollary (Corollary 4.29) we find that the weight of F is given by a linear combination of the coefficients of the normalized Eisenstein series of weight κ for $\mathrm{SL}_2(\mathbb{Z})$.

At the beginning of chapter 5 we determine the Chern class of the Heegner divisor $H(m)$ explicitly. The above properties of ξ_m and Ψ_m imply that $\Psi_m(Z)$ defines a trivialization of the sheaf $\mathcal{L}(H(m))$ associated with $H(m)$, and $e^{\xi_m(Z)/4}$ a Hermitean metric on $\mathcal{L}(H(m))$. Hence the $(1,1)$-form

$$h_m(Z) = \frac{1}{4}\partial\bar{\partial}\xi_m(Z) = \frac{1}{4}\partial\bar{\partial}\Phi_m(Z)$$

represents the Chern class of $H(m)$ in $H^2(\mathcal{X}_L, \mathbb{C})$ via the de Rham isomorphism (Theorem 5.3). Moreover, we show in Theorem 5.5 that $h_m(Z)$ is actually a square integrable harmonic representative.

Together with Theorems 2.12 and 3.9 on the singularities and the Fourier expansion of $\Phi_m(Z)$ this result implies that $\Phi_m(Z)$ is some kind of Green current for the divisor $H(m)$ in the sense of Arakelov geometry (see [SABK]). More precisely this function should define a Green object with log-log-growth in the extended arithmetic intersection theory due to Burgos, Kramer, and Kühn [BKK].

Let $\mathcal{H}^{1,1}(\mathcal{X}_L)$ denote the space of square integrable harmonic $(1,1)$-forms on $\mathcal{X}_L$. By a result of Borel it is a finite dimensional space of automorphic forms [Bl3]. Write $\widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$ for the quotient of $\mathcal{H}^{1,1}(\mathcal{X}_L)$ and the span of the Kähler form $\partial\bar{\partial}\log q(Y)$. It is a consequence of Theorem 0.6 that the assignment

$$H(m) \mapsto h_m(Z)$$

defines a linear map from the subspace of $\widetilde{\mathrm{Cl}}(\mathcal{X}_L)$ generated by the Heegner divisors $H(m)$ to $\widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$. If we compose it with the map η of Theorem 0.4 and tensor with $\mathbb{C}$, we get a homomorphism $\tilde{\vartheta} : S_\kappa^* \to \widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$.

In section 5.1 we show that an automorphic kernel function for $\tilde{\vartheta}$ is essentially given by

$$\Omega(Z,\tau) = \frac{1}{4}\partial\bar{\partial}\log q(Y) + \sum_{n>0} h_{-n}(Z)q^n.$$

To this end we first observe that the function $\Omega(Z,\tau)$ converges normally and is square integrable in Z. Hence it is clear that $\Omega(Z,\tau)$ lies in $\mathcal{H}^{1,1}(\mathcal{X}_L)$ for fixed τ. Using the fact that the Fourier coefficients of the Poincaré series $F_m(\tau, 1-k/2)$ are equal to certain coefficients of holomorphic Poincaré series in S_κ, we may rewrite $\Omega(Z,\tau)$ such that it becomes a sum of Poincaré, Eisenstein, and theta series in the space of modular forms M_κ of weight κ for $\mathrm{SL}_2(\mathbb{Z})$. Hence for fixed Z the function $\Omega(Z,\tau)$ is contained in M_κ (Theorem 5.8). The idea of rewriting an automorphic kernel function in terms of Poincaré series appears already in Zagier's work on the Doi-Naganuma lifting [Za]. It is related to the "Zagier identity" in [RS]. We prove:

Theorem 0.7. *By taking the Petersson scalar product in the τ-variable the function $\Omega(Z,\tau)$ defines a lifting*

$$\vartheta : S_\kappa \longrightarrow \mathcal{H}^{1,1}(\mathcal{X}_L), \qquad f \mapsto \big(f(\tau), \Omega(Z,\tau)\big),$$

with the following properties (see Theorem 5.9 for more details):

1. *Let $f = \sum_n c(n)q^n$ be a cusp form in S_κ. Then the image of f has the Fourier expansion*

$$\vartheta(f)(Z) = \vartheta_0(f)(Y) - C(\kappa) \sum_{\substack{\lambda\in K \\ q(\lambda)<0}} |\lambda|^{-l} \sum_{n|\lambda} n^{l-1} c(-q(\lambda)/n^2) \times \partial\bar{\partial}\mathcal{V}_\kappa\left(\pi|\lambda||Y|, \pi(\lambda,Y)\right) e((\lambda,X)),$$

   where $C(\kappa) = 2^{-\kappa}\pi^{1/2-\kappa}$ is a constant, and $\vartheta_0(f)(Y)$ the 0-th Fourier coefficient, which can also be computed explicitly. Moreover, $\mathcal{V}_\kappa(A,B)$ denotes the generalized Whittaker function defined in (3.25).
2. *The function $\Omega(Z,\tau)$ is an automorphic kernel function for $\tilde{\vartheta}$, that is the diagram*

$$\begin{array}{ccc} S_\kappa & \xrightarrow{\vartheta} & \mathcal{H}^{1,1}(\mathcal{X}_L) \\ \uparrow & & \downarrow \\ S_\kappa^* & \xrightarrow{\tilde{\vartheta}} & \widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L) \end{array}$$

   commutes. Here the left vertical arrow denotes the identification via Petersson scalar product.

The lifting ϑ can be viewed as a generalization of the Doi-Naganuma map (from cusp forms of weight 2 for $\Gamma_0(D)$ to the cohomology of the Hilbert modular surface in question) [DN, Na, Za, Br1]. Note that in the $\mathrm{O}(2,3)$-case of Siegel modular forms of genus 2 such a generalization has been given by Piateskii-Shapiro using representation theoretic methods [PS1, PS2]. It would be interesting to understand the map ϑ in the context of the theory of Kudla and Millson on intersection numbers of cycles on locally symmetric spaces (see for instance [KM1, KM2, KM3]).

The description of ϑ in terms of Fourier expansions given in Theorem 0.7 is vital for answering our initial Question 0.3. It can be used to prove that ϑ is injective in the special case that L splits two hyperbolic planes over $\mathbb{Z}$. (This is automatically true, if L is unimodular.) Hence, by the second statement of Theorem 0.7, $\tilde{\vartheta}$ is also injective, which finally implies that η is injective as well. We obtain the main result of section 5.2 (Theorem 5.12).

Theorem 0.8. *Assume that L splits two hyperbolic planes over $\mathbb{Z}$. Let F be a meromorphic modular form for the group $\Gamma(L)$ (with some multiplier system), whose divisor is a linear combination of Heegner divisors*

$$(F) = \sum_{m<0} c(m) H(m).$$

Then F is a Borcherds product in the sense of Theorem 0.1, i.e. there exists a nearly holomorphic modular form f of weight $1 - l/2$ with principal part $\sum_{m<0} c(m) q^m$, and the Borcherds lifting of f equals F.

We consider the special $\mathrm{O}(2,3)$-case of the paramodular group of level t as an example. By Theorem 0.7 we get a lifting from skew-holomorphic Jacobi forms of index t and weight 3 to square integrable harmonic $(1,1)$-forms on the corresponding quotient, the moduli space of Abelian surfaces with a $(1,t)$-polarization.

Except for Theorem 0.8, we prove all the above results under the mild assumptions that $l > 2$, and that L splits two orthogonal hyperbolic planes over $\mathbb{Q}$. This is always true, if $l > 4$. (We do not require that L is unimodular.) Then the discriminant group, the quotient L'/L of the dual lattice L' modulo L, is usually non-trivial. This leads to several technical complications. For instance, in general we have to replace the space S_κ of elliptic cusp forms by a certain space $S_{\kappa,L}$ of $\mathbb{C}[L'/L]$-valued cusp forms of (possibly half-integral) weight κ for the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$. In the same way we have to consider vector valued nearly holomorphic modular forms and Maass wave forms. Therefore in section 1.2 and 1.3 we collect some basic facts on such modular forms and carry out the standard constructions of Poincaré and Eisenstein series. Moreover, many objects which are indexed by an integer in the above exposition (as $H(m)$ or Φ_m) have to be replaced by objects which are indexed by a tuple (β, m), where $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ (as $H(\beta, m)$ or $\Phi_{\beta,m}$). For the group $\Gamma(L)$ we need to take the discriminant kernel of the orthogonal group of L.

Theorem 0.8 is proved under the assumption that L splits two orthogonal hyperbolic planes over $\mathbb{Z}$. It seems likely that (at least for large l) this result can also be extended to a more general situation. However, this might require additional considerations as some newform theory or Hecke theory for the space $S_{\kappa,L}$ and the lifting ϑ.

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1 Vector valued modular forms for the metaplectic group

In the following chapter we briefly recall some facts about the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$, the Weil representation, and certain vector valued modular forms. We essentially follow the terminology of Borcherds [Bo2]. Moreover, we consider vector valued Poincaré and Eisenstein series in some detail.

1.1 The Weil representation

As usual, we denote by $\mathbb{H} = \{\tau \in \mathbb{C};\ \Im(\tau) > 0\}$ the upper complex half plane. Throughout (unless otherwise specified) we will use τ as a standard variable on $\mathbb{H}$ and write x for its real part and y for its imaginary part, respectively. For $z \in \mathbb{C}$ we put $e(z) = e^{2\pi i z}$, and denote by $\sqrt{z} = z^{1/2}$ the principal branch of the square root, so that $\arg(\sqrt{z}) \in (-\pi/2, \pi/2]$. For any integer k we put $z^{k/2} = (z^{1/2})^k$. More generally if $b \in \mathbb{C}$, then we define $z^b = e^{b\,\mathrm{Log}(z)}$, where $\mathrm{Log}(z)$ denotes the principal branch of the logarithm. If x is a non-zero real number, we let $\mathrm{sgn}(x) = x/|x|$.

We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic group, i.e. the double cover of $\mathrm{SL}_2(\mathbb{R})$, realized by the two choices of holomorphic square roots of $\tau \mapsto c\tau + d$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Thus the elements of $\mathrm{Mp}_2(\mathbb{R})$ are pairs

$$(M, \phi(\tau)),$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, and ϕ denotes a holomorphic function on $\mathbb{H}$ with

$$\phi(\tau)^2 = c\tau + d.$$

The product of $(M_1, \phi_1(\tau)), (M_2, \phi_2(\tau)) \in \mathrm{Mp}_2(\mathbb{R})$ is given by

$$\big(M_1, \phi_1(\tau)\big)\big(M_2, \phi_2(\tau)\big) = \big(M_1 M_2, \phi_1(M_2\tau)\phi_2(\tau)\big),$$

where $M\tau = \frac{a\tau + b}{c\tau + d}$ denotes the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$. The map

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau + d}\right) \tag{1.1}$$

defines a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{Mp}_2(\mathbb{R})$.

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right),$$
$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

One has the relations $S^2 = (ST)^3 = Z$, where

$$Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right)$$

is the standard generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$. We will often use the abbreviations $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$, $\Gamma_\infty = \{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right); \quad n \in \mathbb{Z} \} \leq \Gamma_1$, and

$$\tilde{\Gamma}_\infty := \langle T \rangle = \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1 \right); \quad n \in \mathbb{Z} \right\}.$$

Throughout let L be an even lattice, i.e. a free $\mathbb{Z}$-module of finite rank, equipped with a symmetric $\mathbb{Z}$-valued bilinear form $(\cdot,\cdot)$ such that the associated quadratic form

$$q(x) = \frac{1}{2}(x, x)$$

takes its values in $\mathbb{Z}$. We assume that L is non-degenerated and denote its signature by (b^+, b^-). We write

$$L' = \{ x \in L \otimes \mathbb{Q}; \quad (x, y) \in \mathbb{Z} \text{ for all } y \in L \}$$

for the dual lattice. Then the quotient L'/L is a finite Abelian group, the so-called discriminant group. The modulo 1 reduction of $q(x)$ is a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on L'/L whose associated bilinear form is the modulo 1 reduction of the bilinear form $(\cdot,\cdot)$ on L'.

Recall that there is a unitary representation ϱ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$. If we denote the standard basis of $\mathbb{C}[L'/L]$ by $(\mathfrak{e}_\gamma)_{\gamma \in L'/L}$, then ϱ_L can be defined by the action of the generators $S, T \in \mathrm{Mp}_2(\mathbb{Z})$ as follows (cp. [Bo2]):

$$\varrho_L(T)\mathfrak{e}_\gamma = e(q(\gamma))\mathfrak{e}_\gamma \tag{1.2}$$

$$\varrho_L(S)\mathfrak{e}_\gamma = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} e(-(\gamma, \delta))\mathfrak{e}_\delta. \tag{1.3}$$

This representation is essentially the Weil representation attached to the quadratic module $(L'/L, q)$ (cf. [No]). It factors through the finite group

$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ + b^-$ is even, and through a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ + b^-$ is odd, where N is the smallest integer such that $Nq(\gamma) \in \mathbb{Z}$ for all $\gamma \in L'$. Note that

$$\varrho_L(Z)\mathfrak{e}_\gamma = i^{b^- - b^+}\mathfrak{e}_{-\gamma}. \tag{1.4}$$

We write $\langle \cdot, \cdot \rangle$ for the standard scalar product on $\mathbb{C}[L'/L]$, i.e.

$$\left\langle \sum_{\gamma \in L'/L} \lambda_\gamma \mathfrak{e}_\gamma, \sum_{\gamma \in L'/L} \mu_\gamma \mathfrak{e}_\gamma \right\rangle = \sum_{\gamma \in L'/L} \lambda_\gamma \overline{\mu_\gamma}.$$

For $\gamma, \delta \in L'/L$ and $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ we define the coefficient $\varrho_{\gamma\delta}(M, \phi)$ of the representation ϱ_L by

$$\varrho_{\gamma\delta}(M, \phi) = \langle \varrho_L(M, \phi)\mathfrak{e}_\delta, \mathfrak{e}_\gamma \rangle.$$

Shintani proved the following explicit formula for ϱ_L (cf. [Sn], Prop. 1.6).

Proposition 1.1. *Let $\alpha, \beta \in L'/L$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then the coefficient $\varrho_{\alpha\beta}(\tilde{M})$ is given by*

$$\sqrt{i}^{(b^- - b^+)(1 - \operatorname{sgn}(d))} \delta_{\alpha, a\beta} e(abq(\alpha)), \tag{1.5}$$

if $c = 0$, and by

$$\frac{\sqrt{i}^{(b^- - b^+)\operatorname{sgn}(c)}}{|c|^{(b^- + b^+)/2}\sqrt{|L'/L|}} \sum_{r \in L/cL} e\left(\frac{a(\alpha + r, \alpha + r) - 2(\beta, \alpha + r) + d(\beta, \beta)}{2c}\right), \tag{1.6}$$

if $c \neq 0$. Here, $\delta_{*,*}$ denotes the Kronecker-delta.

Let $\kappa \in \frac{1}{2}\mathbb{Z}$ and f be a $\mathbb{C}[L'/L]$-valued function on $\mathbb{H}$. For $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ we define the Petersson slash operator by

$$\left(f \mid_\kappa (M, \phi)\right)(\tau) = \phi(\tau)^{-2\kappa} \varrho_L(M, \phi)^{-1} f(M\tau). \tag{1.7}$$

As usual, by $f \mapsto f \mid_\kappa (M, \phi)$ an operation of $\mathrm{Mp}_2(\mathbb{Z})$ on functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is defined.

We denote by ϱ_L^* the dual representation of ϱ_L. If we think of $\varrho_L^*(M, \phi)$ ($(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$) as a matrix with entries in $\mathbb{C}$, then $\varrho_L^*(M, \phi)$ is simply the complex conjugate of $\varrho_L(M, \phi)$. We have a "dual operation" of $\mathrm{Mp}_2(\mathbb{Z})$ on functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$, given by

$$\left(f \mid_\kappa^* (M, \phi)\right)(\tau) = \phi(\tau)^{-2\kappa} \varrho_L^*(M, \phi)^{-1} f(M\tau). \tag{1.8}$$

Now assume that $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is a holomorphic function which is invariant under the $\mid_\kappa^*$-operation of $T \in \mathrm{Mp}_2(\mathbb{Z})$. We denote the components of f by f_γ, so that $f = \sum_{\gamma\in L'/L} \mathfrak{e}_\gamma f_\gamma$. The invariance of f under T implies that the functions $e(q(\gamma)\tau)f_\gamma(\tau)$ are periodic with period 1. Thus f has a Fourier expansion

$$f(\tau) = \sum_{\gamma\in L'/L} \sum_{n\in\mathbb{Z}-q(\gamma)} \mathfrak{e}_\gamma c(\gamma, n) e(n\tau),$$

with Fourier coefficients $c(\gamma, n) = \int_0^1 f_\gamma(\tau)e(-n\tau)\, dx$. Using the abbreviation $\mathfrak{e}_\gamma(\tau) := \mathfrak{e}_\gamma e(\tau)$ we may write

$$f(\tau) = \sum_{\gamma\in L'/L} \sum_{n\in\mathbb{Z}-q(\gamma)} c(\gamma, n)\mathfrak{e}_\gamma(n\tau). \tag{1.9}$$

By means of the scalar product in $\mathbb{C}[L'/L]$ the coefficients can be expressed by

$$c(\gamma, n) = \int_0^1 \langle f(\tau), \mathfrak{e}_\gamma(n\bar{\tau})\rangle\, dx. \tag{1.10}$$

Definition 1.2. *Let* $\kappa \in \frac{1}{2}\mathbb{Z}$. *A function* $f : \mathbb{H} \to \mathbb{C}[L'/L]$ *is called a* modular form *of weight* κ *with respect to* ϱ_L^* *and* $\mathrm{Mp}_2(\mathbb{Z})$ *if*

i) $f \mid_\kappa^* (M, \phi) = f$ *for all* $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$,
ii) f *is holomorphic on* $\mathbb{H}$,
iii) f *is holomorphic at the cusp* ∞.

Here condition (iii) means that f *has a Fourier expansion of the form*

$$f(\tau) = \sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}-q(\gamma)\\ n\geq 0}} c(\gamma, n)\mathfrak{e}_\gamma(n\tau).$$

Moreover, if all $c(\gamma, n)$ *with* $n = 0$ *vanish, then* f *is called a* cusp form. *The* $\mathbb{C}$*-vector space of modular forms of weight* κ *with respect to* ϱ_L^* *and* $\mathrm{Mp}_2(\mathbb{Z})$ *is denoted by* $M_{\kappa,L}$, *the subspace of cusp forms by* $S_{\kappa,L}$.

It is easily seen that $M_{\kappa,L}$ is finite dimensional.

Example 1.3. Fix a positive integer t. Let L be the 1-dimensional lattice $\mathbb{Z}$, equipped with the positive definite quadratic form $q(a) = ta^2$. Then $L'/L \cong \mathbb{Z}/2t\mathbb{Z}$, and $M_{\kappa-1/2,L}$ is isomorphic to the space $J_{\kappa,t}$ of Jacobi forms of weight κ and index t (cf. [EZ] Theorem 5.1). If L denotes the lattice $\mathbb{Z}$ with the negative definite quadratic form $q(a) = -ta^2$, then $M_{\kappa-1/2,L}$ is isomorphic to the space $\mathcal{J}_{\kappa,t}$ of skew-holomorphic Jacobi forms of weight κ and index t as defined in [Sk].

1.2 Poincaré series and Eisenstein series

In this section we carry out the standard construction of Poincaré and Eisenstein series for the space $M_{\kappa,L}$. For simplicity we assume that $(b^+, b^-) = (2, l)$, $(1, l-1)$, or $(0, l-2)$ with $l \geq 3$. Moreover, we assume that $\kappa = 1 + l/2$ (as we will only need this case later).

1.2.1 Poincaré series

Let $\beta \in L'/L$ and $m \in \mathbb{Z} - q(\beta)$ with $m > 0$. Then $\mathfrak{e}_\beta(m\tau)$ is a holomorphic function $\mathbb{H} \to \mathbb{C}[L'/L]$ which is invariant under the $|_\kappa^*$-operation of $T \in \mathrm{Mp}_2(\mathbb{Z})$. Moreover, by (1.4) we have

$$\mathfrak{e}_\beta(m\tau) \mid_\kappa^* Z = i^{-2\kappa} i^{l-2} \mathfrak{e}_{-\beta}(m\tau) = \mathfrak{e}_{-\beta}(m\tau).$$

Thus $\mathfrak{e}_\beta(m\tau)$ is also invariant under the action of Z^2.

We define the Poincaré series $P^L_{\beta,m}$ of index (β, m) by

$$P^L_{\beta,m}(\tau) = \frac{1}{2} \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \mathfrak{e}_\beta(m\tau) \mid_\kappa^* (M, \phi) \tag{1.11}$$

(recall that $\tilde{\Gamma}_\infty = \langle T \rangle$). Since $\kappa \geq 5/2$, the usual argument shows that this series converges normally on $\mathbb{H}$ and thereby defines a $\mathrm{Mp}_2(\mathbb{Z})$-invariant holomorphic function $\mathbb{H} \to \mathbb{C}[L'/L]$. We will often omit the superscript L, if it is clear from the context.

Theorem 1.4. *The Poincaré series $P_{\beta,m}$ has the Fourier expansion*

$$P_{\beta,m}(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} - q(\gamma) \\ n > 0}} p_{\beta,m}(\gamma, n) \mathfrak{e}_\gamma(n\tau)$$

with

$$\begin{aligned} p_{\beta,m}(\gamma, n) = {} & \delta_{m,n}(\delta_{\beta,\gamma} + \delta_{-\beta,\gamma}) \\ & + 2\pi \left(\frac{n}{m}\right)^{\frac{\kappa-1}{2}} \sum_{c \in \mathbb{Z} - \{0\}} H_c^*(\beta, m, \gamma, n) J_{\kappa-1}\left(\frac{4\pi}{|c|}\sqrt{mn}\right). \end{aligned} \tag{1.12}$$

Here $H_c^*(\beta, m, \gamma, n)$ denotes the generalized Kloosterman sum

$$H_c^*(\beta, m, \gamma, n) = \frac{e^{-\pi i \,\mathrm{sgn}(c)\kappa/2}}{|c|} \sum_{\substack{d(c)^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty}} \varrho_{\beta\gamma} \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} e\left(\frac{ma + nd}{c}\right), \tag{1.13}$$

and J_ν the Bessel function of the first kind ([AbSt] Chap. 9). In particular $P_{\beta,m} \in S_{\kappa,L}$.

Recall the abbreviations $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right); \quad n \in \mathbb{Z}\}$. The sum in (1.13) runs over all primitive residues d modulo c and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a representative for the double coset in $\Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty$ with lower row $(c\ d')$ and $d' \equiv d \pmod{c}$. Observe that the expression $\varrho_{\beta\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} e\left(\frac{ma+nd}{c}\right)$ does not depend on the choice of the coset representative.

The coefficients $\varrho_{\beta\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}$ are universally bounded, because ϱ_L factors through a finite group. Hence there exists a constant $C > 0$ such that $H_c^*(\beta, m, \gamma, n) < C$ for all $\gamma \in L'/L$, $n \in \mathbb{Z} - q(\gamma)$, and $c \in \mathbb{Z} - \{0\}$. This, together with the asymptotic behavior $J_\nu(t) = O(t^\nu)$ $(t \to 0)$ of the J-Bessel function, implies that the series (1.12) converges absolutely. If we also use the asymptotic property $J_\nu(t) = O(t^{-1/2})$ as $t \to \infty$, we find that $p_{\beta,m}(\gamma, n) = O(n^{\kappa-1})$ for $n \to \infty$. (This is actually a very poor bound. For even l Deligne's theorem, previously the Ramanujan-Petersson conjecture, states that $p_{\beta,m}(\gamma, n) = O(n^{(\kappa-1)/2+\varepsilon})$.)

Proof of Theorem 1.4. Let $\gamma \in L'/L$ and $n \in \mathbb{Z} - q(\gamma)$. According to (1.10) the coefficient $p_{\beta,m}(\gamma, n)$ can be computed as

$$p_{\beta,m}(\gamma, n) = \frac{1}{2} \int_0^1 \left\langle \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \mathfrak{e}_\beta(m\tau) \mid_\kappa^* (M, \phi),\ \mathfrak{e}_\gamma(n\bar{\tau}) \right\rangle dx.$$

Since $\mathfrak{e}_\beta(m\tau)$ is invariant under the action of Z^2, this can be written in the form

$$\begin{aligned}
&\int_0^1 \langle \mathfrak{e}_\beta(m\tau),\ \mathfrak{e}_\gamma(n\bar{\tau}) \rangle\, dx + \int_0^1 \langle \mathfrak{e}_\beta(m\tau) \mid_\kappa^* Z,\ \mathfrak{e}_\gamma(n\bar{\tau}) \rangle\, dx \\
&\quad + \int_0^1 \left\langle \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 \\ c \neq 0}} \mathfrak{e}_\beta(m\tau) \mid_\kappa^* \widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)},\ \mathfrak{e}_\gamma(n\bar{\tau}) \right\rangle dx \\
&= \delta_{m,n}(\delta_{\beta,\gamma} + \delta_{-\beta,\gamma}) \\
&\quad + \sum_{\substack{c \neq 0 \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty}} \int_{-\infty}^{\infty} (c\tau + d)^{-\kappa} \left\langle \varrho_L^* \widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}^{-1} \mathfrak{e}_\beta\left(m \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau\right),\ \mathfrak{e}_\gamma(n\bar{\tau}) \right\rangle dx.
\end{aligned}$$

Since ϱ_L is unitary we have

$$\left\langle \varrho_L^* \widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}^{-1} \mathfrak{e}_\beta\left(m \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau\right),\ \mathfrak{e}_\gamma(n\bar{\tau}) \right\rangle = \varrho_{\beta\gamma}\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} e\left(m \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau\right) e(-n\tau),$$

and thereby

$$p_{\beta,m}(\gamma,n) = \delta_{m,n}(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma}) + \sum_{\substack{c\neq 0\\ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_1/\Gamma_\infty}} \varrho_{\beta\gamma}\widetilde{\begin{pmatrix} a & b\\ c & d\end{pmatrix}} \int_{-\infty}^{\infty} (c\tau+d)^{-\kappa}\, e\Big(m\begin{pmatrix} a & b\\ c & d\end{pmatrix}\tau - n\tau\Big)\,dx. \tag{1.14}$$

We now use the elementary identities $\sqrt{c\tau+d} = \operatorname{sgn}(c)\sqrt{c}\sqrt{\tau+d/c}$ and

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\tau = \frac{a}{c} - \frac{1}{c^2(\tau+d/c)}. \tag{1.15}$$

We find that the latter sum in (1.14) equals

$$\begin{aligned}
&\sum_{\substack{c\neq 0\\ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_1/\Gamma_\infty}} |c|^{-\kappa}\operatorname{sgn}(c)^\kappa \varrho_{\beta\gamma}\widetilde{\begin{pmatrix} a & b\\ c & d\end{pmatrix}}\\
&\qquad\times\int_{-\infty}^{\infty}(\tau+d/c)^{-\kappa}\, e\Big(\frac{ma}{c} - \frac{m}{c^2(\tau+d/c)} - n\tau\Big)\,dx\\
&= \sum_{\substack{c\neq 0\\ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_1/\Gamma_\infty}} |c|^{-\kappa}\operatorname{sgn}(c)^\kappa \varrho_{\beta\gamma}\widetilde{\begin{pmatrix} a & b\\ c & d\end{pmatrix}} e\left(\frac{ma}{c}\right)\\
&\qquad\times\int_{-\infty}^{\infty}\tau^{-\kappa}\, e\Big(-\frac{m}{c^2\tau} - n(\tau-d/c)\Big)\,dx\\
&= \sum_{c\neq 0}|c|^{-\kappa}\operatorname{sgn}(c)^\kappa \sum_{\substack{d(c)^*\\ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_1/\Gamma_\infty}} \varrho_{\beta\gamma}\widetilde{\begin{pmatrix} a & b\\ c & d\end{pmatrix}} e\Big(\frac{ma+nd}{c}\Big)\\
&\qquad\times\int_{-\infty}^{\infty}\tau^{-\kappa}\, e\Big(-\frac{m}{c^2\tau} - n\tau\Big)\,dx.
\end{aligned}$$

We substitute $\tau = iw$ in the integral and obtain for the above expression

$$\begin{aligned}
&2\pi i^{-\kappa}\sum_{c\neq 0}|c|^{-\kappa}\operatorname{sgn}(c)^\kappa \sum_{\substack{d(c)^*\\ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_1/\Gamma_\infty}} \varrho_{\beta\gamma}\widetilde{\begin{pmatrix} a & b\\ c & d\end{pmatrix}} e\Big(\frac{ma+nd}{c}\Big)\\
&\qquad\qquad\times\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} w^{-\kappa}\exp\left(-\frac{2\pi m}{c^2 w} + 2\pi n w\right)dw,
\end{aligned}$$

where C is a positive real constant. We now use $i^{-\kappa}\operatorname{sgn}(c)^\kappa = e^{-\pi i \operatorname{sgn}(c)\kappa/2}$ and the definition of the generalized Kloosterman sum (1.13) and find

$$\begin{aligned} p_{\beta,m}(\gamma,n) &= \delta_{m,n}(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma}) + 2\pi \sum_{c\neq 0} |c|^{1-\kappa} H_c^*(\beta,m,\gamma,n) \\ &\qquad \times \frac{1}{2\pi i}\int\limits_{C-i\infty}^{C+i\infty} w^{-\kappa}\exp\left(-\frac{2\pi m}{c^2 w}+2\pi n w\right)dw. \end{aligned} \tag{1.16}$$

If $n \leq 0$, we consider the limit $C\to\infty$ and deduce that $p_{\beta,m}(\gamma,n)=0$. If $n>0$, then the latter integral is an inverse Laplace transform that can be found in any standard reference on integral transforms. According to [E2] p. 245 (40) one has

$$\frac{1}{2\pi i}\int\limits_{C-i\infty}^{C+i\infty} w^{-\kappa}\exp\left(-\frac{2\pi m}{c^2 w}+2\pi n w\right)dw = \left(\frac{c^2 n}{m}\right)^{\frac{\kappa-1}{2}} J_{\kappa-1}\left(\frac{4\pi}{|c|}\sqrt{mn}\right).$$

If we insert this into (1.16) we obtain the assertion. □

1.2.2 The Petersson scalar product

Let f,g be modular forms in $M_{\kappa,L}$. Then it can be easily checked that

$$\langle f(\tau), g(\tau)\rangle y^\kappa$$

is $\mathrm{Mp}_2(\mathbb{Z})$-invariant. (As before, $\langle\cdot,\cdot\rangle$ denotes the scalar product on $\mathbb{C}[L'/L]$.) If f or g is a cusp form, then $\langle f(\tau), g(\tau)\rangle y^\kappa$ is bounded on the standard fundamental domain

$$\mathcal{F} = \{\tau = x+iy\in\mathbb{H};\quad |x|\leq 1/2,\ |\tau|\geq 1\}$$

for the action of Γ_1 on $\mathbb{H}$. (Hence the invariance property implies that $\langle f(\tau), g(\tau)\rangle y^\kappa$ is bounded on the whole upper half plane $\mathbb{H}$.) We define the Petersson scalar product of f and g by

$$(f,g) = \int\limits_{\mathcal{F}} \langle f(\tau), g(\tau)\rangle y^\kappa \frac{dx\,dy}{y^2}. \tag{1.17}$$

It is easily seen that the integral does not depend on the choice of the fundamental domain. The space $S_{\kappa,L}$ endowed with the scalar product $(\cdot,\cdot)$ is a finite dimensional Hilbert space.

The Petersson coefficient formula for $S_{\kappa,L}$ has the following form.

Proposition 1.5. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} - q(\beta)$ with $m > 0$. Let f be a cusp form in $S_{\kappa,L}$ and denote its Fourier coefficients by $c(\gamma, n)$ (as in (1.9)). Then the scalar product of f with the Poincaré series $P_{\beta,m}$ is given by*

$$(f, P_{\beta,m}) = 2\frac{\Gamma(\kappa-1)}{(4\pi m)^{\kappa-1}} c(\beta, m).$$

Proof. Let $\overline{\Gamma}_1 = \Gamma_1/\{\pm 1\}$. We use the usual unfolding argument and find

$$\begin{aligned}
(f, P_{\beta,m}) &= \frac{1}{2} \int\limits_{\overline{\Gamma}_1 \backslash \mathbb{H}} \left\langle f(\tau), \sum_{(M,\phi)\in\tilde{\Gamma}_\infty \backslash \operatorname{Mp}_2(\mathbb{Z})} \mathfrak{e}_\beta(m\tau) \mid_\kappa^* (M,\phi) \right\rangle y^\kappa \frac{dx\,dy}{y^2} \\
&= \int\limits_{\overline{\Gamma}_1 \backslash \mathbb{H}} \sum_{M\in\Gamma_\infty\backslash\Gamma_1} \langle f(M\tau),\ \mathfrak{e}_\beta(mM\tau)\rangle \Im(M\tau)^\kappa \frac{dx\,dy}{y^2} \\
&= 2 \int\limits_{\Gamma_\infty \backslash \mathbb{H}} \langle f(\tau),\ \mathfrak{e}_\beta(m\tau)\rangle y^{\kappa-2} dx\,dy.
\end{aligned}$$

We insert the Fourier expansion of f and infer:

$$\begin{aligned}
(f, P_{\beta,m}) &= 2\int_0^\infty \int_0^1 \sum_{\substack{n\in\mathbb{Z}-q(\beta)\\ n>0}} c(\beta,n) e(n\tau) e(-mx + miy) y^{\kappa-2}\,dx\,dy \\
&= 2 \int_{y=0}^\infty c(\beta,m) e^{-4\pi m y} y^{\kappa-2} dy \\
&= 2\frac{\Gamma(\kappa-1)}{(4\pi m)^{\kappa-1}} c(\beta,m).
\end{aligned}$$

□

As a corollary we obtain that the Poincaré series $P_{\beta,m}$ generate the space $S_{\kappa,L}$.

1.2.3 Eisenstein series

Let $\beta \in L'/L$ with $q(\beta) \in \mathbb{Z}$. Then the vector $\mathfrak{e}_\beta = \mathfrak{e}_\beta(0) \in \mathbb{C}[L'/L]$, considered as a constant function $\mathbb{H} \to \mathbb{C}[L'/L]$, is invariant under the $|_\kappa^*$-action of $T, Z^2 \in \operatorname{Mp}_2(\mathbb{Z})$. The Eisenstein series

$$E_\beta^L(\tau) = \frac{1}{2} \sum_{(M,\phi)\in\tilde{\Gamma}_\infty \backslash \operatorname{Mp}_2(\mathbb{Z})} \mathfrak{e}_\beta \mid_\kappa^* (M,\phi) \tag{1.18}$$

converges normally on $\mathbb{H}$ and therefore defines a $\operatorname{Mp}_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$.

Theorem 1.6. *The Eisenstein series E_β has the Fourier expansion*

$$E_\beta(\tau) = \sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}-q(\gamma)\\ n\geq 0}} q_\beta(\gamma,n)\mathfrak{e}_\gamma(n\tau)$$

with

$$q_\beta(\gamma,n) = \begin{cases} \delta_{\beta,\gamma}+\delta_{-\beta,\gamma}, & \text{if } n=0,\\ \dfrac{(2\pi)^\kappa n^{\kappa-1}}{\Gamma(\kappa)} \displaystyle\sum_{c\in\mathbb{Z}-\{0\}} |c|^{1-\kappa} H_c^*(\beta,0,\gamma,n), & \text{if } n>0. \end{cases} \tag{1.19}$$

Here, $H_c^*(\beta,0,\gamma,n)$ denotes the generalized Kloosterman sum defined in (1.13). In particular E_β is an element of $M_{\kappa,L}$.

Proof. We proceed in the same way as in the proof of Theorem 1.4. Let $\gamma\in L'/L$ and $n\in\mathbb{Z}-q(\gamma)$. By (1.10) we have

$$q_\beta(\gamma,n) = \frac{1}{2}\int_0^1 \left\langle \sum_{(M,\phi)\in\tilde{\Gamma}_\infty\setminus \mathrm{Mp}_2(\mathbb{Z})} \mathfrak{e}_\beta \mid_\kappa^* (M,\phi),\ \mathfrak{e}_\gamma(n\bar{\tau}) \right\rangle dx.$$

Because $\mathfrak{e}_\beta$ is invariant under the action of Z^2, this equals

$$\delta_{0,n}(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma}) + \sum_{\substack{c\neq 0\\ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\setminus\Gamma_1/\Gamma_\infty}} \int_{-\infty}^{\infty} (c\tau+d)^{-\kappa} \left\langle \varrho_L^* \widetilde{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}^{-1} \mathfrak{e}_\beta,\ \mathfrak{e}_\gamma(n\bar{\tau}) \right\rangle dx$$

$$= \delta_{0,n}(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma}) + \sum_{\substack{c\neq 0\\ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\setminus\Gamma_1/\Gamma_\infty}} \varrho_{\beta\gamma}\widetilde{\begin{pmatrix} a & b\\ c & d\end{pmatrix}} \int_{-\infty}^{\infty} (c\tau+d)^{-\kappa}\, e(-n\tau)\, dx.$$

We now evaluate the integral. Since $\sqrt{c\tau+d} = \operatorname{sgn}(c)\sqrt{c}\sqrt{\tau+d/c}$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} (c\tau+d)^{-\kappa}\, e(-n\tau)\, dx &= |c|^{-\kappa}\operatorname{sgn}(c)^\kappa \int_{-\infty}^{\infty} (\tau+d/c)^{-\kappa} e(-n\tau)\, dx\\ &= |c|^{-\kappa}\operatorname{sgn}(c)^\kappa\, e\left(\frac{nd}{c}\right) \int_{-\infty}^{\infty} \tau^{-\kappa} e(-n\tau)\, dx. \end{aligned}$$

We substitute $\tau = iw$ and find

$$\int_{-\infty}^{\infty} (c\tau + d)^{-\kappa}\, e(-n\tau)\, dx$$
$$= 2\pi i^{-\kappa} |c|^{-\kappa} \operatorname{sgn}(c)^{\kappa}\, e\left(\frac{nd}{c}\right) \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} w^{-\kappa} e^{2\pi n w}\, dw,$$

where C is a positive real constant. If $n \leq 0$, one can consider the limit $C \to \infty$ to infer that the integral is 0. It follows that $q_\beta(\gamma, n) = \delta_{0,n}(\delta_{\beta,\gamma} + \delta_{-\beta,\gamma})$ in this case. Now let $n > 0$. Then the integral is an inverse Laplace transform. According to [E2] p. 238 (1) we have

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} w^{-\kappa} e^{2\pi n w}\, dw = \frac{(2\pi n)^{\kappa-1}}{\Gamma(\kappa)}.$$

(In fact, one can deform the path of integration and essentially obtains the Hankel integral for $1/\Gamma(\kappa)$.) Hence we get

$$q_\beta(\gamma, n) = \frac{(2\pi)^{\kappa} n^{\kappa-1}}{\Gamma(\kappa)} \sum_{\substack{c \neq 0 \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty}} |c|^{-\kappa} i^{-\kappa} \operatorname{sgn}(c)^{\kappa}\, \varrho_{\beta\gamma} \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\, e\left(\frac{nd}{c}\right).$$

If we use $i^{-\kappa} \operatorname{sgn}(c)^{\kappa} = e^{-\pi i \operatorname{sgn}(c)\kappa/2}$ and substitute in the definition of $H_c^*(\beta, 0, \gamma, n)$, we obtain the assertion. □

For computational purposes the formula (1.19) for the coefficients $q_\beta(\gamma, n)$ is not very useful. An explicit formula, involving special values of Dirichlet L-series and finitely many representation numbers modulo prime powers attached to the lattice L, is derived in [BK].

To give an example we briefly consider the case that L is unimodular. Then L'/L is trivial and $M_{\kappa,L}$ is just the space of elliptic modular forms of weight $1 + l/2$. We have one Eisenstein series E with Fourier coefficients $q(n)$ and $q(0) = 2$. For $n > 0$ the above expression for the coefficients can be simplified as follows. First note that $l \equiv 2 \pmod 8$. This implies $e^{-\pi i \operatorname{sgn}(c)\kappa/2} = -1$ and

$$H_c^*(\beta, 0, \gamma, n) = -\frac{1}{|c|} \sum_{d(c)^*} e\left(\frac{nd}{c}\right) = -\frac{1}{|c|} \sum_{\substack{d \mid (c,n) \\ d>0}} \mu(|c|/d) d.$$

Here we have used the evaluation of the Ramanujan sum $\sum_{d(c)^*} e(\frac{nd}{c})$ by means of the Moebius function μ. Thus we find

$$\begin{aligned} q(n) &= -2\frac{(2\pi)^\kappa n^{\kappa-1}}{\Gamma(\kappa)} \sum_{c=1}^{\infty} c^{-\kappa} \sum_{\substack{d|(c,n)\\ d>0}} \mu(c/d)d \\ &= -2\frac{(2\pi)^\kappa n^{\kappa-1}}{\Gamma(\kappa)\zeta(\kappa)} \sum_{d|n} d^{1-\kappa} \\ &= -2\frac{(2\pi)^\kappa}{\Gamma(\kappa)\zeta(\kappa)} \sigma_{\kappa-1}(n), \end{aligned}$$

where $\sigma_{\kappa-1}(n)$ denotes the sum of the $(\kappa-1)$-th powers of the positive divisors of n and $\zeta(s)$ the Riemann zeta function. Using $\zeta(\kappa) = -\frac{(2\pi i)^\kappa}{2\kappa!} B_\kappa$ with the κ-th Bernoulli number B_κ, and the fact that $\kappa \equiv 2 \pmod 4$, we get

$$q(n) = -\frac{4\kappa}{B_\kappa} \sigma_{\kappa-1}(n).$$

Thus in this case the Eisenstein series $E(z)$ is the classical Eisenstein series for Γ_1, normalized such that its constant term equals 2.

Let us now turn back to the general case and determine the number of linearly independent Eisenstein series E_β. The $|^*_\kappa$-invariance of E_β under Z and the identity $\mathfrak{e}_\beta |^*_\kappa Z = \mathfrak{e}_{-\beta}$ imply

$$E_\beta = E_\beta |^*_\kappa Z = \frac{1}{2} \sum_{(M,\phi)\in\tilde{\Gamma}_\infty \setminus \mathrm{Mp}_2(\mathbb{Z})} \mathfrak{e}_\beta |^*_\kappa Z |^*_\kappa (M,\phi) = E_{-\beta}.$$

Thus the space $M^{\mathrm{Eis}}_{\kappa,L}$ spanned by all Eisenstein series E_β equals the space spanned by the E_α where α runs through a set of representatives of $\{\beta \in L'/L;\quad q(\beta) \in \mathbb{Z}\}$ modulo the action of $\{\pm 1\}$. Comparing constant terms one finds that the set of Eisenstein series E_α with α as above is already linearly independent. We obtain

$$\begin{aligned} \dim(M^{\mathrm{Eis}}_{\kappa,L}) &= \#\{\beta \in L'/L;\quad q(\beta) \in \mathbb{Z},\ 2\beta = 0 + L\} \\ &\quad + \frac{1}{2}\#\{\beta \in L'/L;\quad q(\beta) \in \mathbb{Z},\ 2\beta \neq 0 + L\}. \end{aligned} \tag{1.20}$$

Now let $f \in M_{\kappa,L}$ and write the Fourier expansion of f in the form

$$f(\tau) = \sum_{\gamma\in L'/L} \sum_{n\in\mathbb{Z}-q(\gamma)} c(\gamma,n)\mathfrak{e}_\gamma(n\tau).$$

The invariance of f under Z and (1.4) imply that $c(\gamma,n) = c(-\gamma,n)$. Hence

$$f - \frac{1}{2} \sum_{\substack{\beta\in L'/L\\ q(\beta)\in\mathbb{Z}}} c(\beta,0)E_\beta$$

lies in $S_{\kappa,L}$. This shows that $M_{\kappa,L} = M_{\kappa,L}^{\text{Eis}} \oplus S_{\kappa,L}$ and (1.20) is a formula for the codimension of $S_{\kappa,L}$ in $M_{\kappa,L}$.

In later applications we will mainly be interested in the Eisenstein series $E_0(\tau)$ which we simply denote by $E(\tau)$. In the same way we write $q(\gamma, n)$ for the Fourier coefficients $q_0(\gamma, n)$ of $E(\tau)$.

Let us finally cite the following result of [BK]:

Proposition 1.7. *The coefficients $q(\gamma, n)$ of $E(\tau)$ are rational numbers.*

1.3 Non-holomorphic Poincaré series of negative weight

As in the previous section we assume that L is an even lattice of signature $(b^+, b^-) = (2, l)$, $(1, l-1)$, or $(0, l-2)$ with $l \geq 3$. Put $k = 1 - l/2$ and $\kappa = 1 + l/2$. We now construct certain vector valued Maass-Poincaré series for $\mathrm{Mp}_2(\mathbb{Z})$ of weight k. Series of a similar type are well known and appear in many places in the literature (see for instance [He, Ni, Fa]).

Let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ be the usual Whittaker functions as defined in [AbSt] Chap. 13 p. 190 or [E1] Vol. I Chap. 6 p. 264. They are linearly independent solutions of the Whittaker differential equation

$$\frac{d^2 w}{dz^2} + \left(-\frac{1}{4} + \frac{\nu}{z} - \frac{\mu^2 - 1/4}{z^2} \right) w = 0. \tag{1.21}$$

The functions $M_{\nu,\mu}(z)$ and $M_{\nu,-\mu}(z)$ are related by the identity

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \nu)} M_{\nu,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \nu)} M_{\nu,-\mu}(z) \tag{1.22}$$

([AbSt] p. 190 (13.1.34)). This implies in particular $W_{\nu,\mu}(z) = W_{\nu,-\mu}(z)$. As $z \to 0$ one has the asymptotic behavior

$$M_{\nu,\mu}(z) \sim z^{\mu+1/2} \qquad (\mu \notin -\tfrac{1}{2}\mathbb{N}), \tag{1.23}$$

$$W_{\nu,\mu}(z) \sim \frac{\Gamma(2\mu)}{\Gamma(\mu - \nu + 1/2)} z^{-\mu+1/2} \qquad (\mu \geq 1/2). \tag{1.24}$$

If $y \in \mathbb{R}$ and $y \to \infty$ one has

$$M_{\nu,\mu}(y) = \frac{\Gamma(1 + 2\mu)}{\Gamma(\mu - \nu + 1/2)} e^{y/2} y^{-\nu} (1 + O(y^{-1})), \tag{1.25}$$

$$W_{\nu,\mu}(y) = e^{-y/2} y^{\nu} (1 + O(y^{-1})). \tag{1.26}$$

For convenience we put for $s \in \mathbb{C}$ and $y \in \mathbb{R}_{>0}$:

$$\mathcal{M}_s(y) = y^{-k/2} M_{-k/2,\, s-1/2}(y). \tag{1.27}$$

In the same way we define for $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$:

$$\mathcal{W}_s(y) = |y|^{-k/2} W_{k/2\,\mathrm{sgn}(y),\, s-1/2}(|y|). \tag{1.28}$$

If $y < 0$ then equation (1.22) implies that

$$\mathcal{M}_s(|y|) = \frac{\Gamma(1+k/2-s)}{\Gamma(1-2s)} \mathcal{W}_s(y) - \frac{\Gamma(1+k/2-s)\Gamma(2s-1)}{\Gamma(1-2s)\Gamma(s+k/2)} \mathcal{M}_{1-s}(|y|). \tag{1.29}$$

The functions $\mathcal{M}_s(y)$ and $\mathcal{W}_s(y)$ are holomorphic in s. Later we will be interested in certain special s-values. For $y > 0$ we have

$$\mathcal{M}_{k/2}(y) = y^{-k/2} M_{-k/2,\, k/2-1/2}(y) = e^{y/2}, \tag{1.30}$$

$$\mathcal{W}_{1-k/2}(y) = y^{-k/2} W_{k/2,\, 1/2-k/2}(y) = e^{-y/2}. \tag{1.31}$$

Using the standard integral representation

$$\Gamma(1/2-\nu+\mu) W_{\nu,\,\mu}(z) = e^{-z/2} z^{\mu+1/2} \int_0^\infty e^{-tz} t^{-1/2-\nu+\mu} (1+t)^{-1/2+\nu+\mu}\, dt$$

($\Re(\mu-\nu) > -1/2$, $\Re(z) > 0$) of the W-Whittaker function ([E1] Vol. I p. 274 (18)), we find for $y < 0$:

$$\begin{aligned} \mathcal{W}_{1-k/2}(y) &= |y|^{-k/2} W_{-k/2,\, 1/2-k/2}(|y|) \\ &= e^{-|y|/2} |y|^{1-k} \int_0^\infty e^{-t|y|} (1+t)^{-k}\, dt \\ &= e^{|y|/2} |y|^{1-k} \int_1^\infty e^{-t|y|} t^{-k}\, dt. \end{aligned}$$

If we insert the definition of the incomplete Gamma function (cf. [AbSt] p. 81)

$$\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1}\, dt, \tag{1.32}$$

we obtain for $y < 0$ the identity

$$\mathcal{W}_{1-k/2}(y) = e^{-y/2} \Gamma(1-k, |y|). \tag{1.33}$$

The usual Laplace operator of weight k (cf. [Ma1])

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \tag{1.34}$$

acts on smooth functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ component-wise. Let $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$ with $m < 0$. Then it can be easily checked that the function

$$\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)$$

is invariant under the $|_k$-operation of $T \in \mathrm{Mp}_2(\mathbb{Z})$ (as defined in (1.7)) and an eigenfunction of Δ_k with eigenvalue

$$s(1-s) + (k^2 - 2k)/4.$$

Definition 1.8. *With the above notation we define the Poincaré series $F^L_{\beta,m}$ of index (β, m) by*

$$F^L_{\beta,m}(\tau, s) = \frac{1}{2\Gamma(2s)} \sum_{(M,\phi)\in\tilde{\Gamma}_\infty\setminus \mathrm{Mp}_2(\mathbb{Z})} [\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)] \,|_k\, (M,\phi), \tag{1.35}$$

where $\tau = x + iy \in \mathbb{H}$ and $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. If it is clear from the context, to which lattice L the series (1.35) refers, we simply write $F_{\beta,m}(\tau, s)$.

By (1.23) and (1.27) the above series (1.35) has the local majorant

$$\sum_{(M,\phi)\in\tilde{\Gamma}_\infty\setminus \mathrm{Mp}_2(\mathbb{Z})} \Im(M\tau)^\sigma.$$

Hence (1.35) converges normally for $\tau \in \mathbb{H}$, $s \in \mathbb{C}$ and $\sigma > 1$ and thereby defines a $\mathrm{Mp}_2(\mathbb{Z})$-invariant function on $\mathbb{H}$. As in [Ma1] it can be seen that Δ_k commutes with the action of $\mathrm{Mp}_2(\mathbb{Z})$, i.e. that

$$\Delta_k(f \,|_k\, (M,\phi)) = (\Delta_k f) \,|_k\, (M,\phi)$$

for any smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ and any $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$. We may infer that $F_{\beta,m}$ is an eigenfunction of Δ_k:

$$\Delta_k F_{\beta,m} = \big(s(1-s) + (k^2 - 2k)/4\big)\, F_{\beta,m}.$$

(Thus $F_{\beta,m}$ is automatically real analytic as a solution of an elliptic differential equation.) The invariance of $F_{\beta,m}$ under the action of $Z \in \mathrm{Mp}_2(\mathbb{Z})$ implies that $F_{\beta,m} = F_{-\beta,m}$.

Let $f : \mathbb{H} \to \mathbb{C}[L'/L]$ be a continuous function with the property $f \,|_k\, T = f$. By virtue of the same argument as in section 1.1 one finds that f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma\in L'/L} \sum_{n\in\mathbb{Z}+q(\gamma)} c(\gamma, n, y)\mathfrak{e}_\gamma(nx), \tag{1.36}$$

where the coefficients are given by

$$c(\gamma, n, y) = \int_0^1 \langle f(\tau), \mathfrak{e}_\gamma(nx)\rangle \, dx. \tag{1.37}$$

Theorem 1.9. *The Poincaré series* $F_{\beta,m}$ *has the Fourier expansion*

$$\begin{aligned} F_{\beta,m}(\tau,s) &= \frac{\Gamma(1+k/2-s)}{\Gamma(2-2s)\Gamma(s+k/2)}\mathcal{M}_{1-s}(4\pi|m|y)(\mathfrak{e}_\beta(mx)+\mathfrak{e}_{-\beta}(mx)) \\ &\quad + \sum_{\substack{\gamma\in L'/L\\ q(\gamma)\in\mathbb{Z}}} b(\gamma,0,s)y^{1-s-k/2}\mathfrak{e}_\gamma \\ &\quad + \sum_{\gamma\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n>0}} b(\gamma,n,s)\mathcal{W}_s(4\pi ny)\mathfrak{e}_\gamma(nx)+\widetilde{F}_{\beta,m}(\tau,s), \end{aligned}$$

where $\widetilde{F}_{\beta,m}(\tau,s)$ *is the* T*-invariant function*

$$\widetilde{F}_{\beta,m}(\tau,s) = \sum_{\gamma\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n<0}} b(\gamma,n,s)\mathcal{W}_s(4\pi ny)\mathfrak{e}_\gamma(nx),$$

and the Fourier coefficients $b(\gamma,n,s)$ *are given by*

$$\begin{cases} \dfrac{2\pi|n/m|^{\frac{k-1}{2}}}{\Gamma(s+k/2)}\displaystyle\sum_{c\in\mathbb{Z}-\{0\}} H_c(\beta,m,\gamma,n)I_{2s-1}\left(\frac{4\pi}{|c|}\sqrt{|mn|}\right), & \text{if } n>0,\\ \dfrac{4^{1-k/2}\pi^{1+s-k/2}|m|^{s-k/2}}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)}\displaystyle\sum_{c\in\mathbb{Z}-\{0\}}|c|^{1-2s}H_c(\beta,m,\gamma,0), & \text{if } n=0,\\ \dfrac{\Gamma(1+k/2-s)}{\Gamma(2s)\Gamma(1-2s)}\delta_{m,n}(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma}) & \\ \quad +\dfrac{2\pi|n/m|^{\frac{k-1}{2}}}{\Gamma(s-k/2)}\displaystyle\sum_{c\in\mathbb{Z}-\{0\}} H_c(\beta,m,\gamma,n)J_{2s-1}\left(\frac{4\pi}{|c|}\sqrt{|mn|}\right), & \text{if } n<0. \end{cases}$$

Here $H_c(\beta,m,\gamma,n)$ *denotes the generalized Kloosterman sum*

$$H_c(\beta,m,\gamma,n) = \frac{e^{-\pi i\,\mathrm{sgn}(c)k/2}}{|c|}\sum_{\substack{d(c)^*\\ \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_1/\Gamma_\infty}} \varrho_{\gamma\beta}^{-1}\widetilde{\begin{pmatrix}a&b\\c&d\end{pmatrix}}\, e\left(\frac{ma+nd}{c}\right) \tag{1.38}$$

and $J_\nu(z)$*,* $I_\nu(z)$ *the usual Bessel functions as defined in [AbSt] Chap. 9.*

The sum in (1.38) runs over all primitive residues d modulo c and $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is a representative for the double coset in $\Gamma_\infty\backslash\Gamma_1/\Gamma_\infty$ with lower row $(c\ d')$ and $d'\equiv d \pmod c$. Observe that the expression $\varrho_{\gamma\beta}^{-1}\widetilde{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}\, e\left(\frac{ma+nd}{c}\right)$ does not depend on the choice of the coset representative. The fact that $H_c(\beta,m,\gamma,n)$ is universally bounded implies that the series for the coefficients $b(\gamma,n,s)$ converge normally in s for $\sigma>1$.

Proof of Theorem 1.9. We split the sum in equation (1.35) into the sum over $1, Z, Z^2, Z^3 \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})$ and the sum over $(M, \phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \neq 0$. The latter sum will be denoted by $H(\tau, s)$. Since $\mathfrak{e}_\beta(mx) \mid_k Z = \mathfrak{e}_{-\beta}(mx)$, we find that the first term equals

$$\frac{1}{\Gamma(2s)} \mathcal{M}_s(4\pi|m|y)(\mathfrak{e}_\beta(mx) + \mathfrak{e}_{-\beta}(mx)).$$

According to (1.29) this can be rewritten as

$$\frac{\Gamma(1+k/2-s)}{\Gamma(2-2s)\Gamma(s+k/2)} \mathcal{M}_{1-s}(4\pi|m|y)(\mathfrak{e}_\beta(mx) + \mathfrak{e}_{-\beta}(mx)) \\ + \frac{\Gamma(1+k/2-s)}{\Gamma(2s)\Gamma(1-2s)} \mathcal{W}_s(4\pi my)(\mathfrak{e}_\beta(mx) + \mathfrak{e}_{-\beta}(mx)). \quad (1.39)$$

We now calculate the Fourier expansion of $H(\tau, s)$. Since $\mathfrak{e}_\beta(mx)$ is invariant under the action of Z^2, we may write $H(\tau, s)$ in the form

$$\frac{1}{\Gamma(2s)} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 \\ c \neq 0}} [\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)] \mid_k \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}.$$

Let $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ and denote the (γ, n)-th Fourier coefficient of $H(\tau, s)$ by $c(\gamma, n, y)$ (cp. (1.36)). Then we have

$$\begin{aligned} c(\gamma, n, y) &= \int_0^1 \langle H(\tau, s), \mathfrak{e}_\gamma(nx) \rangle \, dx \\ &= \sum_{\substack{c \neq 0 \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty}} \int_{-\infty}^{\infty} \frac{(c\tau+d)^{-k}}{\Gamma(2s)} \mathcal{M}_s\Big(4\pi|m| \tfrac{y}{|c\tau+d|^2}\Big) \\ &\qquad \times \Big\langle \varrho_L^{-1}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \mathfrak{e}_\beta\big(m\Re\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau\big)\big), \ \mathfrak{e}_\gamma(nx) \Big\rangle \, dx. \end{aligned}$$

By (1.15) we find that

$$\begin{aligned} &\Big\langle \varrho_L^{-1}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \mathfrak{e}_\beta\big(m\Re\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau\big)\big), \ \mathfrak{e}_\gamma(nx) \Big\rangle \\ &\qquad = \varrho_{\gamma\beta}^{-1}\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} e\left(\frac{ma}{c} - \Re\left(\frac{m}{c^2(\tau+d/c)}\right)\right) e(-nx), \end{aligned}$$

and therefore

$$c(\gamma,n,y)=\frac{1}{\Gamma(2s)}\sum_{\substack{c\neq 0\\ \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_1/\Gamma_\infty}}\varrho^{-1}_{\gamma\beta}\widetilde{\begin{pmatrix}a&b\\c&d\end{pmatrix}}e\left(\frac{ma}{c}\right)$$
$$\times\int_{-\infty}^{\infty}(c\tau+d)^{-k}\mathcal{M}_s\left(4\pi|m|\frac{y}{|c\tau+d|^2}\right)e\left(-\Re\left(\frac{m}{c^2(\tau+d/c)}\right)-nx\right)dx.$$

If we use $\sqrt{c\tau+d}=\operatorname{sgn}(c)\sqrt{c}\sqrt{\tau+d/c}$ and substitute x by $x-d/c$ in the integral, we obtain

$$\begin{aligned}c(\gamma,n,y)&=\frac{1}{\Gamma(2s)}\sum_{\substack{c\neq 0\\ \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_1/\Gamma_\infty}}\varrho^{-1}_{\gamma\beta}\widetilde{\begin{pmatrix}a&b\\c&d\end{pmatrix}}e\left(\frac{ma+nd}{c}\right)|c|^{-k}\operatorname{sgn}(c)^k\\&\qquad\times\int_{-\infty}^{\infty}\tau^{-k}\mathcal{M}_s\left(4\pi|m|\frac{y}{c^2|\tau|^2}\right)e\left(\frac{-mx}{c^2|\tau|^2}-nx\right)dx\\&=\frac{1}{\Gamma(2s)}\sum_{c\neq 0}|c|^{1-k}i^kH_c(\beta,m,\gamma,n)\\&\qquad\times\int_{-\infty}^{\infty}\tau^{-k}\mathcal{M}_s\left(4\pi|m|\frac{y}{c^2|\tau|^2}\right)e\left(\frac{-mx}{c^2|\tau|^2}-nx\right)dx.\end{aligned}$$

By the definition of $\mathcal{M}_s$ this equals

$$\begin{aligned}&\frac{(4\pi|m|y)^{-k/2}}{\Gamma(2s)}\sum_{c\neq 0}|c|H_c(\beta,m,\gamma,n)\\&\qquad\times\int_{-\infty}^{\infty}\left(\frac{\tau}{-\bar\tau}\right)^{-k/2}M_{-k/2,\,s-1/2}\left(4\pi|m|\frac{y}{c^2|\tau|^2}\right)e\left(\frac{-mx}{c^2|\tau|^2}-nx\right)dx\\&=\frac{(4\pi|m|y)^{-k/2}}{\Gamma(2s)}\sum_{c\neq 0}|c|H_c(\beta,m,\gamma,n)\int_{-\infty}^{\infty}\left(\frac{y-ix}{y+ix}\right)^{-k/2}\\&\qquad\times M_{-k/2,\,s-1/2}\left(\frac{4\pi|m|y}{c^2(x^2+y^2)}\right)e\left(\frac{-mx}{c^2(x^2+y^2)}-nx\right)dx.\qquad(1.40)\end{aligned}$$

We abbreviate the latter integral by I and substitute $x=yu$. Then I is equal to

$$y\int_{-\infty}^{\infty}\left(\frac{1-iu}{1+iu}\right)^{-k/2}M_{-k/2,\,s-1/2}\left(\frac{4\pi|m|}{c^2y(u^2+1)}\right)e\left(\frac{|m|u}{c^2y(u^2+1)}-nyu\right)du;$$

and if we set $A = -ny$ and $B = \frac{|m|}{c^2 y}$, we get

$$I = y \int_{-\infty}^{\infty} \left(\frac{1-iu}{1+iu}\right)^{-k/2} M_{-k/2,\,s-1/2}\left(\frac{4\pi B}{u^2+1}\right) \exp\left(\frac{2\pi i B u}{u^2+1} + 2\pi i A u\right) du.$$

The integral $I/(y\Gamma(2s))$ was evaluated by Hejhal in [He] p. 357. (Notice that $M_{-k/2,\,s-1/2}(y)/\Gamma(2s)$ is the same as Hejhal's function $G(y)$ with $\ell = -k/2$. Moreover, $W_{\nu,\,\mu}(y)$ equals $e^{-y/2}y^{\mu+1/2}\Psi(\mu+1/2-\nu, 1+2\mu, y)$ in Hejhal's notation.) We find

$$\frac{I}{y\Gamma(2s)} = \begin{cases} \dfrac{2\pi\sqrt{|B/A|}}{\Gamma(s-k/2)} W_{-k/2,\,s-1/2}(4\pi|A|) J_{2s-1}\left(4\pi\sqrt{|AB|}\right), & \text{if } A > 0, \\[2ex] \dfrac{4\pi^{1+s}}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)} B^s, & \text{if } A = 0, \\[2ex] \dfrac{2\pi\sqrt{|B/A|}}{\Gamma(s+k/2)} W_{k/2,\,s-1/2}(4\pi|A|) I_{2s-1}\left(4\pi\sqrt{|AB|}\right), & \text{if } A < 0. \end{cases}$$

We resubstitute for A and B and obtain

$$\frac{|c|I}{\Gamma(2s)} = \begin{cases} \dfrac{2\pi\sqrt{|m/n|}}{\Gamma(s-k/2)} (4\pi|n|y)^{k/2} J_{2s-1}\left(\dfrac{4\pi}{|c|}\sqrt{|mn|}\right) \mathcal{W}_s(4\pi ny), & \text{if } n < 0, \\[2ex] \dfrac{4\pi^{1+s}|m|^s|c|^{1-2s}y^{1-s}}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)}, & \text{if } n = 0, \\[2ex] \dfrac{2\pi\sqrt{|m/n|}}{\Gamma(s+k/2)} (4\pi|n|y)^{k/2} I_{2s-1}\left(\dfrac{4\pi}{|c|}\sqrt{|mn|}\right) \mathcal{W}_s(4\pi ny), & \text{if } n > 0. \end{cases}$$

If we insert this into (1.40) we get $c(\gamma, n, y)$ is given by

$$\begin{cases} \dfrac{2\pi|n/m|^{\frac{k-1}{2}}}{\Gamma(s-k/2)} \displaystyle\sum_{c\neq 0} H_c(\beta, m, \gamma, n) J_{2s-1}\left(\dfrac{4\pi}{|c|}\sqrt{|mn|}\right) \mathcal{W}_s(4\pi ny), & n < 0, \\[2ex] \dfrac{4^{1-k/2}\pi^{1+s-k/2}|m|^{s-k/2}}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)} \displaystyle\sum_{c\neq 0} |c|^{1-2s} H_c(\beta, m, \gamma, 0) y^{1-s-k/2}, & n = 0, \\[2ex] \dfrac{2\pi|n/m|^{\frac{k-1}{2}}}{\Gamma(s+k/2)} \displaystyle\sum_{c\neq 0} H_c(\beta, m, \gamma, n) I_{2s-1}\left(\dfrac{4\pi}{|c|}\sqrt{|mn|}\right) \mathcal{W}_s(4\pi ny), & n > 0. \end{cases}$$

Combining this with (1.39) we obtain the assertion. □

Since $\Delta_k F_{\beta,m}(\tau, 1-k/2) = 0$, the Poincaré series $F_{\beta,m}(\tau, s)$ are in particular interesting in the special case $s = 1 - k/2$.

Proposition 1.10. *For $s = 1-k/2$ the Fourier expansion of $F_{\beta,m}(\tau, s)$ given in Theorem 1.9 can be simplified as follows:*

$$F_{\beta,m}(\tau, 1-k/2) = \mathfrak{e}_\beta(m\tau) + \mathfrak{e}_{-\beta}(m\tau) + \sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\geq 0}} b(\gamma, n, 1-k/2)\mathfrak{e}_\gamma(n\tau) + \widetilde{F}_{\beta,m}(\tau, 1-k/2).$$

The function $\widetilde{F}_{\beta,m}(\tau, 1-k/2)$ has the expansion

$$\widetilde{F}_{\beta,m}(\tau, 1-k/2) = \sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n<0}} b(\gamma, n, 1-k/2)\mathcal{W}_{1-k/2}(4\pi ny)\mathfrak{e}_\gamma(nx),$$

and the Fourier coefficients $b(\gamma, n, 1-k/2)$ are given by

$$\begin{cases} 2\pi \left|\dfrac{n}{m}\right|^{\frac{k-1}{2}} \displaystyle\sum_{c\in\mathbb{Z}-\{0\}} H_c(\beta, m, \gamma, n) I_{1-k}\left(\frac{4\pi}{|c|}\sqrt{|mn|}\right), & \text{if } n>0,\\ \dfrac{(2\pi)^{2-k}|m|^{1-k}}{\Gamma(2-k)} \displaystyle\sum_{c\in\mathbb{Z}-\{0\}} |c|^{k-1} H_c(\beta, m, \gamma, 0), & \text{if } n=0,\\ \dfrac{-1}{\Gamma(1-k)}\delta_{m,n}(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma}) \\ \quad + \dfrac{2\pi}{\Gamma(1-k)}\left|\dfrac{n}{m}\right|^{\frac{k-1}{2}} \displaystyle\sum_{c\in\mathbb{Z}-\{0\}} H_c(\beta, m, \gamma, n) J_{1-k}\left(\frac{4\pi}{|c|}\sqrt{|mn|}\right), & \text{if } n<0. \end{cases}$$

Proof. By virtue of (1.30) and (1.31) this immediately follows from Theorem 1.9. □

Definition 1.11. *A function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a* nearly holomorphic modular form *of weight k (with respect to ϱ_L and $\mathrm{Mp}_2(\mathbb{Z})$), if*

i) $f \mid_k (M, \phi) = f$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$,
ii) f is holomorphic on $\mathbb{H}$,
iii) f has a pole in ∞, i.e. f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\gg-\infty}} c(\gamma, n)\mathfrak{e}_\gamma(n\tau).$$

The space of these nearly holomorphic modular forms is denoted by $M^!_{k,L}$. The Fourier polynomial

$$\sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n<0}} c(\gamma, n)\mathfrak{e}_\gamma(n\tau)$$

is called the principal part *of f.*

The following proposition is an analogue of the theorem in [He] on p. 660 (see also [Ni] Theorem 6).

Proposition 1.12. *Let $f(\tau)$ be a nearly holomorphic modular form of weight k and denote its Fourier coefficients by $c(\gamma, n)$ ($\gamma \in L'/L$, $n \in \mathbb{Z}+q(\gamma)$). Then*

$$f(\tau) = \frac{1}{2} \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} c(\gamma, n) F_{\gamma,n}(\tau, 1-k/2).$$

Proof. Consider the function

$$g(\tau) = f(\tau) - \frac{1}{2} \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} c(\gamma, n) F_{\gamma,n}(\tau, 1-k/2).$$

The invariance of f under $Z \in \mathrm{Mp}_2(\mathbb{Z})$ implies that $c(\gamma, n) = c(-\gamma, n)$. According to Proposition 1.10 we have $F_{\gamma,n}(\tau, 1-k/2) = \mathfrak{e}_\gamma(n\tau) + \mathfrak{e}_{-\gamma}(n\tau) + O(1)$ for all γ, n. Hence $g(\tau)$ is bounded as $y \to \infty$. But g also satisfies

$$\begin{aligned} g \mid_k (M, \phi) &= g \qquad ((M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})), \\ \Delta_k g &= 0. \end{aligned}$$

Thus g has to vanish identically (see [He] Chap. 9 Prop. 5.14c). □

Lemma 1.13. *Let $c \in \mathbb{Z} - \{0\}$, $\beta, \gamma \in L'/L$, $m \in \mathbb{Z}+q(\beta)$ and $n \in \mathbb{Z}+q(\beta)$. Then*

$$\overline{H_c(\beta, m, \gamma, n)} = H_{-c}(\beta, m, \gamma, n).$$

Proof. From Proposition 1.1 it can be easily deduced that

$$\overline{\varrho_{\beta\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} = \varrho_{\beta\gamma}\widetilde{\left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)} \qquad \text{and} \qquad \overline{\varrho^{-1}_{\gamma\beta}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} = \varrho^{-1}_{\gamma\beta}\widetilde{\left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)}.$$

Now the assertion immediately follows from the definition of $H_c(\beta, m, \gamma, n)$. □

Remark 1.14. This implies that the Fourier coefficients $b(\gamma, n, 1-k/2)$ of the function $F_{\beta,m}(\tau, 1-k/2)$ are real numbers.

We now compare the Fourier coefficients of $F_{\beta,m}(\tau, 1-k/2)$ with the coefficients of the holomorphic Poincaré series $P_{\gamma,n} \in S_{\kappa,L}$ constructed in the previous section.

Lemma 1.15. *Let $c \in \mathbb{Z} - \{0\}$, $\beta, \gamma \in L'/L$, $m \in \mathbb{Z}+q(\beta)$ and $n \in \mathbb{Z}+q(\beta)$. Then we have*

$$H_c(\beta, m, \gamma, n) = -H^*_{-c}(\gamma, -n, \beta, -m).$$

Proof. We use the fact that $(\tilde{M})^{-1} = \widetilde{M^{-1}}$ for any $M \in \Gamma_1$, and that $\kappa = 2 - k$. We find

$$\begin{aligned} H_c(\beta, m, \gamma, n) &= \frac{e^{-\pi i \operatorname{sgn}(c) k/2}}{|c|} \sum_{\substack{d(c)^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty}} \varrho_{\gamma\beta}^{-1} \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} e\left(\frac{ma + nd}{c}\right) \\ &= -\frac{e^{-\pi i \operatorname{sgn}(-c)(2-k)/2}}{|c|} \sum_{\substack{d(c)^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty}} \varrho_{\gamma\beta}^{-1} \widetilde{\begin{pmatrix} d & b \\ c & a \end{pmatrix}} e\left(\frac{md + na}{c}\right) \\ &= -\frac{e^{-\pi i \operatorname{sgn}(-c)\kappa/2}}{|c|} \sum_{\substack{d(c)^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_1 / \Gamma_\infty}} \varrho_{\gamma\beta} \widetilde{\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}} e\left(\frac{-na - md}{-c}\right) \\ &= -H^*_{-c}(\gamma, -n, \beta, -m). \end{aligned}$$

□

Proposition 1.16. *Let $\beta, \gamma \in L'/L$, $n \in \mathbb{Z} + q(\gamma)$, $m \in \mathbb{Z} + q(\beta)$, and $m < 0$. Denote the (n, γ)-th Fourier coefficient $b(\gamma, n, 1 - k/2)$ of $F_{\beta,m}(\tau, 1 - k/2)$ (see Proposition 1.10) by $b_{\beta,m}(\gamma, n)$. If $n < 0$, then*

$$b_{\beta,m}(\gamma, n) = -\frac{1}{\Gamma(1-k)} p_{\gamma,-n}(\beta, -m),$$

where $p_{\gamma,-n}(\beta, -m)$ is the $(\beta, -m)$-th coefficient of the Poincaré series $P_{\gamma,-n}(\tau) \in S_{\kappa,L}$ (see 1.12). If $n = 0$, then

$$b_{\beta,m}(\gamma, 0) = -q_\gamma(\beta, -m),$$

where $q_\gamma(\beta, -m)$ is the $(\beta, -m)$-th coefficient of the Eisenstein series $E_\gamma(\tau) \in M_{\kappa,L}$ (see 1.19).

Proof. We compare the formulas for the Fourier coefficients given in Proposition 1.10 and Theorem 1.4 (resp. Proposition 1.10 and Theorem 1.6) and apply Lemma 1.15. Note that $k - 1 = 1 - \kappa$. □

For $\gamma \in L'/L$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$ let $a_{\gamma,n} : S_{\kappa,L} \to \mathbb{C}$ denote the functional in the dual space $S^*_{\kappa,L}$ of $S_{\kappa,L}$ which maps a cusp form f to its (γ, n)-th Fourier coefficient $a_{\gamma,n}(f)$. According to Proposition 1.5, $a_{\gamma,n}$ can be described by means of the Petersson scalar product as

$$a_{\gamma,n} = \frac{(4\pi n)^{\kappa-1}}{2\Gamma(\kappa-1)} (\,\cdot\,, P_{\gamma,n}). \tag{1.41}$$

The following theorem can also be proved using Serre duality (see [Bo3] Thm. 3.1). We deduce it from Proposition 1.16.

Theorem 1.17. *There exists a nearly holomorphic modular form $f \in M^!_{k,L}$ with prescribed principal part*

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m)\mathfrak{e}_\beta(m\tau)$$

($c(\beta, m) \in \mathbb{C}$ with $c(\beta, m) = c(-\beta, m)$), if and only if the functional

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m)a_{\beta,-m},$$

equals 0 *in* $S^*_{\kappa,L}$.

Proof. First, suppose that there exists such a nearly holomorphic modular form f. Then by Proposition 1.12

$$f(\tau) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\gamma) \\ m<0}} c(\beta, m)F_{\beta,m}(\tau, 1 - k/2),$$

and in particular

$$\frac{\partial}{\partial \bar{\tau}} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\gamma) \\ m<0}} c(\beta, m)\widetilde{F}_{\beta,m}(\tau, 1 - k/2) = 0.$$

Since

$$\frac{\partial}{\partial \bar{\tau}} \mathcal{W}_{1-k/2}(4\pi n y)\mathfrak{e}_\gamma(nx) \neq 0,$$

this implies that

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\gamma) \\ m<0}} c(\beta, m)b_{\beta,m}(\gamma, n) = 0$$

for all $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. (Here we have used the same notation as in Proposition 1.16.) Applying Proposition 1.16 we find that

$$\begin{aligned} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\gamma) \\ m<0}} c(\beta, m)p_{\gamma,-n}(\beta, -m) &= \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\gamma) \\ m<0}} c(\beta, m)a_{\beta,-m}(P_{\gamma,-n}) \\ &= 0 \end{aligned}$$

for all $\gamma \in L'/L$ and $n \in \mathbb{Z}+q(\gamma)$ with $n < 0$. Since the Poincaré series $P_{\gamma,-n}$ generate the space $S_{\kappa,L}$ we obtain

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m)a_{\beta,-m} = 0. \tag{1.42}$$

Now assume that (1.42) holds. Then we may reverse the above argument to infer that

$$\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\gamma)\\ m<0}} c(\beta,m)\widetilde{F}_{\beta,m}(\tau,1-k/2)=0.$$

According to Proposition 1.10 this implies that

$$\frac{1}{2}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\gamma)\\ m<0}} c(\beta,m)F_{\beta,m}(\tau,1-k/2)$$

is a nearly holomorphic modular form with the right principal part. □

2 The regularized theta lift

In this chapter we consider a certain regularized theta lift of the Maass-Poincaré series studied in the previous chapter.

2.1 Siegel theta functions

We use the same notation as in section 1.1. In particular L denotes an even lattice of signature (b^+, b^-). The bilinear form on L induces a symmetric bilinear form on $V := L \otimes \mathbb{R}$. We write $\mathrm{Gr}(L)$ for the Grassmannian of L, this is the real analytic manifold of b^+-dimensional positive definite subspaces of V. (For $b^+ = 0$ or $b^- = 0$ the Grassmannian is simply a point.) If $v \in \mathrm{Gr}(L)$, we write $v^\perp$ for the orthogonal complement of v in V such that $V = v \oplus v^\perp$. For any vector $x \in V$ let x_v resp. $x_{v^\perp}$ be the orthogonal projection of x to v resp. $v^\perp$. Then $q(x) = \frac{1}{2}(x, x) = q(x_v) + q(x_{v^\perp})$.

The Siegel theta function attached to L is defined by

$$\theta_L(\tau, v) = \sum_{\lambda \in L} e\left(\tau q(\lambda_v) + \bar{\tau} q(\lambda_{v^\perp})\right) \qquad (\tau \in \mathbb{H},\ v \in \mathrm{Gr}(L)).$$

Following Borcherds we introduce a more general theta function. Let $r, t \in V$, $\gamma \in L'/L$, $\tau \in \mathbb{H}$ and $v \in \mathrm{Gr}(L)$. Define

$$\theta_\gamma(\tau, v; r, t) = \sum_{\lambda \in \gamma + L} e\left(\tau q((\lambda + t)_v) + \bar{\tau} q((\lambda + t)_{v^\perp}) - (\lambda + t/2, r)\right) \qquad (2.1)$$

and

$$\Theta_L(\tau, v; r, t) = \sum_{\gamma \in L'/L} \mathfrak{e}_\gamma \theta_\gamma(\tau, v; r, t). \qquad (2.2)$$

If r and t are both 0 we will omit them and simply write $\theta_\gamma(\tau, v)$ resp. $\Theta_L(\tau, v)$. Let Λ be the lattice $\mathbb{Z}^2$ equipped with the standard symplectic form and $W = \Lambda \otimes \mathbb{R}$ (such that $\mathrm{Sp}(\Lambda) = \mathrm{SL}_2(\mathbb{Z})$). Then $W \otimes V$ is a symplectic vector space. The usual theta function attached to the lattice $\Lambda \otimes L \subset W \otimes V$ restricted to the symmetric subspace $\mathbb{H} \times \mathrm{Gr}(L) \subset \mathrm{Sp}(W \otimes V)/K$ (where K denotes a maximal compact subgroup) equals the Siegel theta function $\theta_L(\tau, v)$.

It is well known that $\Theta_L(\tau, v; r, t)$ is a real analytic function in $(\tau, v) \in \mathbb{H} \times \operatorname{Gr}(L)$.

Let

$$\mathrm{O}(V) = \{g \in \mathrm{SL}(V); \quad (gx, gy) = (x, y) \text{ for all } x, y \in V\}$$

be the (special) orthogonal group of V and

$$\mathrm{O}(L) = \{g \in \mathrm{O}(V); \quad gL = L\}.$$

be the orthogonal group of L. We denote by $\mathrm{O}_d(L)$ the discriminant kernel of $\mathrm{O}(L)$. This is the subgroup of finite index of $\mathrm{O}(L)$ consisting of all elements which act trivially on the discriminant group L'/L. Observe that $\mathrm{O}_d(L)$ is functorial in the following sense: If $\tilde{L} \subset L$ is a sublattice, then $\mathrm{O}_d(\tilde{L}) \subset \mathrm{O}_d(L)$.

As a function in $v \in \operatorname{Gr}(L)$ the theta function $\Theta_L(\tau, v)$ is obviously invariant under the action of $\mathrm{O}_d(L)$. As a function in $\tau \in \mathbb{H}$ it transforms like an elliptic modular form under the action of $\mathrm{Mp}_2(\mathbb{Z})$. Using the Poisson summation formula the following transformation law can be established (see [Bo2] Theorem 4.1).

Theorem 2.1. *If $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $\Theta_L(\tau, v; r, t)$ satisfies*

$$\Theta_L(M\tau, v; ar + bt, cr + dt) = \phi(\tau)^{b^+} \overline{\phi(\tau)}^{b^-} \varrho_L(M, \phi) \Theta_L(\tau, v; r, t).$$

Borcherds showed that Θ_L can be written as a Poincaré series involving similar theta series Θ_K which come from a smaller lattice K of signature $(b^+ - 1, b^- - 1)$ (cf. [Bo2] section 5). This is very useful for the evaluation of "theta integrals"

$$\int_{\mathcal{F}} \langle F(\tau), \Theta_L(\tau, v) \rangle \, y^{b^+/2} \frac{dx\, dy}{y^2}$$

by the usual unfolding trick (Rankin-Selberg method).

In the rest of this section we review Borcherds construction in a slightly more explicit way. First we need some basic facts about lattices.

Let $z \in L$ be a primitive norm 0 vector (i.e. $\mathbb{Q}z \cap L = \mathbb{Z}z$ and $q(z) = 0$). Then there exists a $z' \in L'$ with $(z, z') = 1$. Let N be the unique positive integer with $(z, L) = N\mathbb{Z}$. Then we have $z/N \in L'$. Denote by K the lattice

$$K = L \cap z^\perp \cap z'^\perp. \tag{2.3}$$

Over $\mathbb{Q}$ we have

$$L \otimes \mathbb{Q} = (K \otimes \mathbb{Q}) \oplus (\mathbb{Q}z \oplus \mathbb{Q}z'). \tag{2.4}$$

Hence K has signature $(b^+ - 1, b^- - 1)$. If $n \in V = L \otimes \mathbb{R}$ then we write n_K for the orthogonal projection of n to $K \otimes \mathbb{R}$. It can be easily checked that n_K can be computed as

$$n_K = n - (n,z)z' + (n,z)(z',z')z - (n,z')z.$$

If $n \in L'$ then n_K lies in the dual lattice K' of K. (Warning: Notice that K' is not necessarily contained in L'. Thus if $n \in L$, then n_K does not necessarily lie in K.)

Let $\zeta \in L$ be a lattice vector with $(\zeta, z) = N$. It can be uniquely represented as

$$\zeta = \zeta_K + Nz' + Bz \tag{2.5}$$

with $\zeta_K \in K'$ and $B \in \mathbb{Q}$.

Proposition 2.2. *The lattice L can be written as $L = K \oplus \mathbb{Z}\zeta \oplus \mathbb{Z}z$.*

Proof. Let $n \in L$. Then the vector

$$\tilde{n} = n - (n, z/N)\zeta - (n,z')z + (n,z/N)(\zeta,z')z$$

lies in L and an easy calculation shows that $\tilde{n} \perp z$ and $\tilde{n} \perp z'$. Hence $\tilde{n} \in K$ and therefore $n \in K + \mathbb{Z}\zeta + \mathbb{Z}z$. The sum is obviously direct. □

If one uses the fact that $|L'/L|$ is given by the absolute value of the Gram determinant of L, then the above proposition in particular shows that $|L'/L| = N^2|K'/K|$.

Moreover, we may infer that $\gamma \mapsto \gamma - (\gamma,\zeta)z/N$ defines an isometric embedding $K' \to L'$. The kernel of the induced map $K' \to L'/L$ equals $\{\gamma \in K;\ (\gamma,\zeta) \in N\mathbb{Z}\}$.

Consider the sub-lattice

$$L_0' = \{\lambda \in L';\quad (\lambda, z) \equiv 0 \pmod N\} \tag{2.6}$$

of L'. Obviously L is contained in L_0'. Beside the orthogonal projection from L_0' to K' there is another natural projection $L_0' \to K'$: If we combine the linear map $L_0' \to L' \cap z^\perp$, $\lambda \mapsto \lambda - \frac{(\lambda,z)}{N}\zeta$ with the orthogonal projection from $L' \cap z^\perp$ to K', we get a projection

$$p : L_0' \longrightarrow K', \qquad p(\lambda) = \lambda_K - \frac{(\lambda,z)}{N}\zeta_K. \tag{2.7}$$

This map has the property that $p(L) = K$. In fact, if $\lambda = k + a\zeta + bz \in L$ with $k \in K$ and $a, b \in \mathbb{Z}$, then it is easily seen that $p(\lambda) = k$. Thus p induces a surjective map $L_0'/L \to K'/K$ which will also be denoted by p. Note that $L_0'/L = \{\lambda \in L'/L;\quad (\lambda, z) \equiv 0 \pmod N\}$.

We introduce some more notation. If $x \in V$ then we sometimes simply write x^2 instead of (x,x) or $2q(x)$. Furthermore, we abbreviate $|x| := |(x,x)|^{1/2}$. Let $v \in \operatorname{Gr}(L)$. We denote by w the orthogonal complement of z_v in v and by $w^\perp$ the orthogonal complement of $z_{v^\perp}$ in $v^\perp$. Hence one has the orthogonal decomposition

$$V = v \oplus v^\perp = w \oplus \mathbb{R}z_v \oplus w^\perp \oplus \mathbb{R}z_{v^\perp}.$$

If $x \in V$, we write x_w resp. $x_{w^\perp}$ for the orthogonal projection of x to w resp. $w^\perp$. It can be easily verified that w and $w^\perp$ are contained in $(K\otimes\mathbb{R})\oplus\mathbb{R}z$. This implies that the orthogonal projection $V \to K \otimes \mathbb{R}$ induces an isometric isomorphism $w \oplus w^\perp \to w_K \oplus w_K^\perp = K \otimes \mathbb{R}$. In particular we may identify w with w_K and consider w as an element of the Grassmannian $\mathrm{Gr}(K)$.

We denote by μ the vector

$$\mu = -z' + \frac{z_v}{2z_v^2} + \frac{z_{v^\perp}}{2z_{v^\perp}^2} \tag{2.8}$$

in $V \cap z^\perp = (K \otimes \mathbb{R}) \oplus \mathbb{R}z$.

Lemma 2.3. *(cp. [Bo2] Lemma 5.1.) Let $\gamma \in L'/L$. We have*

$$\theta_\gamma(\tau, v) = \frac{1}{\sqrt{2yz_v^2}} \sum_{\lambda\in\gamma+K\oplus\mathbb{Z}\zeta} \sum_{d\in\mathbb{Z}} e\left(\tau q(\lambda_w) + \bar\tau q(\lambda_{w^\perp})\right) \times e\left(-\frac{|(\lambda, z)\tau + d|^2}{4iyz_v^2} - \frac{d(\lambda, z_v - z_{v^\perp})}{2z_v^2}\right).$$

Proof. For $\lambda \in \gamma + K \oplus \mathbb{Z}\zeta$ and $d \in \mathbb{R}$ we define a function

$$g(\lambda, v; d) = e\left(\tau q((\lambda + dz)_v) + \bar\tau q((\lambda + dz)_{v^\perp})\right).$$

Then

$$\theta_\gamma(\tau, v) = \sum_{\lambda\in\gamma+K\oplus\mathbb{Z}\zeta} \sum_{d\in\mathbb{Z}} g(\lambda, v; d).$$

We may apply the Poisson summation formula to rewrite the inner sum. We find

$$\theta_\gamma(\tau, v) = \sum_{\lambda\in\gamma+K\oplus\mathbb{Z}\zeta} \sum_{d\in\mathbb{Z}} \hat g(\lambda, v; d),$$

where $\hat g(\lambda, v; d)$ denotes the partial Fourier transform with respect to the variable d. We now compute $\hat g(\lambda, v; d)$. Using $q(z_v) + q(z_{v^\perp}) = 0$ we get

$$\begin{aligned} g(\lambda, v; d) &= e\left(d^2(\tau - \bar\tau)q(z_v) + d(\tau(\lambda, z_v) + \bar\tau(\lambda, z_{v^\perp})) + \tau q(\lambda_v) + \bar\tau q(\lambda_{v^\perp})\right) \\ &= e(Ad^2 + Bd + C), \end{aligned}$$

with $A = (\tau - \bar\tau)q(z_v)$, $B = (\tau(\lambda, z_v) + \bar\tau(\lambda, z_{v^\perp}))$, and $C = \tau q(\lambda_v) + \bar\tau q(\lambda_{v^\perp})$. The Fourier transform of $e(Ax^2 + Bx + C)$ equals $(\frac{2A}{i})^{-1/2} e\left(-\frac{(x+B)^2}{4A} + C\right)$ (see for instance [Bo2] Cor. 3.3). Hence we find that $\hat g(\lambda, v; d)$ is equal to

$$\frac{1}{\sqrt{2yz_v^2}}\, e\left(\frac{-d^2 - 2d(\tau(\lambda, z_v) + \bar\tau(\lambda, z_{v^\perp})) - (\tau(\lambda, z_v) + \bar\tau(\lambda, z_{v^\perp}))^2}{2(\tau - \bar\tau)z_v^2}\right) \times e\left(\tau q(\lambda_v) + \bar\tau q(\lambda_{v^\perp})\right).$$

Using $q(\lambda_v) = q(\lambda_w) + (\lambda, z_v)^2/2z_v^2$ and $q(\lambda_{v^\perp}) = q(\lambda_{w^\perp}) + (\lambda, z_{v^\perp})^2/2z_{v^\perp}^2$ we obtain

$$\hat{g}(\lambda, v; d) = \frac{1}{\sqrt{2yz_v^2}} e\left(\frac{\tau(\lambda, z_v)^2 - \bar{\tau}(\lambda, z_{v^\perp})^2}{2z_v^2} + \tau q(\lambda_w) + \bar{\tau} q(\lambda_{w^\perp})\right)$$
$$\times\, e\left(\frac{-d^2 - 2d(\tau(\lambda, z_v) + \bar{\tau}(\lambda, z_{v^\perp})) - (\tau(\lambda, z_v) + \bar{\tau}(\lambda, z_{v^\perp}))^2}{2(\tau - \bar{\tau})z_v^2}\right). \tag{2.9}$$

Since

$$2d(\tau(\lambda, z_v) + \bar{\tau}(\lambda, z_{v^\perp})) = d\left[(\lambda, z_v - z_{v^\perp})(\tau - \bar{\tau}) + (\lambda, z)(\tau + \bar{\tau})\right],$$

the right hand side of (2.9) can be written as

$$\frac{1}{\sqrt{2yz_v^2}} e\left(\tau q(\lambda_w) + \bar{\tau} q(\lambda_{w^\perp}) - \frac{d(\lambda, z_v - z_{v^\perp})}{2z_v^2}\right)$$
$$\times\, e\left(-\frac{d^2 + d(\lambda, z)\tau + d(\lambda, z)\bar{\tau} + (\lambda, z)^2\tau\bar{\tau}}{2(\tau - \bar{\tau})z_v^2}\right),$$

which is equal to

$$\frac{1}{\sqrt{2yz_v^2}} e\left(\tau q(\lambda_w) + \bar{\tau} q(\lambda_{w^\perp}) - \frac{d(\lambda, z_v - z_{v^\perp})}{2z_v^2} - \frac{|d + (\lambda, z)\tau|^2}{4iyz_v^2}\right).$$

This implies the assertion. □

Theorem 2.4. *(cp. [Bo2] Theorem 5.2.) Let $\gamma \in L'$. We have*

$$\theta_{L+\gamma}(\tau, v) = \frac{1}{\sqrt{2yz_v^2}} \sum_{\substack{c,d \in \mathbb{Z} \\ c \equiv (\gamma, z)\ (N)}} e\left(-\frac{|c\tau + d|^2}{4iyz_v^2} - d(\gamma, z') + cdq(z')\right)$$
$$\times\, \theta_{K + p(\gamma - cz')}(\tau, w; d\mu, -c\mu),$$

where p denotes the projection $L_0' \to K'$ defined in (2.7).

Remark 2.5. Regarding the definition (2.1) of Θ_K we should write more correctly μ_K instead of μ (note that $\mu_K = \mu - (\mu, z')z$). Since $\mu_w = (\mu_K)_w = z_w'$ and $(\mu, z) = (\mu_K, z)$, we permit ourselves this abuse of notation.

Proof. We want to apply Lemma 2.3. Write $\gamma = \gamma_K + a_\gamma z' + b_\gamma z$ with $\gamma_K \in K'$, $a_\gamma = (\gamma, z) \in \mathbb{Z}$ and $b_\gamma = (\gamma, z') - (\gamma, z){z'}^2 \in \mathbb{Q}$ and observe that every $\lambda \in \gamma + K \oplus \mathbb{Z}\zeta$ can be uniquely written as

$$\lambda = \lambda_K + cz' + f(c, \gamma)z$$

with $c \in \mathbb{Z}$, $c \equiv a_\gamma \pmod N$, and $\lambda_K \in K + \gamma_K + \zeta_K(c - a_\gamma)/N$. The function $f(c,\gamma)$ is defined by $f(c,\gamma) = b_\gamma + (c - a_\gamma)B/N$, and ζ_K, B are given by (2.5). It can be easily checked that $\gamma_K + \zeta_K(c - a_\gamma)/N = p(\gamma - cz')$.

We find that $\theta_{L+\gamma}(\tau, v)$ is given by

$$\frac{1}{\sqrt{2yz_v^2}} \sum_{\substack{c,d\in\mathbb{Z} \\ c\equiv(\gamma,z)\ (N)}} \sum_{\lambda_K\in K+p(\gamma-cz')} e\left(\tau q((\lambda_K + cz')_w) + \bar\tau q((\lambda_K + cz')_{w^\perp})\right)$$
$$\times\, e\left(-\frac{|(\lambda_K + cz', z)\tau + d|^2}{4iyz_v^2} - \frac{d(\lambda_K + cz' + f(c,\gamma)z,\, z_v - z_{v^\perp})}{2z_v^2}\right),$$

where we have used $z_w = z_{w^\perp} = 0$ and $q(z) = 0$. Regarding the definition (2.1) of $\Theta_K(\tau, w; d\mu, -c\mu)$, we see that we have to prove

$$-\frac{d(\lambda_K + cz' + f(c,\gamma)z,\, z_v - z_{v^\perp})}{2z_v^2} \equiv -(\lambda_K - c\mu/2,\, d\mu) - d(\gamma, z') + cdq(z')$$

modulo 1. Since $(z, z_v - z_{v^\perp})/2z_v^2 = 1$, we find

$$\begin{aligned} -d\frac{(f(c,\gamma)z,\, z_v - z_{v^\perp})}{2z_v^2} &= -df(c,\gamma) \\ &= -db_\gamma + \frac{(\gamma,z) - c}{N} dB \\ &= -d(\gamma, z') + d(\gamma,z){z'}^2 + \frac{(\gamma,z) - c}{N} dB. \end{aligned}$$

From $(\zeta, z') \in \mathbb{Z}$ it follows that $B \equiv -N{z'}^2 \pmod 1$, and this implies that

$$-d\frac{(f(c,\gamma)z,\, z_v - z_{v^\perp})}{2z_v^2} \equiv -d(\gamma, z') + cd{z'}^2 \pmod 1.$$

Thus it suffices to show

$$-d\frac{(\lambda_K + cz',\, z_v - z_{v^\perp})}{2z_v^2} = -(\lambda_K - c\mu/2,\, d\mu) - cdq(z'). \tag{2.10}$$

The left hand side of (2.10) equals

$$\begin{aligned} &-d\left(\lambda_K + cz',\, z_v/2z_v^2 + z_{v^\perp}/2z_{v^\perp}^2\right) \\ &\quad = -d\left(\lambda_K + c(z' - z_v/2z_v^2 - z_{v^\perp}/2z_{v^\perp}^2)/2,\, z_v/2z_v^2 + z_{v^\perp}/2z_{v^\perp}^2\right) \\ &\qquad - \frac{dc}{2}\left(z',\, z_v/2z_v^2 + z_{v^\perp}/2z_{v^\perp}^2\right) \\ &\qquad - \frac{dc}{2}\left(z_v/2z_v^2 + z_{v^\perp}/2z_{v^\perp}^2,\, z_v/2z_v^2 + z_{v^\perp}/2z_{v^\perp}^2\right). \end{aligned} \tag{2.11}$$

Since $q(z) = 0$, the third term on the right hand side of (2.11) is 0. The second term can be written as

$$-d\left(\lambda_K + c(z' - z_v/2z_v^2 - z_{v^\perp}/2z_{v^\perp}^2)/2,\ -z'\right) - \frac{dc}{2}(z', z').$$

Thus the left hand side of (2.10) equals

$$\begin{aligned}-d\left(\lambda_K + c(z' - z_v/2z_v^2 + z_{v^\perp}/2z_{v^\perp}^2)/2,\ -z' + z_v/2z_v^2 + z_{v^\perp}/2z_{v^\perp}^2\right) - dcq(z')\\ = -d(\lambda_K - c\mu/2,\ \mu) - cdq(z').\end{aligned}$$

□

Let $k \in \frac{1}{2}\mathbb{Z}$ and $F_L(\tau) = \sum_{\delta\in L'/L} \mathfrak{e}_\delta f_\delta(\tau)$ be a function $\mathbb{H} \to \mathbb{C}[L'/L]$ with $F_L \mid_k (M,\phi) = F_L$ for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$. Let r, t be given integers. We define a function

$$F_K(\tau; r, t) = \sum_{\gamma\in K'/K} \mathfrak{e}_\gamma f_\gamma(\tau; r, t), \tag{2.12}$$

by putting

$$f_\gamma(\tau; r, t) = \sum_{\substack{\lambda\in L_0'/L\\ p(\lambda)=\gamma}} e\left(-r(\lambda, z') - rtq(z')\right) f_{L+\lambda+tz'}(\tau)$$

for $\gamma \in K'/K$. (This is just a different notation for the definition given in [Bo2] on page 512.) Let $\gamma \in K'$. Then as a set of representatives for $\lambda \in L_0'/L$ with $p(\lambda) = \gamma + K$ we may take $\gamma - (\gamma,\zeta)z/N + bz/N$ where b runs modulo N. A set of representatives for L_0'/L is given by $\lambda = \gamma - (\gamma,\zeta)z/N + bz/N$, where γ runs through a set of representatives for K'/K and b runs modulo N. (Since the index of L_0' in L' equals N, we find again that $|L'/L| = N^2|K'/K|$.)

Theorem 2.6. *We use the same notation as above. Let $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$, $M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$. Then $F_K(\tau; r, t)$ satisfies*

$$F_K(M\tau; ar + bt, cr + dt) = \phi(\tau)^{2k} \varrho_K(M,\phi) F_K(\tau; r, t).$$

Proof. See [Bo2] Theorem 5.3. □

For the Poincaré series $F^L_{\beta,m}(\tau, s)$ (see Definition 1.8) we denote the $\mathbb{C}[K'/K]$-valued function constructed in (2.12) by $F^K_{\beta,m}(\tau, s; r, t)$.

Proposition 2.7. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$.*

i). If $(\beta, z) \not\equiv 0 \pmod N$, then $F^K_{\beta,m}(\tau, s; 0, 0) = 0$.

ii). If $(\beta, z) \equiv 0 \pmod N$, then the function $F^K_{\beta,m}(\tau, s; 0, 0)$ equals the Poincaré series $F^K_{p(\beta),m}(\tau, s)$ of index $(p(\beta), m)$ attached to the lattice K (see Definition 1.35).

Proof. i). Assume that $\sigma \geq 1 - k/2$. From the definition of $F^K_{\beta,m}(\tau, s; 0, 0)$, the Fourier expansion of $F^L_{\beta,m}(\tau, s)$ (see Theorem 1.9), and the asymptotic property (1.26) of the Whittaker function $W_{\nu,\mu}(y)$ it follows that

$$F^K_{\beta,m}(\tau, s; 0, 0) = O(1)$$

for $y \to \infty$. Since $F^K_{\beta,m}(\tau, s; 0, 0)$ also satisfies

$$F^K_{\beta,m}(\tau, s; 0, 0) \mid_k (M, \phi) = F^K_{\beta,m}(\tau, s; 0, 0), \qquad (M, \phi) \in \mathrm{Mp}_2(\mathbb{Z}),$$
$$\Delta_k F^K_{\beta,m}(\tau, s; 0, 0) = \big(s(1-s) + (k^2 - 2k)/4\big)\, F^K_{\beta,m}(\tau, s; 0, 0),$$

we may infer that it vanishes identically (cf. [He] Chap. 9 Prop. 5.14c). Because $F^K_{\beta,m}(\tau, s; 0, 0)$ is holomorphic in s, we find that it vanishes for all s.

ii) In the same way as in (i) it follows that

$$F^K_{\beta,m}(\tau, s; 0, 0)$$
$$= \frac{\Gamma(1 + k/2 - s)}{\Gamma(2 - 2s)\Gamma(s + k/2)} \mathcal{M}_{1-s}(4\pi|m|y)(\mathfrak{e}_{p(\beta)}(mx) + \mathfrak{e}_{-p(\beta)}(mx)) + O(1)$$

for $y \to \infty$, and thereby

$$F^K_{\beta,m}(\tau, s; 0, 0) - F^K_{p(\beta),m}(\tau, s) = O(1) \qquad (y \to \infty).$$

The assertion can be deduced as in (i). □

2.2 The theta integral

As in section 1.3 let L be an even lattice of signature $(b^+, b^-) = (2, l), (1, l-1)$, or $(0, l-2)$ with $l \geq 3$. Put $k = 1 - l/2$ and

$$\sigma_0 = \max\{1, b^+/2 - k/2\}.$$

Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. We consider the theta integral

$$\Phi^L_{\beta,m}(v, s) = \int_{\mathcal{F}} \langle F^L_{\beta,m}(\tau, s),\, \Theta_L(\tau, v)\rangle y^{b^+/2} \frac{dx\, dy}{y^2}, \qquad (2.13)$$

where $F_{\beta,m}(\tau, s)$ denotes the Maass-Poincaré series of index (β, m) defined in (1.35) and $\Theta_L(\tau, v)$ the Siegel theta function (2.2). If it is clear from the context, to which lattice L the function (2.13) refers, we simply write $\Phi_{\beta,m}(v, s)$. According to Theorem 2.1, the integrand

$$\langle F_{\beta,m}(\tau, s),\, \Theta_L(\tau, v)\rangle y^{b^+/2}$$

is $\mathrm{Mp}_2(\mathbb{Z})$-invariant. Because $\Theta_L(\tau, v)$ is invariant under the action of $\mathrm{O}_d(L)$, the function $\Phi_{\beta,m}(v, s)$ is formally invariant under $\mathrm{O}_d(L)$, too.

Unfortunately, since $|F_{\beta,m}(\tau,s)|$ increases exponentially as $y \to \infty$, the integral (2.13) diverges. However, we shall show that it can be regularized as follows. Let $\mathcal{F}_u$ be the truncated fundamental domain

$$\mathcal{F}_u = \{\tau = x + iy \in \mathcal{F}; \quad y \le u\} \qquad (u \in \mathbb{R}_{>0}). \tag{2.14}$$

Then for $s = \sigma + it \in \mathbb{C}$ with $\sigma > \sigma_0$ we may define the regularized theta lift of $F_{\beta,m}(\tau,s)$ by

$$\Phi^L_{\beta,m}(v,s) = \lim_{u\to\infty} \int\limits_{\mathcal{F}_u} \langle F^L_{\beta,m}(\tau,s),\, \Theta_L(\tau,v)\rangle y^{b^+/2} \frac{dx\,dy}{y^2}. \tag{2.15}$$

Roughly speaking this means that we first carry out the integration over x and afterwards the integration over y. We will show that $\Phi_{\beta,m}(v,s)$ as defined in (2.15) is holomorphic in s for $\sigma > \sigma_0$. It can be continued to a holomorphic function on $\{s \in \mathbb{C};\ \sigma > 1,\ s \neq b^+/2 - k/2\}$ with a simple pole at $s = b^+/2 - k/2$. Hence for $\sigma > 1$ and $s \neq b^+/2 - k/2$ the function $\Phi_{\beta,m}(v,s)$ is given by holomorphic continuation. For $s = b^+/2 - k/2$ we may define the regularized theta integral to be the constant term of the Laurent expansion in s of $\Phi_{\beta,m}(v,s)$ at the point $s = b^+/2 - k/2$.

This construction is very similar to the definition of the regularized theta lift due to Harvey, Moore, and Borcherds [HM, Bo2]. Following Borcherds, one could also define $\Phi_{\beta,m}(v,s)$ to be the constant term of the Laurent expansion in s' of

$$\lim_{u\to\infty} \int\limits_{\mathcal{F}_u} \langle F_{\beta,m}(\tau,s),\, \Theta_L(\tau,v)\rangle y^{b^+/2-s'} \frac{dx\,dy}{y^2} \tag{2.16}$$

at $s' = 0$. Here s' is an additional complex variable. The above expression makes sense if $\Re(s') \gg 0$. Note that the integrand in (2.16) is only invariant under $\mathrm{Mp}_2(\mathbb{Z})$ if $s' = 0$. We will prove in Proposition 2.11 that (2.15) and (2.16) essentially yield to the same $\Phi_{\beta,m}(v,s)$. However, for our purposes the definition of the regularized theta lift given in (2.15) is more natural.

As in [Bo2] we will show that $\Phi_{\beta,m}(v,s)$ is a real analytic function on $\mathrm{Gr}(L)$ with singularities along smaller sub-Grassmannians. To make this more precise, we define a subset $H(\beta,m)$ of the Grassmannian $\mathrm{Gr}(L)$ by

$$H(\beta,m) = \bigcup_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \lambda^\perp. \tag{2.17}$$

Here $\lambda^\perp$ means the orthogonal complement of λ in $\mathrm{Gr}(L)$, i.e. the set of all positive definite b^+-dimensional subspaces $v \subset V$ with $v \perp \lambda$.[1] It is known that for any subset U with compact closure $\overline{U} \subset \mathrm{Gr}(L)$ the set

$$\mathcal{S}(\beta,m,U) = \{\lambda \in \beta + L; \quad q(\lambda) = m,\ \exists v \in U \text{ with } v \perp \lambda\} \tag{2.18}$$

[1] If $b^+ = 0$ we put $H(\beta,m) = \emptyset$.

is finite. Hence $H(\beta,m)$ is locally a finite union of codimension b^+ sub-Grassmannians. It will turn out that $\Phi_{\beta,m}(v,s)$ is a real analytic function on $\operatorname{Gr}(L)-H(\beta,m)$.

Proposition 2.8. *Let $v\in\operatorname{Gr}(L)-H(\beta,m)$ and $s\in\mathbb{C}$ with $\sigma>\sigma_0$. Then the regularized theta integral (2.15) converges and defines a holomorphic function in s. It can be continued holomorphically to*

$$\{s\in\mathbb{C};\quad \sigma>1,\ s\neq b^+/2-k/2\},$$

and has a pole of first order with residue $b(0,0,b^+/2-k/2)$ at $s=b^+/2-k/2$ (supposed that $b^+/2-k/2>1$).

Proof. Recall that $F_{\beta,m}(\tau,s)$ is holomorphic in s for $\sigma>1$. The integral

$$\int\limits_{\mathcal{F}_1}\langle F_{\beta,m}(\tau,s),\,\Theta_L(\tau,v)\rangle y^{b^+/2}\,\frac{dx\,dy}{y^2}$$

over the compact set $\mathcal{F}_1$ is clearly holomorphic in s for $\sigma>1$. Thus it suffices to consider the function

$$\varphi(v,s)=\int\limits_{y=1}^{\infty}\int\limits_{x=0}^{1}\langle F_{\beta,m}(\tau,s),\,\Theta_L(\tau,v)\rangle y^{b^+/2-2}\,dx\,dy. \tag{2.19}$$

We insert the Fourier expansions

$$F_{\beta,m}(\tau,s)=\sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)}c(\gamma,n;y,s)\mathfrak{e}_\gamma(nx)$$
$$\Theta_L(\tau,v)=\sum_{\lambda\in L'}e(iyq(\lambda_v)-iyq(\lambda_{v^\perp}))\mathfrak{e}_\lambda(q(\lambda)x)$$

into (2.19), carry out the integral over x, and obtain

$$\begin{aligned}\varphi(v,s)&=\int\limits_{y=1}^{\infty}\sum_{\lambda\in L'}c(\lambda,q(\lambda);y,s)\exp(-2\pi yq(\lambda_v)+2\pi yq(\lambda_{v^\perp}))y^{b^+/2-2}\,dy\\&=\int\limits_{y=1}^{\infty}c(0,0;y,s)y^{b^+/2-2}dy\\&\quad+\int\limits_{y=1}^{\infty}\sum_{\lambda\in L'-0}c(\lambda,q(\lambda);y,s)\exp(-4\pi yq(\lambda_v)+2\pi yq(\lambda))y^{b^+/2-2}\,dy.\end{aligned}$$

According to Theorem 1.9 the Fourier expansion of $F_{\beta,m}(\tau,s)$ can be written more explicitly in the form

$$F_{\beta,m}(\tau,s) = \frac{1}{\Gamma(2s)}\mathcal{M}_s(4\pi|m|y)(\mathfrak{e}_\beta(mx) + \mathfrak{e}_{-\beta}(mx)) + H(\tau,s),$$

where

$$H(\tau,s) = \sum_{\substack{\gamma\in L'/L\\ q(\gamma)\in\mathbb{Z}}} \tilde{b}(\gamma,0,s)y^{1-s-k/2} + \sum_{\substack{\gamma\in L'/L\\ n\in\mathbb{Z}+q(\gamma)\\ n\neq 0}} \tilde{b}(\gamma,n,s)\mathcal{W}_s(4\pi ny)\mathfrak{e}_\gamma(nx),$$

$$\tilde{b}(\gamma,n,s) = b(\gamma,n,s) - \frac{\Gamma(1+k/2-s)}{\Gamma(2s)\Gamma(1-2s)}\delta_{m,n}(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma}).$$

Observe that all Fourier coefficients $\tilde{b}(\gamma,n,s)$ are holomorphic in s for $\sigma > 1$. (This is not true for $b(\pm\beta,m,s)$. That is why we have introduced the $\tilde{b}(\gamma,n,s)$.) We find that $\varphi(v,s)$ is given by

$$\begin{aligned}
&b(0,0,s)\int_1^\infty y^{b^+/2-k/2-s-1}\,dy\\
&+\int_1^\infty \sum_{\substack{\lambda\in L'-0\\ q(\lambda)=0}} b(\lambda,0,s)\exp(-4\pi yq(\lambda_v))y^{b^+/2-k/2-s-1}dy\\
&+\int_1^\infty \sum_{\substack{\lambda\in L'\\ q(\lambda)\neq 0}} \tilde{b}(\lambda,q(\lambda),s)\mathcal{W}_s(4\pi q(\lambda)y)\exp(2\pi yq(\lambda_{v^\perp})-2\pi yq(\lambda_v))y^{b^+/2-2}dy\\
&+\frac{2}{\Gamma(2s)}\int_1^\infty \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \mathcal{M}_s(4\pi|m|y)\exp(2\pi ym-4\pi yq(\lambda_v))y^{b^+/2-2}dy. \qquad (2.20)
\end{aligned}$$

The first integral obviously converges for $\sigma > b^+/2 - k/2$ and equals

$$\frac{1}{s-b^+/2+k/2}.$$

Thus the first summand in (2.20) has a holomorphic continuation to $\{s\in\mathbb{C};\quad \sigma>1,\ s\neq b^+/2-k/2\}$. If $b^+/2-k/2>1$ it has a simple pole at $s=b^+/2-k/2$ with residue $b(0,0,b^+/2-k/2)$.

Therefore it suffices to show that the remaining terms in (2.20) converge locally uniformly absolutely for $\sigma > 1$.

For the convergence of the second term, notice that $q(\lambda)=0$ implies $q(\lambda_v)=\frac{1}{2}q(\lambda_v)-\frac{1}{2}q(\lambda_{v^\perp})$. The latter quadratatic form is a multiple of the (positive definite) majorant associated with v. Thus the sum over λ is dominated by a subseries of a theta series attached to a definite lattice.

To prove the convergence of the third term we also use the majorant associated with v, and in addition the asymptotic property (1.26) of the Whittaker function $\mathcal{W}_{\nu,\mu}(y)$ and the estimates

$$b(\gamma, n, s) = O\left(\exp(4\pi\sqrt{|mn|}\,)\right), \qquad n \to +\infty, \tag{2.21}$$

$$b(\gamma, n, s) = O\left(|n|^{\sigma+k/2-1}\right), \qquad n \to -\infty \tag{2.22}$$

(locally uniformly in s for $\sigma > 1$). These bounds can be easily obtained from the asymptotic behavior of the I_ν- resp. J_ν-Bessel function and the fact that $H_c(\beta, m, \gamma, n)$ is universally bounded.

If $b^+ = 0$, then L is a definite lattice, and $\lambda_v = 0$ for all λ. The convergence of the last term in (2.20) follows from the asymptotic property (1.25) of the M-Whittaker function.

To prove the convergence of the last term for $b^+ = 1, 2$ we again use the asymptotic property (1.25) and the following fact: For any $C \geq 0$ and any compact subset $B \subset \mathrm{Gr}(L)$ the set

$$\{\lambda \in L'; \quad q(\lambda) = m, \, \exists v \in B \text{ with } q(\lambda_v) \leq C\}$$

is finite. By assumption $v \notin H(\beta, m)$. Hence there exists an $\varepsilon > 0$ such that $q(\lambda_v) > \varepsilon$ for all $\lambda \in L'$ with $q(\lambda) = m$. It follows that

$$\begin{aligned}
&\sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \left|\mathcal{M}_s(4\pi|m|y)\exp(-4\pi y q(\lambda_v) + 2\pi y m) y^{b^+/2-2}\right| \\
&\qquad \ll \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \exp(-4\pi y q(\lambda_v)) \\
&\qquad \ll e^{-2\pi\varepsilon y} \sum_{\substack{\lambda\in L'\\ q(\lambda)=m}} e^{-\pi(m+q(\lambda_v)-q(\lambda_{v^\perp}))}
\end{aligned}$$

uniformly for $y \in [1, \infty)$ and locally uniformly in s. This implies the desired convergence statement. □

Since $F_{\beta,m} = F_{-\beta,m}$, we have $\Phi_{\beta,m}(v, s) = \Phi_{-\beta,m}(v, s)$.

Definition 2.9. *Let $D \subset \mathbb{C}$ be an open subset, $a \in D$, and f a meromorphic function on D. We denote the constant term of the Laurent expansion of f at $s = a$ by $\mathcal{C}_{s=a}[f(s)]$.*

Definition 2.10. *For $v \in \mathrm{Gr}(L) - H(\beta, m)$ we define*

$$\Phi_{\beta,m}(v) = \mathcal{C}_{s=1-k/2}\left[\Phi_{\beta,m}(v, s)\right].$$

If $b^+ = 0$ or $b^+ = 1$, then $\Phi_{\beta,m}(v, s)$ is holomorphic at $s = 1 - k/2$ and we may also write $\Phi_{\beta,m}(v) = \Phi_{\beta,m}(v, 1 - k/2)$.

Proposition 2.11. *Borcherds' regularization of the theta integral as described in (2.16) equals*

$$\begin{cases} \Phi_{\beta,m}(v,s), & \text{if } s \neq b^+/2-k/2,\\ \mathcal{C}_{s=b^+/2-k/2}\left[\Phi_{\beta,m}(v,s)\right] - b'(0,0,b^+/2-k/2), & \text{if } s = b^+/2-k/2. \end{cases}$$

Here $b'(0,0,b^+/2-k/2)$ means the derivative of the coefficient $b(0,0,s)$ at $s=b^+/2-k/2$.

Proof. The same argument as in Proposition 2.8 shows that Borcherds' regularization of the theta integral can be written as

$$\int_{\mathcal{F}_1} \langle F_{\beta,m}(\tau,s),\, \Theta_L(\tau,v)\rangle y^{b^+/2}\frac{dx\,dy}{y^2} + \mathcal{C}_{s'=0}\left[\frac{b(0,0,s)}{s'+s-b^+/2+k/2}\right]$$
$$+\int_1^\infty \sum_{\lambda\in L'-0} c(\lambda,q(\lambda);y,s)\exp(-4\pi y q(\lambda_v)+2\pi y q(\lambda)) y^{b^+/2-2}\,dy.$$

If we compare this with the expression for $\varphi(v,s)$ given in Proposition 2.8, we immediately obtain the assertion in the case $s\neq b^+/2-k/2$. For $s=b^+/2-k/2$ we find that the difference of our regularization $\mathcal{C}_{s=b^+/2-k/2}\left[\Phi_{\beta,m}(v,s)\right]$ and Borcherds' regularization is given by

$$\mathcal{C}_{s=b^+/2-k/2}\left[\frac{b(0,0,s)}{s-b^+/2+k/2}\right] - b(0,0,b^+/2-k/2)\mathcal{C}_{s'=0}\left[{s'}^{-1}\right]$$
$$= b'(0,0,b^+/2-k/2).$$

□

For the rest of this section we assume that $b^+=1,2$. In Proposition 2.8 we considered $\Phi_{\beta,m}(v,s)$ for a fixed v and varying s. Now we show that for fixed s the function $\Phi_{\beta,m}(v,s)$ is real analytic in v on $\mathrm{Gr}(L)-H(\beta,m)$.

Let $U\subset \mathrm{Gr}(L)$ be an open subset and f, g functions on a dense open subset of U. Then we write

$$f\approx g,$$

if $f-g$ can be continued to a real analytic function on U. In this case we will say that f has a singularity of type g.

Theorem 2.12. *i) For fixed $s\in\{s\in\mathbb{C};\ \sigma>1,\ s\neq b^+/2-k/2\}$ the function $\Phi_{\beta,m}(v,s)$ is real analytic in v on $\mathrm{Gr}(L)-H(\beta,m)$.*

ii) Let $U\subset\mathrm{Gr}(L)$ be an open subset with compact closure $\overline{U}\subset\mathrm{Gr}(L)$. The function $\Phi_{\beta,m}(v)$ is real analytic on $\mathrm{Gr}(L)-H(\beta,m)$. On U it has a singularity of type

$$-2\sum_{\lambda\in\mathcal{S}(\beta,m,U)}\log q(\lambda_v),$$

if $b^+=2$, and of type

$$-4\sqrt{2}\pi\sum_{\lambda\in\mathcal{S}(\beta,m,U)}|\lambda_v|,$$

if $b^+=1$.

Proof. We use the same argument as in [Bo2] Theorem 6.2. We only give a proof for (ii), because (i) can be treated similarly.

Since $\Theta_L(\tau, v)$ is real analytic and $F_{\beta,m}(\tau, s)$ is holomorphic in s, the integral

$$\int_{\mathcal{F}_1} \langle F_{\beta,m}(\tau, s),\, \Theta_L(\tau, v)\rangle y^{b^+/2} \frac{dx\,dy}{y^2}$$

is holomorphic in s and real analytic in v. As in the proof of Proposition 2.8 it suffices to consider the function $\varphi(v, s)$ defined by (2.19). We insert the Fourier expansions of $\Theta_L(\tau, v)$ and $F_{\beta,m}(\tau, s)$ (as given in Theorem 1.9) and infer as in the proof of Proposition 2.8 that $\varphi(v, s)$ is equal to

$$\begin{aligned}
&\frac{b(0,0,s)}{b^+/2 - k/2 - s} + \int_1^\infty \sum_{\substack{\lambda\in L'-0\\ q(\lambda)=0}} b(\lambda, 0, s)\exp(-4\pi y q(\lambda_v)) y^{b^+/2-k/2-s-1} dy\\
&+ \int_1^\infty \sum_{\substack{\lambda\in L'\\ q(\lambda)\neq 0}} b(\lambda, q(\lambda), s)\mathcal{W}_s(4\pi q(\lambda) y)\exp(-2\pi y q(\lambda_v) + 2\pi y q(\lambda_{v^\perp})) y^{b^+/2-2} dy\\
&+ \frac{2\Gamma(1+k/2-s)}{\Gamma(2-2s)\Gamma(s+k/2)} \int_1^\infty \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \mathcal{M}_{1-s}(4\pi |m| y)\\
&\qquad\qquad \times \exp(-4\pi y q(\lambda_v) + 2\pi y m) y^{b^+/2-2} dy. \qquad (2.23)
\end{aligned}$$

The first quantity does not depend on v. The second and the third term are holomorphic in s near $s = 1 - k/2$ and real analytic in $v \in \mathrm{Gr}(L)$. Hence they do not contribute to the singularity. The last term is holomorphic in s near $s = 1 - k/2$ and real analytic for $v \in \mathrm{Gr}(L) - H(\beta, m)$. We find that

$$\begin{aligned}
\Phi_{\beta,m}(v) &\approx \mathcal{C}_{s=1-k/2}\left[\varphi(v, s)\right]\\
&\approx 2 \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \int_1^\infty \mathcal{M}_{k/2}(4\pi|m|y)\exp(-4\pi y q(\lambda_v) + 2\pi y m) y^{b^+/2-2}\, dy\\
&\approx 2 \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \int_1^\infty \exp(-4\pi y q(\lambda_v)) y^{b^+/2-2}\, dy.
\end{aligned}$$

In the last line we have used $\mathcal{M}_{k/2}(y) = e^{y/2}$. The integrals in the last line only become singular if $q(\lambda_v) = 0$. Thus for $v \in U$ we get

$$\begin{aligned}\Phi_{\beta,m}(v) &\approx 2 \sum_{\lambda\in\mathcal{S}(\beta,m,U)} \int_1^\infty \exp(-4\pi y q(\lambda_v)) y^{b^+/2-2}\, dy \\ &\approx 2 \sum_{\lambda\in\mathcal{S}(\beta,m,U)} (4\pi q(\lambda_v))^{1-b^+/2}\, \Gamma(b^+/2-1, 4\pi q(\lambda_v)),\end{aligned}$$

where $\Gamma(a,x)$ denotes the incomplete Gamma function defined in (1.32) (cf. [AbSt] p. 81). Now the singularity can be read off from [Bo2] Lemma 6.1. □

Note that the functions $\Gamma(0,x)$ and $\Gamma(-1/2,x)$ that occur at the end of the proof of Theorem 2.12 can be described more explicitly as follows: According to [E1] Vol. II p. 143 (5) one has

$$\Gamma(0,x) = -\gamma - \log(x) - \sum_{n=1}^\infty \frac{(-x)^n}{n!\, n}, \tag{2.24}$$

where γ denotes the Euler-Mascheroni constant. Using the recurrence relation

$$\Gamma(a+1,x) = a\Gamma(a,x) + x^a e^{-x}$$

([E1] Vol. II p. 134 (3)) and the identity

$$\Gamma(1/2,x^2) = \sqrt{\pi}(1-\operatorname{erf}(x))$$

([AbSt] p. 82 (6.5.17)), one finds

$$\Gamma(-1/2,x^2) = 2|x|^{-1}e^{-x^2} - 2\sqrt{\pi}(1-\operatorname{erf}(x)). \tag{2.25}$$

Here $\operatorname{erf}(x) = \frac{2}{\pi}\int_0^x e^{-t^2}\,dt$ denotes the error function (cf. [AbSt] chapter 7). It has the expansion

$$\operatorname{erf}(x) = \frac{2}{\pi}\sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{n!(2n+1)}.$$

In particular, from (2.24) and (2.25) we can also read off the singularities of $\Gamma(0,x)$ and $\Gamma(-1/2,x)$.

2.3 Unfolding against $F_{\beta,m}$

As in section 1.3 let L be an even lattice of signature $(b^+,b^-) = (2,l)$, $(1,l-1)$, or $(0,l-2)$ with $l\geq 3$, and put $k = 1-l/2$. In this section we find a nice invariant expression for $\Phi_{\beta,m}(v,s)$ by unfolding the theta integral $\Phi_{\beta,m}(v,s)$ against the Poincaré series $F_{\beta,m}(\tau,s)$. In [Br1] we started with such an invariant expression for $\Phi_{\beta,m}(v,s)$, calculated its Fourier expansion, and thereby obtained a connection to certain elliptic modular forms.

Lemma 2.13. *The Siegel theta function satisfies*

$$\theta_\beta(\tau, v) = O(y^{-b^+/2-b^-/2}) \qquad (y \to 0)$$

uniformly in x.

Proof. This can be proved in the same way as the analogous statement for holomorphic elliptic modular forms. □

Let $F(a,b,c;z)$ denote the Gauss hypergeometric function

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{2.26}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ (cf. [AbSt] Chap. 15 or [E1] Vol. I Chap. 2). The circle of convergence of the series (2.26) is the unit circle $|z| = 1$. Using the estimate ([E1] Vol. I p. 57 (5))

$$\frac{(a)_n (b)_n}{(c)_n\, n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} n^{a+b-c-1} \left(1 + O(n^{-1})\right), \qquad n \to \infty,$$

one finds that (2.26) converges absolutely for $|z| = 1$, if $\Re(c-a-b) > 0$. Then the value for $z = 1$ is given by ([E1] Vol. I p. 104 (46))

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (c \neq 0, -1, -2, \dots). \tag{2.27}$$

Theorem 2.14. *Let* $\beta \in L'/L$ *and* $m \in \mathbb{Z} + q(\beta)$ *with* $m < 0$. *We have the identity*

$$\begin{aligned} \Phi_{\beta,m}(v,s) = 2\frac{\Gamma(b^-/4 + b^+/4 + s - 1)}{\Gamma(2s)(4\pi|m|)^{k/2-s}} &\sum_{\substack{\lambda \in \beta + L \\ q(\lambda) = m}} (4\pi |q(\lambda_{v^\perp})|)^{1-b^-/4-b^+/4-s} \\ &\times F(b^-/4 + b^+/4 + s - 1,\ s - b^-/4 + b^+/4,\ 2s;\ m/q(\lambda_{v^\perp})). \end{aligned}$$

The series converges normally for $v \in \mathrm{Gr}(L) - H(\beta,m)$ *and* $\sigma > b^-/4 + b^+/4$. *In particular, if* $(b^+, b^-) = (0, l-2)$, *then*

$$\Phi_{\beta,m}(v,s) = \frac{8\pi|m|}{(s + l/4 - 3/2)\Gamma(s + 3/2 - l/4)} \cdot \#\{\lambda \in \beta + L; \quad q(\lambda) = m\}.$$

Proof. Let $v \in \mathrm{Gr}(L) - H(\beta,m)$ and $s \in \mathbb{C}$ with $\sigma > b^-/4 + b^+/4$. By definition the function $\Phi_{\beta,m}(v,s)$ is equal to

$$\frac{1}{\Gamma(2s)} \int_{\mathcal{F}} \sum_{M \in \Gamma_\infty \backslash \Gamma_1} \left\langle [\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)] \mid_k \tilde{M},\ \Theta_L(\tau,v) \right\rangle y^{b^+/2} \frac{dx\, dy}{y^2},$$

where the integral is to be understood in the regularized sense (cp. (2.15)). According to the theta transformation formula (Theorem 2.1) this can be written as

$$\begin{aligned}
&\frac{1}{\Gamma(2s)} \int\limits_{\mathcal{F}} \sum_{M\in\Gamma_\infty\backslash\Gamma_1} \mathcal{M}_s(4\pi|m|\Im(M\tau))(\Im M\tau)^{b^+/2} \\
&\times \langle \mathfrak{e}_\beta(m\Re(M\tau)),\, \Theta_L(M\tau,v)\rangle \,\frac{dx\,dy}{y^2} \\
&= \frac{2}{\Gamma(2s)} \int\limits_{\mathcal{F}} \mathcal{M}_s(4\pi|m|y)y^{b^+/2}e(mx)\overline{\theta_\beta(\tau,v)}\,\frac{dx\,dy}{y^2} \\
&+ \frac{1}{\Gamma(2s)} \int\limits_{\mathcal{F}} \sum_{\substack{M=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_1 \\ c\neq 0}} \mathcal{M}_s(4\pi|m|\Im(M\tau))(\Im M\tau)^{b^+/2} \\
&\times e(m\Re(M\tau))\overline{\theta_\beta(M\tau,v)}\,\frac{dx\,dy}{y^2}.
\end{aligned} \tag{2.28}$$

The second integral on the right hand side of (2.28) can be evaluated by the usual unfolding trick. It equals

$$\frac{2}{\Gamma(2s)} \int\limits_{\mathcal{G}} \mathcal{M}_s(4\pi|m|y)y^{b^+/2}e(mx)\overline{\theta_\beta(\tau,v)}\,\frac{dx\,dy}{y^2}, \tag{2.29}$$

where

$$\mathcal{G} = \{\tau\in\mathbb{H};\quad |x|\le 1/2,\ |\tau|\le 1\} \tag{2.30}$$

is a fundamental domain for the action of Γ_∞ on $\mathbb{H}-\bigcup_{M\in\Gamma_\infty} M\mathcal{F}$. By virtue of (1.23) and Lemma 2.13, the integral in (2.29) converges absolutely if $\sigma > 1 + b^-/4 + b^+/4$, and therefore the unfolding is justified. Combining (2.29) with the first integral on the right hand side of (2.28), we obtain

$$\Phi_{\beta,m}(v,s) = \frac{2}{\Gamma(2s)} \int\limits_{y=0}^{\infty}\int\limits_{x=0}^{1} \mathcal{M}_s(4\pi|m|y)y^{b^+/2-2}e(mx)\overline{\theta_\beta(\tau,v)}\,dx\,dy.$$

We now insert the Fourier expansion

$$\theta_\beta(\tau,v) = \sum_{\lambda\in\beta+L} \exp(-2\pi yq(\lambda_v) + 2\pi yq(\lambda_{v^\perp}))e(q(\lambda)x)$$

of $\theta_\beta(\tau,v)$ and carry out the integration over x. We find

$$\begin{aligned}\Phi_{\beta,m}(v,s) &= \frac{2}{\Gamma(2s)} \int\limits_{y=0}^{\infty} \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \mathcal{M}_s(4\pi|m|y) y^{b^+/2-2} e^{-4\pi y q(\lambda_v)+2\pi y m}\, dy \\ &= \frac{2(4\pi|m|)^{-k/2}}{\Gamma(2s)} \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \int\limits_{0}^{\infty} M_{-k/2,\, s-1/2}(4\pi|m|y) y^{b^-/4+b^+/4-2} \\ &\qquad \times e^{-4\pi y q(\lambda_v)+2\pi y m}\, dy. \end{aligned} \tag{2.31}$$

The integral in (2.31) is a Laplace transform. If $\lambda_v \neq 0$, it equals (cp. [E2] p. 215 (11))

$$\frac{(4\pi|m|)^s \Gamma(b^-/4+b^+/4+s-1)}{(4\pi|q(\lambda_{v^\perp})|)^{b^-/4+b^+/4+s-1}} \times F\left(b^-/4+b^+/4+s-1,\, s-b^-/4+b^+/4,\, 2s;\, m/q(\lambda_{v^\perp})\right). \tag{2.32}$$

In the case $(b^+,b^-) = (0, l-2)$ the projection λ_v is always 0, and the integral in (2.31) is

$$\int\limits_0^\infty M_{-k/2,\, s-1/2}(4\pi|m|y) y^{l/4-5/2} e^{2\pi y m}\, dy. \tag{2.33}$$

By the asymptotic behavior of the M-Whittaker function, this integral converges absolutely for $\sigma > 3/2 - l/4$. It can be easily checked that (2.32) still holds:

$$\begin{aligned}(2.33) &= \lim_{\varepsilon\to 0+} \int\limits_0^\infty M_{-k/2,\, s-1/2}(4\pi|m|y) y^{l/4-5/2} e^{2\pi y m-\varepsilon y}\, dy \\ &= (4\pi|m|)^{\frac{3}{2}-\frac{l}{4}} \Gamma(s+l/4-3/2) F(s+l/4-3/2,\, s+1/2-l/4,\, 2s;\, 1).\end{aligned}$$

Using (2.27) we find

$$(2.33) = (4\pi|m|)^{3/2-l/4} \frac{\Gamma(s+l/4-3/2)\Gamma(2s)}{\Gamma(s-l/4+3/2)\Gamma(s+l/4-1/2)}. \tag{2.34}$$

If we substitute (2.32) resp. (2.34) into (2.31), we obtain the assertion. □

2.4 Unfolding against Θ_L

Throughout the following section, let L be an indefinite even lattice of signature $(b^+,b^-) = (2,l)$ or $(1,l-1)$ with $l \geq 3$, and put $k = 1 - l/2$. Recall the notation introduced in section 2.1.

Assume that L contains a primitive norm 0 vector z. Then we may use the method of Borcherds to evaluate the theta integral $\Phi_{\beta,m}(v,s)$ (cf. [Bo2] Chap. 7). Thereby we obtain the Fourier expansion of $\Phi_{\beta,m}(v,s)$ in terms of the Fourier coefficients of $F_{\beta,m}(\tau,s)$.

We denote the components of $F_{\beta,m}(\tau,s)$ by $f_\gamma(\tau,s)$, so that

$$F_{\beta,m}(\tau,s) = \sum_{\gamma\in L'/L} \mathfrak{e}_\gamma f_\gamma(\tau,s).$$

Here we write the Fourier expansion of $f_\gamma(\tau,s)$ in the form

$$f_\gamma(\tau,s) = \sum_{n\in\mathbb{Z}+q(\gamma)} c(\gamma,n;y,s)\mathfrak{e}_\gamma(nx).$$

The coefficients $c(\gamma,n;y,s)$ were calculated in section 1.3. For instance, for $\gamma\in L'/L$ with $q(\gamma)\in\mathbb{Z}$ we found:

$$\begin{aligned} c(\gamma,0;y,s) &= b(\gamma,0,s)y^{1-s-k/2},\\ b(\gamma,0,s) &= \frac{4^{1-k/2}\pi^{1+s-k/2}|m|^{s-k/2}}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)} \sum_{c\in\mathbb{Z}-\{0\}} |c|^{1-2s} H_c(\beta,m,\gamma,0). \end{aligned}$$

Theorem 2.15. *(cp. [Bo2] Theorem 7.1.) Let $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$, $m<0$. If $v\in\operatorname{Gr}(L)-H(\beta,m)$ and $z_v^2<\frac{1}{4|m|}$, then $\Phi^L_{\beta,m}(v,s)$ is equal to*

$$\begin{aligned} &\frac{1}{\sqrt{2}|z_v|}\Phi^K_{\beta,m}(w,s) + \frac{2}{\sqrt{\pi}}\left(\frac{2z_v^2}{\pi}\right)^{s-b^+/4-b^-/4} \Gamma(s+1/2-b^+/4-b^-/4)\\ &\times \sum_{\ell\ (N)} b(\ell z/N,0,s)\sum_{n\geq 1} e(\ell n/N) n^{b^+/2+b^-/2-1-2s}\\ &+\frac{\sqrt{2}}{|z_v|}\sum_{\lambda\in K'-0}\sum_{n\geq 1} e(n(\lambda,\mu)) \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e(n(\delta,z'))\\ &\times\int_0^\infty c(\delta,q(\lambda);y,s)y^{b^+/2-5/2}\exp\left(\frac{-\pi n^2}{2z_v^2 y} - 2\pi y q(\lambda_w) + 2\pi y q(\lambda_{w^\perp})\right)dy. \end{aligned} \tag{2.35}$$

Here $\Phi^K_{\beta,m}(w,s)$ is defined by

$$\Phi^K_{\beta,m}(w,s) = \begin{cases} 0, & \text{if } (\beta,z)\not\equiv 0 \pmod N,\\ \Phi^K_{p(\beta),m}(w,s), & \text{if } (\beta,z)\equiv 0 \pmod N. \end{cases}$$

The third summand in the above expression for $\Phi^L_{\beta,m}(v,s)$ converges normally in s for $\sigma>1$.

Proof. We use Theorem 2.4 to rewrite $\Phi^L_{\beta,m}(v,s)$ as

$$\frac{1}{\sqrt{2z_v^2}}\int\limits_{\mathcal{F}}\sum_{\gamma\in L'/L} f_\gamma(\tau,s)\sum_{\substack{c,d\in\mathbb{Z}\\ c\equiv(\gamma,z)\ (N)}} e\left(-\frac{|c\tau+d|^2}{4iyz_v^2}+d(\gamma,z')-cdq(z')\right)$$
$$\times\,\overline{\theta}_{K+p(\gamma-cz')}(\tau,w;d\mu,-c\mu)y^{(b^+-1)/2}\,\frac{dx\,dy}{y^2}$$
$$=\frac{1}{\sqrt{2z_v^2}}\int\limits_{\mathcal{F}}\sum_{\substack{\gamma\in L'/L\\ (\gamma,z)\equiv 0\ (N)}} f_\gamma(\tau,s)\overline{\theta}_{K+p(\gamma)}(\tau,w)y^{(b^+-1)/2}\,\frac{dx\,dy}{y^2}$$
$$+\frac{1}{\sqrt{2z_v^2}}\int\limits_{\mathcal{F}}\sideset{}{'}\sum_{c,d\in\mathbb{Z}}\sum_{\substack{\gamma\in L'/L\\ (\gamma,z)\equiv c\ (N)}} e\left(d(\gamma,z')-cdq(z')\right)f_\gamma(\tau,s)$$
$$\times\,e\left(-\frac{|c\tau+d|^2}{4iyz_v^2}\right)\overline{\theta}_{K+p(\gamma-cz')}(\tau,w;d\mu,-c\mu)y^{(b^+-1)/2}\,\frac{dx\,dy}{y^2}. \tag{2.36}$$

Here $\sum'_{c,d\in\mathbb{Z}}$ means that the sum runs over all pairs $(c,d)\in\mathbb{Z}^2$ with $(c,d)\neq(0,0)$. If we substitute in the definition (2.12) of $F^K_{\beta,m}(\tau,s;0,0)$, we obtain for the first term on the right hand side of (2.36):

$$\frac{1}{\sqrt{2z_v^2}}\int\limits_{\mathcal{F}}\left\langle F^K_{\beta,m}(\tau,s;0,0),\,\Theta_K(\tau,w)\right\rangle y^{(b^+-1)/2}\,\frac{dx\,dy}{y^2}.$$

By Proposition 2.7 this equals $\frac{1}{\sqrt{2z_v^2}}\Phi^K_{\beta,m}(w,s)$.

In the second term on the right hand side of (2.36) we change γ to $\gamma+cz'$ and substitute in the definition (2.12) of $F^K_{\beta,m}(\tau,s;-d,c)$. We find that it is equal to

$$\frac{1}{\sqrt{2z_v^2}}\int\limits_{\mathcal{F}}\sideset{}{'}\sum_{c,d\in\mathbb{Z}} e\left(-\frac{|c\tau+d|^2}{4iyz_v^2}\right)\left\langle F^K_{\beta,m}(\tau,s;-d,c),\,\Theta_K(\tau,w;d\mu,-c\mu)\right\rangle$$
$$\times\,y^{(b^+-1)/2}\,\frac{dx\,dy}{y^2}$$
$$=\frac{1}{\sqrt{2z_v^2}}\int\limits_{\mathcal{F}}\sum_{(c,d)=1}\sum_{n\geq 1} e\left(-\frac{n^2|c\tau+d|^2}{4iyz_v^2}\right)$$
$$\times\left\langle F^K_{\beta,m}(\tau,s;-nd,nc),\,\Theta_K(\tau,w;nd\mu,-nc\mu)\right\rangle y^{(b^+-1)/2}\,\frac{dx\,dy}{y^2}. \tag{2.37}$$

By Theorem 2.1 and Theorem 2.6 we get

$$\frac{1}{\sqrt{2z_v^2}} \int\limits_{\mathcal{F}} \sum_{M\in\Gamma_\infty\backslash\Gamma_1} \sum_{n\geq 1} \exp\left(\frac{-\pi n^2}{2z_v^2\Im(M\tau)}\right) \\ \times \left\langle F^K_{\beta,m}(M\tau,s;-n,0),\, \Theta_K(M\tau,w;n\mu,0)\right\rangle \Im(M\tau)^{(b^+-1)/2}\,\frac{dx\,dy}{y^2}.$$

As in the proof of Theorem 2.14 we split this up into

$$\frac{\sqrt{2}}{|z_v|} \int\limits_{\mathcal{F}} \sum_{n\geq 1} \exp\left(\frac{-\pi n^2}{2z_v^2 y}\right) \left\langle F^K_{\beta,m}(\tau,s;-n,0),\, \Theta_K(\tau,w;n\mu,0)\right\rangle y^{(b^+-1)/2}\frac{dx\,dy}{y^2} \\ + \frac{\sqrt{2}}{|z_v|} \int\limits_{\mathcal{F}} \sum_{\substack{M=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\overline{\Gamma}_1\\ c\neq 0}} \sum_{n\geq 1} \exp\left(\frac{-\pi n^2}{2z_v^2\Im(M\tau)}\right) \\ \times \left\langle F^K_{\beta,m}(M\tau,s;-n,0),\, \Theta_K(M\tau,w;n\mu,0)\right\rangle \Im(M\tau)^{(b^+-1)/2}\,\frac{dx\,dy}{y^2}, \tag{2.38}$$

where $\overline{\Gamma}_1 = \Gamma/\{\pm 1\}$. The first integral on the right hand side of (2.38) is to be understood in the regularized sense. As in Lemma 2.13 it can be seen that

$$F^K_{\beta,m}(\tau,s;-n,0) = O(e^{2\pi|m|/y}) \tag{2.39}$$

for $y\to 0$. This implies that the second integral converges absolutely if $z_v^2 < \frac{1}{4|m|}$. By the usual unfolding argument it can be written in the form

$$\frac{\sqrt{2}}{|z_v|} \int\limits_{\mathcal{G}} \sum_{n\geq 1} \exp\left(\frac{-\pi n^2}{2z_v^2 y}\right) \left\langle F^K_{\beta,m}(\tau,s;-n,0),\, \Theta_K(\tau,w;n\mu,0)\right\rangle y^{(b^+-1)/2}\,\frac{dx\,dy}{y^2}, \tag{2.40}$$

with $\mathcal{G}$ as in (2.30). By (2.39) the latter integral also converges absolutely for $z_v^2 < \frac{1}{4|m|}$. We find that (2.38) equals

$$\frac{\sqrt{2}}{|z_v|} \int\limits_0^\infty \int\limits_0^1 \sum_{n\geq 1} \exp\left(\frac{-\pi n^2}{2z_v^2 y}\right) \left\langle F^K_{\beta,m}(\tau,s;-n,0),\, \Theta_K(\tau,w;n\mu,0)\right\rangle y^{\frac{b^+}{2}-\frac{5}{2}}\,dx\,dy.$$

We insert the Fourier expansions

$$\Theta_K(\tau,w;n\mu,0) = \sum_{\lambda\in K'} e(iyq(\lambda_w) - iyq(\lambda_{w^\perp}) - (\lambda,n\mu))\mathfrak{e}_\lambda(q(\lambda)x)$$

$$F^K_{\beta,m}(\tau,s;-n,0) = \sum_{\gamma\in K'/K} \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\gamma}} \sum_{r\in\mathbb{Z}+q(\delta)} e(n(\delta,z'))c(\delta,r;y,s)\mathfrak{e}_\gamma(rx)$$

and carry out the integration over x to get

$$\frac{\sqrt{2}}{|z_v|}\int_{y=0}^{\infty}\sum_{n\geq 1}\exp\left(\frac{-\pi n^2}{2z_v^2 y}\right)\sum_{\lambda\in K'}\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e(n(\delta,z'))c(\delta,q(\lambda);y,s)$$
$$\times\, e\left(iyq(\lambda_w)-iyq(\lambda_{w^\perp})+(\lambda,n\mu)\right)y^{b^+/2-5/2}\,dy. \quad (2.41)$$

In the above sum we consider the $\lambda=0$ term

$$\frac{\sqrt{2}}{|z_v|}\int_{y=0}^{\infty}\sum_{n\geq 1}\exp\left(\frac{-\pi n^2}{2z_v^2 y}\right)\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=0+K}} e(n(\delta,z'))c(\delta,0;y,s)y^{b^+/2-5/2}\,dy$$

separately. A set of representatives for $\delta\in L_0'/L$ with $p(\delta)=0+K$ is given by $\ell z/N$ where ℓ runs modulo N. We substitute in $c(\delta,0;y,s)=b(\delta,0,s)y^{1-s-k/2}$ and obtain

$$\frac{\sqrt{2}}{|z_v|}\sum_{\ell\ (N)} b(\ell z/N,0,s)\sum_{n\geq 1} e(n\ell/N)\int_{y=0}^{\infty}\exp\left(\frac{-\pi n^2}{2z_v^2 y}\right)y^{b^+/4+b^-/4-3/2-s}\,dy.$$

Since the latter integral equals

$$\left(\frac{2z_v^2}{\pi n^2}\right)^{s+1/2-b^+/4-b^-/4}\Gamma(s+1/2-b^+/4-b^-/4),$$

we find for the $\lambda=0$ term:

$$\frac{2}{\sqrt{\pi}}\left(\frac{2z_v^2}{\pi}\right)^{s-b^+/4-b^-/4}\Gamma(s+1/2-b^+/4-b^-/4)$$
$$\times\sum_{\ell\ (N)} b(\ell z/N,0,s)\sum_{n\geq 1} e(\ell n/N)n^{b^+/2+b^-/2-1-2s}.$$

Regarding the remaining sum over $\lambda\in K'-0$ in (2.41) we observe the following: Using the explicit formulas for the coefficients $c(\delta,r;y,s)$ (Theorem 1.9) and the asymptotic properties of the Whittaker functions, it can be seen that the sum of the absolute values of the terms with $\lambda\neq 0$ is integrable for $\sigma>1$. Hence we may interchange integration and summation and obtain the third term on the right hand side of (2.35). The resulting sum converges normally in s. (If $(\beta,z)\equiv 0 \pmod N$ and if there is an $\lambda\in p(\beta)+K$ with $q(\lambda)=m$ and $\lambda_w=0$, then this λ gives a singular contribution in (2.41). However, this singularity cancels out with the singularity of $\frac{1}{\sqrt{2}|z_v|}\Phi_{\beta,m}^K(w,s)$.)

□

3 The Fourier expansion of the theta lift

We now determine the Fourier expansion of the regularized theta integral more explicitly. We find that it consists of two different contributions. In the case of signature $(2, l)$ the first part gives rise to a certain "generalized Borcherds product", whereas the second part carries some cohomological information.

3.1 Lorentzian lattices

Throughout this section we assume that L is Lorentzian, i.e. has signature $(1, l-1)$. Then $\Phi_{\beta,m}(v, s)$ is holomorphic in s at $1 - k/2$. Therefore the function $\Phi_{\beta,m}(v)$ (see Definition 2.10) is simply given by $\Phi_{\beta,m}(v, 1 - k/2)$.

Moreover, $\mathrm{Gr}(L)$ is real hyperbolic space of dimension $l-1$, and $H(\beta, m)$ is a union of hyperplanes of codimension 1. The set of points $\mathrm{Gr}(L) - H(\beta, m)$ where $\Phi_{\beta,m}(v)$ is real analytic is not connected. We call its components the *Weyl chambers* of $\mathrm{Gr}(L)$ of index (β, m).

Observe that the smaller lattice K is negative definite. Hence the Grassmannian of K is a point, and the projection λ_w is 0 for any $\lambda \in K'$.

We now compute the Fourier expansion of $\Phi^L_{\beta,m}(v)$ more explicitly. We show that it can be written as the sum of two functions $\psi^L_{\beta,m}(v)$ and $\xi^L_{\beta,m}(v)$, where $\xi^L_{\beta,m}(v)$ is real analytic on the whole $\mathrm{Gr}(L)$ and $\psi^L_{\beta,m}(v)$ is the restriction of a continuous piecewise linear function on V. Both, $\psi^L_{\beta,m}(v)$ and $\xi^L_{\beta,m}(v)$ are eigen functions of the hyperbolic Laplacian on $\mathrm{Gr}(L)$.

We need some basic properties of Bernoulli polynomials (cf. [E1] Vol. I Chap. 1.13). For $r \in \mathbb{N}_0$ the r-th Bernoulli polynomial $B_r(x)$ is defined by the generating series

$$\frac{ze^{xz}}{e^z - 1} = \sum_{r=0}^{\infty} \frac{B_r(x)}{r!} z^r \qquad (|z| < 2\pi).$$

The first Bernoulli polynomials are

$$B_0(x) = 1, \qquad B_1(x) = x - 1/2, \qquad B_2(x) = x^2 - x + 1/6.$$

The identity

$$B'_r(x) = rB_{r-1}(x)$$

implies that $B_r(x)$ is a polynomial of degree r. Let $\mathbb{B}_r(x)$ be the 1-periodic function with $\mathbb{B}_r(x) = B_r(x)$ for $0 \leq x < 1$. The Fourier expansion of $\mathbb{B}_r(x)$ can be easily determined (cf. [E1] Vol. I p. 37). For $r \geq 2$ we have

$$\mathbb{B}_r(x) = -r! \sum_{n \in \mathbb{Z} - \{0\}} \frac{e(nx)}{(2\pi i n)^r}. \tag{3.1}$$

If $f(\lambda, \beta)$ is a function depending on $\lambda \in K'$ and $\delta \in L'/L$, then we use the abbreviation

$$\sum_{\lambda \in \pm p(\delta) + K} f(\lambda, \pm\delta) = \begin{cases} \sum_{\lambda \in p(\delta)+K} f(\lambda, \delta) + \sum_{\lambda \in -p(\delta)+K} f(\lambda, -\delta), & \text{if } \delta \in L'_0/L, \\ 0, & \text{if } \delta \notin L'_0/L. \end{cases} \tag{3.2}$$

Recall that the projection p is only defined on L'_0/L.

Proposition 3.1. *Let $v \in \mathrm{Gr}(L) - H(\beta, m)$ with $z_v^2 < \frac{1}{4|m|}$. Then*

$$\begin{aligned} \Phi^L_{\beta,m}(v) = {} & \frac{1}{\sqrt{2}|z_v|} \Phi^K_{\beta,m} + 4\sqrt{2}\pi |z_v| \sum_{\ell \, (N)} b(\ell z/N, 0) \mathbb{B}_2(\ell/N) \\ & + 4\sqrt{2}\pi |z_v| \sum_{\substack{\lambda \in p(\beta)+K \\ q(\lambda) = m}} \mathbb{B}_2((\lambda, \mu) + (\beta, z')) \\ & + 4\sqrt{2}(\pi/|z_v|)^{-k} \sum_{\lambda \in K' - 0} \sum_{\substack{\delta \in L'_0/L \\ p(\delta) = \lambda + K}} b(\delta, q(\lambda)) |\lambda|^{1-k} \\ & \times \sum_{n \geq 1} n^{-k-1} e(n(\lambda, \mu) + n(\delta, z')) K_{1-k}(2\pi n |\lambda|/|z_v|), \end{aligned} \tag{3.3}$$

where $\Phi^K_{\beta,m}$ denotes the constant $\Phi^K_{\beta,m}(w, 1 - k/2)$, and $b(\gamma, n) = b(\gamma, n, 1 - k/2)$ denote the Fourier coefficients of the Poincaré series $F^L_{\beta,m}(\tau, 1 - k/2)$. The third term on the right hand side is to be interpreted as 0 if $(\beta, z) \not\equiv 0 \pmod N$. As usual K_ν is the modified Bessel function of the third kind (cf. [AbSt] Chap. 9).

Proof. We apply Theorem 2.15. Since L is Lorentzian, the first and the second term on the right hand side of (2.35) are holomorphic in s near $1 - k/2$. We find

$$\Phi^L_{\beta,m}(v) = \frac{1}{\sqrt{2}|z_v|}\Phi^K_{\beta,m} + \frac{2\sqrt{2}|z_v|}{\pi}\sum_{\ell\ (N)} b(\ell z/N,0)\sum_{n\geq 1} e(\ell n/N)n^{-2}$$
$$+\frac{\sqrt{2}}{|z_v|}\sum_{\lambda\in K'-0}\sum_{n\geq 1} e(n(\lambda,\mu))\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e(n(\delta,z'))$$
$$\times\int_0^\infty c(\delta,q(\lambda);y,1-k/2)y^{-2}\exp\left(\frac{-\pi n^2}{2z_v^2 y}+2\pi y q(\lambda)\right)dy. \quad (3.4)$$

According to Proposition 1.10 we have

$$c(\gamma,q(\lambda);y,1-k/2)$$
$$=(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma})\delta_{m,q(\lambda)}e^{-2\pi m y}+b(\gamma,q(\lambda))\mathcal{W}_{1-k/2}(4\pi q(\lambda)y)$$

for $\lambda\in K'-0$. Moreover, we know that $\mathcal{W}_{1-k/2}(y)=e^{-y/2}\Gamma(1-k,|y|)$ for $y<0$ by (1.33). Hence the last term in (3.4) can be rewritten as

$$\frac{\sqrt{2}}{|z_v|}\sum_{\substack{\lambda\in\pm p(\beta)+K\\ q(\lambda)=m}}\sum_{n\geq 1} e(n(\lambda,\mu)+n(\pm\beta,z'))\int_0^\infty \exp\left(\frac{-\pi n^2}{2z_v^2 y}\right)y^{-2}\,dy$$
$$+\frac{\sqrt{2}}{|z_v|}\sum_{\lambda\in K'-0}\sum_{n\geq 1} e(n(\lambda,\mu))\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e(n(\delta,z'))b(\delta,q(\lambda))$$
$$\times\int_0^\infty \Gamma(1-k,\,4\pi|q(\lambda)|y)\exp\left(\frac{-\pi n^2}{2z_v^2 y}\right)y^{-2}dy. \quad (3.5)$$

The first integral in (3.5) equals $\frac{2z_v^2}{\pi n^2}$. If we replace the sum over $n\geq 1$ by a sum over $n\neq 0$ and substitute in identity (3.1) for $\mathbb{B}_2(x)$, we find

$$\Phi^L_{\beta,m}(v) = \frac{1}{\sqrt{2}|z_v|}\Phi^K_{\beta,m} + 4\sqrt{2}\pi|z_v|\sum_{\ell\ (N)} b(\ell z/N,0)\mathbb{B}_2(\ell/N)$$
$$+4\sqrt{2}\pi|z_v|\sum_{\substack{\lambda\in p(\beta)+K\\ q(\lambda)=m}}\mathbb{B}_2((\lambda,\mu)+(\beta,z'))$$
$$+\frac{\sqrt{2}}{|z_v|}\sum_{\lambda\in K'-0}\sum_{n\geq 1} e(n(\lambda,\mu))\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e(n(\delta,z'))$$
$$\times b(\delta,q(\lambda))\int_0^\infty \Gamma(1-k,\,4\pi|q(\lambda)|y)\exp\left(\frac{-\pi n^2}{2z_v^2 y}\right)y^{-2}dy. \quad (3.6)$$

We now compute the latter integral, which we abbreviate by

$$I(A,B) = \int_0^\infty \Gamma(1-k,\, Ay) e^{-B/y} y^{-2} dy \qquad (A, B > 0).$$

The integral representation (1.32) of the incomplete Gamma function obviously implies

$$\Gamma(a,x)' = -x^{a-1}e^{-x}.$$

If we integrate by parts, we find

$$I(A,B) = A^{1-k}B^{-1} \int_0^\infty e^{-B/y-Ay} y^{-k}\, dy.$$

Using the standard integral representation

$$\int_0^\infty e^{-Ay-B/y} y^{\nu-1}\, dy = 2(B/A)^{\nu/2} K_\nu(2\sqrt{AB}\,) \tag{3.7}$$

for the K-Bessel function ([E2] Vol. I p. 313 (6.3.17)), we get

$$I(A,B) = 2A^{(1-k)/2} B^{-(1+k)2} K_{1-k}(2\sqrt{AB}\,).$$

If we put this into (3.6), we obtain the assertion. □

The set of norm 1 vectors in $V = L \otimes \mathbb{R}$ has two components, one being given by

$$V_1 = \{v_1 \in V; \quad v_1^2 = 1,\ (z, v_1) > 0\}.$$

We may identify $\mathrm{Gr}(L)$ with V_1 via

$$V_1 \longrightarrow \mathrm{Gr}(L), \quad v_1 \mapsto \mathbb{R}v_1.$$

This is the “hyperboloid model” of hyperbolic space. In this identification we have

$$\begin{aligned} z_v &= (z, v_1)v_1, \\ |z_v| &= (z, v_1). \end{aligned}$$

The vector μ defined in (2.8) equals

$$\begin{aligned} \mu &= -z' + \frac{(z,v_1)v_1}{2(z,v_1)^2} - \frac{z-(z,v_1)v_1}{2(z,v_1)^2} \\ &= \frac{v_1}{(z,v_1)} - z' - \frac{z}{2(z,v_1)^2}. \end{aligned}$$

After a short calculation one finds that v_1 can be expressed in terms of (z, v_1) and the projection $\mu_K = \mu - (\mu, z')z \in K \otimes \mathbb{R}$ by

$$v_1 = (z, v_1)\mu_K + (z, v_1)z' + bz. \tag{3.8}$$

Here $b \in \mathbb{R}$ is uniquely determined by the condition $v_1^2 = 1$. Conversely every $x_0 > 0$, $\mu \in K \otimes \mathbb{R}$ defines a unique $v_1 \in V_1$ as in (3.8). To be precise we may identify

$$\mathbb{R}_{>0} \times (K \otimes \mathbb{R}) \longrightarrow V_1, \quad (x_0, \mu) \mapsto \mu/x_0 + z'/x_0 + bz = v_1, \tag{3.9}$$

where b is again determined by $v_1^2 = 1$. Thereby we have realized $\operatorname{Gr}(L)$ as the subset

$$\mathcal{H} = \{(x_0, \dots, x_{l-2}) \in \mathbb{R}^{l-1}; \quad x_0 > 0\}$$

of $\mathbb{R}^{l-1} = \mathbb{R} \times (K \otimes \mathbb{R})$. This is known as the "upper half space model" of hyperbolic space.

Let $\varrho \in V$ with $\varrho = \varrho_K + \varrho_1 z' + \varrho_2 z$, $\varrho_K \in K \otimes \mathbb{R}$, and $\varrho_1, \varrho_2 \in \mathbb{R}$. The scalar product of $(x_0, \mu) \in \mathcal{H}$ with ϱ is given by

$$((x_0, \mu), \varrho) = \frac{1}{2x_0}\left(2(\mu, \varrho_K) + \varrho_1 {z'}^2 + 2\varrho_2 - \varrho_1 \mu^2\right) + \varrho_1 x_0/2. \tag{3.10}$$

Let us now consider the Weyl chambers of index (β, m), i.e. the connected components of $\operatorname{Gr}(L) - H(\beta, m)$, in the hyperboloid model. For a Weyl chamber $W \subset V_1$ and $\lambda \in L'$ we write $(\lambda, W) > 0$, if λ has positive inner product with all elements in the interior of W.

Lemma 3.2. *Let $W \subset V_1$ be a Weyl chamber of index (β, m) and assume that $\lambda \in L'$ with $q(\lambda) \geq 0$ or $q(\lambda) = m$, $\lambda + L = \pm\beta$. Suppose that $(\lambda, v_1) > 0$ for a $v_1 \in W$. Then $(\lambda, W) > 0$.*

Proof. Obviously $\lambda \neq 0$. Assume that there is a $v_2 \in W$ with $(\lambda, v_2) \leq 0$. Then, since W is connected, there exists a $v_0 \in W$ satisfying $(\lambda, v_0) = 0$, i.e. $\lambda \in v_0^\perp$. Because $v_0^\perp \subset V$ is a negative definite subspace, it follows that $q(\lambda) < 0$. Without any restriction we may conclude that $q(\lambda) = m$ and $\lambda + L = \beta$. But then $(\lambda, v_0) = 0$ implies $v_0 \in H(\beta, m)$, a contradiction. □

In the coordinates of V_1 the Fourier expansion of $\Phi_{\beta,m}^L$ is given by

$$\begin{aligned}
\Phi^L_{\beta,m}(v_1) &= \frac{1}{\sqrt{2}(z,v_1)}\Phi^K_{\beta,m} + 4\sqrt{2}\pi(z,v_1)\sum_{\ell\,(N)} b(\ell z/N,0)\mathbb{B}_2(\ell/N)\\
&\quad + 4\sqrt{2}\pi(z,v_1)\sum_{\substack{\lambda\in p(\beta)+K\\ q(\lambda)=m}} \mathbb{B}_2\left(\frac{(\lambda,v_1)}{(z,v_1)} + (\beta,z')\right)\\
&\quad + 4\sqrt{2}(\pi/(z,v_1))^{-k}\sum_{\lambda\in K'-0}\ \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} b(\delta,q(\lambda))|\lambda|^{1-k}\\
&\quad \times\sum_{n\geq 1} n^{-k-1} e\left(n\frac{(\lambda,v_1)}{(z,v_1)} + n(\delta,z')\right) K_{1-k}(2\pi n|\lambda|/(z,v_1)).
\end{aligned} \tag{3.11}$$

Definition 3.3. *Using the above notation we define two functions* $\xi^L_{\beta,m}$, $\psi^L_{\beta,m}$ *on* $\mathrm{Gr}(L)$ *by*

$$\begin{aligned}
\xi^L_{\beta,m}(v_1) &= \frac{\Phi^K_{\beta,m}}{\sqrt{2}}\left(\frac{1}{(z,v_1)} - 2(z',v_1)\right) + \frac{4\sqrt{2}\pi}{(z,v_1)}\sum_{\substack{\lambda\in p(\beta)+K\\ q(\lambda)=m}} (\lambda,v_1)^2\\
&\quad + 4\sqrt{2}(\pi/(z,v_1))^{-k}\sum_{\lambda\in K'-0}\ \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} b(\delta,q(\lambda))|\lambda|^{1-k}\\
&\quad \times\sum_{n\geq 1} n^{-k-1} e\left(n\frac{(\lambda,v_1)}{(z,v_1)} + n(\delta,z')\right) K_{1-k}(2\pi n|\lambda|/(z,v_1))
\end{aligned} \tag{3.12}$$

and

$$\psi^L_{\beta,m}(v_1) = \Phi^L_{\beta,m}(v_1) - \xi^L_{\beta,m}(v_1).$$

Note that this definition depends on the choice of the vectors z and z'. It is easily checked that $\xi^L_{\beta,m} = \xi^L_{-\beta,m}$.

By Theorem 2.14, $\Phi^L_{\beta,m}$ is a real valued function and $\Phi^K_{\beta,m}$ a real constant. According to Lemma 1.13 the coefficients $b(\delta,q(\lambda))$ are real numbers. Hence $\xi^L_{\beta,m}$ and $\psi^L_{\beta,m}$ are real-valued functions.

Using the boundedness of the coefficients $b(\delta,n)$ with $n<0$ (cf. (2.22)) and the asymptotic property

$$K_\nu(y) \sim \sqrt{\frac{\pi}{2y}}e^{-y} \qquad (y\to\infty) \tag{3.13}$$

of the K-Bessel function, one can show that $\xi^L_{\beta,m}(v_1)$ is real analytic on the whole $\mathrm{Gr}(L)$.

Theorem 3.4. *The function* $\psi^L_{\beta,m}$ *is the restriction of a continuous piecewise linear function on* V. *Its only singularities lie on* $H(\beta,m)$. *For* $v_1\in V_1$ *with* $(z,v_1)^2 < \frac{1}{4|m|}$ *it equals*

$$\psi^L_{\beta,m}(v_1) = \sqrt{2}(z',v_1)\Phi^K_{\beta,m} + 4\sqrt{2}\pi(z,v_1)\sum_{\ell\ (N)} b(\ell z/N,0)\mathbb{B}_2(\ell/N)$$
$$+ 4\sqrt{2}\pi(z,v_1)\sum_{\substack{\lambda\in p(\beta)+K\\ q(\lambda)=m}}\left(\mathbb{B}_2\left(\frac{(\lambda,v_1)}{(z,v_1)}+(\beta,z')\right) - \frac{(\lambda,v_1)^2}{(z,v_1)^2}\right). \tag{3.14}$$

Proof. Equality (3.14) immediately follows from (3.11) and the definition of $\psi^L_{\beta,m}(v_1)$. Using the fact that

$$\mathbb{B}_2(x) = B_2(x-[x]) = x^2 - (2[x]+1)x + [x]^2 + [x] + 1/6,$$

one finds that the quadratic terms in the sum on the right hand side of (3.14) cancel out. Hence $\psi^L_{\beta,m}(v_1)$ is piecewise linear on $\{v_1 \in V_1;\ (z,v_1)^2 < \frac{1}{4|m|}\}$.

Since $\xi^L_{\beta,m}(v_1)$ is real analytic on $\mathrm{Gr}(L)$, the only singularities of $\psi^L_{\beta,m}(v_1)$ lie on $H(\beta,m)$ by Theorem 2.12. Moreover, $\psi^L_{\beta,m}(v_1)$ is continuous on $\mathrm{Gr}(L)$.

Let W_1 and W_2 be two different Weyl chambers of index (β,m) such that the interior of $\overline{W_1}\cup\overline{W_2}$ is connected. Then there is a $\lambda\in\beta+L$ with $q(\lambda)=m$ and $\overline{W_1}\cap\overline{W_2}\subset\lambda^\perp$. If Φ_1 resp. Φ_2 is a real analytic function on $\overline{W_1}\cup\overline{W_2}$ with $\Phi_1|_{W_1} = \Phi^L_{\beta,m}|_{W_1}$ resp. $\Phi_2|_{W_2} = \Phi^L_{\beta,m}|_{W_2}$ then Theorem 2.12 implies that

$$\Phi_1(v_1) - \Phi_2(v_1) = \mathrm{const}\cdot(\lambda,v_1).$$

The functions $\psi_1 = \Phi_1 - \xi^L_{\beta,m}$ and $\psi_2 = \Phi_2 - \xi^L_{\beta,m}$ are real analytic on $\overline{W_1}\cup\overline{W_2}$, differ by a constant multiple of (λ,v_1), and satisfy $\psi_1|_{W_1} = \psi^L_{\beta,m}|_{W_1}$ resp. $\psi_2|_{W_2} = \psi^L_{\beta,m}|_{W_2}$. We may conclude: If $\psi^L_{\beta,m}|_{W_1}$ is linear then $\psi^L_{\beta,m}|_{W_2}$ is linear, too.

This fact, together with the first observation, implies that $\psi^L_{\beta,m}(v_1)$ is piecewise linear on the whole Grassmannian V_1. ☐

Definition 3.5. *Let W be a Weyl chamber of index (β,m). Then we write $\varrho_{\beta,m}(W)$ for the unique vector in V with the property*

$$\psi^L_{\beta,m}(v_1) = 8\sqrt{2}\pi(v_1,\varrho_{\beta,m}(W))$$

on W. We call $\varrho_{\beta,m}(W)$ the Weyl vector *of W.*

Later we will need the following theorem.

Theorem 3.6. *(cp. [Bo2] Theorem 10.3.) Let*

$$\xi^L(v_1) = \sum_{\beta,m} c_{\beta,m}\xi^L_{\beta,m}(v_1) \qquad (c_{\beta,m}\in\mathbb{C})$$

be a finite linear combination of the $\xi^L_{\beta,m}$. If $\xi^L(v_1)$ is a rational function on V_1, then it vanishes identically.

Remark 3.7. The condition that ξ^L is rational is obviously equivalent to saying that

$$\sum_{\beta,m} c_{\beta,m} b(\delta, q(\lambda)) = 0$$

for all $\lambda \in K' - 0$ and $\delta \in L_0'/L$ with $p(\delta) = \lambda + K$.

Proof of Theorem 3.6. We use the following notation: Let f and g be functions on V_1. Then we write

$$f \doteq g,$$

if $f - g$ is the restriction of a piecewise linear function on V.

We first prove that $\xi^L(v_1) \doteq 0$. Let

$$\Phi^L(v_1) = \sum_{\beta,m} c_{\beta,m} \Phi^L_{\beta,m}(v_1).$$

By Theorem 3.4 we have $\Phi^L(v_1) \doteq \xi^L(v_1)$. Hence it suffices to show that $\Phi^L(v_1) \doteq 0$.

The assumption that ξ^L is rational implies that

$$\Phi^L(v_1) \doteq \frac{1}{\sqrt{2}} \sum_{\beta,m} c_{\beta,m} \left[\frac{\Phi^K_{\beta,m}}{(z, v_1)} - 2(z', v_1)\Phi^K_{\beta,m} + \frac{8\pi}{(z, v_1)} \sum_{\substack{\lambda \in p(\beta)+K \\ q(\lambda)=m}} (\lambda, v_1)^2 \right]. \tag{3.15}$$

This formula for $\Phi^L(v_1)$ depends on the choice of a primitive norm 0 vector $z \in L$. We compute $\Phi^L(v_1)$ using a different norm 0 vector and compare the result with (3.15).

The vector $z' - \frac{z'^2}{2} z \in L \otimes \mathbb{Q}$ has norm 0. Let a be the unique positive integer such that

$$\tilde{z} = a\left(z' - \tfrac{z'^2}{2} z\right)$$

is a primitive element of L. Define $\tilde{z}' \in L'$, $\tilde{K} = L \cap \tilde{z}^\perp \cap \tilde{z}'^\perp$, and $\tilde{p}$ analogous to z', K, and p (see section 2.1). Since $\Phi^L(v_1)$ is the sum of a rational function and a piecewise linear function, we find by Proposition 3.1 and Theorem 3.4 that

$$\Phi^L(v_1) \doteq \frac{1}{\sqrt{2}(\tilde{z}, v_1)} \sum_{\beta,m} c_{\beta,m} \left[\Phi^{\tilde{K}}_{\beta,m} + 8\pi \sum_{\substack{\lambda \in \tilde{p}(\beta)+\tilde{K} \\ q(\lambda)=m}} (\lambda, v_1)^2 \right]. \tag{3.16}$$

It is easily seen that $(L \otimes \mathbb{Q}) \cap \tilde{z}^\perp = (K \otimes \mathbb{Q}) \oplus \mathbb{Q}\tilde{z}$. Thus $\tilde{K} \otimes \mathbb{Q} \subset (K \otimes \mathbb{Q}) \oplus \mathbb{Q}\tilde{z}$. Hence every $\lambda \in \tilde{K}'$ in the sum in (3.16) can be written as $\lambda = \lambda_K + b\tilde{z}$ with $\lambda_K \in K \otimes \mathbb{Q}$ and $b \in \mathbb{Q}$. We have

$$\frac{(\lambda, v_1)^2}{(\tilde{z}, v_1)} \doteq \frac{(\lambda_K, v_1)^2}{(\tilde{z}, v_1)},$$

and (3.16) can be rewritten in the form

$$\Phi^L(v_1) \doteq \frac{1}{\sqrt{2}(\tilde{z}, v_1)} \sum_{\beta,m} c_{\beta,m} \left[\Phi^{\tilde{K}}_{\beta,m} + 8\pi \sum_{\lambda \in \tilde{K}'} (\lambda_K, v_1)^2 \right]. \tag{3.17}$$

We now compare (3.15) and (3.17) in the upper half space model $\mathcal{H}$. If v_1 is represented by (x_0, μ) as in (3.9), we have

$$\begin{aligned} (\tilde{z}, v_1) &= \frac{a(x_0^2 - \mu^2)}{2x_0}, \\ \frac{\mu^2}{x_0} &= \frac{1}{(z, v_1)} - 2(z', v_1). \end{aligned} \tag{3.18}$$

We find

$$\begin{aligned} \frac{1}{x_0} \sum_{\beta,m} c_{\beta,m} & \left[\mu^2 \Phi^K_{\beta,m} + 8\pi \sum_{\lambda \in K'} (\lambda, \mu)^2 \right] \\ & \doteq \frac{2x_0}{a(x_0^2 - \mu^2)} \sum_{\beta,m} c_{\beta,m} \left[\Phi^{\tilde{K}}_{\beta,m} + \frac{8\pi}{x_0^2} \sum_{\lambda \in \tilde{K}'} (\lambda_K, \mu)^2 \right]. \end{aligned}$$

We multiply this equation by $(x_0^2 - \mu^2)$ and consider x_0 as a complex variable. Putting $x_0 = i|\mu|$ we obtain

$$8\pi \sum_{\beta,m} c_{\beta,m} \sum_{\lambda \in \tilde{K}'} (\lambda_K, \mu)^2 = \mu^2 \sum_{\beta,m} c_{\beta,m} \Phi^{\tilde{K}}_{\beta,m}.$$

Inserting this into (3.17) we find

$$\Phi^L(v_1) \doteq \frac{\sqrt{2}}{a x_0} \sum_{\beta,m} c_{\beta,m} \Phi^{\tilde{K}}_{\beta,m} = \frac{\sqrt{2}(z, v_1)}{a} \sum_{\beta,m} c_{\beta,m} \Phi^{\tilde{K}}_{\beta,m} \doteq 0$$

and thereby $\xi^L(v_1) \doteq 0$.

In the upper half space model we have

$$\xi^L((x_0, \mu)) = \frac{1}{\sqrt{2} x_0} \sum_{\beta,m} c_{\beta,m} \left[\mu^2 \Phi^K_{\beta,m} + 8\pi \sum_{\lambda \in K'} (\lambda, \mu)^2 \right]. \tag{3.19}$$

Comparing coefficients in (3.19) and (3.10), we may infer that $\xi^L((x_0, \mu)) \doteq 0$ already implies $\xi^L((x_0, \mu)) = 0$. □

3.1.1 The hyperbolic Laplacian

We now consider the functions $\psi_{\beta,m}^{L}$ and $\xi_{\beta,m}^{L}$ in the upper half space model $\mathcal{H}$ and determine the action of the of the hyperbolic Laplacian. For $(x_0,\mu) \in \mathcal{H} = \mathbb{R}_{>0} \times (K \otimes \mathbb{R})$ we have

$$\begin{aligned}\xi_{\beta,m}^{L}((x_0,\mu)) &= \frac{\mu^2}{\sqrt{2}x_0}\Phi_{\beta,m}^{K} + \frac{4\sqrt{2}\pi}{x_0}\sum_{\substack{\lambda\in p(\beta)+K\\ q(\lambda)=m}}(\lambda,\mu)^2\\ &\quad + 4\sqrt{2}(\pi x_0)^{l/2-1}\sum_{\lambda\in K'-0}\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} b(\delta,q(\lambda))|\lambda|^{l/2}\\ &\quad \times \sum_{n\geq 1} n^{l/2-2} e\left(n(\lambda,\mu)+n(\delta,z')\right) K_{l/2}(2\pi n|\lambda|x_0). \qquad (3.20)\end{aligned}$$

In the coordinates of $\mathcal{H}$ the hyperbolic Laplace operator has the form (cp. [Ma2])

$$\Delta_L = -x_0^{l-1}\sum_{j=0}^{l-2}\frac{\partial}{\partial x_j}x_0^{3-l}\frac{\partial}{\partial x_j} = -x_0^2\sum_{j=0}^{l-2}\frac{\partial^2}{\partial x_j^2} + (l-3)x_0\frac{\partial}{\partial x_0}.$$

It is invariant under the action of $\mathrm{SO}(1,l-1)$ on $\mathcal{H}$.

Theorem 3.8. *The functions $\psi_{\beta,m}^{L}((x_0,\mu))$ and $\xi_{\beta,m}^{L}((x_0,\mu))$ are eigenfunctions of the hyperbolic Laplacian with eigenvalue $1-l$.*

Proof. First we consider $\psi_{\beta,m}^{L}$. Since it is piecewise linear, it suffices to show that

$$\Delta_L((x_0,\mu),\varrho) = (1-l)((x_0,\mu),\varrho)$$

for any $\varrho \in V$. Using (3.10) this can be verified by a straightforward computation.

To prove the statement for $\xi_{\beta,m}^{L}$ we compute the action of Δ_L on the different terms of the Fourier expansion (3.20). It is easily seen that

$$\Delta_L\frac{\mu^2}{x_0} = (1-l)\frac{\mu^2}{x_0} + 2(l-2)x_0. \qquad (3.21)$$

Regarding the second term we find

$$\begin{aligned}\Delta_L\frac{4\sqrt{2}\pi}{x_0}\sum_{\substack{\lambda\in p(\beta)+K\\ q(\lambda)=m}}(\lambda,\mu)^2 &= (1-l)\frac{4\sqrt{2}\pi}{x_0}\sum_{\substack{\lambda\in p(\beta)+K\\ q(\lambda)=m}}(\lambda,\mu)^2\\ &\quad - 16\sqrt{2}\pi|m|x_0\cdot\#\{\lambda\in p(\beta)+K;\quad q(\lambda)=m\}. \qquad (3.22)\end{aligned}$$

Moreover, a lengthy but trivial calculation shows that

$$\Delta_L x_0^{l/2-1} K_{l/2}(2\pi n|\lambda|x_0) e(n(\lambda,\mu)) = (1-l) x_0^{l/2-1} K_{l/2}(2\pi n|\lambda|x_0) e(n(\lambda,\mu)) \tag{3.23}$$

for $\lambda \in K' - 0$. Here one has to use that K_ν is a solution of the differential equation

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} - (z^2+\nu^2) f = 0.$$

If we put (3.21), (3.22), and (3.23) together, we see that

$$\begin{aligned}\Delta_L \xi^L_{\beta,m}((x_0,\mu)) &= (1-l)\xi^L_{\beta,m}((x_0,\mu)) + (l-2)\sqrt{2}x_0 \Phi^K_{\beta,m}\\ &\quad - 16\sqrt{2}\pi|m|x_0 \cdot \#\{\lambda \in p(\beta)+K;\quad q(\lambda)=m\}. \end{aligned}\tag{3.24}$$

But Theorem 2.14 tells us that

$$\Phi^K_{\beta,m} = \frac{8\pi|m|}{l/2-1} \cdot \#\{\lambda \in p(\beta)+K;\quad q(\lambda)=m\}.$$

Thus the last two terms in (3.24) cancel and

$$\Delta_L \xi^L_{\beta,m}((x_0,\mu)) = (1-l)\xi^L_{\beta,m}((x_0,\mu)).$$

This proves the assertion. □

3.2 Lattices of signature $(2, l)$

For the rest of this book let L denote a lattice of signature $(2,l)$ with $l \geq 3$. Then the Grassmannian $\mathrm{Gr}(L)$ is a Hermitean symmetric space. Let $z \in L$ be a primitive norm 0 vector and choose a $z' \in L'$ with $(z,z') = 1$. Then $K = L \cap z^\perp \cap z'^\perp$ is a Lorentzian lattice. We assume that K also contains a primitive norm 0 vector. We set $k = 1 - l/2$ and $\kappa = 1 + l/2$.

If $v \in \mathrm{Gr}(L)$, then we write w for the corresponding element in the Grassmannian $\mathrm{Gr}(K)$ as in section 2.1 and w_1 for its realization in the hyperboloid model.

We shall determine the Fourier expansion of the function $\Phi_{\beta,m}(v)$ (see Definition 2.10) more explicitly. In the same way as in the previous section, we show that it can be written as the sum of two real valued functions functions $\psi^L_{\beta,m}(v)$ and $\xi^L_{\beta,m}(v)$. Here $\xi^L_{\beta,m}(v)$ is real analytic on the whole $\mathrm{Gr}(L)$, and $\psi^L_{\beta,m}(v)$ is essentially the logarithm of the absolute value of a holomorphic function on $\mathrm{Gr}(L)$ whose only zeros lie on $H(\beta,m)$. These functions $\psi^L_{\beta,m}$, $\xi^L_{\beta,m}$ generalize the functions ψ_m and ξ_m considered in [Br1, Br2].

In the Fourier expansion of $\xi^L_{\beta,m}(v)$ the following special function will occur. For $A, B \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ with $\kappa > 1$ we define

$$\mathcal{V}_\kappa(A,B) = \int_0^\infty \Gamma(\kappa-1, A^2 y) e^{-B^2 y - 1/y} y^{-3/2}\, dy. \tag{3.25}$$

Let $\varepsilon > 0$. Then it can be easily checked that

$$\mathcal{V}_\kappa(A,B) \ll_\varepsilon e^{-2(1-\varepsilon)\sqrt{A^2+B^2}}. \tag{3.26}$$

For $\kappa \in \mathbb{N}$ the function $\mathcal{V}_\kappa(A,B)$ can be expressed as a finite sum of K-Bessel functions: If $n \in \mathbb{N}_0$ then by repeated integration by parts one finds that $\Gamma(1+n,x) = n! e^{-x} e_n(x)$, where $e_n(x)$ denotes the truncated exponential series

$$e_n(x) = \sum_{r=0}^{n} \frac{x^r}{r!}$$

(also see [E1] Vol. II p. 136 (18)). Hence we have for integral κ:

$$\begin{aligned}\mathcal{V}_\kappa(A,B) &= (\kappa-2)! \int_0^\infty e_{\kappa-2}(A^2 y) e^{-(A^2+B^2)y-1/y} y^{-3/2}\, dy \\ &= (\kappa-2)! \sum_{r=0}^{\kappa-2} \frac{A^{2r}}{r!} \int_0^\infty e^{-(A^2+B^2)y-1/y} y^{r-3/2}\, dy.\end{aligned}$$

By virtue of (3.7) we obtain

$$\mathcal{V}_\kappa(A,B) = 2(\kappa-2)! \sum_{r=0}^{\kappa-2} \frac{A^{2r}}{r!} (A^2+B^2)^{1/4-r/2} K_{r-1/2}(2\sqrt{A^2+B^2}).$$

Theorem 3.9. *Let $v \in \operatorname{Gr}(L) - H(\beta,m)$ with $z_v^2 < \frac{1}{2|m|}$. Then $\Phi^L_{\beta,m}(v)$ is equal to*

$$\begin{aligned}&\frac{1}{\sqrt{2}|z_v|}\Phi^K_{\beta,m}(w,1-k/2) + C_{\beta,m} + b(0,0)\log(z_v^2) \\ &- 2 \sum_{\substack{\lambda\in\pm p(\beta)+K \\ q(\lambda)=m}} \operatorname{Log}\left(1 - e((\pm\beta,z') + (\lambda,\mu) + i|\lambda_w|/|z_v|)\right) \\ &- 2 \sum_{\substack{\lambda\in K'-0 \\ q(\lambda)\geq 0}} \sum_{\substack{\delta\in L_0'/L \\ p(\delta)=\lambda+K}} b(\delta,q(\lambda)) \operatorname{Log}\left(1 - e((\delta,z') + (\lambda,\mu) + i|\lambda_w|/|z_v|)\right) \\ &+ \frac{2}{\sqrt{\pi}} \sum_{\substack{\lambda\in K' \\ q(\lambda)<0}} \sum_{\substack{\delta\in L_0'/L \\ p(\delta)=\lambda+K}} b(\delta,q(\lambda)) \sum_{n\geq 1} \frac{1}{n} e(n(\delta,z') + n(\lambda,\mu)) \\ &\qquad\qquad \times \mathcal{V}_{2-k}\left(\pi n|\lambda|/|z_v|,\, \pi n|\lambda_w|/|z_v|\right),\end{aligned}$$

with

$$\begin{aligned}C_{\beta,m} = &-b(0,0)\left(\log(2\pi) + \Gamma'(1)\right) + b'(0,0) \\ &- 2 \sum_{\substack{\ell\ (N) \\ \ell\not\equiv 0\ (N)}} b(\ell z/N,0) \log|1 - e(\ell/N)|.\end{aligned}$$

Here $b(\gamma,n)=b(\gamma,n,1-k/2)$ denote the Fourier coefficients of the Poincaré series $F^L_{\beta,m}(\tau,1-k/2)$ as in Proposition 1.10, and $b'(0,0)$ means the derivative of $b(0,0,s)$ at $s=1-k/2$. The sum over λ in the second line is to be interpreted as in (3.2).

Proof. We first assume $z_v^2<\frac{1}{4|m|}$ and apply Theorem 2.15. The only term on the right hand side of (2.35) that is not holomorphic in s near $1-k/2$ is the second one

$$h(s)=\frac{2}{\sqrt{\pi}}\left(2z_v^2/\pi\right)^{s-1/2-l/4}\Gamma(s-l/4)\sum_{\ell\ (N)}b(\ell z/N,0,s)\sum_{n\geq 1}e(\ell n/N)n^{l/2-2s}.$$

Here the summand with $\ell\equiv 0\pmod N$ gives the singular contribution

$$\frac{2}{\sqrt{\pi}}\left(2z_v^2/\pi\right)^{s-1/2-l/4}\Gamma(s-l/4)b(0,0,s)\zeta(2s-l/2), \tag{3.27}$$

where $\zeta(s)$ denotes the Riemann zeta function. The remaining terms with $\ell\not\equiv 0\pmod N$ converge for $\sigma>l/4$ and are therefore holomorphic. Their value at $s=1-k/2=1/2+l/4$ is

$$-2\sum_{\substack{\ell\ (N)\\ \ell\not\equiv 0\ (N)}}b(\ell z/N,0)\log|1-e(\ell/N)|. \tag{3.28}$$

To compute the constant term in the Laurent expansion of (3.27) at $s=1-k/2$ we note that

$$\begin{aligned}\zeta(2s-l/2)&=\frac{1}{2}(s-1/2-l/4)^{-1}-\Gamma'(1)+\dots,\\ \left(2z_v^2/\pi\right)^{s-1/2-l/4}&=1+(s-1/2-l/4)\log\left(2z_v^2/\pi\right)+\dots,\\ \Gamma(s-l/4)&=\sqrt{\pi}+\Gamma'(1/2)(s-1/2-l/4)+\dots,\\ b(0,0,s)&=b(0,0)+b'(0,0)(s-1/2-l/4)+\dots.\end{aligned}$$

(For the first expansion for instance see [E1] p. 34 (17); note that $-\Gamma'(1)=\gamma=$ Euler-Mascheroni constant.) We may infer that it is equal to

$$b(0,0)\left(\log\left(\frac{2z_v^2}{\pi}\right)+\frac{\Gamma'(1/2)}{\sqrt{\pi}}-2\Gamma'(1)\right)+b'(0,0).$$

From the duplication formula $\Gamma(\nu)\Gamma(\nu+1/2)=2^{1-2\nu}\sqrt{\pi}\Gamma(2\nu)$ for the gamma function it follows that $\Gamma'(1/2)/\sqrt{\pi}=\Gamma'(1)-2\log 2$. Thus the above expression for the constant term in the Laurent expansion of (3.27) can be simplified to

$$b(0,0)\left(\log(z_v^2)-\log(2\pi)-\Gamma'(1)\right)+b'(0,0). \tag{3.29}$$

Putting (3.28) and (3.29) together, we get

$$\mathcal{C}_{s=1-k/2}[h(s)] = b(0,0)\log(z_v^2) + C_{\beta,m}.$$

Hence the function $\Phi^L_{\beta,m}(v) = \mathcal{C}_{s=1-k/2}[\Phi^L_{\beta,m}(v,s)]$ is given by

$$\begin{aligned}
&\frac{1}{\sqrt{2}|z_v|}\Phi^K_{\beta,m}(w,1-k/2) + b(0,0)\log(z_v^2) + C_{\beta,m}\\
&+\frac{\sqrt{2}}{|z_v|}\sum_{\lambda\in K'-0}\sum_{n\geq 1} e(n(\lambda,\mu)) \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e(n(\delta,z'))\\
&\times\int_0^\infty c(\delta,q(\lambda);y,1-k/2)\exp\left(\frac{-\pi n^2}{2z_v^2 y} - 4\pi y q(\lambda_w) + 2\pi y q(\lambda)\right) y^{-3/2}\,dy.
\end{aligned} \tag{3.30}$$

We now insert the explicit formulas for the Fourier coefficients $c(\gamma,q(\lambda);y,1-k/2)$ given in Proposition 1.10. We saw that $c(\gamma,q(\lambda);y,1-k/2)$ equals

$$\begin{cases} b(\gamma,q(\lambda))e^{-2\pi q(\lambda)y}, & \text{if } q(\lambda)\geq 0,\\ b(\gamma,q(\lambda))\mathcal{W}_{1-k/2}(4\pi q(\lambda)y), & \text{if } q(\lambda)<0,\ q(\lambda)\neq m,\\ (\delta_{\beta,\gamma}+\delta_{-\beta,\gamma})e^{-2\pi m y} + b(\gamma,q(\lambda))\mathcal{W}_{1-\frac{k}{2}}(4\pi q(\lambda)y), & \text{if } q(\lambda)=m. \end{cases}$$

Furthermore, according to (1.33) we have

$$\mathcal{W}_{1-k/2}(y) = e^{-y/2}\Gamma(1-k,|y|)$$

for $y<0$. Hence the last summand in (3.30) can be rewritten in the form

$$\begin{aligned}
&\frac{\sqrt{2}}{|z_v|}\sum_{\substack{\lambda\in K'-0\\ q(\lambda)\geq 0}}\sum_{n\geq 1} e(n(\lambda,\mu)) \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e(n(\delta,z'))b(\delta,q(\lambda))\\
&\qquad\times\int_0^\infty \exp\left(\frac{-\pi n^2}{2z_v^2 y} - 4\pi y q(\lambda_w)\right)\frac{dy}{y^{3/2}}\\
&+\frac{\sqrt{2}}{|z_v|}\sum_{\substack{\lambda\in K'\\ \lambda+K=\pm\beta\\ q(\lambda)=m}}\sum_{n\geq 1} e(n(\lambda,\mu)+n(\pm\beta,z'))\int_0^\infty \exp\left(\frac{-\pi n^2}{2z_v^2 y} - 4\pi y q(\lambda_w)\right)\frac{dy}{y^{3/2}}\\
&+\frac{\sqrt{2}}{|z_v|}\sum_{\substack{\lambda\in K'\\ q(\lambda)<0}}\sum_{n\geq 1} e(n(\lambda,\mu)) \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e(n(\delta,z'))b(\delta,q(\lambda))\\
&\qquad\times\int_0^\infty \Gamma(1-k,4\pi|q(\lambda)|y)\exp\left(\frac{-\pi n^2}{2z_v^2 y} - 4\pi y q(\lambda_w)\right)\frac{dy}{y^{3/2}}.
\end{aligned} \tag{3.31}$$

Here the sum over λ in the second term is to be understood as in (3.2).

We now compute the integrals in the above expression. Using (3.7) and the identity

$$\sqrt{\frac{2z}{\pi}}K_{-1/2}(z) = e^{-z} \tag{3.32}$$

(cf. [AbSt] p. 160 (10.2.17)), we obtain for $A, B > 0$:

$$\int_0^\infty e^{-Ay-B/y}y^{-3/2}\,dy = \sqrt{\pi}B^{-1/2}e^{-2\sqrt{AB}}.$$

For instance by considering the limit $A \to 0$ it can be shown that this formula still holds for $A = 0$. We may conclude that

$$\int\limits_0^\infty \exp\left(\frac{-\pi n^2}{2z_v^2 y} - 4\pi y q(\lambda_w)\right) y^{-3/2}\,dy = \frac{\sqrt{2}|z_v|}{n}e^{-2\pi n|\lambda_w|/|z_v|}.$$

In the second integral in (3.31)

$$\int\limits_0^\infty \Gamma(1-k, 4\pi|q(\lambda)|y)\exp\left(\frac{-\pi n^2}{2z_v^2 y} - 4\pi y q(\lambda_w)\right) y^{-3/2}\,dy$$

we substitute $u = \frac{2z_v^2}{\pi n^2}y$ and find that it is equal to

$$\frac{\sqrt{2}|z_v|}{\sqrt{\pi}n}\mathcal{V}_{2-k}\left(\pi n|\lambda|/|z_v|,\, \pi n|\lambda_w|/|z_v|\right).$$

We may rewrite (3.31) in the form

$$\begin{aligned}
&2\sum_{\substack{\lambda\in K'-0\\ q(\lambda)\geq 0}}\ \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} b(\delta, q(\lambda))\sum_{n\geq 1}\frac{1}{n}e(n(\lambda,\mu)+n(\delta,z'))e^{-2\pi n|\lambda_w|/|z_v|}\\
&+2\sum_{\substack{\lambda\in\pm p(\beta)+K\\ q(\lambda)=m}}\ \sum_{n\geq 1}\frac{1}{n}e(n(\lambda,\mu)+n(\pm\beta,z'))e^{-2\pi n|\lambda_w|/|z_v|}\\
&+\frac{2}{\sqrt{\pi}}\sum_{\substack{\lambda\in K'\\ q(\lambda)<0}}\ \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} b(\delta, q(\lambda))\sum_{n\geq 1}\frac{1}{n}e(n(\lambda,\mu)+n(\delta,z'))\\
&\qquad\times\mathcal{V}_{2-k}\left(\pi n|\lambda|/|z_v|,\, \pi n|\lambda_w|/|z_v|\right).
\end{aligned} \tag{3.33}$$

If we insert the power series expansion of the principal branch of the logarithm and put everything together, we obtain the stated identity for $z_v^2 < \frac{1}{4|m|}$.

Using the estimates (2.21) and (2.22) it can be seen that (3.33) even converges and defines a real analytic function on $z_v^2 < \frac{1}{2|m|}$. Hence the assertion follows by real analytic continuation. □

We now introduce a complex structure on the Grassmannian $\mathrm{Gr}(L)$. Let $V(\mathbb{C}) = V \otimes \mathbb{C}$ be the complexification of $V = L \otimes \mathbb{R}$, and extend the bilinear form $(\cdot,\cdot)$ on V to a $\mathbb{C}$-bilinear form on $V(\mathbb{C})$. Let $P(V(\mathbb{C}))$ be the associated projective space and denote the canonical projection by

$$V(\mathbb{C}) \to P(V(\mathbb{C})), \qquad Z_L \mapsto [Z_L].$$

In the zero-quadric

$$\mathcal{N} = \{[Z_L] \in P(V(\mathbb{C})); \quad (Z_L, Z_L) = 0\}$$

we consider the open subset

$$\mathcal{K} = \left\{[Z_L] \in \mathcal{N}; \quad (Z_L, \bar{Z}_L) > 0\right\}.$$

It is easily seen that $\mathcal{K}$ is a complex manifold that consists of 2 connected components. The action of the orthogonal group $\mathrm{O}(V)$ on V induces an action on $\mathcal{K}$. The connected component $\mathrm{O}^+(V)$ of the identity (i.e. the subgroup of elements with positive spinor norm) preserves the components of $\mathcal{K}$, whereas $\mathrm{O}(V) \setminus \mathrm{O}^+(V)$ interchanges them. We choose one fixed component of $\mathcal{K}$ and denote it by $\mathcal{K}^+$.

If we write $Z_L = X_L + iY_L$ with $X_L, Y_L \in V$, then $[Z_L] \in \mathcal{K}$ is equivalent to

$$X_L \perp Y_L, \quad \text{and} \quad X_L^2 = Y_L^2 > 0. \tag{3.34}$$

But this means that X_L and Y_L span a two dimensional positive definite subspace of V and thereby define an element of $\mathrm{Gr}(L)$. Conversely for a given $v \in \mathrm{Gr}(L)$ we may choose an orthogonal base X_L, Y_L as in (3.34) and obtain a unique $[Z_L] = [X_L + iY_L]$ in $\mathcal{K}^+$. (Then $[\bar{Z}_L] \in \mathcal{K}$ corresponds to the same $v \in \mathrm{Gr}(L)$.) We get a bijection between $\mathrm{Gr}(L)$ and $\mathcal{K}^+$ and hereby a complex structure on $\mathrm{Gr}(L)$.

We may realize $\mathcal{K}^+$ as a tube domain in the following way. Suppose that $z \in L$ is a primitive norm 0 vector and $z' \in L'$ with $(z, z') = 1$. As in section 2.1, define a sub-lattice K by $K = L \cap z^\perp \cap z'^\perp$. Then K is Lorentzian and

$$V = (K \otimes \mathbb{R}) \oplus \mathbb{R}z' \oplus \mathbb{R}z.$$

If $Z_L \in L \otimes \mathbb{C}$ and $Z_L = Z + az' + bz$ with $Z \in K \otimes \mathbb{C}$ and $a, b \in \mathbb{C}$, then we briefly write $Z_L = (Z, a, b)$.

We have a map from the set

$$\{Z = X + iY \in K \otimes \mathbb{C}; \quad Y^2 > 0\} \tag{3.35}$$

of vectors in $K \otimes \mathbb{C}$ with positive imaginary part to $\mathcal{K}$ given by

$$Z \mapsto [Z_L] = [(Z, 1, -q(Z) - q(z'))]. \tag{3.36}$$

Conversely assume that $[Z_L] \in \mathcal{K}$, $Z_L = X_L + iY_L$. From the fact that X_L, Y_L span a two dimensional positive definite subspace of V it follows that $(Z_L, z) \neq 0$. Thus $[Z_L]$ has a unique representative of the form $(Z, 1, b)$. The condition $q(Z_L) = 0$ implies $b = -q(Z) - q(z')$, and thereby $[Z_L] = [(Z, 1, -q(Z) - q(z'))]$. Moreover, from $(Z_L, \bar{Z}_L) > 0$ one easily deduces $Y^2 > 0$. We may infer that the map (3.36) is biholomorphic.

The set (3.35) has two components. Let $\mathbb{H}_l$ be the component which is mapped to $\mathcal{K}^+$ under (3.36). Then $\mathbb{H}_l$ is a realization of $\mathcal{K}^+$ as a tube domain and can be viewed as a generalized upper half plane.

The cone $iC = \mathbb{H}_l \cap i(K \otimes \mathbb{R})$ is given by one of the two components of the set $\{Y \in K \otimes \mathbb{R};\quad Y^2 > 0\}$ of positive norm vectors of $K \otimes \mathbb{R}$ and $\mathbb{H}_l = K \otimes \mathbb{R} + iC$. Without any restriction we may assume that $\{Y/|Y|;\ Y \in C\}$ coincides with the hyperboloid model of $\operatorname{Gr}(K)$ that we used in section 3.1.

Let $Z = X + iY$ be a point of $\mathbb{H}_l$ and $v = \mathbb{R}X_L + \mathbb{R}Y_L$ the corresponding element of the Grassmannian. We have

$$\begin{aligned} Z_L &= X_L + iY_L = (Z, 1, -q(Z) - q(z')),\\ X_L &= (X, 1, q(Y) - q(X) - q(z')),\\ Y_L &= (Y, 0, -(X, Y)). \end{aligned}$$

Using $X_L^2 = Y_L^2 = Y^2$ we find

$$\begin{aligned} z_v &= (z, X_L)X_L/X_L^2 + (z, Y_L)Y_L/Y_L^2 = X_L/Y^2,\\ z_v^2 &= 1/Y^2. \end{aligned}$$

The vector

$$\mu = -z' + \frac{z_v}{2z_v^2} + \frac{z_{v^\perp}}{2z_{v^\perp}^2} = -z' + \frac{z_v}{z_v^2} - \frac{z}{2z_v^2}$$

is given by

$$\mu = (X, 0, -q(X) - q(z'))$$

and its orthogonal projection μ_K by $\mu_K = \mu - (\mu, z')z = X$. The subspace $w \subset V$, i.e. the orthogonal complement of z_v in v, is equal to $\mathbb{R}Y_L$. Recall from section 2.1 that we have identified w with its orthogonal projection $w_K \subset K \otimes \mathbb{R}$. Obviously $w_K = \mathbb{R}Y$, $w_1 = Y/|Y|$, and thereby

$$\begin{aligned} \lambda_w &= (\lambda, Y)Y/Y^2,\\ |\lambda_w| &= |(\lambda, Y)||z_v|. \end{aligned}$$

The intersection

$$\Gamma(L) = \mathrm{O}^+(V) \cap \mathrm{O}_d(L) \tag{3.37}$$

of the discriminant kernel $\mathrm{O}_d(L)$ with $\mathrm{O}^+(V)$ is a discrete subgroup of $\mathrm{O}^+(V)$. We will be interested in its action on $\mathcal{K}^+$ and $\mathbb{H}_l$.

Let $\lambda \in L'$ be a vector of negative norm. If we write $\lambda = \lambda_K + az' + bz$ with $\lambda_K \in K'$, $a \in \mathbb{Z}$, and $b \in \mathbb{Q}$, then

$$\lambda^\perp = \{Z \in \mathbb{H}_l; \quad aq(Z) - (Z, \lambda_K) - aq(z') - b = 0\}$$

in the coordinates of $\mathbb{H}_l$. This set defines a prime divisor on $\mathbb{H}_l$. The sum

$$\sum_{\substack{\lambda \in \beta + L \\ q(\lambda) = m}} \lambda^\perp$$

is a $\Gamma(L)$-invariant divisor on $\mathbb{H}_l$ with support $H(\beta, m)$. Following Borcherds we call it the *Heegner divisor* of discriminant (β, m). The proof of the next lemma will be left to the reader.

Lemma 3.10. *Let $\lambda \in L'$ with $q(\lambda) = m < 0$ and $(\lambda, z) = a \neq 0$. Assume that $Z \in \lambda^\perp \subset \mathbb{H}_l$. Then $q(Y) < |m|/a^2$.*

Assume that $(\beta, z) \equiv 0 \pmod{N}$. Let W be a Weyl chamber of $\operatorname{Gr}(K)$ of index $(p(\beta), m)$. Then we also call the subset

$$\{Z = X + iY \in \mathbb{H}_l; \quad Y/|Y| \in W\} \subset \mathbb{H}_l$$

a *Weyl chamber* of index (β, m) and denote it by W. For the corresponding Weyl vector in $K \otimes \mathbb{R}$ we write $\varrho_{\beta,m}(W)$.

If $(\beta, z) \not\equiv 0 \pmod{N}$, then by a Weyl chamber of index (β, m) we simply mean $\mathbb{H}_l$ and put $\varrho_{\beta,m}(W) = 0$. (In this case there is no contribution to $\Phi^L_{\beta,m}$ from the smaller lattice K.)

We may consider $\Phi^L_{\beta,m}$ as a function on $\mathbb{H}_l$. By construction it is invariant under the action of $\Gamma(L)$.

Let $W \subset \mathbb{H}_l$ be a Weyl chamber of index (β, m). Suppose that $Z \in W$ with $q(Y) > |m|$. Then Lemma 3.10 implies that $Z \notin H(\beta, m)$. According to Theorem 3.9 the Fourier expansion of $\Phi^L_{\beta,m}(Z)$ is given by

$$\begin{aligned}
&\frac{|Y|}{\sqrt{2}} \Phi^K_{\beta,m}(Y/|Y|, 1 - k/2) + C_{\beta,m} - b(0,0) \log(Y^2) \\
&- 4 \sum_{\substack{\lambda \in \pm p(\beta) + K \\ (\lambda, W) > 0 \\ q(\lambda) = m}} \log |1 - e((\pm\beta, z') + (\lambda, Z))| \\
&- 4 \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0 \\ q(\lambda) \geq 0}} \sum_{\substack{\delta \in L_0'/L \\ p(\delta) = \lambda + K}} b(\delta, q(\lambda)) \log |1 - e((\delta, z') + (\lambda, Z))| \\
&+ \frac{2}{\sqrt{\pi}} \sum_{\substack{\lambda \in K' \\ q(\lambda) < 0}} \sum_{\substack{\delta \in L_0'/L \\ p(\delta) = \lambda + K}} b(\delta, q(\lambda)) \sum_{n \geq 1} \frac{1}{n} e(n(\delta, z') + n(\lambda, X)) \\
&\qquad \times \mathcal{V}_{2-k} \left(\pi n |\lambda| |Y|, \, \pi n (\lambda, Y)\right).
\end{aligned}$$

Here we have used $b(\gamma, n) = b(-\gamma, n)$ and the fact that $|\lambda_w| = (\lambda, Y)|z_v|$ for any $\lambda \in K'$ with $(\lambda, W) > 0$.

Definition 3.11. *We define a function* $\xi^L_{\beta,m} : \mathbb{H}_l \to \mathbb{R}$ *by*

$$\xi^L_{\beta,m}(Z) = \frac{|Y|}{\sqrt{2}}\,\xi^K_{\beta,m}(Y/|Y|) - b(0,0)\log(Y^2) + \frac{2}{\sqrt{\pi}} \sum_{\substack{\lambda\in K' \\ q(\lambda)<0}} \sum_{\substack{\delta\in L_0'/L \\ p(\delta)=\lambda+K}} b(\delta, q(\lambda))$$

$$\times \sum_{n\geq 1} \frac{1}{n} e(n(\delta, z') + n(\lambda, X))\mathcal{V}_{2-k}\left(\pi n|\lambda||Y|,\, \pi n(\lambda, Y)\right)$$

and a function $\psi^L_{\beta,m} : \mathbb{H}_l - H(\beta, m) \to \mathbb{R}$ *by*

$$\psi^L_{\beta,m}(Z) = \Phi^L_{\beta,m}(Z) - \xi^L_{\beta,m}(Z).$$

Here $\xi^K_{\beta,m}$ *is given by*

$$\xi^K_{\beta,m} = \begin{cases} 0, & \text{if } (\beta, z) \not\equiv 0 \pmod{N}, \\ \xi^K_{p(\beta),m}, & \text{if } (\beta, z) \equiv 0 \pmod{N} \text{ (see Def. 3.3).} \end{cases}$$

Note that this definition depends on the choice of z and z'. It is easily checked that $\xi^L_{\beta,m} = \xi^L_{-\beta,m}$ and $\psi^L_{\beta,m} = \psi^L_{-\beta,m}$

Using the boundedness of the coefficients $b(\delta, n)$ with $n < 0$ (cf. (2.22)) and the asymptotic property (3.26) of the function $\mathcal{V}_{2-k}(A, B)$, one can show that $\xi^L_{\beta,m}$ is real analytic on the whole $\mathbb{H}_l$.

According to Lemma 1.13 the coefficients $b(\delta, n)$ are real numbers. We may infer that $\xi^L_{\beta,m}$ and $\psi^L_{\beta,m}$ are real-valued functions.

Definition 3.12. *Let D be a domain in $\mathbb{C}^l$ and $f : D \to \mathbb{R}$ a twice continuously differentiable function. Then f is called* pluriharmonic, *if*

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} f = 0$$

for all $1 \leq i, j \leq l$.

Lemma 3.13. *Let $D \subset \mathbb{C}^l$ be a simply connected domain and $f : D \to \mathbb{R}$ a twice continuously differentiable function. Then f is pluriharmonic if and only if there is a holomorphic function $h : D \to \mathbb{C}$ with $f = \Re(h)$.*

Proof. See [GR] chapter IX section C. □

Now let $W \subset \mathbb{H}_l$ be a fixed Weyl chamber of index (β, m) and $\varrho_{\beta,m}(W)$ the corresponding Weyl vector. For $Z \in W$ with $q(Y) > |m|$ we may write the Fourier expansion of $\psi^L_{\beta,m}(Z)$ in the form

$$C_{\beta,m} + 8\pi(\varrho_{\beta,m}(W), Y) - 4 \sum_{\substack{\lambda\in\pm p(\beta)+K\\(\lambda,W)>0\\q(\lambda)=m}} \log|1-e((\pm\beta,z')+(\lambda,Z))|$$

$$-4\sum_{\substack{\lambda\in K'\\(\lambda,W)>0\\q(\lambda)\geq 0}}\sum_{\substack{\delta\in L_0'/L\\p(\delta)=\lambda+K}} b(\delta,q(\lambda))\log|1-e((\delta,z')+(\lambda,Z))|\,. \tag{3.38}$$

Definition 3.14. *For $Z\in\mathbb{H}_l$ with $q(Y)>|m|$ we define*

$$\Psi_{\beta,m}(Z) = e\big((\varrho_{\beta,m}(W),Z)\big) \prod_{\substack{\lambda\in\pm p(\beta)+K\\(\lambda,W)>0\\q(\lambda)=m}} \big(1-e((\pm\beta,z')+(\lambda,Z))\big)$$

$$\times \prod_{\substack{\lambda\in K'\\(\lambda,W)>0\\q(\lambda)\geq 0}}\prod_{\substack{\delta\in L_0'/L\\p(\delta)=\lambda+K}} \big(1-e((\delta,z')+(\lambda,Z))\big)^{b(\delta,q(\lambda))}. \tag{3.39}$$

Lemma 3.15. *The infinite product (3.39) converges normally for $Z\in\mathbb{H}_l$ with $q(Y)>|m|$.*

Proof. It suffices to show that the two series

$$\sum_{\substack{\lambda\in\pm p(\beta)+K\\(\lambda,W)>0\\q(\lambda)=m}} e^{-2\pi(\lambda,Y)} \qquad \text{and} \qquad \sum_{\substack{\lambda\in K'\\(\lambda,Y)>0\\q(\lambda)\geq 0}}\sum_{\substack{\delta\in L_0'/L\\p(\delta)=\lambda+K}} b(\delta,q(\lambda))e^{-2\pi(\lambda,Y)}$$

converge normally. The convergence of the second series for $q(Y)>|m|$ follows from the asymptotic behavior (2.21) of the coefficients $b(\gamma,n)$.

It is easily seen that the first series converges normally for $Z\in W$. Now suppose that U is an arbitrary open subset of $\mathbb{H}_l$ with compact closure $\overline{U}\subset\mathbb{H}_l$. Then there are only finitely many $\lambda\in K'$ with $q(\lambda)=m$ and $\lambda+K=\pm p(\beta)$ such that $(\lambda,W)>0$ and $(\lambda,Y)\leq 0$ for some $Z=X+iY\in U$. Using this fact, one immediately obtains the convergence of the first series on U. □

By Lemma 3.15 the function $\Psi_{\beta,m}(Z)$ is holomorphic on $\{Z\in\mathbb{H}_l;\ q(Y)>|m|\}$ and satisfies

$$\log|\Psi_{\beta,m}(Z)| = -\frac{1}{4}(\psi^L_{\beta,m}(Z)-C_{\beta,m}) \tag{3.40}$$

on the complement of $H(\beta,m)$.

Theorem 3.16. *The function $\Psi_{\beta,m}(Z)$ has a holomorphic continuation to $\mathbb{H}_l$, and (3.40) holds on $\mathbb{H}_l - H(\beta,m)$. Let $U\subset\mathbb{H}_l$ be an open subset with compact closure $\overline{U}\subset\mathbb{H}_l$ and denote by $\mathcal{S}(\beta,m,U)$ the finite set*

$$\mathcal{S}(\beta,m,U)=\{\lambda\in\beta+L;\quad q(\lambda)=m,\ \exists Z\in U\ \text{with}\ (Z_L,\lambda)=0\}\,.$$

Then

$$\Psi_{\beta,m}(Z)\prod_{\lambda\in\mathcal{S}(\beta,m,U)}(\lambda,Z_L)^{-1} \tag{3.41}$$

is a holomorphic function without any zeros on U.

Proof. It suffices to show that $\Psi_{\beta,m}(Z)$ can be continued holomorphically to any simply connected open set U with compact closure $\overline{U}\subset\mathbb{H}_l$ and nonempty intersection $U\cap\{Z\in\mathbb{H}_l;\ q(Y)>|m|\}$, and that (3.40), (3.41) hold on U.

According to Theorem 2.12 we may consider

$$\psi^L_{\beta,m}(Z)+4\sum_{\lambda\in\mathcal{S}(\beta,m,U)}\log|(\lambda,Z_L)| \tag{3.42}$$

as a real analytic function on U. Moreover, the Fourier expansion (3.38) of $\psi^L_{\beta,m}$ on $\{Z\in W;\ q(Y)>|m|\}$ implies that (3.42) is even pluriharmonic (by real analytic continuation on U). By Lemma 3.13 there exists a holomorphic function $f:U\to\mathbb{C}$ with

$$\psi^L_{\beta,m}(Z)+4\sum_{\lambda\in\mathcal{S}(\beta,m,U)}\log|(\lambda,Z_L)|=\Re(f(Z)).$$

On the non-empty intersection of the open sets U and $\{Z\in\mathbb{H}_l;\ q(Y)>|m|\}$ we therefore have

$$\Re\left(\operatorname{Log}\left(\Psi_{\beta,m}(Z)\prod_{\lambda\in\mathcal{S}(\beta,m,U)}(\lambda,Z_L)^{-1}\right)-C_{\beta,m}/4\right)=-\frac{1}{4}\Re(f(Z)).$$

But now $-\frac{1}{4}f$ can only differ by an additive constant from the expression in brackets on the left-hand side. We may assume that this constant equals zero and find

$$\Psi_{\beta,m}(Z)\prod_{\lambda\in\mathcal{S}(\beta,m,U)}(\lambda,Z_L)^{-1}=e^{C_{\beta,m}/4}e^{-f(Z)/4}.$$

Since $e^{-f(Z)/4}$ is a holomorphic function without any zeros on U, we obtain the assertion. □

If $2\beta=0$ in L'/L, then all zeros of $\Psi_{\beta,m}(Z)$ have order 2, because $\lambda\in\mathcal{S}(\beta,m,U)$ implies $-\lambda\in\mathcal{S}(\beta,m,U)$. If $2\beta\neq 0$ in L'/L, then the zeros of $\Psi_{\beta,m}$ have order 1, and $\Psi_{\beta,m}=\Psi_{-\beta,m}$.

According to (3.40) the function $\Psi_{\beta,m}$ is independent of the choice of W up to multiplication by a constant of absolute value 1.

3.3 Modular forms on orthogonal groups

In this section we review a few basic facts on modular forms for orthogonal groups. The theory can be developed similarly as in [Fr1], [Fr2].

Recall the realizations $\mathcal{K}^+$ and $\mathbb{H}_l$ of the Hermitean symmetric space $\mathrm{Gr}(L)$ introduced in the previous section. Let

$$\widetilde{\mathcal{K}}^+ = \{W \in V(\mathbb{C}) - \{0\}; \quad [W] \in \mathcal{K}^+\} \subset V(\mathbb{C}) \tag{3.43}$$

be the cone over $\mathcal{K}^+ \subset P(V(\mathbb{C}))$. The map

$$Z \mapsto Z_L = (Z, 1, -q(Z) - q(z')) \tag{3.44}$$

defines an embedding of $\mathbb{H}_l$ into $\widetilde{\mathcal{K}}^+$, and the assignment $Z \mapsto [Z_L] \in P(V(\mathbb{C}))$ induces an isomorphism $\mathbb{H}_l \to \mathcal{K}^+$. The linear action of the orthogonal group $\mathrm{O}^+(V)$ on $\mathcal{K}^+$ induces an action on $\mathbb{H}_l$ such that the diagram

$$\begin{array}{ccc} \mathcal{K}^+ & \xrightarrow{[W]\mapsto[\sigma W]} & \mathcal{K}^+ \\ {\scriptstyle Z\mapsto[Z_L]}\uparrow & & \uparrow \\ \mathbb{H}_l & \xrightarrow{Z\mapsto\sigma Z} & \mathbb{H}_l \end{array} \tag{3.45}$$

commutes for all $\sigma \in \mathrm{O}^+(V)$. It is easily verified that σZ is the unique element of $\mathbb{H}_l$ with the property

$$(\sigma(Z_L), z)(\sigma Z)_L = \sigma(Z_L). \tag{3.46}$$

The function

$$j(\sigma, Z) = (\sigma(Z_L), z)$$

on $\mathrm{O}^+(V) \times \mathbb{H}_l$ is an automorphy factor for $\mathrm{O}^+(V)$, i.e. it satisfies the cocycle relation

$$j(\sigma_1\sigma_2, Z) = j(\sigma_1, \sigma_2 Z) j(\sigma_2, Z)$$

and does not vanish on $\mathbb{H}_l$.

Let r be a rational number. If $\sigma \in \mathrm{O}^+(V)$ and $Z \in \mathbb{H}_l$, we define

$$j(\sigma, Z)^r := e^{r \operatorname{Log} j(\sigma, Z)},$$

where $\operatorname{Log} j(\sigma, Z)$ denotes a fixed holomorphic logarithm of $j(\sigma, Z)$. There exists a map w_r from $\mathrm{O}^+(V) \times \mathrm{O}^+(V)$ to the set of roots of unity (of order bounded by the denominator of r) such that

$$j(\sigma_1\sigma_2, Z)^r = w_r(\sigma_1, \sigma_2) j(\sigma_1, \sigma_2 Z)^r j(\sigma_2, Z)^r.$$

This map obviously only depends on r modulo $\mathbb{Z}$.

Definition 3.17. *Let $\Gamma \leq \mathrm{O}^+(V)$ be a subgroup and $r \in \mathbb{Q}$ as above. By a* multiplier system *of weight r for Γ we mean a map*

$$\chi : \Gamma \to S^1 = \{t \in \mathbb{C}; \quad |t| = 1\}$$

satisfying

$$\chi(\sigma_1\sigma_2) = w_r(\sigma_1, \sigma_2)\chi(\sigma_1)\chi(\sigma_2).$$

If $r \in \mathbb{Z}$, then χ is actually a character[1] of Γ. If χ is a multiplier system of weight r for Γ, then $\chi(\sigma)j(\sigma, Z)^r$ is a cocycle of Γ.

We now define modular forms for congruence subgroups of $\mathrm{O}^+(V)$.

Definition 3.18. *Let $\Gamma \leq \Gamma(L)$ be a subgroup of finite index and χ a multiplier system for Γ of weight $r \in \mathbb{Q}$. A meromorphic function F on $\mathbb{H}_l$ is called a* meromorphic modular form *of weight r and multiplier system χ with respect to Γ, if*

$$F(\sigma Z) = \chi(\sigma)j(\sigma, Z)^r F(Z) \tag{3.47}$$

for all $\sigma \in \Gamma$. If F is even holomorphic on $\mathbb{H}_l$ then it is called a holomorphic modular form. *(Since $l \geq 3$, then the Koecher principle ensures that F is also holomorphic on the Satake boundary.)*

Remark 3.19. Since $l \geq 3$, the Lie group $\mathrm{O}^+(V)$ has no almost simple factor of real rank 1. This implies that the factor group of Γ as in Definition 3.18 modulo its commutator subgroup is finite (see [Mar] Proposition 6.19 on p. 333). Thus any multiplier system of Γ has finite order.

Sometimes it is convenient to consider modular forms as homogeneous functions on $\widetilde{\mathcal{K}}^+$. Here we temporarily have to assume that $r \in \mathbb{Z}$. (Otherwise one would have to work with covers of $\mathcal{K}^+$.)

If F is a meromorphic modular form on $\mathbb{H}_l$ of weight r, then we may define a meromorphic function $\tilde{F}$ on $\widetilde{\mathcal{K}}^+$ by

$$\tilde{F}(tZ_L) = t^{-r}F(Z) \qquad (t \in \mathbb{C} - 0).$$

It is easily seen that:

i) $\tilde{F}$ is homogeneous of degree $-r$, i.e. $\tilde{F}(tZ_L) = t^{-r}\tilde{F}(Z_L)$ for any $t \in \mathbb{C} - 0$;
ii) $\tilde{F}$ is invariant under Γ, i.e. $\tilde{F}(\sigma Z_L) = \chi(\sigma)\tilde{F}(Z_L)$ for any $\sigma \in \Gamma$.

Conversely, if G is any meromorphic function on $\widetilde{\mathcal{K}}^+$ satisfying (i) and (ii), then the function $F(Z) = G(Z_L)$ is a meromorphic modular form of weight r on $\mathbb{H}_l$ in the sense of Definition 3.18. Thus modular forms can also be viewed as meromorphic functions on $\widetilde{\mathcal{K}}^+$ satisfying (i) and (ii).

Let $u \in V = L \otimes \mathbb{R}$ be an isotropic vector and $v \in V$ be orthogonal to u. The *Eichler transformation* $E(u, v)$, defined by

[1] Here multiplier systems and characters are always unitary.

$$E(u,v)(a) = a - (a,u)v + (a,v)u - q(v)(a,u)u \tag{3.48}$$

for $a \in V$, is an element of $\mathrm{O}^+(V)$. If u, v belong to the lattice L, then $E(u,v)$ lies in the discriminant kernel $\Gamma(L)$.

For any $k \in K$ the special Eichler transformation $E(z,k) \in \Gamma(L)$ acts on $\mathbb{H}_l$ as the translation $Z \mapsto Z - k$ and the corresponding automorphy factor equals 1. This implies that any holomorphic modular form F of weight $r \in \mathbb{Q}$ for the group $\Gamma(L)$ is periodic with period lattice K in the following sense: There exists a vector $\varrho \in K \otimes \mathbb{Q}$ (which is unique modulo K') such that

$$F(Z+k) = e((\varrho,k))F(Z)$$

for all $k \in K$. Thus F has a Fourier expansion of the form

$$F(Z) = \sum_{\lambda \in \varrho + K'} a(\lambda) e((\lambda, Z)).$$

If F has integral weight and trivial multiplier system, then $\varrho = 0$. The Koecher principle tells us that $a(\lambda) \neq 0$, only if λ lies in the closure of the positive cone $C \subset K \otimes \mathbb{R}$ (see page 77), in particular $q(\lambda) \geq 0$. A precise version will be stated later in Proposition 4.15.

The following lemma generalizes the elementary formula $\Im(M\tau) = \frac{y}{|c\tau+d|^2}$ for $\tau \in \mathbb{H}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.

Lemma 3.20. *If $\sigma \in \mathrm{O}^+(V)$ and $Z = X + iY \in \mathbb{H}_l$, then*

$$q(\Im(\sigma Z)) = \frac{q(Y)}{|j(\sigma,Z)|^2}.$$

Proof. We use the formula $q(Y) = \frac{1}{4}(Z_L, \overline{Z}_L)$ and (3.46). One has

$$\begin{aligned} q(\Im(\sigma Z)) &= \frac{1}{4}\left(\frac{\sigma(Z_L)}{(\sigma Z_L, z)}, \frac{\sigma(\overline{Z}_L)}{(\sigma \overline{Z}_L, z)}\right) \\ &= \frac{1}{4|(\sigma Z_L, z)|^2}\big(\sigma(Z_L), \sigma(\overline{Z}_L)\big) \\ &= \frac{q(Y)}{|j(\sigma,Z)|^2}. \end{aligned}$$

□

Lemma 3.21. *(cp. [Bo2] Lemma 13.1.) Let $r \in \mathbb{Q}$ and $\Gamma \leq \Gamma(L)$ be subgroup of finite index. Suppose that Ψ is a meromorphic function on $\mathbb{H}_l$ for which $|\Psi(Z)| q(Y)^{r/2}$ is invariant under Γ. Then there exists a multiplier system χ of weight r for Γ such that Ψ is a meromorphic modular form of weight r and multiplier system χ with respect to Γ.*

Proof. Let $\sigma \in \Gamma$. By assumption we have

$$|\Psi(\sigma Z)| q(\Im(\sigma Z))^{r/2} = |\Psi(Z)| q(Y)^{r/2}.$$

By Lemma 3.20 this means that the function $\frac{|\Psi(\sigma Z)|}{|\Psi(Z)|} |j(\sigma, Z)|^{-r}$ is constant with value 1. Hence, according to the maximum modulus principle, there exists a constant $\chi(\sigma)$ of absolute value 1 such that

$$\frac{\Psi(\sigma Z)}{\Psi(Z)} j(\sigma, Z)^{-r} = \chi(\sigma).$$

The function $\chi : \Gamma \to \mathbb{C}$ is a multiplier system of weight r. □

3.4 Borcherds products

Theorem 3.16 states that $\Psi_{\beta,m}(Z)$ is a holomorphic function on $\mathbb{H}_l$ with divisor $H(\beta, m)$. Observe that $\Psi_{\beta,m}(Z)$ is not necessarily automorphic; by construction we only know that

$$|\Psi_{\beta,m}(Z)| e^{-\xi^L_{\beta,m}(Z)/4}$$

is invariant under $\Gamma(L)$. However, taking suitable finite products of the $\Psi_{\beta,m}$ one can attain that the main parts of the $\xi^L_{\beta,m}$ cancel out. Thereby one finds a new explanation of [Bo2] Theorem 13.3 and [Bo3] from a cohomological point of view.

Let $f : \mathbb{H} \to \mathbb{C}[L'/L]$ be a nearly holomorphic modular form for $\mathrm{Mp}_2(\mathbb{Z})$ of weight $k = 1 - l/2$ with principal part

$$\sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} c(\gamma, n) \mathfrak{e}_\gamma(n\tau).$$

Then we call the components of

$$\mathrm{Gr}(K) - \bigcup_{\gamma \in L_0'/L} \bigcup_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0 \\ c(\gamma,n)\neq 0}} H(p(\gamma), n)$$

the *Weyl chambers* of $\mathrm{Gr}(K)$ with respect to f. Let W be such a Weyl chamber. For every $\gamma \in L_0'/L$, $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$ and $c(\gamma, n) \neq 0$ there is a Weyl chamber $W_{\gamma,n}$ of $\mathrm{Gr}(K)$ of index $(p(\gamma), n)$ (as in section 3.1) such that $W \subset W_{\gamma,n}$. Then

$$W = \bigcap_{\gamma \in L_0'/L} \bigcap_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0 \\ c(\gamma,n)\neq 0}} W_{\gamma,n}.$$

We define the *Weyl vector* $\varrho_f(W)$ attached to W and f by

$$\varrho_f(W) = \frac{1}{2} \sum_{\gamma \in L_0'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} c(\gamma, n) \varrho_{p(\gamma),n}(W_{\gamma,n}).$$

As before, for $\lambda \in K'$ we write $(\lambda, W) > 0$, if λ has positive inner product with all elements in the interior or W. We will also call the subset $\{Z = X + iY \in \mathbb{H}_l;\quad Y/|Y| \in W\} \subset \mathbb{H}_l$ a Weyl chamber with respect to f.

Theorem 3.22 (Borcherds). *Let L be an even lattice of signature $(2,l)$ with $l \geq 3$, and $z \in L$ a primitive isotropic vector. Let $z' \in L'$, $K = L \cap z^\perp \cap z'^\perp$, and the projection p be defined as in section 2.1. Moreover, assume that K also contains an isotropic vector.*

Let f be a nearly holomorphic modular form of weight $k = 1 - l/2$ whose Fourier coefficients $c(\gamma, n)$ are integral for $n < 0$. Then

$$\Psi(Z) = \prod_{\beta \in L'/L} \prod_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} \Psi_{\beta,m}(Z)^{c(\beta,m)/2}$$

is a meromorphic function on $\mathbb{H}_l$ with the following properties:

i) It is a meromorphic modular form of (rational) weight $c(0,0)/2$ for the orthogonal group $\Gamma(L)$ with some multiplier system χ of finite order. If $c(0,0) \in 2\mathbb{Z}$, then χ is a character.

ii) The divisor of $\Psi(Z)$ on $\mathbb{H}_l$ is given by

$$(\Psi) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) H(\beta, m).$$

(The multiplicities of $H(\beta, m)$ are 2, if $2\beta = 0$ in L'/L, and 1, if $2\beta \neq 0$ in L'/L. Note that $c(\beta, m) = c(-\beta, m)$ and $H(\beta, m) = H(-\beta, m)$.)

iii) Let $W \subset \mathbb{H}_l$ be a Weyl chamber with respect to f and $m_0 = \min\{n \in \mathbb{Q};\ c(\gamma, n) \neq 0\}$. On the set of $Z \in \mathbb{H}_l$, which satisfy $q(Y) > |m_0|$, and which belong to the complement of the set of poles of $\Psi(Z)$, the function $\Psi(Z)$ has the normally convergent Borcherds product expansion

$$\Psi(Z) = Ce\big((\varrho_f(W), Z)\big) \prod_{\substack{\lambda \in K' \\ (\lambda, W)>0}} \prod_{\substack{\delta \in L_0'/L \\ p(\delta)=\lambda+K}} \big(1 - e((\delta, z') + (\lambda, Z))\big)^{c(\delta, q(\lambda))}.$$

Here C is a constant of absolute value 1, and $\varrho_f(W) \in K \otimes \mathbb{R}$ denotes the Weyl vector attached to W and f. (Usually $\varrho_f(W)$ can be computed explicitly using Theorem 3.4.)

Proof. Throughout the proof we write $b_{\beta,m}(\gamma,n)$ instead of $b(\gamma,n,1-k/2)$ for the (γ,n)-th Fourier coefficient of the Poincaré series $F_{\beta,m}(\tau,1-k/2)$ to emphasize the dependence on (β,m). Note that

$$f(\tau)=\frac{1}{2}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\gamma)\\ m<0}}c(\beta,m)F_{\beta,m}(\tau,1-k/2)$$

by Proposition 1.12. Thereby the Fourier coefficients $c(\gamma,n)$ of f with $n\geq 0$ are given by

$$c(\gamma,n)=\frac{1}{2}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\gamma)\\ m<0}}c(\beta,m)b_{\beta,m}(\gamma,n). \tag{3.49}$$

In particular, by Lemma 1.13 the coefficients $c(\gamma,n)$ are real numbers. Define two functions

$$\begin{aligned}\Phi(Z)&=\frac{1}{2}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\gamma)\\ m<0}}c(\beta,m)\Phi^{L}_{\beta,m}(Z),\\ \xi(Z)&=\frac{1}{2}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\gamma)\\ m<0}}c(\beta,m)\xi^{L}_{\beta,m}(Z).\end{aligned}$$

i) First, we show that the existence of the nearly holomorphic modular form f implies that the main terms in $\xi(Z)$ cancel. By virtue of Proposition 1.16 we may replace the coefficients $b(\delta,q(\lambda))$ in the Fourier expansion of $\xi^{L}_{\beta,m}$ (see Definition 3.11) by

$$-\frac{1}{\Gamma(1-k)}p_{\delta,-q(\lambda)}(\beta,-m),$$

where $p_{\delta,-n}(\beta,-m)$ denotes the $(\beta,-m)$-th coefficient of the Poincaré series $P_{\delta,-n}\in S_{\kappa,L}$. Hence the Fourier expansion of $\xi(Z)$ can be written in the form

$$\begin{aligned}\xi(Z)=&\frac{1}{2}\sum_{\substack{\beta\in L'/L\\ m\in\mathbb{Z}+q(\gamma)\\ m<0}}c(\beta,m)\left(\frac{|Y|}{\sqrt{2}}\xi^{K}_{\beta,m}(Y/|Y|)-b_{\beta,m}(0,0)\log(Y^2)\right)\\ &-\frac{1}{\sqrt{\pi}\Gamma(1-k)}\sum_{\substack{\lambda\in K'\\ q(\lambda)<0}}\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}}\sum_{\substack{\beta\in L'/L\\ m\in\mathbb{Z}+q(\gamma)\\ m<0}}c(\beta,m)p_{\delta,-q(\lambda)}(\beta,-m)\\ &\times\sum_{n\geq 1}\frac{1}{n}e(n(\delta,z')+n(\lambda,X))\mathcal{V}_{2-k}\left(\pi n|\lambda||Y|,\,\pi n(\lambda,Y)\right).\end{aligned}$$

But Theorem 1.17 implies that the sums

$$\sum_{\beta,m} c(\beta,m) p_{\delta,-q(\lambda)}(\beta,-m)$$

vanish. Thus $\xi(Z)$ can be simplified:

$$\xi(Z) = -\frac{1}{2}\log(Y^2)\sum_{\beta,m} c(\beta,m) b_{\beta,m}(0,0) + \frac{|Y|}{2\sqrt{2}}\sum_{\beta,m} c(\beta,m)\xi^K_{\beta,m}(Y/|Y|).$$

If we apply the same argument to the Fourier expansion of the $\xi^K_{\beta,m}$, we find that the function $\sum_{\beta,m} c(\beta,m)\xi^K_{\beta,m}$ is rational. Hence, according to Theorem 3.6, it vanishes identically. Using (3.49) we finally obtain

$$\xi(Z) = -c(0,0)\log(Y^2)$$

and hereby

$$\Phi(Z) = -c(0,0)\log(Y^2) + \frac{1}{2}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\gamma)\\ m<0}} c(\beta,m)\psi^L_{\beta,m}(Z). \tag{3.50}$$

By Theorem 3.16 there is a real constant C such that

$$|\Psi(Z)| = C\exp\left(-\frac{1}{8}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\gamma)\\ m<0}} c(\beta,m)\psi^L_{\beta,m}(Z)\right).$$

Inserting (3.50) we get

$$|\Psi(Z)| q(Y)^{c(0,0)/4} = C' e^{-\Phi(Z)/4}.$$

Since $\Phi(Z)$ is invariant under $\Gamma(L)$, the function $|\Psi(Z)|q(Y)^{c(0,0)/4}$ is invariant, too. Using Lemma 3.21 we may infer that $\Psi(Z)$ is a modular form of weight $\frac{c(0,0)}{2}$ with some multiplier system χ for $\Gamma(L)$.

ii) This immediately follows from Theorem 3.16.

iii) It is a consequence of Theorem 3.16 and equation (3.49) that

$$\log|\Psi(Z)| = -2\pi(\varrho_f(W),Y) + \sum_{\substack{\lambda\in K'\\ (\lambda,W)>0}}\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} c(\delta,q(\lambda))\log|1-e((\delta,z')+(\lambda,Z))|.$$

This implies the last assertion. □

Remark 3.23. According to Proposition 1.16 the coefficient $c(0,0)$ is equal to the linear combination

$$-\frac{1}{2}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\gamma)\\ m<0}} c(\beta,m)q(\beta,-m)$$

of the coefficients $q(\gamma, n)$ of the Eisenstein series $E \in M_{\kappa,L}$. In particular, the assumption that the $c(\gamma, n)$ are integral for $n < 0$ implies that $c(0,0) \in \mathbb{Q}$ by Proposition 1.7.

The fact that χ has finite order implies that the Weyl vector $\varrho_f(W)$ lies in $K \otimes \mathbb{Q}$. This can also be proved directly for the individual Weyl vectors $\varrho_{\gamma,n}(W)$ (see Definition 3.5) using Theorem 3.4 and Theorem 2.14.

3.4.1 Examples

Let H be the lattice $\mathbb{Z}^2$ with the quadratic form $q((a,b)) = ab$. Any lattice isomorphic to H is called a *hyperbolic plane*.

1. Let $L = H \oplus H$ be the even unimodular lattice of signature $(2,2)$. Strictly speaking this case is not covered by our approach to Borcherds products because $l = 2$. However, if we define all Poincaré and Eisenstein series in section 1.2 using Hecke's trick, then our argument still works. There are two minor difficulties: To get all nearly holomorphic modular forms of weight 0 (Proposition 1.12) we also have to consider Eisenstein series of weight 0. Moreover, the Eisenstein series E of weight 2 (Theorem 1.6) is in general no longer holomorphic. Some steps of the proof of Theorem 3.22 can be simplified. For instance one could use Theorem 2.14 and the fact that $F(1,1,2;z) = -z^{-1}\log(1-z)$ to compute the Weyl vector $\varrho_f(W)$ directly. The results of section 3.1 are not needed in this case.

The space $M^!_{0,L}$ of nearly holomorphic modular forms of weight 0 is equal to the polynomial ring $\mathbb{C}[j(\tau)]$, where

$$j(\tau) = q^{-1} + 744 + 196884q + \dots$$

is the usual j-function, and $q = e(\tau)$. In particular any Fourier polynomial $\sum_{n<0} c(n)q^n$ is the principal part of a nearly holomorphic modular form. This is equivalent to the fact that there are no elliptic cusp forms of weight 2, i.e. $S_{2,L} = \{0\}$. We consider the nearly homomorphic modular form

$$J(\tau) = j(\tau) - 744 = \sum_{n\geq -1} c(n)q^n$$

in $M^!_{0,L}$. Modular forms for the orthogonal group $\Gamma(L)$ can be identified with modular forms on $\mathbb{H}\times\mathbb{H}$ for the group $\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SL}_2(\mathbb{Z})$. The Heegner divisors $H(m) = \frac{1}{2}H(0,m)$ are just the Hecke correspondences $T(|m|)$ on $\mathbb{H}\times\mathbb{H}$. In particular $H(-1)$ is given by the translates of the diagonal $\{(\tau,\tau);\ \tau\in\mathbb{H}\}$. In this case Theorem 3.22 (when extended to $l = 2$) tells us that there exists a (on $\mathbb{H}\times\mathbb{H}$) holomorphic modular form Ψ for $\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SL}_2(\mathbb{Z})$ of weight 0 with divisor $T(1)$. It is easily checked that Ψ equals the function $j(\tau_1)-j(\tau_2)$. We obtain the product expansion

$$j(\tau_1) - j(\tau_2) = p^{-1} \prod_{\substack{m>0 \\ n\in\mathbb{Z}}} (1 - p^m q^n)^{c(mn)},$$

where $p = e(\tau_1)$ and $q = e(\tau_2)$. The product converges for $\Im(\tau_1)\Im(\tau_2) > 1$. This is the famous denominator formula for the monster Lie algebra [Bo4]. It was proved independently in the 80's by Koike, Norton, and Zagier.

2. This example is due to Gritsenko and Nikulin [GN]. Let D be the negative definite lattice $\mathbb{Z}$ with the quadratic form $q(a) = -a^2$ and L the lattice $D\oplus H\oplus H$ of signature $(2,3)$. In this case the space $M^!_{-1/2,L}$ of nearly holomorphic modular forms of weight $-1/2$ can be identified with the space $\widetilde{J}_{0,1}$ of weak Jacobi forms of weight 0 and index 1 in the sense of [EZ]. The obstruction space $S_{5/2,L}$ can be identified in the same way with the space of skew holomorphic Jacobi forms of weight 3 and index 1. This space equals $\{0\}$. Therefore every Fourier polynomial is the principal part of a weak Jacobi form in $\widetilde{J}_{0,1}$. We consider the special weak Jacobi form

$$\begin{aligned}\phi_{0,1}(\tau,z) = \frac{\phi_{12,1}(\tau,z)}{\Delta(\tau)} &= \sum_{\substack{n\geq 0 \\ r\in\mathbb{Z}}} c(n,r) q^n \zeta^r \\ &= (\zeta + 10 + \zeta^{-1}) + q(10\zeta^{-2} - 64\zeta^{-1} + 108 - 64\zeta + 10\zeta^2) + O(q^2)\end{aligned}$$

in $\widetilde{J}_{0,1}$. Here we have used the notation of [EZ] §9, in particular $\zeta = e(z)$.

Modular forms for the orthogonal group $\Gamma(L)$ can be identified with Siegel modular forms for the symplectic group $\mathrm{Sp}(4,\mathbb{Z})$ of genus 2. The Heegner divisor $H(*,m)$ only depends on m and is equal to the Humbert surface of discriminant $|4m|$ in the Siegel upper half plane $\mathbb{H}_S$ of genus 2. For $Z = \left(\begin{smallmatrix}\tau & z \\ z & \tau'\end{smallmatrix}\right) \in \mathbb{H}_S$ we write $q = e(\tau)$, $r = e(z)$, and $s = e(\tau')$. According to Theorem 3.22 there exists a holomorphic modular form $\Psi(Z)$ of weight 5 for the group $\mathrm{Sp}(4,\mathbb{Z})$, whose divisor is the Humbert surface of discriminant 1. It is known that the product

$$\Delta_0^{(2)} = \prod \vartheta_{a,b}(Z)$$

of the ten even theta constants is a Siegel modular form of weight 5 with the same divisor (see [Fr1] chapter 3.1). Thus Ψ has to be a constant multiple of $\Delta_0^{(2)}$. It easily checked that the right constant factor is 1/64. We obtain the following product expansion for $\Delta_0^{(2)}$:

$$\frac{1}{64}\Delta_0^{(2)} = (qrs)^{1/2} \prod_{\substack{i,j,k\in\mathbb{Z} \\ i-j/2+k>0}} (1 - q^i r^j s^k)^{c(ik,j)}.$$

The product converges on $\det(Y) > 1/4$, where $Y = \Im(Z)$. Maass showed in [Ma3] that $\Delta_0^{(2)}$ can also be constructed as a Saito-Kurokawa lifting.

3. Let K be the even unimodular lattice of signature $(1, 33)$ and $L = K \oplus H$. Then $k = -16$, $\kappa = 18$, and $S_{18,L} = \mathbb{C}g(\tau)$, where $g(\tau)$ is the unique normalized elliptic cusp form of weight 18, i.e.

$$\begin{aligned} g(\tau) &= \Delta(\tau)E_6(\tau) = q - 528q^2 - 4284q^3 + 147712q^4 + \dots \\ &= \sum_{n=1}^{\infty} a(n)q^n. \end{aligned}$$

Here $\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24}$ and $E_r(\tau)$ denotes the Eisenstein series of weight r normalized such that its constant term equals 1. The nearly holomorphic modular form

$$\begin{aligned} f(\tau) &= E_8(\tau)/\Delta(\tau)^2 = q^{-2} + 528q^{-1} + 86184 + 4631872q + \dots \\ &= \sum_{n=-2}^{\infty} c(n)q^n \end{aligned}$$

belongs to $M^!_{-16,L}$. Write $H(m) = \frac{1}{2}H(0,m)$. Theorem 3.22 implies that there is a holomorphic modular form Ψ of weight 43092 for the orthogonal group $\Gamma(L)$ with divisor $H(-2) + 528H(-1)$ and product expansion

$$\Psi(Z) = e((\varrho_f(W), Z)) \prod_{\substack{\lambda\in K \\ (\lambda,W)>0}} (1 - e((\lambda, Z)))^{c(q(\lambda))}$$

for a suitable choice of W and $\varrho_f(W) \in K \otimes \mathbb{Q}$. The product converges for $q(Y) > 2$. Since $S_{18,L} \neq \{0\}$, in this case *not* any Fourier polynomial is the principal part of a nearly holomorphic modular form. The obstructions are given by the coefficients $a(n)$ of g.

A conjecture of Lehmer says that $a(n) \neq 0$ for all positive integers n. (This conjecture was originally stated for the coefficients of $\Delta(\tau)$ but was generalized later.) Freitag pointed out that Lehmer's conjecture can be restated in terms of the geometry of the modular variety $\mathcal{X}_L = \mathbb{H}_l/\Gamma(L)$. The vanishing of $a(n)$ implies by Theorem 3.22 that the Heegner divisor $H(-n)$ vanishes in the modified divisor class group $\widetilde{\mathrm{Cl}}(\mathcal{X}_L) \otimes \mathbb{Q}$ of $\mathcal{X}_L$ (see chapter 5). On the other hand it is a consequence of our converse-theorem 5.12 that the vanishing of $H(-n)$ in $\widetilde{\mathrm{Cl}}(\mathcal{X}_L) \otimes \mathbb{Q}$ implies that $a(n) = 0$.

4 Some Riemann geometry on $\mathrm{O}(2,l)$

The aim of this chapter is to prove Theorem 4.23. To this end we have to exploit the Riemann structure of the quotient $\mathbb{H}_l/\Gamma(L)$.

4.1 The invariant Laplacian

In this section we consider the $\mathrm{O}^+(V)$-invariant Laplacian which acts on functions on the generalized upper half plane $\mathbb{H}_l$. General references are the books [GH, Hel, Wel].

As in section 3.2 let $z \in L$ be a primitive norm 0 vector and $z' \in L'$ with $(z,z') = 1$. Then $\tilde{z} = z' - q(z')z$ is an isotropic vector in $L \otimes \mathbb{Q}$. Put $K = L \cap z^\perp \cap z'^\perp$ and recall the definition of $\mathbb{H}_l$ and $\mathcal{K}^+$.

First, we introduce suitable coordinates on $\mathbb{H}_l$. Let $d \in K$ be a primitive norm 0 vector and $d' \in K'$ with $(d,d') = 1$. Then $D = K \cap d^\perp \cap d'^\perp$ is a negative definite lattice and $K \otimes \mathbb{Q} = (D \otimes \mathbb{Q}) \oplus \mathbb{Q}d' \oplus \mathbb{Q}d$. The vector $\tilde{d} = d' - q(d')d \in K \otimes \mathbb{Q}$ has norm 0 and satisfies $(\tilde{d}, d) = 1$. The lattice $D \oplus \mathbb{Z}\tilde{d} \oplus \mathbb{Z}d$ is commensurable with K.

Let $d_3, \dots, d_l$ be an orthogonal basis of $D \otimes \mathbb{R}$ normalized such that $d_3^2 = \dots = d_l^2 = -2$. Then $\tilde{d}, d, d_3, \dots, d_l$ is a basis of $K \otimes \mathbb{R}$. If $Z = z_1\tilde{d} + z_2 d + z_3 d_3 + \dots + z_l d_l \in K \otimes \mathbb{C}$, we briefly write $Z = (z_1, z_2, z_3, \dots, z_l)$ and denote its real part by $X = (x_1, x_2, x_3, \dots, x_l)$, resp. its imaginary part by $Y = (y_1, y_2, y_3, \dots, y_l)$. We obviously have

$$q(Y) = y_1 y_2 - y_3^2 - y_4^2 - \dots - y_l^2$$

and

$$\mathbb{H}_l = \{Z = (z_1, \dots, z_l) \in K \otimes \mathbb{C}; \quad y_1 > 0,\ y_1 y_2 - y_3^2 - y_4^2 - \dots - y_l^2 > 0\}.$$

Moreover, sometimes we just write $Z = (z_1, z_2, Z_D)$ for $Z = (z_1, \dots, z_l) \in \mathbb{H}_l$, where $Z_D = z_3 d_3 + \dots + z_l d_l \in D \otimes \mathbb{C}$. The real part of Z_D is denoted by X_D, the imaginary part by Y_D.

The generalized upper half plane $\mathbb{H}_l$ can be viewed as a realization of the irreducible Hermitean symmetric space $\mathrm{O}^+(V)/H$, where $H \cong \mathrm{SO}(2) \times \mathrm{SO}(l)$ denotes a maximal compact subgroup. The Killing form on the Lie algebra of

$O^+(V)$ induces an $O^+(V)$-invariant Riemann metric g on $\mathbb{H}_l$. According to [Hel] Chap. VIII §5 any $O^+(V)$-invariant Riemann metric on $\mathbb{H}_l$ is a constant multiple of g. In this sense the geometry on $\mathbb{H}_l$ is determined uniquely. By [Hel] Chap. VIII §4 the metric g is Kählerian.

We will now describe g more explicitly. Lemma 3.20 implies that the $(1,1)$-form

$$\omega = -\frac{i}{2}\partial\bar{\partial}\log(q(Y))$$

is invariant under $O^+(V)$. Moreover, ω is positive, that is

$$\omega = \frac{i}{2}\sum_{\mu,\nu} h_{\mu\nu}(Z)dz_\mu \wedge d\bar{z}_\nu,$$

where $h(Z) = (h_{\mu\nu}(Z))$ is a positive definite Hermitean matrix for any $Z \in \mathbb{H}_l$. In fact, since ω is $O^+(V)$-invariant it suffices to check this at the special point $I = i(1,1,0,\dots,0) \in \mathbb{H}_l$. A somewhat lengthy but trivial computation shows that[1]

$$h(I) = \frac{1}{4}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & 2 \end{pmatrix}.$$

Furthermore, ω is obviously closed. Hence it defines an $O^+(V)$-invariant Kähler metric on $\mathbb{H}_l$. Its underlying Riemann metric has to be a constant multiple of g. It is easily seen that the attached invariant volume element equals

$$\frac{dX\,dY}{q(Y)^l},$$

where $dX = dx_1 \cdots dx_l$ and $dY = dy_1 \cdots dy_l$. The invariant Laplace operator is given by

$$-(d * d * + * d * d).$$

Here $*$ denotes the Hodge star operator for the Riemann metric g. Nakajima [Nak] showed that, up to a constant multiple, the invariant Laplace operator is equal to

$$\Omega = \sum_{\mu,\nu=1}^{l} y_\mu y_\nu \partial_\mu \bar{\partial}_\nu - q(Y)\left(\partial_1\bar{\partial}_2 + \bar{\partial}_1\partial_2 - \frac{1}{2}\sum_{\mu=3}^{l}\partial_\mu\bar{\partial}_\mu\right), \tag{4.1}$$

where

[1] Zero matrix entries are often omitted.

$$\partial_\mu = \frac{\partial}{\partial z_\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\mu} - i\frac{\partial}{\partial y_\mu}\right),$$
$$\bar{\partial}_\mu = \frac{\partial}{\partial \bar{z}_\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\mu} + i\frac{\partial}{\partial y_\mu}\right).$$

(Note that there is a misprint in [Nak] in the definition of Δ_1.) The proof relies on the fact that there is up to a constant multiple only one $\mathrm{O}^+(V)$-invariant second order differential operator on $\mathbb{H}_l$. Therefore one only has to verify the invariance of Ω.

We will now show that for $\sigma > 1 - k/2$ and $Z \notin H(\beta, m)$ the function $\Phi^L_{\beta,m}(Z, s)$ is an eigenfunction of Ω. First, we prove some lemmas.

Lemma 4.1. *If $Z = X + iY \in \mathbb{H}_l$ and $s \in \mathbb{C}$, then*

$$\Omega q(Y)^s = \left(\frac{s^2}{2} - \frac{sl}{4}\right) q(Y)^s, \tag{4.2}$$

$$\Omega \log(q(Y)) = -l/4. \tag{4.3}$$

Proof. This can be verified by a straightforward computation. □

Recall the definition (1.34) of the Laplace operator Δ_k of weight k. We define the usual Maass differential operators on smooth functions $\mathbb{H} \to \mathbb{C}[L'/L]$ by

$$R_k = 2i\frac{\partial}{\partial \tau} + ky^{-1}, \tag{4.4}$$

$$L_k = 2iy^2\frac{\partial}{\partial \bar{\tau}}. \tag{4.5}$$

For any smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ and $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ we have

$$(R_k f) \mid_{k+2} (M, \phi) = R_k \left(f \mid_k (M, \phi)\right),$$
$$(L_k f) \mid_{k-2} (M, \phi) = L_k \left(f \mid_k (M, \phi)\right).$$

Hence R_k raises the weight, whereas L_k lowers it. The operator Δ_k can be expressed in terms of R_k and L_k by

$$\Delta_k = L_{k+2} R_k - k = R_{k-2} L_k.$$

Lemma 4.2. *Let $f, g : \mathbb{H} \to \mathbb{C}[L'/L]$ be smooth functions satisfying $f \mid_k (M, \phi) = f$ and $g \mid_{k+2} (M, \phi) = g$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$. Then*

$$\int_{\mathcal{F}_u} \langle f, (L_{k+2} g)\rangle y^{k-2}\, dx\, dy - \int_{\mathcal{F}_u} \langle (R_k f), g\rangle y^k\, dx\, dy = \int_{-1/2}^{1/2} \left[\langle f, g\rangle y^k\right]_{y=u} dx.$$

Here $\mathcal{F}_u$ denotes the truncated fundamental domain (2.14).

Proof. The assumptions imply that $\omega = y^k\langle f(\tau), g(\tau)\rangle d\bar\tau$ is a $\mathrm{SL}_2(\mathbb{Z})$-invariant 1-form on $\mathbb{H}$. By Stokes' theorem we have

$$\begin{aligned}\int_{\partial\mathcal{F}_u} y^k\langle f(\tau), g(\tau)\rangle d\bar\tau &= \int_{\mathcal{F}_u} d\left(y^k\langle f(\tau), g(\tau)\rangle d\bar\tau\right)\\ &= \int_{\mathcal{F}_u}\left(-\frac{\partial}{\partial y} y^k\langle f(\tau), g(\tau)\rangle - i\frac{\partial}{\partial x} y^k\langle f(\tau), g(\tau)\right) dx\wedge dy\\ &= \int_{\mathcal{F}_u}\left(y^{k-2}\langle f, (L_{k+2}g)\rangle - y^k\langle (R_k f), g\rangle\right) dx\wedge dy.\end{aligned}$$

In the integral over $\partial\mathcal{F}_u$ on the left hand side the contributions from $\mathrm{SL}_2(\mathbb{Z})$-equivalent boundary pieces cancel. Thus

$$\int_{\partial\mathcal{F}_u} y^k\langle f(\tau), g(\tau)\rangle d\bar\tau = \int_{-1/2}^{1/2}\left[y^k\langle f(\tau), g(\tau)\rangle\right]_{y=u} dx.$$

This implies the assertion. □

Lemma 4.3. *Let $f, g : \mathbb{H} \to \mathbb{C}[L'/L]$ be smooth functions satisfying $f\mid_k (M,\phi) = f$ and $g\mid_k (M,\phi) = g$ for all $(M,\phi)\in \mathrm{Mp}_2(\mathbb{Z})$. Then*

$$\begin{aligned}&\int_{\mathcal{F}_u}\langle(\Delta_k f), g\rangle y^{k-2}\,dx\,dy - \int_{\mathcal{F}_u}\langle f, (\Delta_k g)\rangle y^{k-2}\,dx\,dy\\ &\qquad= \int_{-1/2}^{1/2}\left[\langle f, (L_k g)\rangle y^{k-2}\right]_{y=u} dx - \int_{-1/2}^{1/2}\left[\langle (L_k f), g\rangle y^{k-2}\right]_{y=u} dx.\end{aligned}$$

Proof. We write $\Delta_k = R_{k-2}L_k$ and apply the previous lemma twice. □

Lemma 4.4. *For $v\in \mathrm{Gr}(L) - H(\beta, m)$ and $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1 - k/2$ we have the identity*

$$\begin{aligned}&\lim_{u\to\infty}\int_{\mathcal{F}_u}\langle F_{\beta,m}(\tau, s),\, \Delta_k(\Theta_L(\tau, v)y^{l/2})\rangle y^k\,\frac{dx\,dy}{y^2}\\ &\qquad= (l/4 + 1/2 - s)(l/4 - 1/2 + s)\Phi_{\beta,m}(v, s).\end{aligned}$$

Proof. Since

$$\Delta_k F_{\beta,m}(\tau, s) = (l/4 + 1/2 - s)(l/4 - 1/2 + s)F_{\beta,m}(\tau, s),$$

Lemma 4.3 implies that

$$\int_{\mathcal{F}_u} \langle F_{\beta,m}(\tau,s),\, \Delta_k(\Theta_L(\tau,v)y^{l/2})\rangle y^k \,\frac{dx\,dy}{y^2}$$
$$- (s-1/2-l/4)(s-1/2+l/4)\int_{\mathcal{F}_u} \langle F_{\beta,m}(\tau,s),\, \Theta_L(\tau,v)y^{l/2}\rangle y^k \,\frac{dx\,dy}{y^2}$$
$$= \int_{-1/2}^{1/2} \Big[\langle (L_k F_{\beta,m}(\tau,s)),\, \Theta_L(\tau,v)y^{l/2}\rangle y^{k-2}\Big]_{y=u} dx$$
$$- \int_{-1/2}^{1/2} \Big[\langle F_{\beta,m}(\tau,s),\, (L_k\Theta_L(\tau,v)y^{l/2})\rangle y^{k-2}\Big]_{y=u} dx. \tag{4.6}$$

Thus it suffices to show that the integral on the right hand side tends to 0 as $u \to \infty$. We have

$$\int_{-1/2}^{1/2} \langle (L_k F_{\beta,m}(\tau,s)),\, \Theta_L(\tau,v)y^{l/2}\rangle y^{k-2} dx$$
$$= i\int_{-1/2}^{1/2} y\left\langle \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) F_{\beta,m}(\tau,s),\, \Theta_L(\tau,v)\right\rangle dx. \tag{4.7}$$

If we insert the Fourier expansions

$$F_{\beta,m}(\tau,s) = \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)} c(\gamma,n;y,s)\mathfrak{e}_\gamma(nx),$$
$$\Theta_L(\tau,v) = \sum_{\lambda\in L'} \exp(-4\pi y q(\lambda_v) + 2\pi y q(\lambda))\mathfrak{e}_\lambda(q(\lambda)x),$$

and carry out the integration, we find that (4.7) equals

$$-\sum_{\lambda\in L'} y e^{-4\pi y q(\lambda_v)+2\pi y q(\lambda)}\left(2\pi q(\lambda)c(\lambda,q(\lambda);y,s) + \frac{\partial}{\partial y}c(\lambda,q(\lambda);y,s)\right).$$

Using the explicit formulas for the coefficients $c(\gamma,n;y,s)$ given in Theorem 1.9 and the asymptotic behavior of the Whittaker functions we may infer that

$$\lim_{y\to\infty}(4.7) = -\lim_{y\to\infty} y\frac{\partial}{\partial y}c(0,0;y,s) = -b(0,0,s)\lim_{y\to\infty} y\frac{\partial}{\partial y}y^{1-s-k/2}.$$

Since $\sigma > 1-k/2$ this is equal to 0. In the same way it can be seen that

$$\lim_{y\to\infty}\int_{-1/2}^{1/2} \langle F_{\beta,m}(\tau,s),\, (L_k\Theta_L(\tau,v)y^{l/2})\rangle y^{k-2} dx = 0.$$

Taking the limit $u \to \infty$ in (4.6), we obtain the assertion. □

The following proposition is crucial for our argument.

Proposition 4.5. *The Siegel theta function $\Theta_L(\tau, Z)$, considered as a function on $\mathbb{H} \times \mathbb{H}_l$, satisfies the differential equation*

$$\Delta_k \Theta_L(\tau, Z) y^{l/2} = -2\Omega \Theta_L(\tau, Z) y^{l/2}.$$

Proof. For $\tau \in \mathbb{H}$ and $v \in \mathrm{Gr}(L)$ we define a function on $V = L \otimes \mathbb{R}$ by

$$F_{\tau,v} : V \to \mathbb{C}, \qquad \lambda \mapsto F_{\tau,v}(\lambda) = e(\tau q(\lambda_v) + \bar{\tau} q(\lambda_{v^\perp})).$$

If $Z \in \mathbb{H}_l$ corresponds to v via the isomorphism $\mathbb{H}_l \cong \mathrm{Gr}(L)$, then we also write $F_{\tau,Z}$ instead of $F_{\tau,v}$. For any fixed $\lambda \in V$ we may consider $F_{\tau,Z}(\lambda)$ as a function in τ and Z. To prove the proposition it suffices to show that

$$\Delta_k F_{\tau,Z}(\lambda) y^{l/2} = -2\Omega F_{\tau,Z}(\lambda) y^{l/2}.$$

This essentially follows from [Sn] Lemma 1.5 as we will now indicate. As far as possible we use the same notation as in [Sn].

We choose a basis for L and identify V with $\mathbb{R}^{l+2}$. Denote by Q the Gram matrix of L with respect to this basis, so that $(x, y) = x^t Q y$. Let $g \mapsto r_0(g, Q)$ be the Weil representation of $\mathrm{Mp}_2(\mathbb{R}) \times \mathrm{O}(V)$ in $L^2(V)$ as defined in [Sn].

For fixed $\tau \in \mathbb{H}$ and $Z \in \mathbb{H}_l$ the function $F_{\tau,Z}$ is rapidly decreasing. According to [Sn] Lemma 1.2 we have

$$\begin{aligned}
(r_0(\sigma, Q)F_{\tau,Z})(\lambda) &= \phi(\tau)^{-2}\overline{\phi(\tau)}^{-l} F_{M\tau,Z}(\lambda), \qquad && \sigma = (M, \phi) \in \mathrm{Mp}_2(\mathbb{R}),\\
(r_0(g, Q)F_{\tau,Z})(\lambda) &= F_{\tau,Z}(g^{-1}\lambda) = F_{\tau,gZ}(\lambda), \qquad && g \in \mathrm{O}^+(V).
\end{aligned}$$

It is easily seen that $F_{i,Z}$ satisfies the condition (1.19) in [Sn] with $m = 2 - l$. If we put

$$\sigma_\tau = \left(\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}, 1 \right) \in \mathrm{Mp}_2(\mathbb{R})$$

for $\tau = x + iy \in \mathbb{H}$, then $r_0(\sigma_\tau, Q)F_{i,Z} = y^{(2+l)/4} F_{\tau,Z}$.

The map (1.1) gives a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{Mp}_2(\mathbb{R})$. Thus for any element u of the universal enveloping algebra U of the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ we may consider $r_0(u, Q)$ as a differential operator on functions $V \to \mathbb{C}$. Let C be the Casimir element in U and $C_Q = r_0(C, Q)$. Then by [Sn] Lemma 1.4 we may infer

$$-4\Delta_k y^{l/2} F_{\tau,Z}(\lambda) = (1 - l^2/4) y^{l/2} F_{\tau,Z}(\lambda) + y^{-k/2} \left(r_0(\sigma_\tau, Q) C_Q F_{i,Z} \right)(\lambda). \tag{4.8}$$

Let ϱ denote the representation of $\mathrm{O}^+(V)$ on $C^\infty(\mathbb{H}_l)$ given by

$$(\varrho(g)f)(Z) = f(g^{-1}Z), \qquad g \in \mathrm{O}^+(V).$$

For $\lambda \in V$ we obviously have

$$\varrho(g)(F_{\tau,Z}(\lambda)) = \left(r_0(g^{-1}, Q)F_{\tau,Z}\right)(\lambda). \tag{4.9}$$

Let D be the Casimir element of the universal enveloping algebra of the Lie algebra of $\mathrm{O}^+(V)$. Then $D_Q = r_0(D, Q)$ is a differential operator on functions $V \to \mathbb{C}$, and $\varrho(D)$ a differential operator on functions $\mathbb{H}_l \to \mathbb{C}$. According to [Hel] p. 451 ex. 5 the operator $\varrho(D)$ equals (up to the sign) the Laplace operator on $\mathbb{H}_l$. By (4.9) we have $\varrho(D)(F_{\tau,Z}(\lambda)) = (-D_Q F_{\tau,Z})(\lambda)$. Hence there exists a non-zero constant c_l, only depending on l, such that

$$c_l \Omega(F_{\tau,Z}(\lambda)) = (D_Q F_{\tau,Z})(\lambda) \tag{4.10}$$

for all $\tau \in \mathbb{H}$ and $\lambda \in V$. According to [Sn] Lemma 1.5 for any smooth rapidly decreasing function F on V the identity

$$C_Q F = (D_Q + l^2/4 - 1)F \tag{4.11}$$

holds. Combining (4.8), (4.10), and (4.11), we find

$$-4\Delta_k y^{l/2} F_{\tau,Z}(\lambda) = c_l \Omega y^{l/2} F_{\tau,Z}(\lambda) \tag{4.12}$$

for any $\lambda \in V$.

To determine the constant c_l we consider the above identity (4.12) for $\lambda = d \in K$. In this case we have

$$y^{l/2} F_{\tau,Z}(d) = y^{l/2} \exp\left(-\pi y \frac{x_1^2 + y_1^2}{q(Y)}\right),$$

and also (4.12) becomes much easier. (Here we have to be careful with our notation: $y = \Im(\tau)$ and $Y = (y_1, \dots, y_l) = \Im(Z)$.) A straightforward computation shows that $c_l = 8$. This completes the proof. □

Theorem 4.6. *For $Z \in \mathbb{H}_l - H(\beta, m)$ and $\sigma > 1 - k/2$ the function $\Phi_{\beta,m}(Z, s)$ is an eigenfunction of the invariant Laplacian Ω on $\mathbb{H}_l$. More precisely we have*

$$\Omega \Phi_{\beta,m}(Z, s) = \frac{1}{2}(s - 1/2 - l/4)(s - 1/2 + l/4)\Phi_{\beta,m}(Z, s).$$

Proof. Arguing as in section 2.2 we find that all iterated partial derivatives with respect to Z of

$$\int_{\mathcal{F}_u} \langle F_{\beta,m}(\tau, s),\, \Theta_L(\tau, Z) y^{l/2} \rangle y^k \, \frac{dx\, dy}{y^2}$$

converge locally uniformly in Z as $u \to \infty$. Therefore we have

$$\Omega\Phi_{\beta,m}(Z,s) = \lim_{u\to\infty} \int_{\mathcal{F}_u} \langle F_{\beta,m}(\tau,s),\, \Omega\Theta_L(\tau,Z)y^{l/2}\rangle y^k \, \frac{dx\,dy}{y^2}.$$

By Proposition 4.5 this equals

$$-\frac{1}{2}\lim_{u\to\infty} \int_{\mathcal{F}_u} \langle F_{\beta,m}(\tau,s),\, \Delta_k(\Theta_L(\tau,Z)y^{l/2})\rangle y^k \, \frac{dx\,dy}{y^2}.$$

Using Lemma 4.4, we obtain the assertion. □

Theorem 4.7. *For $Z \in \mathbb{H}_l - H(\beta,m)$ the regularized theta integral $\Phi_{\beta,m}(Z)$ satisfies*

$$\Omega\Phi_{\beta,m}(Z) = \frac{l}{4}\, b_{\beta,m}(0,0).$$

Here $b_{\beta,m}(0,0) = b_{\beta,m}(0,0,1-k/2)$ denotes the $(0,0)$-th Fourier coefficient of the Poincaré series $F_{\beta,m}(\tau, 1-k/2)$.

Proof. Define a function

$$R(Y,s) = \frac{2}{\sqrt{\pi}}(\pi q(Y))^{1/2+l/4-s}\Gamma(s-l/4)b_{\beta,m}(0,0,s)\zeta(2s-l/2).$$

In the proof of Theorem 3.9 we saw that

$$\tilde{\Phi}_{\beta,m}(Z,s) := \Phi_{\beta,m}(Z,s) - R(Y,s)$$

is holomorphic in s near $1-k/2$. Using Theorem 2.15, it can be shown in the same way that $\tilde{\Phi}_{\beta,m}(Z,s)$ is twice continuously differentiable as a function in (Z,s) for $Z \in \mathbb{H}_l - H(\beta,m)$ and s varying in a neighborhood of $1-k/2$. This implies that

$$\begin{aligned}\Omega\Phi_{\beta,m}(Z) &= \Omega\mathcal{C}_{s=1-k/2}\left[\Phi_{\beta,m}(Z,s)\right]\\ &= \mathcal{C}_{s=1-k/2}\left[\Omega\tilde{\Phi}_{\beta,m}(Z,s)\right] + \Omega\mathcal{C}_{s=1-k/2}\left[R(Y,s)\right]. \end{aligned} \tag{4.13}$$

By Theorem 4.6 and Lemma 4.1 we have

$$\begin{aligned}\Omega\tilde{\Phi}_{\beta,m}(Z,s) &= \Omega\Phi_{\beta,m}(Z,s) - \Omega R(Y,s)\\ &= \frac{1}{2}(s-1/2-l/4)(s-1/2+l/4)\tilde{\Phi}_{\beta,m}(Z,s)\end{aligned}$$

and therefore

$$\mathcal{C}_{s=1-k/2}\left[\Omega\tilde{\Phi}_{\beta,m}(Z,s)\right] = 0. \tag{4.14}$$

In the proof of Theorem 3.9 we found that

$$\mathcal{C}_{s=1-k/2}\left[R(Y,s)\right] = -b_{\beta,m}(0,0)\log(q(Y)) + \text{const}.$$

Thus, using Lemma 4.1, we get

$$\Omega\mathcal{C}_{s=1-k/2}\left[R(Y,s)\right] = \frac{l}{4}b_{\beta,m}(0,0). \tag{4.15}$$

If we insert (4.14) and (4.15) into (4.13), we obtain the assertion. □

Remark 4.8. Theorem 4.6 and Theorem 4.7, together with a regularity result for elliptic differential operators on analytic Riemann manifolds, also imply that the functions $\Phi_{\beta,m}(Z,s)$, $\Phi_{\beta,m}(Z)$, and $\xi_{\beta,m}(Z)$ are real analytic (see [Rh] §34).

4.2 Reduction theory and L^p-estimates

Recall that the invariant volume element on the upper half plane $\mathbb{H}_l$ is given by $\frac{dX\,dY}{q(Y)^l}$. Let $\Gamma \leq \Gamma(L)$ be a subgroup of finite index. As usual we write $L^p(\mathbb{H}_l/\Gamma)$ for the Banach space of equivalence classes of measurable Γ-invariant functions $f : \mathbb{H}_l \to \mathbb{C}$ with

$$\int\limits_{\mathbb{H}_l/\Gamma} |f(Z)|^p \frac{dX\,dY}{q(Y)^l} < \infty. \tag{4.16}$$

To decide whether a function satisfies this integrability condition we need a fundamental domain for the action of Γ on $\mathbb{H}_l$.

We fix a coordinate system on $\mathbb{H}_l$ as in section 4.1 page 93. (This depends on the choice of the vectors z, z', d, d' and the orthogonal basis $d_3, \dots, d_l$ of $D \otimes \mathbb{R}$.) We use the notation of section 4.1.

Let

$$\mathrm{O}^+_{\mathbb{Q}}(L) = \{g \in \mathrm{O}^+(V);\quad g(L\otimes\mathbb{Q}) = L\otimes\mathbb{Q}\} \tag{4.17}$$

be the rational orthogonal group of the lattice L.

Definition 4.9. *Let $t > 0$. We define the* Siegel domain $\mathcal{S}_t$ *in $\mathbb{H}_l$ to be the subset of $Z = X + iY \in \mathbb{H}_l$ satisfying*

$$x_1^2 + x_2^2 + |q(X_D)| < t^2, \tag{4.18a}$$
$$1/t < y_1, \tag{4.18b}$$
$$y_1^2 < t^2 q(Y), \tag{4.18c}$$
$$|q(Y_D)| < t^2 y_1^2. \tag{4.18d}$$

Proposition 4.10. *Let $\Gamma \leq \Gamma(L)$ be a subgroup of finite index.*

i) For any $t > 0$ and any $g \in \mathrm{O}^+_{\mathbb{Q}}(L)$ the set

$$\{\sigma \in \Gamma;\quad \sigma g\mathcal{S}_t \cap \mathcal{S}_t \neq \emptyset\}$$

is finite.

ii) There exist a $t > 0$ and finitely many rational transformations $g_1, \dots, g_a \in \mathrm{O}^+_{\mathbb{Q}}(L)$ such that

$$\mathcal{S} = g_1\mathcal{S}_t \cup \dots \cup g_a\mathcal{S}_t$$

is a fundamental domain for the action of Γ on $\mathbb{H}_l$, i.e. $\Gamma\mathcal{S} = \mathbb{H}_l$.

Proof. This can be deduced from the general reduction theory for algebraic groups (cf. [Bl1], [Bl2]). We briefly indicate the argument. See [Me] for a more detailed treatment.

For any algebraic group A we denote by A^+ the component of the identity.

There is a basis $e_3, \dots, e_l$ of $D \otimes \mathbb{Q}$ such that the restriction of q to $D \otimes \mathbb{Q}$ has the diagonal Gram matrix $C = \mathrm{diag}(-c_3, \dots, -c_l)$ with positive rational numbers c_j. Then, with respect to the basis $z, d, e_3, \dots, e_l, \tilde{d}, \tilde{z}$ of $L \otimes \mathbb{Q}$, the quadratic form q has the Gram matrix

$$F = \begin{pmatrix} 0 & 0 & J \\ 0 & C & 0 \\ J & 0 & 0 \end{pmatrix},$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The real orthogonal group

$$\mathrm{O}(V) = \{M \in \mathrm{SL}_{l+2}(\mathbb{R}); \quad M^t F M = F\}$$

of V is defined over $\mathbb{Q}$. The group $G = \mathrm{O}_{\mathbb{Q}}(L)$ of rational points is simply the subgroup of matrices with rational entries, and Γ is an arithmetic subgroup of G.

Let I_L denote the special point $z + id + i\tilde{d} + \tilde{z} \in V(\mathbb{C})$ and $[I_L]$ the corresponding Element of $\mathcal{K}^+ \subset P(V(\mathbb{C}))$. The upper half plane $\mathbb{H}_l$ is identified with $\mathcal{K}^+$ via $Z \mapsto [-q(Z)z + Z + \tilde{z}]$ as in section 3.2. Here the point $I = (i, i, 0) \in \mathbb{H}_l$ is mapped to $[I_L]$. The group $\mathrm{O}^+(V) = G_{\mathbb{R}}^+$ acts transitively on $\mathcal{K}^+$ and the stabilizer K of $[I_L]$ is a maximal compact subgroup[2]. The assignment $Kg \mapsto g^{-1}[I_L]$ identifies $K \backslash \mathrm{O}^+(V)$ with $\mathcal{K}^+$.

A minimal parabolic $\mathbb{Q}$-subgroup of G is given by

$$P = \left\{ \begin{pmatrix} H_0 & H_1 & H_2 \\ 0 & B & H_3 \\ 0 & 0 & H_4 \end{pmatrix} \right\},$$

where H_0, H_4 are upper triangular and B is an orthogonal transformation in $\mathrm{O}_{\mathbb{Q}}(D)$, with additional relations that ensure $P \subset G$. The unipotent radical U of P is the set of matrices H in P, where H_0, H_4 are unipotent, $B = E$, and

$$H_4 = H_0^{-1}, \tag{4.19}$$

$$CH_3 + H_1^t J H_4 = 0, \tag{4.20}$$

$$H_4^t J H_2 + H_2^t J H_4 + H_3^t C H_3 = 0. \tag{4.21}$$

The inverse of a matrix $H \in U$ equals

[2] Only in this proof K denotes a maximal compact subgroup. Usually it denotes a sublattice of L.

$$H^{-1} = \begin{pmatrix} H_4 & JH_3^tC & JH_2^tJ \\ 0 & E & C^{-1}H_1^tJ \\ 0 & 0 & H_0 \end{pmatrix}.$$

If we write

$$H_0 = \begin{pmatrix} 1 & h_{12} \\ 0 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} h_{13} & \dots & h_{1l} \\ h_{23} & \dots & h_{2l} \end{pmatrix}, \quad H_2 = \begin{pmatrix} h_{1,l+1} & h_{1,l+2} \\ h_{2,l+1} & h_{2,l+2} \end{pmatrix},$$

then H is completely determined by $h_{12}, \dots, h_{1,l+1}$, $h_{23}, \dots, h_{2l}$, and these may be freely chosen. Thus $U_{\mathbb{R}}$ is homeomorphic to $\mathbb{R}^{2l-2}$. Equality (4.21) implies in particular that

$$h_{2,l+1} = \frac{1}{2}(c_3^{-1}h_{23}^2 + \dots + c_l^{-1}h_{2l}^2).$$

For any $t > 0$ the set

$$\omega_t = \left\{ H \in U; \quad h_{12}^2 + h_{1,l+1}^2 + \tfrac{1}{2}\textstyle\sum_{j=3}^l c_j^{-1}h_{1j}^2 < t^2 \text{ and } h_{2,l+1} < t^2 \right\}$$

is a relatively compact open neighborhood of the identity in $U_{\mathbb{R}}$. This can be checked using the above relations. Moreover, we have $U_{\mathbb{R}} = \bigcup_{t>0} \omega_t$.

A maximal $\mathbb{Q}$-split torus S in P is given by the set of diagonal matrices

$$\begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & E & & \\ & & & \lambda_2^{-1} & \\ & & & & \lambda_1^{-1} \end{pmatrix}.$$

The set of simple roots of G with respect to S and P is equal to $\Delta = \{\alpha_1, \alpha_2\}$, where

$$\alpha_1(T) = \lambda_1/\lambda_2, \qquad \alpha_2(T) = \lambda_2$$

for $T \in S$ as above (cf. [Bl1] p. 80, [Bl2] p. 16). Let $t > 0$. We denote by A_t the subset

$$A_t = \{T \in S_{\mathbb{R}}^+; \quad \alpha_1(T) < t, \ \alpha_2(T) < t\}$$

of $S_{\mathbb{R}}^+$. Then the open subset

$$\mathfrak{S}_t = K \cdot A_t \cdot \omega_t$$

of $G_{\mathbb{R}}$ is a Siegel domain (with respect to K, P, S) in the sense of definition 12.3 in [Bl1]. (Note the remark at the end of §12.3 in [Bl1] and the fact that S is also a maximal split torus over $\mathbb{R}$.) Now, by a straightforward calculation it can be verified that $\mathfrak{S}_t$ precisely corresponds to $\mathcal{S}_t$ under the identification $K\backslash \mathrm{O}^+(V) \to \mathbb{H}_l$, $Kg \mapsto g^{-1} \circ I$. (Caution: $\mathcal{S}_t$ is defined with respect to the basis $z, d, d_3, \dots, d_l, \tilde{d}, \tilde{z}$ of $L \otimes \mathbb{R}$, whereas A_t and ω_t are defined with respect to the basis $z, d, e_3, \dots, e_l, \tilde{d}, \tilde{z}$.) Therefore assertion (i) follows from [Bl1] Theorem 15.4 and (ii) from [Bl1] Theorem 13.1. □

Let $f : \mathbb{H}_l \to \mathbb{C}$ be a measurable Γ-invariant function. Proposition 4.10 immediately implies that $f \in L^p(\mathbb{H}_l/\Gamma)$ if and only if $\int_{\mathcal{S}} |f(Z)|^p \frac{dX\,dY}{q(Y)^l} < \infty$.

Let $t > 0$. In the next few lemmas we derive some useful estimates that hold on $\mathcal{S}_t$.

Lemma 4.11. *Suppose that $p < l/2$. Then*

$$\int_{\mathcal{S}_t} q(Y)^p \frac{dX\,dY}{q(Y)^l} < \infty.$$

Proof. Since the set of X defined by (4.18a) is compact, it suffices to prove that

$$\int_{\mathcal{R}_t} q(Y)^{p-l}\,dY < \infty,$$

where $\mathcal{R}_t$ denotes the set of Y defined by (4.18b)–(4.18d). It is easily seen that

$$\frac{y_1 y_2}{1+t^4} < q(Y) < y_1 y_2 \tag{4.22}$$

for $Y \in \mathcal{R}_t$. Thus it suffices to show that

$$I = \int\limits_{\substack{y_1 > 1/t \\ t^2 q(Y) > y_1^2 \\ |q(Y_D)| < t^2 y_1^2}} (y_1 y_2)^{p-l}\,dy_1\,dy_2\,dY_D$$

is finite. We have

$$\begin{aligned} I &\ll_t \int\limits_{\substack{y_1 > 1/t \\ y_2/y_1 > 1/t^2 \\ |q(Y_D)| < t^2 y_1^2}} (y_1 y_2)^{p-l}\,dy_1\,dy_2\,dY_D \\ &\ll_t \int\limits_{\substack{y_1 > 1/t \\ y_2/y_1 > 1/t^2}} (y_1 y_2)^{p-l} y_1^{l-2}\,dy_1\,dy_2 \\ &\ll_t \int\limits_{\substack{y_1 > 1/t \\ u > 1/t^2}} y_1^{2p-l-1} u^{p-l}\,dy_1\,du. \end{aligned}$$

Since $l \geq 2$, the latter integral is finite for $p < l/2$. □

In particular any Siegel domain has finite volume.

Lemma 4.12. *Let $p < l - 1$. Then*

$$\int_{\mathcal{S}_t} |(\nu, Y)|^p \, \frac{dX\, dY}{q(Y)^l} < \infty.$$

for any $\nu \in K \otimes \mathbb{R}$.

Proof. Using Minkowski's inequality and Lemma 4.11, we find that it suffices to prove that $\int_{\mathcal{S}_t} y_2^p q(Y)^{-l}\, dX\, dY < \infty$. This can be done as in Lemma 4.11. □

Lemma 4.13. *There exists an $\varepsilon > 0$ such that*

$$2(\nu, Y)^2 - Y^2\nu^2 > \varepsilon \left(y_2^2\nu_1^2 + y_1^2\nu_2^2 + q(Y)|\nu_D^2| \right)$$

for any $\nu = (\nu_1, \nu_2, \nu_D) \in K'$ and any $Y \in \mathcal{R}_t$. Here $\mathcal{R}_t$ denotes the set of Y defined by (4.18b)–(4.18d).

Proof. Let $h(\nu, Y)$ denote the function

$$h(\nu, Y) = 2(\nu, Y)^2 - Y^2\nu^2$$

($\nu \in K \otimes \mathbb{R}$ and $iY \in \mathbb{H}_l$). For any real orthogonal transformation $g \in \mathrm{O}^+(K \otimes \mathbb{R})$ we have $h(g\nu, gY) = h(\nu, Y)$. Moreover, if Y is of the form $(y_1, y_2, 0)$, then

$$h(\nu, Y) = 2y_2^2\nu_1^2 + 2y_1^2\nu_2^2 + Y^2|\nu_D^2|. \tag{4.23}$$

Recall that for any isotropic vector $u \in K \otimes \mathbb{R}$ and any $v \in K \otimes \mathbb{R}$ with $(u, v) = 0$ one has the Eichler transformation $E(u, v) \in \mathrm{O}^+(K \otimes \mathbb{R})$, defined by

$$E(u, v)(a) = a - (a, u)v + (a, v)u - q(v)(a, u)u.$$

For a given $Y = (y_1, y_2, Y_D) \in \mathcal{R}_t$ we consider the special Eichler transformations

$$E = E(d, Y_D/y_1),$$
$$\tilde{E} = E(\tilde{d}, Y_D/y_2).$$

It is easily checked that

$$E(Y) = (y_1,\, y_2 + q(Y_D)/y_1,\, 0)\,,$$
$$E(\nu) = \left(\nu_1,\, *,\, \nu_D - \tfrac{\nu_1}{y_1} Y_D \right),$$
$$\tilde{E}(Y) = (y_1 + q(Y_D)/y_2,\, y_2,\, 0)\,,$$
$$\tilde{E}(\nu) = \left(*,\, \nu_2,\, \nu_D - \tfrac{\nu_2}{y_2} Y_D \right).$$

Using 4.23 we obtain the following estimate:

$$\begin{aligned}h(\nu,Y) &= \tfrac{1}{2}h(E\nu,EY)+\tfrac{1}{2}h(\tilde{E}\nu,\tilde{E}Y)\\ &\geq (y_2+q(Y_D)/y_1)^2\nu_1^2+(y_1+q(Y_D)/y_2)^2\nu_2^2\\ &\quad +q(Y)\left|(\nu_D-\tfrac{\nu_1}{y_1}Y_D)^2\right|+q(Y)\left|(\nu_D-\tfrac{\nu_2}{y_2}Y_D)^2\right|.\end{aligned}$$

Let $\varepsilon\in(0,1)$. Then

$$\begin{aligned}&q(Y)\left|(\nu_D-\tfrac{\nu_1}{y_1}Y_D)^2\right|+q(Y)\left|(\nu_D-\tfrac{\nu_2}{y_2}Y_D)^2\right|\\ &\geq -\varepsilon q(Y)\left((\nu_D-\tfrac{\nu_1}{y_1}Y_D)^2+(\nu_D-\tfrac{\nu_2}{y_2}Y_D)^2\right)\\ &= -\varepsilon q(Y)\left(\nu_D^2+\left(\nu_D-\frac{\nu_1y_2+\nu_2y_1}{y_1y_2}Y_D\right)^2-2\frac{\nu_1\nu_2}{y_1y_2}Y_D^2\right)\\ &\geq \varepsilon q(Y)|\nu_D^2|-2\varepsilon|\nu_1\nu_2||Y_D^2|.\end{aligned}$$

Hence we find that

$$\begin{aligned}h(\nu,Y)\geq (y_2+q(Y_D)/y_1)^2\nu_1^2+(y_1+q(Y_D)/y_2)^2\nu_2^2\\ +\varepsilon q(Y)|\nu_D^2|-2\varepsilon|\nu_1\nu_2||Y_D^2|.\end{aligned}\tag{4.24}$$

It is easily seen that $|q(Y_D)|<\frac{t^4}{t^4+1}y_1y_2$ for $Y\in\mathcal{R}_t$. This implies

$$\begin{aligned}y_2+q(Y_D)/y_1\geq y_2/(t^4+1),\\ y_1+q(Y_D)/y_2\geq y_1/(t^4+1).\end{aligned}$$

If we insert this into (4.24), we get

$$\begin{aligned}h(\nu,Y)\geq \frac{1}{2(t^4+1)^2}(y_2^2\nu_1^2+y_1^2\nu_2^2)+\varepsilon q(Y)|\nu_D^2|\\ +\frac{1}{2(t^4+1)^2}\left(y_2^2\nu_1^2-8\varepsilon(t^4+1)^2y_1y_2|\nu_1\nu_2|+y_1^2\nu_2^2\right).\end{aligned}$$

Thus for $\varepsilon<\frac{1}{4(t^4+1)^2}$ we finally obtain

$$h(\nu,Y)\geq\varepsilon(y_2^2\nu_1^2+y_1^2\nu_2^2+q(Y)|\nu_D^2|).$$

□

Corollary 4.14. *There exists an $\varepsilon>0$ with the following property: For any $\nu\in K'$ with $q(\nu)\geq 0$ and any $Y\in\mathcal{R}_t$ the inequality*

$$|(\nu,Y)|\geq\varepsilon(y_2|\nu_1|+y_1|\nu_2|)$$

holds. Here $\mathcal{R}_t$ denotes the set of Y defined by (4.18b)–(4.18d).

Proof. For $q(\nu) \geq 0$ one has $2(\nu, Y)^2 \geq 2(\nu, Y)^2 - \nu^2 Y^2$. Hence, according to Lemma 4.13 there is an $\varepsilon > 0$ such that

$$(\nu, Y)^2 \geq \varepsilon(y_2^2\nu_1^2 + y_1^2\nu_2^2 + q(Y)|\nu_D^2|) \geq \varepsilon(y_2^2\nu_1^2 + y_1^2\nu_2^2)$$

for all $\nu \in K'$ with $q(\nu) \geq 0$ and all $Y \in \mathcal{R}_t$. This implies the assertion. □

Proposition 4.15 (Koecher principle). *Let $f : \mathbb{H}_l \to \mathbb{C}$ be a holomorphic function that satisfies*

i) $f(Z + k) = f(Z)$, for $k \in K$,
ii) $f(\sigma Z) = f(Z)$, for $\sigma \in \Gamma(L) \cap \mathrm{O}(K)$, where $\mathrm{O}(K)$ is considered as a subgroup of $\mathrm{O}(L)$.

Then f has a Fourier expansion of the form

$$f(Z) = \sum_{\substack{\nu=(\nu_1,\nu_2,\nu_D)\in K' \\ \nu_1 \geq 0;\; q(\nu)\geq 0}} a(\nu) e((\nu, Z)).$$

In particular, f is bounded on the Siegel domain $\mathcal{S}_t$.

Proof. The first assertion can be proved in the same way as the Koecher principle for Siegel modular forms (cf. [Fr1] chapter I Hilfssatz 3.5). Note that the assumption $l \geq 3$ is crucial. The group $\Gamma(L) \cap \mathrm{O}(K)$ plays the role of $\mathrm{SL}(n, \mathbb{Z})$ in [Fr1]. Then the second assertion can be deduced using Lemma 4.13 or Corollary 4.14. □

Let $t > 0$ and $g \in \mathrm{O}^+_{\mathbb{Q}}(L)$. Let $f : \mathbb{H}_l \to \mathbb{C}$ be a Γ-invariant measurable function. We now consider the integral

$$I = \int_{g\mathcal{S}_t} |f(Z)|^p \frac{dX\, dY}{q(Y)^l}$$

over the translated Siegel domain $g\mathcal{S}_t$. It is convenient to use the identification $\mathbb{H}_l \to \mathcal{K}^+$, $Z \mapsto [Z + \tilde{z} - q(Z)z]$, and to rewrite I as the integral

$$\int_{g\tilde{\mathcal{S}}_t} |F(W)|^p \, d\omega,$$

where $\tilde{\mathcal{S}}_t = \{W = [Z + \tilde{z} - q(Z)z]; \quad Z \in \mathcal{S}_t\}$, F denotes the function on $\mathcal{K}^+$ corresponding to f, and $d\omega$ denotes the $\mathrm{O}^+(V)$-invariant measure on $\mathcal{K}^+$. We will work with different coordinate systems on $\mathcal{K}^+$.

A 4-tuple $\Lambda = (\lambda_1, \dots, \lambda_4)$ of vectors in $L \otimes \mathbb{Q}$ is called an *admissible index-tuple*, if

i) $q(\lambda_j) = 0$ for $j = 1, \dots, 4$,
ii) $(\lambda_1, \lambda_2) = 1$ and $(\lambda_3, \lambda_4) = 1$,

iii) $(\lambda_i,\lambda_j)=0$ for $i=1,2$ and $j=3,4$,
iv) $\Im((W,\lambda_3)/(W,\lambda_1))>0$ for $W\in\mathcal{K}^+$.

Then $\mathbb{Q}\lambda_1+\mathbb{Q}\lambda_2$ and $\mathbb{Q}\lambda_3+\mathbb{Q}\lambda_4$ are orthogonal hyperbolic planes in $L\otimes\mathbb{Q}$. The intersection of $(\mathbb{Q}\lambda_1+\mathbb{Q}\lambda_2)^\perp$ and L is a Lorentzian sublattice of L, which will be denoted by $K(\Lambda)$. The intersection of $(\mathbb{Q}\lambda_1+\cdots+\mathbb{Q}\lambda_4)^\perp$ and L is a negative definite sublattice, denoted by $D(\Lambda)$. If $X\in L\otimes\mathbb{R}$, then we write $X_{K(\Lambda)}$ resp. $X_{D(\Lambda)}$ for the orthogonal projection of X to $K(\Lambda)\otimes\mathbb{R}$ resp. $D(\Lambda)\otimes\mathbb{R}$.

In the same way as in section 3.2 it can be seen that $\mathcal{K}^+$ is biholomorphic equivalent to

$$\mathbb{H}_l(\Lambda)=\{X+iY\in K(\Lambda)\otimes\mathbb{C};\quad X,Y\in K(\Lambda)\otimes\mathbb{R},\ (Y,\lambda_3)>0,\ q(Y)>0\},$$

the isomorphism being given by

$$\mathbb{H}_l(\Lambda)\to\mathcal{K}^+,\quad Z\mapsto[Z+\lambda_2-q(Z)\lambda_1].$$

Let $t>0$ and $\Lambda=(\lambda_1,\dots,\lambda_4)$ be an admissible index-tuple. We denote by $\mathcal{S}_t(\Lambda)$ the subset of $Z=X+iY\in\mathbb{H}_l(\Lambda)$ satisfying

$$(X,\lambda_3)^2+(X,\lambda_4)^2-q(X_{D(\Lambda)})<t^2, \tag{4.25a}$$
$$1/t<(Y,\lambda_3), \tag{4.25b}$$
$$(Y,\lambda_3)^2<t^2q(Y), \tag{4.25c}$$
$$-q(Y_{D(\Lambda)})<t^2(Y,\lambda_3)^2. \tag{4.25d}$$

We define the *Siegel domain* $\tilde{\mathcal{S}}_t(\Lambda)$ attached to Λ and t by

$$\tilde{\mathcal{S}}_t(\Lambda)=\{[Z+\lambda_2-q(Z)\lambda_1];\quad Z\in\mathcal{S}_t(\Lambda)\}\subset\mathcal{K}^+.$$

Obviously $\Lambda_0=(z,\tilde{z},d,\tilde{d})$ is an admissible index-tuple. We have $\mathbb{H}_l(\Lambda_0)=\mathbb{H}_l$ and $\mathcal{S}_t(\Lambda_0)=\mathcal{S}_t$. It is easily seen that the rational orthogonal group $\mathrm{O}^+_{\mathbb{Q}}(L)$ acts transitively on admissible index-tuples. For $g\in\mathrm{O}^+_{\mathbb{Q}}(V)$ the identity

$$g\tilde{\mathcal{S}}_t(\lambda_1,\dots,\lambda_4)=\tilde{\mathcal{S}}_t(g\lambda_1,\dots,g\lambda_4) \tag{4.26}$$

holds.

In order to prove that $F\in L^p(\mathcal{K}^+/\Gamma)$, it suffices to show that for any $t>0$ and any admissible index-tuple Λ the integral

$$\int\limits_{\tilde{\mathcal{S}}_t(\Lambda)}|F(W)|^p d\omega \tag{4.27}$$

is finite.

Lemma 4.16. *Let $t > 0$ and $\Lambda = (\lambda_1, \dots, \lambda_4)$ be an admissible index-tuple. For $a, b \in \mathbb{Q}_{>0}$ there exist $t_1, t_2 \in \mathbb{R}$ with $0 < t_1 < t < t_2$, such that*

$$\tilde{\mathcal{S}}_{t_1}(\Lambda) \subset \tilde{\mathcal{S}}_t(a\lambda_1, \tfrac{1}{a}\lambda_2, b\lambda_3, \tfrac{1}{b}\lambda_4) \subset \tilde{\mathcal{S}}_{t_2}(\Lambda).$$

Proof. The proof is left to the reader. □

Lemma 4.17. *Let $\lambda_1, \lambda_2 \in L \otimes \mathbb{Q}$ with $q(\lambda_1) = q(\lambda_2) = 0$ and $(\lambda_1, \lambda_2) = 1$. Let $t' > 0$ and Λ' be an admissible index-tuple of the form $(\lambda_1, *, *, *)$. There exist a $t > 0$ and an admissible index-tuple Λ of the form $(\lambda_1, \lambda_2, *, *)$ such that*

$$\tilde{\mathcal{S}}_{t'}(\Lambda') \subset \tilde{\mathcal{S}}_t(\Lambda).$$

Proof. Write $\Lambda' = (\lambda_1, \lambda_2', \lambda_3', \lambda_4')$ and put $u = \lambda_2 - \lambda_2'$. Then $u \in L \otimes \mathbb{Q}$ with $(u, \lambda_1) = 0$. Consider the Eichler transformation $E(\lambda_1, u) : L \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$,

$$E(\lambda_1, u)(a) = a - (a, \lambda_1)u + (a, u)\lambda_1 - q(u)(a, \lambda_1)\lambda_1.$$

It is easily checked that $E(\lambda_1, u) \in \mathrm{O}^+_{\mathbb{Q}}(L)$. Put $\Lambda = E(\lambda_1, u)^{-1}\Lambda' = E(\lambda_1, -u)\Lambda'$. Then Λ is clearly an admissible index-tuple. We have

$$\begin{aligned} E(\lambda_1, u)(\lambda_1) &= \lambda_1, \\ E(\lambda_1, u)(\lambda_2) &= \lambda_2', \\ E(\lambda_1, u)(k) &= k - (k, \lambda_2')\lambda_1, \quad k \in K(\Lambda). \end{aligned}$$

The induced action of $E(\lambda_1, u)$ on $\mathbb{H}_l(\Lambda)$ is simply the translation

$$E(\lambda_1, u)Z = Z + (\lambda_2')_{K(\Lambda)} \qquad (Z \in \mathbb{H}_l(\Lambda)).$$

Hence there is a $t > 0$ such that $E(\lambda_1, u)\mathcal{S}_{t'}(\Lambda) \subset \mathcal{S}_t(\Lambda)$. This implies

$$\tilde{\mathcal{S}}_{t'}(\Lambda') = \tilde{\mathcal{S}}_{t'}(E(\lambda_1, u)\Lambda) = E(\lambda_1, u)\tilde{\mathcal{S}}_{t'}(\Lambda) \subset \tilde{\mathcal{S}}_t(\Lambda).$$

□

Lemma 4.18. *Let $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be an admissible index-tuple. Moreover, let $t' > 0$ and Λ' be an admissible index-tuple of the form $(\lambda_1, \lambda_2, \lambda_3, *)$. Then there exists a $t > 0$ such that*

$$\tilde{\mathcal{S}}_{t'}(\Lambda') \subset \tilde{\mathcal{S}}_t(\Lambda).$$

Proof. This can be proved similarly as Lemma 4.17. □

4.3 Modular forms whose zeros and poles lie on Heegner divisors

Let F be a meromorphic modular form for $\Gamma(L)$ of weight r. Suppose that its divisor is a linear combination of Heegner divisors[3] $H(\beta,m)$. Then there are integral coefficients $c(\beta,m)$ with $c(\beta,m)=c(-\beta,m)$ such that

$$(F)=\frac{1}{2}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\beta)\\ m<0}} c(\beta,m)H(\beta,m).$$

The function $f(Z)=\log(|F(Z)|q(Y)^{r/2})$ is $\Gamma(L)$-invariant and has logarithmic singularities along Heegner divisors. In this section we show that up to an additive constant f is equal to the regularized theta integral

$$-\frac{1}{8}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}+q(\beta)\\ m<0}} c(\beta,m)\Phi_{\beta,m}(Z).$$

In the proof we use a result due to S.-T. Yau on subharmonic functions on a complete Riemann manifold that satisfy certain integrability conditions.

Let M be a *complete* connected Riemann manifold. Write Δ for the Laplace operator on M and $(\cdot,\cdot)$ for the inner product on square integrable differential forms on M.

Proposition 4.19. *i) Let u be a smooth function on M. Suppose that u and Δu are square integrable. Then du is square integrable, too.*

ii) The Laplace operator is symmetric in the following sense: Let u and v be smooth square integrable functions on M. Suppose that Δu and Δv are also square integrable. Then

$$(\Delta u,v)=(u,\Delta v).$$

Proof. In the following proof we use some ideas of [Rh] §34, to simplify the argument in [Ro].

i) Since M is complete, there exists a proper C^∞ function $\varrho: M\to[0,\infty)$ and a constant $C>0$ such that

$$|d\varrho(x)|<C$$

for all $x\in M$. Such a function can be obtained by regularization from the function $x\mapsto d(x,o)$, where $d(x,o)$ denotes the geodesic distance of x from a fixed point $o\in M$ (cf. [Rh] §34, §15). For any $r>0$ the set $B_r=\{x\in M;\;\; \varrho(x)\leq r\}$ is compact and $M=\bigcup_{r>0}B_r$. Let $\mu:[0,\infty)\to[0,1]$ be a C^∞ function equal to 1 on $[0,1]$ and 0 on $[2,\infty)$. Then the function

[3] For a precise definition of the Heegner divisor $H(\beta,m)$ see chapter 5.

$\sigma_r(x) = \mu(\varrho(x)/r)$ has its values in $[0, 1]$. Moreover, it satisfies $\sigma_r(x) = 1$ for $x \in B_r$, $\sigma_r(x) = 0$ for $x \notin B_{2r}$, and

$$|d\sigma_r(x)| < D/r, \tag{4.28}$$

where $D > 0$ is a constant which is independent of r.

Because σ_r has compact support, we have

$$(\Delta u, u\sigma_r^2) = (du, \sigma_r^2 du) + (\sigma_r du, 2ud\sigma_r).$$

Using the inequality $|(f, g)| \leq \frac{1}{2}(f, f) + \frac{1}{2}(g, g)$, which follows from the Schwarz inequality, we find

$$\begin{aligned}
(\sigma_r du, \sigma_r du) &\leq (\Delta u, u\sigma_r^2) + \tfrac{1}{2}(\sigma_r du, \sigma_r du) + 2(ud\sigma_r, ud\sigma_r)\\
(\sigma_r du, \sigma_r du) &\leq 2(\Delta u, u\sigma_r^2) + 4(ud\sigma_r, ud\sigma_r)\\
&\leq 2(\Delta u, u\sigma_r^2) + 4D^2 r^{-2}(u, u).
\end{aligned}$$

In the last line we have used (4.28). As $r \to \infty$ we obtain $(du, du) \leq 2(\Delta u, u)$. So du is square integrable.

ii) In the same way as in (i), since σ_r has compact support, we have

$$(\Delta u, \sigma_r v) = (du, \sigma_r dv) + (du, vd\sigma_r).$$

By (4.28), as $r \to \infty$, the second term on the right hand side tends to zero. This implies $(\Delta u, v) = (du, dv)$. □

Theorem 4.20. *Let u be a harmonic form on M. Suppose that $u \in L^p(M)$ for some $p > 1$. Then u is closed and coclosed.*

Proof. See [Yau] Proposition 1 on p. 663. The case $p = 2$ is also treated in [Rh] Theorem 26. □

A continuous function u on M is called *subharmonic*, if it satisfies the local harmonic maximum principle, i.e. if for any point $a \in M$ and any smooth harmonic function h defined in a neighborhood of a, the function $u - h$ has a local maximum at a only if it is constant in a neighborhood of a (cf. [GW]). If u is smooth, then it is subharmonic if and only if $\Delta u \geq 0$ everywhere on M. The following theorem is also due to S.-T. Yau (see [Yau] Theorem 3 on p. 663 and Appendix (ii)).

Theorem 4.21. *Let u be a non-negative continuous subharmonic function on M. Suppose that $u \in L^p(M)$ $(p > 1)$. Then u is constant.*

Note that Theorem 4.20 for functions immediately follows from Theorem 4.21. If u is a harmonic function, then $|u|$ is subharmonic in the above sense.

Corollary 4.22. *Let M be a complete connected Riemann manifold of finite volume. Let $f \in L^p(M)$ $(p > 1)$ be a smooth function that satisfies $\Delta f = c$, where c is a real constant. Then f is constant.*

Proof. Assume that f is a solution of $\Delta f = c$. Without any restriction we may assume that $c \geq 0$. So f is subharmonic. Let $d > 0$ be a large real number. The function $f+d$ is also subharmonic. Because M has finite volume, $f + d \in L^p(M)$. Consider the function $g = \max(f + d, 0)$. It is continuous, nonnegative, and belongs to $L^p(M)$. Moreover, it is subharmonic in the above sense, since it is the maximum of two subharmonic functions (cf. [GW]). According to Theorem 4.21, g is constant. This implies that f is constant on $\{x \in M; \quad f(x) \geq -d\}$. Since d was arbitrary, we find that f is constant on M.

In the case $p = 2$ we may also argue as follows: By Proposition 4.19 we have

$$c^2 \operatorname{vol}(M) = (c, c) = (\Delta f, c) = (f, \Delta c) = 0.$$

Hence f is a square integrable harmonic function. According to Theorem 4.20 it is constant. □

Theorem 4.23. *Let F be a meromorphic modular form of weight r with respect to $\Gamma(L)$, whose divisor is a linear combination of Heegner divisors*

$$(F) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) H(\beta, m). \tag{4.29}$$

Define $f(Z) = \log(|F(Z)| q(Y)^{r/2})$ (with $Y = \Im(Z)$) and

$$\Phi(Z) = -\frac{1}{8} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) \Phi_{\beta,m}(Z).$$

Then $f - \Phi$ is constant.

Proof. Let us temporarily assume $l \geq 4$. First, we will prove that $f - \Phi \in L^2(\mathcal{K}^+/\Gamma(L))$. It suffices to show that

$$\int_{\tilde{\mathcal{S}}_t(\Lambda)} |f - \Phi|^2 \, d\omega < \infty$$

for any $t > 0$ and any admissible index-tuple Λ. According to Lemma 4.16–4.18 we may assume that $\Lambda = (z, \tilde{z}, d, \tilde{d})$, where $z \in L$ is a primitive norm 0 vector, $z' \in L'$ with $(z', z) = 1$, $\tilde{z} = z' - q(z')z$, and d is a primitive norm 0 vector in $K = L \cap z^\perp \cap z'^\perp$, $d' \in K'$ with $(d', d) = 1$, and $\tilde{d} = d' - q(d')d$. This is precisely the situation we considered in the previous sections. We may view F, f, and Φ as functions on $\mathbb{H}_l = \mathbb{H}_l(\Lambda)$ and use the Fourier expansion of Φ which was determined in section 3.2. We have to prove that

$$\int_{\mathcal{S}_t} |f(Z) - \Phi(Z)|^2 \frac{dX \, dY}{q(Y)^l} < \infty.$$

Define a meromorphic function Ψ on $\mathbb{H}_l$ by

$$\Psi(Z) = \prod_{\beta \in L'/L} \prod_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} \Psi_{\beta,m}(Z)^{c(\beta,m)/2}.$$

By Theorem 3.16 the quotient F/Ψ is a holomorphic function on $\mathbb{H}_l$ without any zeros. Let

$$G = \operatorname{Log}(F/\Psi)$$

be a holomorphic logarithm and $g = \Re(G) = \log|F/\Psi|$. Since F is a modular form for $\Gamma(L)$ (with some multiplier system), the absolute value of F is K-periodic, i.e. $|F(Z+\lambda)| = |F(Z)|$ for any $\lambda \in K$. The Fourier expansions (3.38) of the functions $\psi^L_{\beta,m}(Z)$ imply that $\log|\Psi|$ is also K-periodic. Thus g is K-periodic, too. By the maximum modulus principle we may infer that there is a $\lambda_0 \in K \otimes \mathbb{R}$, such that

$$G(Z+\lambda) = G(Z) + 2\pi i(\lambda_0, \lambda)$$

for any $\lambda \in K$. Hence the function

$$\tilde{G}(Z) = G(Z) - 2\pi i(\lambda_0, Z)$$

is holomorphic on $\mathbb{H}_l$ and K-periodic.

Let $\sigma \in \Gamma(L) \cap \mathrm{O}(K)$. Then σ acts linearly on $\mathbb{H}_l$ and $j(\sigma, Z) = 1$. This implies $|F(\sigma Z)| = |F(Z)|$. According to Theorem 3.16 we have

$$\Re(\tilde{G}(Z)) = \log|F(Z)| + 2\pi(\lambda_0, Y) + \frac{1}{8}\sum_{\beta,m} c(\beta,m)\left(\psi^L_{\beta,m}(Z) - C_{\beta,m}\right).$$

In view of the Fourier expansion (3.38) of the $\psi^L_{\beta,m}$ we find

$$\Re(\tilde{G}(\sigma Z)) = \Re(\tilde{G}(Z)) + \ell(Y),$$

where $\ell(Y)$ is (a priori) a piecewise linear real function of Y. Because $\Re(\tilde{G}(Z))$ and $\Re(\tilde{G}(\sigma Z))$ are real analytic, $\ell(Y)$ is even linear on $\mathbb{H}_l$. Thus there exists a $\varrho_0 \in K \otimes \mathbb{R}$ with

$$\Re(\tilde{G}(\sigma Z)) = \Re(\tilde{G}(Z)) + 2\pi(\varrho_0, Y).$$

Again, by the maximum modulus principle, there is a real constant C such that

$$\tilde{G}(\sigma Z) = \tilde{G}(Z) - 2\pi i(\varrho_0, Z) + iC. \tag{4.30}$$

Since $\tilde{G}(Z)$ is K-periodic, $\tilde{G}(\sigma Z)$ is K-periodic, too. Using (4.30), one easily deduces that $\varrho_0 = 0$. If we insert the Fourier expansion

$$\tilde{G}(Z) = \sum_{\lambda \in K'} a(\lambda) e((\lambda, Z)), \qquad a(\lambda) \in \mathbb{C},$$

of $\tilde{G}$ into (4.30) and compare the constant terms on both sides, we find $C = 0$ and thereby

$$\tilde{G}(\sigma Z) = \tilde{G}(Z).$$

But now, by the Koecher principle (Proposition 4.15), $\tilde{G}$ is bounded on $\mathcal{S}_t$. We obtain

$$|g(Z)| = |\Re(\tilde{G}(Z)) - 2\pi(\lambda_0, Y)| \leq 2\pi|(\lambda_0, Y)| + \text{const.} \tag{4.31}$$

Let us now consider the function

$$\xi(Z) = -\frac{1}{8} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) \xi^L_{\beta,m}(Z).$$

Using the asymptotic behavior (3.26) of the function $\mathcal{V}_\kappa$, the boundedness (2.22) of the coefficients $b(\gamma, n)$ for $n < 0$, and Lemma 4.13, one easily obtains that the sum over $\lambda \in K'$ in the definition of $\xi^L_{\beta,m}(Z)$ (Def. 3.11) is bounded on $\mathcal{S}_t$. By means of the Fourier expansion (3.12) of $\xi^K_{\beta,m}$ it can be deduced that there is a $\nu \in K \otimes \mathbb{R}$ with

$$|Y| \left|\xi^K_{\beta,m}(Y/|Y|)\right| \leq |(\nu, Y)|$$

on $\mathcal{S}_t$. Hence there exists a $\nu_0 \in K \otimes \mathbb{R}$ such that

$$|\xi(Z)| \leq |B \log q(Y)| + |(\nu_0, Y)| + \text{const.} \tag{4.32}$$

on $\mathcal{S}_t$, where

$$B = -\frac{1}{8} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) b_{\beta,m}(0, 0).$$

By virtue of (4.31) and (4.32) we find

$$\begin{aligned} |f(Z) - \Phi(Z)| &= \left|\log|F(Z)/\Psi(Z)| - \xi(Z) + \tfrac{r}{2}\log(q(Y))\right| \\ &\leq |g(Z)| + |\xi(Z)| + |\tfrac{r}{2}\log(q(Y))| \\ &\ll |(\lambda_0, Y)| + |(\nu_0, Y)| + (|B| + \tfrac{|r|}{2})|\log(q(Y))| \end{aligned} \tag{4.33}$$

on $\mathcal{S}_t$. Using Lemma 4.11 and Lemma 4.12, we obtain that $f - \Phi \in L^2(\mathcal{K}^+/\Gamma(L))$.

Let us now consider the action of the invariant Laplace operator Ω on $f - \Phi$. According to Theorem 4.7 we have

$$\Omega\Phi(Z) = \frac{l}{4}B.$$

Moreover, Lemma 4.1 implies that $\Omega f(Z) = -\frac{rl}{8}$. Hence

$$\Omega(f(Z) - \Phi(Z)) = -\frac{rl}{8} - \frac{l}{4}B$$

is constant. Since $\mathbb{H}_l/\Gamma(L)$ is a *complete* Riemann manifold of finite volume, we may apply Corollary 4.22. We find that $f - \Phi$ is constant.

In the case $l = 3$ the same argument shows that $f - \Phi \in L^p(\mathcal{K}^+/\Gamma(L))$ for $p < 2$. Again, by Corollary 4.22 it has to be constant. □

Corollary 4.24. *Let F be a meromorphic modular form of weight r with respect to $\Gamma(L)$. Suppose that its divisor is a linear combination of Heegner divisors as in (4.29). Then*

$$r = -\frac{1}{4} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) q(\beta, -m),$$

where the $q(\gamma, n)$ denote the Fourier coefficients of the Eisenstein series E with constant term $2\mathfrak{e}_0$ in $M_{\kappa,L}$.

Proof. We use the same notation as in the proof of Theorem 4.23. By Theorem 4.23 we know that $f - \Phi$ is constant. Thus

$$\Re(\tilde{G}(Z)) - 2\pi(\lambda_0, Y) - \xi(Z) + \tfrac{r}{2}\log(q(Y)) \tag{4.34}$$

is constant. Comparing coefficients we find that the 0-th Fourier coefficient

$$-2\pi(\lambda_0, Y) + \frac{|Y|}{8\sqrt{2}} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) \xi^K_{\beta,m}(Y/|Y|) + (\tfrac{r}{2} + B)\log(q(Y))$$

of (4.34) is constant.

This implies in particular that the function

$$-2\pi(\lambda_0, Y/|Y|) + \frac{|Y|}{8\sqrt{2}} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) \xi^K_{\beta,m}(Y/|Y|)$$

is invariant under the action of the real orthogonal group $\mathrm{O}^+(K \otimes \mathbb{R})$. Hence it is constant.

We may infer that there are constants $C, C' \in \mathbb{R}$ such that

$$C|Y| + (\tfrac{r}{2} + B)\log(q(Y)) = C'.$$

But this is only possible, if $C = 0$ and $\frac{r}{2} = -B$. Using Proposition 1.16, we obtain the assertion. □

Remark 4.25. Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. If there is no $\lambda \in L'$ with $(\lambda, q(\lambda)) = (\beta, m)$, then $H(\beta, m) = 0$ in the divisor group of $\mathbb{H}_l/\Gamma(L)$. In this case Theorem 4.23 implies that $\Phi_{\beta,m}$ is constant.

5 Chern classes of Heegner divisors

Let X be a normal irreducible complex space. By a divisor D on X we mean a formal linear combination

$$D = \sum n_Y Y \qquad (n_Y \in \mathbb{Z})$$

of irreducible closed analytic subsets Y of codimension 1 such that the support $\operatorname{supp}(D) = \bigcup_{n_Y \neq 0} Y$ is a closed analytic subset of everywhere pure codimension 1. Then for any compact subset $K \subset X$ there are only finitely many Y with $Y \cap K \neq \emptyset$ and $n_Y \neq 0$. We denote the group of divisors on X by $D(X)$.

If Γ is a group of biholomorphic transformations of X acting properly discontinuously, we may consider the inverse image $\pi^*(D)$ of a divisor D on X/Γ under the canonical projection $\pi : X \to X/\Gamma$. For any irreducible component Y of the inverse image of $\operatorname{supp}(D)$ the multiplicity of Y with respect to $\pi^*(D)$ equals the multiplicity of $\pi(Y)$ with respect to D. Then $\pi^*(D)$ is a Γ-invariant divisor on X. (So we do not take account of possibly occurring ramification in the definition of π^*.)

Now let $X = \mathbb{H}_l$ be the generalized upper half plane as in section 3.2, and let Γ be the orthogonal group $\Gamma(L)$ or a subgroup of finite index. If $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$, then

$$\sum_{\substack{\lambda \in \beta + L \\ q(\lambda) = m}} \lambda^\perp$$

is a Γ-invariant divisor on $\mathbb{H}_l$ with support $H(\beta, m)$ (see (2.17)). It is the inverse image under the canonical projection of an algebraic divisor on X/Γ, which we call the *Heegner divisor* of discriminant (β, m). For simplicity we denote it by $H(\beta, m)$, too. The multiplicities of all irreducible components equal 2, if $2\beta = 0$, and 1, if $2\beta \neq 0$. Notice that $H(\beta, m) = H(-\beta, m)$.

We will use the following *modified divisor class group* $\widetilde{\operatorname{Cl}}(X/\Gamma)$: Any meromorphic modular form f with multiplier system χ with respect to Γ defines via its zeros and poles a Γ-invariant divisor in $D(X)$, which is the inverse image of an algebraic divisor (f) in $D(X/\Gamma)$. We denote the subgroup generated by these divisors (f) by $\widetilde{H}(X/\Gamma)$ and put

$$\widetilde{\mathrm{Cl}}(X/\Gamma) = D(X/\Gamma)/\widetilde{H}(X/\Gamma).$$

Moreover, we write $\mathrm{Cl}(X/\Gamma)$ for the quotient of $D(X/\Gamma)$ modulo the subgroup of divisors coming from meromorphic modular forms of weight 0 with trivial character. If Γ acts freely, then $\mathrm{Cl}(X/\Gamma)$ coincides with the usual notion of the divisor class group.

We now give an algebraic interpretation of Theorem 3.22. First we need a rational structure on the space $S_{\kappa,L}$ of cusp forms of weight $\kappa = 1 + l/2$.

Throughout this section let

$$N = \min\{n \in \mathbb{N}; \quad nq(\gamma) \in \mathbb{Z} \text{ for all } \gamma \in L'\}$$

denote the level of the lattice L. The representation ϱ_L is trivial on

$$\Gamma(N) = \{(M,\varphi) \in \mathrm{Mp}_2(\mathbb{Z}); \quad M \equiv 1 \pmod N\}$$

and thereby factors through the finite group

$$\mathrm{Mp}_2(\mathbb{Z}/N\mathbb{Z}) = \mathrm{Mp}_2(\mathbb{Z})/\Gamma(N).$$

If $f = \sum_{\gamma\in L'/L} \mathfrak{e}_\gamma f_\gamma \in S_{\kappa,L}$, then each component f_γ lies in the space $S_\kappa(N)$ of cusp forms of weight κ for the group $\Gamma(N)$. In other words

$$S_{\kappa,L} \subset S_\kappa(N) \otimes \mathbb{C}[L'/L].$$

Let ζ_N be a primitive N-th root of unity. It is easily seen that the coefficients of ϱ_L are contained in $\mathbb{Q}(\zeta_N)$. Indeed, it suffices to check this on the generators T and S. For T the assertion is trivial. For S one uses the evaluation of the Gauss sum

$$\sum_{\gamma\in L'/L} e(q(\gamma)) = \sqrt{|L'/L|}\sqrt{i}^{b^+-b^-}$$

due to Milgram (cf. [Bo2] Corollary 4.2). The following Lemma is a slight improvement of [Bo3] Lemma 4.2.

Lemma 5.1. *If κ is integral, then the space $S_{\kappa,L}$ has a basis of cusp forms whose Fourier coefficients all lie in $\mathbb{Z}[\zeta_N]$. If κ is half-integral, then $S_{\kappa,L}$ has a basis of cusp forms whose coefficients all lie in $\mathbb{Z}[\zeta_{N'}]$, where N' is the least common multiple of N and 8.*

Proof. First, assume that κ is integral. We use the following well known results on integral weight modular forms:

i) The space $S_\kappa(N)$ has a basis of cusp forms with coefficients in $\mathbb{Z}[\zeta_N]$ (cf. [Sh] section 5 or [DI] Corollary 12.3.9).

ii) If $f \in S_\kappa(N)$ with coefficients in $\mathbb{Z}[1/N,\zeta_N]$ and $(M,\varphi) \in \Gamma(N)$, then $f|_\kappa(M,\varphi)$ also lies in $S_\kappa(N)$ and has coefficients in $\mathbb{Z}[1/N,\zeta_N]$. Here

$f|_{\boldsymbol{\kappa}}(M,\varphi)$ denotes the usual Petersson operator of weight κ acting on functions $\mathbb{H} \to \mathbb{C}$. This follows from the q-expansion principle (see [Ka] §1.6 or [DI] section 12.3).

Consider the trace map

$$\mathrm{tr}: S_\kappa(N) \otimes \mathbb{C}[L'/L] \to S_{\kappa,L}, \qquad f \mapsto \sum_{(M,\varphi)\in \mathrm{Mp}_2(\mathbb{Z}/N\mathbb{Z})} f|_\kappa^*(M,\varphi).$$

It is obviously surjective. The image of $f = \sum_{\gamma\in L'/L} \mathfrak{e}_\gamma f_\gamma \in S_\kappa(N)\otimes\mathbb{C}[L'/L]$ can also be written in the form

$$\mathrm{tr}(f) = \sum_{(M,\varphi)\in \mathrm{Mp}_2(\mathbb{Z}/N\mathbb{Z})} \varrho_L^*(M,\varphi)^{-1} \sum_{\gamma\in L'/L} \mathfrak{e}_\gamma f_\gamma|_{\boldsymbol{\kappa}}(M,\varphi).$$

If f has coefficients in $\mathbb{Z}[\zeta_N]$, then by (ii) $\mathrm{tr}(f)$ has coefficients in $\mathbb{Q}(\zeta_N)$. Hence by virtue of (i) we may infer that $S_{\kappa,L}$ has a set of generators with coefficients in $\mathbb{Q}(\zeta_N)$. After multiplying by a common denominator, we obtain a basis with coefficients in $\mathbb{Z}[\zeta_N]$.

Now assume that κ is half integral. We claim that (i) and (ii) also hold in the half integral weight case with N replaced by N'.

To prove the claim we argue similarly as in [SeSt] §5. Let $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}$ be the Jacobi theta series of weight 1/2. It is well known that $\theta \in S_{1/2}(8)$. Moreover, for any $(M,\varphi) \in \mathrm{Mp}_2(\mathbb{Z})$ the Fourier coefficients of $\theta|_{\mathbf{1/2}}(M,\varphi)$ lie in $\mathbb{Z}[\zeta_8]$. The space $S_\kappa(N)$ can be embedded into $S_{2\kappa}(N')$ by multiplying with $\theta^{2\kappa}$. Hence it suffices to show that the image can be defined by linear equations with coefficients in $\mathbb{Q}(\zeta_{N'})$. Then the claim follows from (i), (ii) applied to the space $S_{2\kappa}(N')$.

Since θ has no zeros on $\mathbb{H}$, a cusp form g belongs to the image if and only if it vanishes with prescribed multiplicities at the cusps of $\Gamma(N')$, i.e. if certain coefficients of g in the expansions at the various cusps vanish. By (ii) these conditions are given by linear equations over $\mathbb{Q}(\zeta_{N'})$.

Now we can proceed as in the integral weight case. □

Let $f \in S_{\kappa,L}$ be a cusp form with Fourier coefficients $c(\gamma,n)$ and $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ an automorphism of $\mathbb{C}$. Then we define the σ-conjugate of f by

$$f^\sigma(\tau) = \sum_{\gamma\in L'/L} \sum_{n\in\mathbb{Z}-q(\gamma)} c^\sigma(\gamma,n)\mathfrak{e}_\gamma(n\tau),$$

where $c^\sigma(\gamma,n)$ denotes the conjugate of $c(\gamma,n)$. If κ is integral (resp. half-integral) and $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(\zeta_N))$ (resp. $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(\zeta_{N'}))$), then Lemma 5.1 implies that f^σ is also a cusp form in $S_{\kappa,L}$. For general $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ we only have $f^\sigma \in S_\kappa(N)\otimes\mathbb{C}[L'/L]$. In this context it would be interesting to know, if $S_{\kappa,L}$ always has a basis with rational Fourier coefficients.

Let

$$\mathcal{S}_{\kappa,L} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q})\cdot S_{\kappa,L}$$

be the space of all Galois conjugates of the elements of $S_{\kappa,L}$ and analogously $\mathcal{M}_{\kappa,L} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \cdot M_{\kappa,L}$. If R is a subring of $\mathbb{C}$ then we write $\mathcal{S}_{\kappa,L}(R)$ resp. $S_{\kappa,L}(R)$ for the R-module of cusp forms in $\mathcal{S}_{\kappa,L}$ resp. $S_{\kappa,L}$, whose coefficients all lie in R. Observe that Lemma 5.1 implies

$$\mathcal{S}_{\kappa,L} = \begin{cases} [\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cdot S_{\kappa,L}(\mathbb{Q}(\zeta_N))] \otimes \mathbb{C}, & \text{for } \kappa \in \mathbb{Z}, \\ [\mathrm{Gal}(\mathbb{Q}(\zeta_{N'})/\mathbb{Q}) \cdot S_{\kappa,L}(\mathbb{Q}(\zeta_{N'}))] \otimes \mathbb{C}, & \text{for } \kappa \in \frac{1}{2}\mathbb{Z}. \end{cases}$$

In particular $\mathcal{S}_{\kappa,L}$ is finite dimensional.

By construction $\mathcal{S}_{\kappa,L}$ has a basis of cusp forms whose coefficients are rational integers. Hence $\mathcal{S}_{\kappa,L}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{S}_{\kappa,L}$. We will also consider the dual $\mathbb{Z}$-module $\mathcal{S}^*_{\kappa,L}(\mathbb{Z})$ of $\mathcal{S}_{\kappa,L}(\mathbb{Z})$. Special elements are the functionals

$$a_{\beta,m} : \mathcal{S}_{\kappa,L}(\mathbb{Z}) \to \mathbb{Z}; \qquad f = \sum_{\gamma,n} c(\gamma,n)\mathfrak{e}_\gamma(n\tau) \mapsto a_{\beta,m}(f) = c(\beta,m)$$

($\beta \in L'/L$ and $m \in \mathbb{Z} - q(\beta)$, $m > 0$). We write $\mathcal{A}_{\kappa,L}(\mathbb{Z})$ for the submodule of $\mathcal{S}^*_{\kappa,L}(\mathbb{Z})$ consisting of finite linear combinations $\sum_{\beta,m} c(\beta,m)a_{\beta,m}$, where $c(\beta,m) \in \mathbb{Z}$ and $c(\beta,m) = c(-\beta,m)$. It is easily seen that $\mathcal{A}_{\kappa,L}(\mathbb{Z})$ has finite index in $\mathcal{S}^*_{\kappa,L}(\mathbb{Z})$ and

$$\mathcal{A}_{\kappa,L}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{S}^*_{\kappa,L},$$

where $\mathcal{S}^*_{\kappa,L}$ denotes the $\mathbb{C}$-dual space of $\mathcal{S}_{\kappa,L}$.

Theorem 5.2. *Let* $\mathcal{X}_L = \mathbb{H}_l/\Gamma(L)$. *The assignment* $a_{\beta,m} \mapsto \frac{1}{2}H(\beta,-m)$ *defines a homomorphism*

$$\eta : \mathcal{A}_{\kappa,L}(\mathbb{Z}) \longrightarrow \widetilde{\mathrm{Cl}}(\mathcal{X}_L). \tag{5.1}$$

Proof. We have to show that η is well defined. Suppose that

$$a = \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}-q(\beta) \\ m>0}} c(\beta,m)a_{\beta,m}$$

($c(\beta,m) \in \mathbb{Z}$ and $c(\beta,m) = c(-\beta,m)$) is a finite linear combination which is equal to 0 in $\mathcal{A}_{\kappa,L}(\mathbb{Z})$. Then we may also consider a as an element of the dual space of $S_{\kappa,L}$. According to Theorem 1.17 there exists a nearly holomorphic modular form f of weight $k = 1 - l/2$ with principal part

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}-q(\beta) \\ m>0}} c(\beta,m)\mathfrak{e}_\beta(-m\tau).$$

Now Theorem 3.22 implies that there is a meromorphic modular form of rational weight $c(0,0)/2$ for $\Gamma(L)$ with divisor

$$\frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}-q(\beta) \\ m>0}} c(\beta,m)H(\beta,-m),$$

in other words $\eta(a) = 0$ in $\widetilde{\mathrm{Cl}}(\mathcal{X}_L)$. □

As before, let $X = \mathbb{H}_l$ be the generalized upper half plane and Γ be the orthogonal group $\Gamma(L)$ or a subgroup of finite index. Write $\mathcal{X}_L$ for the quotient $\mathbb{H}_l/\Gamma(L)$.

Let us briefly recall some basic facts on Chern classes and the cohomology of $\mathcal{X}_L$. For any divisor D on X/Γ one has a corresponding sheaf $\mathcal{L}(D)$. The sections of $\mathcal{L}(D)$ over an open subset $U \subset X/\Gamma$ are meromorphic functions f with $(f) \geq -D$ on U.

We now temporarily assume that Γ acts fixed point freely on X. Then X/Γ is an analytic manifold and every divisor D on X/Γ a Cartier divisor, i.e. $\mathcal{L}(D)$ is a line bundle. The *Chern class*

$$c(D) = c(\mathcal{L}(D)) \in H^2(X/\Gamma, \mathbb{C})$$

of $\mathcal{L}(D)$ can be constructed as follows: One chooses a meromorphic function f on X such that (f) equals the inverse image $\pi^*(D)$ of D under the canonical projection π. Then

$$J(\gamma, z) = \frac{f(\gamma(z))}{f(z)} \qquad (\gamma \in \Gamma)$$

is an automorphy factor, i.e. a 1-cocycle of Γ in the ring of holomorphic invertible functions on X. Hence a Hermitean metric on the bundle $\mathcal{L}(D)$ is given by a positive C^∞-function $h : X \to \mathbb{R}$ with

$$h(\gamma z) = |J(\gamma, z)| h(z) \qquad \text{for all } \gamma \in \Gamma.$$

Then the differential form $\omega = \partial\bar{\partial}\log(h)$ is Γ-invariant and closed. It defines (via de Rham isomorphism) a cohomology class in $H^2(X/\Gamma, \mathbb{C})$. This is the Chern class of D in the case that Γ acts fixed point freely on X.

In the general case one chooses a normal subgroup $\Gamma_0 \leq \Gamma$ of finite index and obtains the Chern class $c(D)$ by the isomorphism $H^2(X/\Gamma, \mathbb{C}) \cong H^2(X/\Gamma_0, \mathbb{C})^{\Gamma/\Gamma_0}$.

The construction of the Chern class gives rise to a homomorphism

$$c : \mathrm{Cl}(\mathcal{X}_L) \longrightarrow H^2(\mathcal{X}_L, \mathbb{C})$$

into the second cohomology. Using the results of the previous chapters, we may determine the images of the divisors $H(\beta, m)$ explicitly.

Before stating the theorem let us give an easy example. Let $F : \mathbb{H}_l \to \mathbb{C}$ be a modular form of weight r with respect to $\Gamma(L)$. Then $|F(Z)| q(Y)^{r/2}$ is invariant under $\Gamma(L)$. We may view F as a trivialization of the inverse image $\mathcal{L}(\pi^*((F)))$ of the sheaf attached to $(F) \in D(\mathcal{X}_L)$, and $q(Y)^{-r/2}$ as a Hermitean metric on $\mathcal{L}((F))$. Hence the Chern class of the divisor (F) is given by

$$c((F)) = -\frac{r}{2}\partial\bar{\partial}\log q(Y),$$

a constant multiple of the Kähler class.

Theorem 5.3. *The* $(1,1)$*-form*

$$h_{\beta,m}(Z) = \frac{1}{4}\partial\bar\partial \xi^L_{\beta,m}(Z)$$

is a representative of the Chern class $c(H(\beta,m))$ *of the Heegner divisor* $H(\beta,m)$*. It has the Fourier expansion*

$$\begin{aligned} h_{\beta,m}(Z) &= \partial\bar\partial \frac{|Y|}{4\sqrt{2}}\,\xi^K_{\beta,m}(Y/|Y|) - \frac{b(0,0)}{4}\partial\bar\partial\log(Y^2)\\ &\quad+\frac{1}{2\sqrt{\pi}}\sum_{\substack{\lambda\in K'\\ q(\lambda)<0}}\ \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} b(\delta,q(\lambda))\\ &\quad\times\sum_{n\geq 1}\frac{1}{n}e(n(\delta,z'))\partial\bar\partial\mathcal{V}_{2-k}\left(\pi n|\lambda||Y|,\,\pi n(\lambda,Y)\right)e(n(\lambda,X)). \end{aligned}$$

Here $b(\gamma,n)=b(\gamma,n,1-k/2)$ *denote the Fourier coefficients of the Poincaré series* $F^L_{\beta,m}(\tau,1-k/2)$ *as in Proposition 1.10.*

Proof. By Theorem 3.16 the function $\Psi_{\beta,m}$ is meromorphic on $\mathbb{H}_l$. Its divisor equals $\pi^*(H(\beta,m))$. Since $\xi^L_{\beta,m}$ is real valued and real analytic, the function $e^{\xi^L_{\beta,m}/4}$ is real analytic and positive. Moreover, by construction

$$|\Psi_{\beta,m}(Z)|e^{-\xi^L_{\beta,m}(Z)/4} = e^{C_{\beta,m}/4}e^{-\Phi^L_{\beta,m}(Z)/4}$$

is invariant under $\Gamma(L)$. Thus $e^{\xi^L_{\beta,m}/4}$ defines a Hermitean metric on the sheaf $\mathcal{L}(H(\beta,m))$. This implies the assertion. □

Remark 5.4. The above theorem, together with Theorem 2.12 on the singularities of $\Phi_{\beta,m}(Z)$, and Theorem 3.9 on its Fourier expansion, imply that $\Phi_{\beta,m}(Z)$ is something as a Green current for the divisor $H(\beta,m)$ in the sense of Arakelov geometry (see [SABK]). More precisely this function should define a Green object with log-log-growth in the extended arithmetic intersection theory due to Burgos, Kramer, and Kühn [BKK]. It will be interesting to investigate this connection in future.

5.1 A lifting into the cohomology

Denote by $\tilde H^2(\mathcal{X}_L,\mathbb{C})$ the quotient

$$\tilde H^2(\mathcal{X}_L,\mathbb{C}) = H^2(\mathcal{X}_L,\mathbb{C})\big/\mathbb{C}\partial\bar\partial\log q(Y)$$

of the second cohomology $H^2(\mathcal{X}_L,\mathbb{C})$ and the span of the Kähler class $\partial\bar\partial\log q(Y)$. The Chern class map $\operatorname{Cl}(\mathcal{X}_L)\to H^2(\mathcal{X}_L,\mathbb{C})$ induces a homomorphism

$$c : \widetilde{\mathrm{Cl}}(\mathcal{X}_L) \longrightarrow \tilde{H}^2(\mathcal{X}_L, \mathbb{C}).$$

If we combine c with the map η constructed in Theorem 5.2, we obtain a homomorphism

$$\mathcal{A}_{\kappa,L}(\mathbb{Z}) \longrightarrow \tilde{H}^2(\mathcal{X}_L, \mathbb{C}).$$

After tensoring with $\mathbb{C}$ one gets a linear map

$$\mathcal{S}^*_{\kappa,L} \longrightarrow \tilde{H}^2(\mathcal{X}_L, \mathbb{C}), \tag{5.2}$$

which is characterized by $a_{\beta,m} \mapsto \frac{1}{2} c(H(\beta, -m))$. The image is the subspace spanned by the classes of Heegner divisors. Define a function $\Omega(\tau)$ by

$$\Omega(\tau) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z} - q(\beta) \\ m > 0}} c(H(\beta, -m)) \mathfrak{e}_\beta(m\tau). \tag{5.3}$$

Then $\Omega(\tau)$ is a priori a formal power series with coefficients in the subspace of $\tilde{H}^2(\mathcal{X}_L, \mathbb{C})$ spanned by the Heegner divisors. It is a consequence of the existence of the Borcherds lift (Theorem 3.22), Serre duality (as in Theorem 1.17), and Lemma 5.1 that $\Omega(\tau)$ is in fact a modular form in $\mathcal{M}_{\kappa,L}$ with values in $\tilde{H}^2(\mathcal{X}_L, \mathbb{C})$. This argument is due to Borcherds (see [Bo3]). Hence $\Omega(\tau)$ can be viewed as a kernel function for the map (5.2). The image of a functional $a \in \mathcal{S}^*_{\kappa,L}$ is given by

$$a(\Omega(\tau)) \in \tilde{H}^2(\mathcal{X}_L, \mathbb{C}).$$

The purpose of this section is to refine the map (5.2) and to describe it more precisely. First we will replace the image space $\tilde{H}^2(\mathcal{X}_L, \mathbb{C})$ by a certain space of harmonic $(1,1)$-forms.

Let $\widetilde{\mathrm{Cl}}_{\mathrm{H}}(\mathcal{X}_L)$ denote the subgroup of $\widetilde{\mathrm{Cl}}(\mathcal{X}_L)$ generated by the classes of the Heegner divisors $H(\beta, m)$. Write

$$\mathcal{H}^{1,1}(\mathcal{X}_L) \tag{5.4}$$

for the vector space of square integrable harmonic $(1,1)$-forms on $\mathcal{X}_L$. According to [Bl3] (Theorem 6.2 and §7) $\mathcal{H}^{1,1}(\mathcal{X}_L)$ is a finite dimensional space of automorphic forms in the sense of [Ha]. In particular its elements are $\mathcal{Z}$-finite, where $\mathcal{Z}$ denotes the center of the universal enveloping algebra of the Lie algebra of $\mathrm{O}(V)$. By Theorem 4.20 any square integrable harmonic form on a complete Riemann manifold is closed and thereby defines a cohomology class via de Rham isomorphism.

Recall the basic facts on the Riemann geometry of $\mathbb{H}_l$ summarized in section 4.1. We saw that the $l \times l$ matrix

$$h(Z) = h(Y) = -\frac{1}{4} \left(\frac{\partial^2}{\partial y_i \partial y_j} \log(q(Y)) \right)_{i,j}$$

defines the (up to a constant multiple unique) $\mathrm{O}^+(V)$-invariant Hermitean metric on $\mathbb{H}_l$ and thereby a Hermitean metric on $\mathcal{X}_L$. Let $h^{-1}(Y) = \big(h^{ij}(Y)\big)_{i,j}$ be the inverse of $h(Y)$. A computation shows that

$$h^{-1}(Y) = 4\begin{pmatrix} y_1y_1 & \cdots\cdots & y_1y_l \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ y_ly_1 & \cdots\cdots & y_ly_l \end{pmatrix} + 2\begin{pmatrix} 0 & -2q(Y) & & & \\ -2q(Y) & 0 & & & \\ & & q(Y) & & \\ & & & \ddots & \\ & & & & q(Y) \end{pmatrix}.$$

The natural scalar product for $(1,1)$-forms on $\mathcal{X}_L$ is given by

$$(f(Z)\,dz_i \wedge d\bar{z}_j,\ g(Z)\,dz_m \wedge d\bar{z}_n) = 4\int_{\mathcal{X}_L} f(Z)\overline{g(Z)}h^{im}(Y)h^{jn}(Y)\frac{dX\,dY}{q(Y)^l}.$$

On the Siegel domain $\mathcal{S}_t$ in $\mathbb{H}_l$ (see Definition 4.9) the components of $h^{-1}(Y)$ satisfy the estimate

$$h^{ij}(Y) \ll q(Y), \quad \text{if } (i,j) \neq (2,2).$$

The Kähler form $\partial\bar{\partial}\log q(Y)$ is harmonic by (4.3). We leave it to the reader to show that it is square integrable. We put

$$\widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L) = \mathcal{H}^{1,1}(\mathcal{X}_L)/\mathbb{C}\partial\bar{\partial}\log q(Y). \tag{5.5}$$

Notice that the natural map $\widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L) \to \tilde{H}^2(\mathcal{X}_L, \mathbb{C})$ is in general not injective.

The following strengthening of Theorem 5.3 is crucial for our argument.

Theorem 5.5. *i) The $(1,1)$-form $h_{\beta,m}(Z) = \frac{1}{4}\partial\bar{\partial}\xi^L_{\beta,m}(Z)$ is a harmonic square integrable representative of the Chern class of $H(\beta,m)$.*

ii) The assignment

$$H(\beta,m) \mapsto h_{\beta,m}(Z)$$

defines a homomorphism $\widetilde{\mathrm{Cl}}_{\mathrm{H}}(\mathcal{X}_L) \to \widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$ such that the diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Cl}}_{\mathrm{H}}(\mathcal{X}_L) & \longrightarrow & \widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L) \\ & \searrow^{c} & \downarrow \\ & & \tilde{H}^2(\mathcal{X}_L, \mathbb{C}) \end{array}$$

commutes.

Proof. i) Since $\partial\bar\partial\xi^L_{\beta,m}(Z) = \partial\bar\partial\Phi^L_{\beta,m}(Z)$, the $(1,1)$-form $\partial\bar\partial\xi^L_{\beta,m}(Z)$ is invariant under the action of $\Gamma(L)$. By Theorem 4.7 we know that $\Omega\xi^L_{\beta,m}$ is constant. Because $\mathcal{X}_L$ is Kählerian, the Laplace operator commutes with ∂ and $\bar\partial$. Thus $\partial\bar\partial\xi^L_{\beta,m}$ is harmonic.

We now prove that $\partial\bar\partial\xi^L_{\beta,m}$ is square integrable. In the same way as in the proof of Theorem 4.23 it suffices to show that $\partial\bar\partial\xi^L_{\beta,m}$ is square integrable on any Siegel domain $\mathcal{S}_t(\Lambda)$ (where Λ is an admissible index tuple and $t>0$). By Lemmas 4.16–4.18 we may assume that $\Lambda = (z,\tilde z, d, \tilde d)$, where $z\in L$ is a primitive norm 0 vector, $z'\in L'$ with $(z',z)=1$, $\tilde z = z' - q(z')z$, and d is a primitive norm 0 vector in $K = L\cap z^\perp \cap z'^\perp$, $d'\in K'$ with $(d',d)=1$, and $\tilde d = d' - q(d')d$. We put $D = K\cap d^\perp\cap d'^\perp$ and use the coordinates on $\mathbb{H}_l = \mathbb{H}_l(\Lambda)$ introduced in section 4.1 on page 93.

We consider the Fourier expansion of $\partial\bar\partial\xi^L_{\beta,m}$ (see Theorem 5.3). The constant term involves the function $\xi^K_{\beta,m}(Y/|Y|)$ (see Definition 3.3). Here the contribution $\Phi^D_{\beta,m}$ can be evaluated by Theorem 2.14. If we put everything together we find that $\partial\bar\partial\xi^L_{\beta,m}(Z)$ is equal to

$$\begin{aligned}
&\frac{2}{\sqrt{\pi}} \sum_{\substack{\lambda\in K'\\ q(\lambda)<0}} \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} b(\delta,q(\lambda)) \sum_{n\geq 1} \frac{1}{n} e(n(\delta,z'))\\
&\qquad\qquad \times \partial\bar\partial\mathcal{V}_\kappa\left(\pi n|\lambda||Y|,\, \pi n(\lambda,Y)\right) e(n(\lambda,X))\\
&- b(0,0)\partial\bar\partial\log(Y^2) + 4 \sum_{\substack{\lambda\in D'\\ \lambda+D=p^2(\beta)\\ q(\lambda)=m}} \pi\partial\bar\partial\frac{1}{y_1}\left[(\lambda,Y_D)^2 - \frac{\lambda^2Y_D^2}{l-2}\right]\\
&+ \frac{4}{\pi} \sum_{\lambda\in D'-0} \sum_{\substack{\delta\in K_0'/K\\ p(\delta)=\lambda+D}} b(\delta,q(\lambda)) \sum_{n\geq 1}\frac{1}{n^2}\partial\bar\partial y_1\left(\frac{\pi n|\lambda||Y|}{y_1}\right)^{\kappa-1}\\
&\qquad\qquad \times K_{\kappa-1}\left(\frac{2\pi n|\lambda||Y|}{y_1}\right) e\left(n\frac{(\lambda,Y_D)}{y_1} + n(\delta,d')\right).
\end{aligned} \tag{5.6}$$

(Since the computation is similar to the argument in the proof of Theorem 5.8 we omit the details.)

According to (2.22) the coefficients $b(\delta,n)$ satisfy

$$b(\delta,n) = O(1), \qquad n\to-\infty. \tag{5.7}$$

Moreover, we will use the following estimates. Let $\varepsilon>0$ and $a,b\in\mathbb{N}_0$. Then

$$\left(\frac{\partial}{\partial A}\right)^a \left(\frac{\partial}{\partial B}\right)^b \mathcal{V}_\kappa(A,B) \ll_{\varepsilon,a,b} e^{-2(1-\varepsilon)\sqrt{A^2+B^2}} \tag{5.8}$$

uniformly on $\sqrt{A^2+B^2}>\varepsilon$ (compare (3.26)), and

$$\left(\frac{\partial}{\partial y}\right)^a K_{\kappa-1}(y) \ll_{\varepsilon,a} e^{-y} \tag{5.9}$$

uniformly on $y > \varepsilon$ (see [AbSt] chapter 9).

By means of (5.7), (5.8), and Lemma 4.13 one can show that the first sum over $\lambda \in K'$ in (5.6) is rapidly decreasing on $\mathcal{S}_t$. Hence it is square integrable over $\mathcal{S}_t$. The term $b(0,0)\partial\bar{\partial}\log(Y^2)$ is a multiple of the Kähler form. It is also square integrable.

We now prove that the third term in (5.6) is square integrable over $\mathcal{S}_t$. For $l = 3$ it vanishes identically. Hence we may temporarily assume that $l \geq 4$. It suffices to show that

$$\frac{\partial^2}{\partial y_i \partial y_j} \frac{1}{y_1} \left[(\lambda, Y_D)^2 - \frac{\lambda^2 Y_D^2}{l-2} \right] dz_i \wedge d\bar{z}_j$$

is square integrable for all $i, j = 1, \dots, l$. On $\mathcal{S}_t$ the estimate

$$\frac{\partial^2}{\partial y_i \partial y_j} \frac{1}{y_1} \left[(\lambda, Y_D)^2 - \frac{\lambda^2 Y_D^2}{l-2} \right] \ll \begin{cases} 1/y_1, & \text{if } i \neq 2 \text{ and } j \neq 2, \\ 0, & \text{if } i = 2 \text{ or } j = 2, \end{cases}$$

holds. Thus it suffices to show that the integral

$$I = \int_{\mathcal{S}_t} y_1^{-2} h^{ii}(Y) h^{jj}(Y) \frac{dX\, dY}{q(Y)^l}$$

is finite for all $i \neq 2$ and $j \neq 2$. We have

$$I \ll \int_{\mathcal{S}_t} y_1^{-2} q(Y)^2 \frac{dX\, dY}{q(Y)^l} \ll \int_{\mathcal{S}_t} y_2^2 \frac{dX\, dY}{q(Y)^l}.$$

The latter integral is finite by Lemma 4.12.

Let us now consider the last term in (5.6). We prove that for any $i, j = 1, \dots, l$ the $(1,1)$-form

$$\begin{aligned} f_{ij} dz_i \wedge d\bar{z}_j = &\sum_{\lambda \in D' - 0} \sum_{\substack{\delta \in K_0'/K \\ p(\delta) = \lambda + D}} b(\delta, q(\lambda)) \sum_{n \geq 1} \frac{1}{n^2} \frac{\partial^2}{\partial y_i \partial y_j} y_1 \left(\frac{\pi n |\lambda||Y|}{y_1} \right)^{\kappa-1} \\ &\times K_{\kappa-1}\left(\frac{2\pi n |\lambda||Y|}{y_1} \right) e\left(n \frac{(\lambda, Y_D)}{y_1} + n(\delta, d') \right) dz_i \wedge d\bar{z}_j \end{aligned} \tag{5.10}$$

is square integrable over $\mathcal{S}_t$. There is a $\delta > 0$ such that

$$|\lambda||Y|/y_1 > \delta \tag{5.11}$$

for all $Z \in \mathcal{S}_t$ and all $\lambda \in D' - 0$. Let $b \in \mathbb{Z}$. It is a consequence of (5.9) and (5.11) that there exists a polynomial $P(Y, \lambda)$ of degree r in Y and degree r' in λ such that

$$\left| \frac{\partial^2}{\partial y_i \partial y_j} y_1 \left(\frac{|\lambda||Y|}{y_1} \right)^{\kappa-1} K_{\kappa-1}\left(\frac{2\pi|\lambda||Y|}{y_1} \right) e((\lambda, Y_D)/y_1) \right| \ll \frac{|P(Y,\lambda)|}{y_1^{r+1}} \left(\frac{y_1^2}{|\lambda^2|Y^2} \right)^b e^{-\pi|\lambda||Y|/y_1}$$

for all $Z \in \mathcal{S}_t$ and all $\lambda \in D' - 0$. Using (4.22) we find that the right hand side satisfies the estimate

$$\begin{aligned} \frac{|P(Y,\lambda)|}{y_1^{r+1}} \left(\frac{y_1^2}{|\lambda^2|Y^2} \right)^b e^{-\pi|\lambda||Y|/y_1} & \\ &\ll \frac{1}{y_1} \frac{|P(Y,\lambda)|}{y_2^r |\lambda|^{r'}} \left(\frac{y_1}{y_2} \right)^{b-r} \left(\frac{1}{|\lambda^2|} \right)^{b-r'/2} e^{-\pi|\lambda||Y|/y_1} \\ &\ll \frac{1}{y_1} \left(\frac{y_1}{y_2} \right)^{b-r} \left(\frac{1}{|\lambda^2|} \right)^{b-r'/2} e^{-\pi|\lambda||Y|/y_1}. \end{aligned}$$

Hence the series in (5.10) has (up to a multiplicative constant) the majorant

$$\sum_{\lambda \in D'-0} \sum_{n \geq 1} \frac{1}{n^2} \frac{1}{y_1} \left(\frac{y_1}{y_2} \right)^{b-r} \left(\frac{1}{n^2|\lambda^2|} \right)^{b-r'/2} e^{-\pi n|\lambda||Y|/y_1}$$

on $\mathcal{S}_t$. If we choose b large enough a priori, then this is bounded by y_1/y_2^2 on $\mathcal{S}_t$. Thus we find

$$\begin{aligned} \int_{\mathcal{S}_t} |f_{ij}(Z)|^2 h^{ii}(Y) h^{jj}(Y) \frac{dX\,dY}{q(Y)^l} &\ll \int_{\mathcal{S}_t} \frac{y_1^2}{y_2^4} h^{ii}(Y) h^{jj}(Y) \frac{dX\,dY}{q(Y)^l} \\ &\ll \int_{\mathcal{S}_t} q(Y) \frac{dX\,dY}{q(Y)^l}. \end{aligned}$$

The latter integral is finite for $l \geq 3$ by Lemma 4.11. This concludes the proof of the first assertion.

ii) In order to prove that the above homomorphism is well defined it suffices to show: For any meromorphic modular form F whose divisor

$$(F) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) H(\beta, m)$$

($c(\beta, m) \in \mathbb{Z}$ with $c(\beta, m) = c(-\beta, m)$) is a linear combination of Heegner divisors, the corresponding linear combination

$$\sum_{\beta, m} c(\beta, m) \partial\bar{\partial} \xi^L_{\beta,m}(Z)$$

equals zero in $\widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$.

By Theorem 4.23 we have

$$\log\left(|F(Z)|q(Y)^{r/2}\right) = -\frac{1}{8}\sum_{\beta,m} c(\beta,m)\Phi^L_{\beta,m}(Z),$$

where r denotes the weight of F. This implies that

$$\begin{aligned}\sum_{\beta,m} c(\beta,m)\partial\bar\partial\xi^L_{\beta,m}(Z) &= \sum_{\beta,m} c(\beta,m)\partial\bar\partial\Phi^L_{\beta,m}(Z)\\ &= -8\partial\bar\partial\log\left(|F(Z)|q(Y)^{r/2}\right)\\ &= -4r\partial\bar\partial\log\left(q(Y)\right)\end{aligned}$$

is a multiple of the Kähler class. □

We now consider the generating series

$$\Omega^L(Z,\tau) = \frac{1}{4}\partial\bar\partial\log(Y^2)\mathfrak{e}_0 + \frac{1}{2}\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}-q(\beta)\\ m>0}} h_{\beta,-m}(Z)\mathfrak{e}_\beta(m\tau) \qquad (5.12)$$

of the special representatives $h_{\beta,-m}(Z)$ of the Chern classes of the Heegner divisors. (The presence of the constant term will become clear in the proof of Theorem 5.8.) By a similar argument as in the proof of Theorem 5.5 it can be shown that $\Omega^L(Z,\tau)$ converges normally and is square integrable in Z. Hence we may consider it as an element of $\mathcal{H}^{1,1}(\mathcal{X}_L)\otimes\mathbb{C}[L'/L]$ for fixed τ.

In the rest of this section we shall show that for fixed Z the components of $\Omega^L(Z,\tau)$ are contained in the space $M_{\kappa,L}$. Thus $\Omega^L(Z,\tau)$ can be viewed as the kernel function of a lifting

$$\vartheta: S_{\kappa,L}\longrightarrow \mathcal{H}^{1,1}(\mathcal{X}_L);\qquad f\mapsto \big(f(\tau),\Omega^L(Z,\tau)\big)_\tau$$

from elliptic cusp forms to $\mathcal{H}^{1,1}(\mathcal{X}_L)$. The idea of the proof is the same as in [Za]: We write $\Omega^L(Z,\tau)$ as an explicit linear combination of Poincaré and Eisenstein series in $M_{\kappa,L}$. This description is crucial for our further argument. It can be used to evaluate the scalar product of $\Omega^L(Z,\tau)$ and a cusp form $f\in S_{\kappa,L}$ explicitly. Thereby one obtains the Fourier expansion of $\vartheta(f)$ in terms of the Fourier coefficients of f. The map ϑ is one possible generalization of the Doi-Naganuma map [DN, Na, Za]. In the $\mathrm{O}(2,3)$-case of Siegel modular forms of genus 2 such a generalization was given by Piateskii-Shapiro using representation theoretic methods ([PS1, PS2], see also [Wei]).

For technical reasons we first introduce two linear operators between the spaces $S_{\kappa,L}$ and $S_{\kappa,K}$ where K is a sublattice of L as in section 2.1.

Lemma 5.6. *Let* $F = \sum_{\gamma\in K'/K} \mathfrak{e}_\gamma f_\gamma$ *be a modular form in* $M_{\kappa,K}$. *Define a function* $F \mid\uparrow_K^L : \mathbb{H} \to \mathbb{C}[L'/L]$ *by*

$$F \mid\uparrow_K^L = \sum_{\beta\in L_0'/L} \mathfrak{e}_\beta f_{p(\beta)}.$$

Then $F \mid\uparrow_K^L \in M_{\kappa,L}$.

Proof. Denote the components of $F \mid\uparrow_K^L$ by g_β ($\beta \in L'/L$). It suffices to check the transformation behavior for the generators T, S of $\mathrm{Mp}_2(\mathbb{Z})$. For T this is immediately verified. Regarding S we have to show that

$$g_\beta(-1/\tau) = \frac{\sqrt{\tau}^{2\kappa}\sqrt{i}^{l-2}}{\sqrt{|L'/L|}} \sum_{\delta\in L'/L} e((\beta,\delta))g_\delta(\tau).$$

The right hand side obviously equals

$$\frac{\sqrt{\tau}^{2\kappa}\sqrt{i}^{l-2}}{\sqrt{|L'/L|}} \sum_{\delta\in L_0'/L} e((\beta,\delta))f_{p(\delta)}(\tau). \tag{5.13}$$

A set of representatives for L_0'/L is given by $\lambda = \gamma - (\gamma,\zeta)z/N + bz/N$, where γ runs through a set of representatives for K'/K and b runs modulo N. We find

$$(5.13) = \frac{\sqrt{\tau}^{2\kappa}\sqrt{i}^{l-2}}{\sqrt{|L'/L|}} \sum_{\gamma\in K'/K} e\left((\beta, \gamma - (\gamma,\zeta)z/N)\right) f_\gamma(\tau) \sum_{b\ (N)} e((\beta,z)b/N).$$

The latter sum is 0 if $\beta \notin L_0'/L$, and N if $\beta \in L_0'/L$. In the second case we obtain

$$\begin{aligned}(5.13) &= \frac{\sqrt{\tau}^{2\kappa}\sqrt{i}^{l-2}}{\sqrt{|K'/K|}} \sum_{\gamma\in K'/K} e\left((p(\beta),\gamma)\right) f_\gamma(\tau)\\ &= f_{p(\beta)}(-1/\tau) = g_\beta(-1/\tau).\end{aligned}$$

□

Lemma 5.7. *Let* $F = \sum_{\beta\in L'/L} \mathfrak{e}_\beta f_\beta$ *be a modular form in* $M_{\kappa,L}$. *Define a function* $F \mid\downarrow_K^L : \mathbb{H} \to \mathbb{C}[K'/K]$ *by*

$$F \mid\downarrow_K^L = \sum_{\gamma\in K'/K} \mathfrak{e}_\gamma \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\gamma}} f_\delta.$$

Then $F \mid\downarrow_K^L \in M_{\kappa,K}$.

Proof. This can be proved in the same way as Theorem 2.6. □

It is easily seen that the operators $\uparrow_K^L$ and $\downarrow_K^L$ are adjoint with respect to the Petersson scalar product. Moreover, they take cusp forms to cusp forms.

For the rest of this section we fix a decomposition of the lattice L. Let $z \in L$ be a primitive norm 0 vector and define z', N, L_0', p as in section 2.1. Then the lattice $K = L \cap z'^{\perp} \cap z^{\perp}$ has signature $(1, l-1)$ and $L \otimes \mathbb{R} = (K \otimes \mathbb{R}) \oplus \mathbb{R}z' \oplus \mathbb{R}z$. We assume that there also exists a primitive norm 0 vector $d \in K$ and define d', M, K_0', p analogous to z', N, L_0', p. Then the lattice $D = K \cap d'^{\perp} \cap d^{\perp}$ is negative definite and $K \otimes \mathbb{R} = (D \otimes \mathbb{R}) \oplus \mathbb{R}d' \oplus \mathbb{R}d$. If $\lambda \in L \otimes \mathbb{R}$ then we write λ_K resp. λ_D for the orthogonal projection of λ to $K \otimes \mathbb{R}$ resp. $D \otimes \mathbb{R}$. Recall that $k = 1 - l/2$ and $\kappa = 1 + l/2$.

Theorem 5.8. *The function $\Omega^L(Z,\tau)$ defined in (5.12) can be written in the form*

$$\begin{aligned}\Omega^L(Z,\tau) &= \Omega^K(Y,\tau) \mid\uparrow_K^L + \frac{1}{8}\partial\bar\partial \log(Y^2) E^L(\tau) \\ &\quad - \frac{1}{4\sqrt{\pi}\Gamma(\kappa-1)} \sum_{\substack{\lambda \in K' \\ q(\lambda)<0}} \sum_{\substack{\delta \in L_0'/L \\ p(\delta)=\lambda+K}} \sum_{n\geq 1} \frac{1}{n} e(n(\delta, z')) \\ &\quad \times \partial\bar\partial \mathcal{V}_\kappa\left(\pi n|\lambda||Y|, \pi n(\lambda, Y)\right) e(n(\lambda, X)) P^L_{\delta,-q(\lambda)}(\tau), \end{aligned} \tag{5.14}$$

where

$$\Omega^K(Y,\tau) = \frac{1}{8\sqrt{2}} \sum_{\beta \in K'/K} \sum_{\substack{m \in \mathbb{Z}-q(\beta) \\ m>0}} \partial\bar\partial |Y| \xi^K_{\beta,-m}(Y/|Y|) \mathfrak{e}_\beta(m\tau). \tag{5.15}$$

The function $\Omega^K(Y,\tau)$ is equal to

$$\begin{aligned}&\pi\partial\bar\partial(d,Y)^{-1}\Theta^D(Y_D,\tau) \mid\uparrow_D^K - \frac{1}{2\pi\Gamma(\kappa-1)}\partial\bar\partial(d,Y) \sum_{\lambda\in D'-0} \sum_{\substack{\delta\in K_0'/K \\ p(\delta)=\lambda+D}} \sum_{n\geq1}\frac{1}{n^2} \\ &\times \left(\frac{\pi n|\lambda||Y|}{(d,Y)}\right)^{\kappa-1} K_{\kappa-1}\left(\frac{2\pi n|\lambda||Y|}{(d,Y)}\right) e\left(n\frac{(\lambda,Y)}{(d,Y)} + n(\delta,d')\right) P^K_{\delta,-q(\lambda)}(\tau).\end{aligned} \tag{5.16}$$

Here $\Theta^D(Y_D,\tau) \in S_{\kappa,D}$ denotes the theta series

$$\Theta^D(Y_D,\tau) = \frac{1}{2}\sum_{\beta\in D'/D}\sum_{\lambda\in\beta+D}\left[(\lambda,Y_D)^2 - \frac{\lambda^2 Y_D^2}{l-2}\right]\mathfrak{e}_\beta(-q(\lambda)\tau)$$

attached to the negative definite lattice D and the harmonic polynomial $(\lambda, Y_D)^2 - \frac{\lambda^2 Y_D^2}{l-2}$. Furthermore, $P^K_{\delta,n}$ resp. $P^L_{\delta,n}$ denote the Poincaré series *in $S_{\kappa,K}$ resp. $S_{\kappa,L}$, and E^L the Eisenstein series with constant term $2\mathfrak{e}_0$ in $M_{\kappa,L}$ (cf. section 1.2).*

In particular, for fixed Z the components of $\Omega^K(Y,\tau)$ lie in $S_{\kappa,K}$. Similarly $\Omega^L(Z,\tau)$ can be viewed as an element of $\mathcal{H}^{1,1}(\mathcal{X}_L)\otimes M_{\kappa,L}$.

Proof. Throughout the proof we write $b^L_{\beta,m}(\gamma,n)$ instead of $b(\gamma,n,1-k/2)$ for the (γ,n)-th Fourier coefficient of the Poincaré series $F^L_{\beta,m}(\tau,1-k/2)$ to emphasize the dependence on L and (β,m).

To prove (5.14) we insert the Fourier expansions of the differential forms $h_{\beta,-m}(Z)$ into (5.12) and exchange the order of summation. We find

$$\begin{aligned}\Omega^L(Z,\tau) &= \frac{1}{8\sqrt{2}}\sum_{\beta\in L_0'/L}\sum_{\substack{m\in\mathbb{Z}-q(\beta)\\ m>0}}\partial\bar\partial|Y|\xi^K_{p(\beta),-m}(Y/|Y|)\mathfrak{e}_\beta(m\tau)\\ &\quad+\frac{1}{4}\partial\bar\partial\log(Y^2)\mathfrak{e}_0-\frac{1}{8}\partial\bar\partial\log(Y^2)\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}-q(\beta)\\ m>0}}b^L_{\beta,-m}(0,0)\mathfrak{e}_\beta(m\tau)\\ &\quad+\frac{1}{4\sqrt{\pi}}\sum_{\substack{\lambda\in K'\\ q(\lambda)<0}}\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}}\sum_{n\geq1}\frac{1}{n}e(n(\delta,z'))\partial\bar\partial\mathcal{V}_\kappa\left(\pi n|\lambda||Y|,\pi n(\lambda,Y)\right)\\ &\quad\times e(n(\lambda,X))\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}-q(\beta)\\ m>0}}b^L_{\beta,-m}(\delta,q(\lambda))\mathfrak{e}_\beta(m\tau). \end{aligned}\tag{5.17}$$

According to Proposition 1.16 one has

$$2\mathfrak{e}_0-\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}-q(\beta)\\ m>0}}b^L_{\beta,-m}(0,0)\mathfrak{e}_\beta(m\tau)=E^L(\tau),$$

$$\sum_{\beta\in L'/L}\sum_{\substack{m\in\mathbb{Z}-q(\beta)\\ m>0}}b^L_{\beta,-m}(\delta,q(\lambda))\mathfrak{e}_\beta(m\tau)=-\frac{1}{\Gamma(\kappa-1)}P^L_{\delta,-q(\lambda)}(\tau).$$

If we put this into (5.17), we obtain (5.14).

Using (3.12) it can be seen in the same way that $\Omega^K(Y,\tau)$ is equal to

$$\frac{1}{16}\partial\bar\partial \sum_{\beta\in K_0'/K} \sum_{\substack{m\in\mathbb{Z}-q(\beta)\\ m>0}} \Bigg[\left(\frac{Y^2}{(d,Y)} - 2(d',Y)\right) \Phi^D_{p(\beta),-m} + \frac{8\pi}{(Y,d)} \sum_{\substack{\lambda\in D'\\ \lambda+D=p(\beta)\\ q(\lambda)=-m}} (\lambda,Y)^2 \Bigg] \mathfrak{e}_\beta(m\tau)$$
$$- \frac{1}{2\pi\Gamma(\kappa-1)}\partial\bar\partial \sum_{\lambda\in D'-0} \sum_{\substack{\delta\in K_0'/K\\ p(\delta)=\lambda+D}} \sum_{n\geq 1} \frac{(d,Y)}{n^2}\left(\frac{\pi n|\lambda||Y|}{(d,Y)}\right)^{\kappa-1}$$
$$\times K_{\kappa-1}\left(\frac{2\pi n|\lambda||Y|}{(d,Y)}\right) e\left(n\frac{(\lambda,Y)}{(d,Y)} + n(\delta,d')\right) P^K_{\delta,-q(\lambda)}(\tau). \tag{5.18}$$

Since the $\partial\bar\partial$-operator annihilates all terms which are linear in Y, the first summand in (5.18) is equal to

$$\frac{1}{16}\partial\bar\partial(d,Y)^{-1} \sum_{\beta\in K_0'/K} \sum_{\substack{m\in\mathbb{Z}-q(\beta)\\ m>0}} \Bigg[Y_D^2\Phi^D_{p(\beta),-m} + 8\pi \sum_{\substack{\lambda\in D'\\ \lambda+D=p(\beta)\\ q(\lambda)=-m}} (\lambda,Y_D)^2 \Bigg] \mathfrak{e}_\beta(m\tau). \tag{5.19}$$

According to Theorem 2.14 we have

$$\Phi^D_{p(\beta),-m} = \frac{16\pi m}{l-2} \#\left\{\lambda\in p(\beta)+D;\quad q(\lambda)=-m\right\}.$$

Thus (5.19) can be rewritten as

$$\frac{\pi}{2}\partial\bar\partial(d,Y)^{-1} \sum_{\beta\in K_0'/K} \sum_{\substack{m\in\mathbb{Z}-q(\beta)\\ m>0}} \sum_{\substack{\lambda\in D'\\ \lambda+D=p(\beta)\\ q(\lambda)=-m}} \left[(\lambda,Y_D)^2 - \frac{\lambda^2Y_D^2}{l-2}\right]\mathfrak{e}_\beta(m\tau)$$
$$= \frac{\pi}{2}\partial\bar\partial(d,Y)^{-1} \sum_{\beta\in K_0'/K} \sum_{\lambda\in p(\beta)+D} \left[(\lambda,Y_D)^2 - \frac{\lambda^2Y_D^2}{l-2}\right]\mathfrak{e}_\beta(-q(\lambda)\tau)$$
$$= \pi\partial\bar\partial(d,Y)^{-1}\Theta^D(Y_D,\tau)\mid\uparrow^K_D .$$

Putting this into (5.18) we get (5.16). The fact that $\Theta^D(Y_D,\tau)$ is a cusp form in $S_{\kappa,D}$ follows from the usual theta transformation formula ([Bo2] Theorem 4.1). □

Theorem 5.9. *The lifting*

$$\vartheta: S_{\kappa,L} \longrightarrow \mathcal{H}^{1,1}(\mathcal{X}_L);\qquad f\mapsto \left(f(\tau),\Omega^L(Z,\tau)\right)_\tau$$

has the following properties:

i) Let $f=\sum_{\gamma,n} c(\gamma,n)\mathfrak{e}_\gamma(n\tau)$ *be a cusp form in* $S_{\kappa,L}$*. Then the image of* f *has the Fourier expansion*

$$\vartheta(f)(Z)=\vartheta_0(f)(Y)-2^{-\kappa}\pi^{1/2-\kappa}\sum_{\substack{\lambda\in K'\\ q(\lambda)<0}}\sum_{n\geq 1}|\lambda|^{2-2\kappa}n^{-1}\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e(n(\delta,z'))$$
$$\times\, c(\delta,-q(\lambda))\partial\bar\partial\mathcal{V}_\kappa\left(\pi n|\lambda||Y|,\pi n(\lambda,Y)\right)e(n(\lambda,X)),$$

where the 0*-th Fourier coefficient* $\vartheta_0(f)$ *is given by*

$$\vartheta_0(f)(Y)=\left(f(\tau)\,|\!\downarrow^L_K,\,\Omega^K(Y,\tau)\right)_\tau .$$

ii) The diagram

$$\begin{array}{ccccc} S^*_{\kappa,L} & \longrightarrow & S_{\kappa,L} & \xrightarrow{\vartheta} & \mathcal{H}^{1,1}(\mathcal{X}_L) \\ \uparrow & & & & \downarrow \\ \mathcal{S}^*_{\kappa,L} & \xrightarrow{\eta} & \widetilde{\mathrm{Cl}}_{\mathrm{H}}\otimes\mathbb{C} & \longrightarrow & \widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L) \end{array} \tag{5.20}$$

commutes. Here $\mathcal{S}^*_{\kappa,L}\to S^*_{\kappa,L}$ *is given by the restriction of functionals, and* $S^*_{\kappa,L}$ *is identified with* $S_{\kappa,L}$ *by means of the Petersson scalar product* $(\cdot,f)\mapsto f$*. The right vertical arrow denotes the canonical projection.*

Proof. i) This is an immediate consequence of Theorem 5.8 and Proposition 1.5.

ii) By Theorem 5.2 and Theorem 5.5 the image of the functional $a_{\beta,m}\in\mathcal{S}^*_{\kappa,L}$ under the maps in the lower line is given by $\frac{1}{2}h_{\beta,-m}(Z)$. On the other hand its image in $S_{\kappa,L}$ equals

$$\frac{(4\pi m)^{\kappa-1}}{2\Gamma(\kappa-1)}P_{\beta,m}.$$

According to Theorem 5.8, for fixed Z the function $\Omega(Z,\tau)-\frac{1}{8}E(\tau)\partial\bar\partial\log(Y^2)$ is a cusp form. Using Proposition 1.5 we find that the image of $a_{\beta,m}$ in $\mathcal{H}^{1,1}(\mathcal{X}_L)$ is equal to

$$\frac{1}{2}h_{\beta,-m}(Z)+\frac{1}{8}b_{\beta,-m}(0,0)\partial\bar\partial\log(Y^2).$$

This implies the assertion. □

5.1.1 Comparison with the classical theta lift

We now compare the map ϑ constructed above with the theta lifting from elliptic cusp forms to holomorphic cusp forms on the orthogonal group $\mathrm{O}^+(V)$ due to Oda [Od] and Rallis-Schiffmann [RS]. We use the description of

Borcherds [Bo2], because it fits most easily into our setting. Let us first recall how holomorphic cusp forms of weight l contribute to the cohomology of $\mathcal{X}_L$.

It is easily verified that the determinant of the Jacobi matrix of the transformation $Z \mapsto \sigma(Z)$ ($\sigma \in \mathrm{O}^+(V)$) at $Z \in \mathbb{H}_l$ is equal to

$$j(\sigma, Z)^{-l}$$

(see [Fr1] chapter I Hilfssatz 1.6 for the case of Siegel modular forms). Thus, if f is a holomorphic modular form of weight l for $\Gamma(L)$, then

$$\omega_f = f\, dz_1 \wedge \dots \wedge dz_l$$

is a holomorphic differential form of type $(l, 0)$, which is invariant under $\Gamma(L)$. If f is a cusp form, then ω_f is square integrable.

If $r \in \mathbb{Q}$, we denote by $S_r(\Gamma(L))$ the space of holomorphic cusp forms of weight r for the group $\Gamma(L)$. Moreover, we write $\mathcal{H}^{l,0}(\mathcal{X}_L)$ for the space of square integrable holomorphic l-forms on the quotient $\mathcal{X}_L$.

Lemma 5.10. *The assignment $f \mapsto \omega_f$ defines an isomorphism*

$$S_l(\Gamma(L)) \xrightarrow{\sim} \mathcal{H}^{l,0}(\mathcal{X}_L).$$

Proof. This can be proved in the same way as Satz 2.6 in chapter III of [Fr1]. □

Note that the natural map $\mathcal{H}^{l,0}(\mathcal{X}_L) \to H^l(\mathcal{X}_L, \mathbb{C})$ to the middle cohomology of $\mathcal{X}_L$ is injective.

We write $-L$ for the lattice L as a $\mathbb{Z}$-module, but equipped with the quadratic form $-q(\cdot)$. Then the representation ϱ_{-L} is the dual representation of ϱ_L. This follows immediately from the definition.

The classical theta lift of Oda and Rallis-Schiffmann in weight $\kappa = 1 + l/2$ is a map from $S_{\kappa,-L}$ to $S_l(\Gamma(L))$. It can be viewed as a lifting

$$\iota : S_{\kappa,-L} \longrightarrow \mathcal{H}^{l,0}(\mathcal{X}_L)$$

or as a lifting to the middle cohomology of $\mathcal{X}_L$. According to Theorem 14.3 in [Bo2], if $f \in S_{\kappa,-L}$ with Fourier coefficients $c(\gamma, n)$ ($\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$), then there is a square integrable holomorphic l-form $\iota(f) \in \mathcal{H}^{l,0}(\mathcal{X}_L)$ with Fourier expansion

$$\iota(f)(Z) = \sum_{\substack{\lambda \in K' \\ q(\lambda)>0 \\ (\lambda,d)>0}} \sum_{n|\lambda} n^{l-1} \sum_{\substack{\delta \in L_0'/L \\ p(\delta) = \lambda/n + K}} e(n(\delta, z')) c(\delta, q(\lambda)/n^2) e((\lambda, Z))\, dz_1 \dots dz_l.$$

Here the second sum $\sum_{n|\lambda}$ runs over all $n \in \mathbb{N}$ with $\lambda/n \in K'$. The map

$$\vartheta : S_{\kappa,L} \longrightarrow \mathcal{H}^{1,1}(\mathcal{X}_L)$$

(cf. Theorem 5.9) can be regarded as a lifting to the second cohomology of $\mathcal{X}_L$. If $f \in S_{\kappa,L}$ with Fourier coefficients $c(\gamma, n)$ ($\gamma \in L'/L$ and $n \in \mathbb{Z} - q(\gamma)$), then there exists a square integrable harmonic $(1,1)$-form $\vartheta(f)$ with Fourier expansion

$$\vartheta(f)(Z) = \vartheta_0(f)(Y) - 2^{-\kappa}\pi^{1/2-\kappa} \sum_{\substack{\lambda \in K' \\ q(\lambda)<0}} |\lambda|^{-l} \sum_{n|\lambda} n^{l-1} \sum_{\substack{\delta \in L_0'/L \\ p(\delta)=\lambda/n+K}} e(n(\delta, z')) \\ \times c(\delta, -q(\lambda)/n^2) \partial\bar{\partial}\mathcal{V}_\kappa\left(\pi|\lambda||Y|, \pi(\lambda, Y)\right) e((\lambda, X)).$$

The similarity of the Fourier expansions of the two liftings is striking. In both cases the arithmetic part of the coefficient with index λ (and $q(\lambda) \neq 0$) is given by

$$\sum_{n|\lambda} n^{l-1} \sum_{\substack{\delta \in L_0'/L \\ p(\delta)=\lambda/n+K}} e(n(\delta, z'))c(\delta, |q(\lambda)|/n^2).$$

We see that in some cases an elliptic modular form $f \in S_{\kappa,L}$ can contribute to both, the middle cohomology and the second cohomology of $\mathcal{X}_L$. For instance, if the level of L is 1 or 2, then $\varrho_L = \varrho_{-L}$ and therefore $S_{\kappa,L} = S_{\kappa,-L}$.

In a series of papers Kudla and Millson constructed liftings from the cohomology with compact support of locally symmetric spaces associated to orthogonal (or unitary) groups to classical holomorphic Siegel modular forms on congruence subgroups [KM1, KM2, KM3]. Serious difficulties arise, if one tries to generalize their construction to obtain liftings from the cohomology with arbitrary support to classical modular forms. In particular the lifts are no longer holomorphic and the geometric meaning of the Fourier coefficients of the non-holomorphic part remains unclear. A first step towards such a generalization was made by J. Funke in his thesis [Fu]. In our case the (generalized) lifting of Kudla and Millson would go from $H^2(\mathcal{X}_L, \mathbb{C})$ to (non-holomorphic) elliptic modular forms of weight κ. One should expect that it is in some sense adjoint to our map ϑ. It would be interesting to understand this connection in detail.

5.2 Modular forms whose zeros and poles lie on Heegner divisors II

We pick up again our study of meromorphic modular forms whose divisor is a linear combination of Heegner divisors. In section 4.3 we saw that such a modular form F is (up to an additive constant) equal to a regularized theta lift Φ of a linear combination of Maass-Poincaré series $F_{\beta,m}$.

In this section we show, how Theorem 5.9 can be used to answer the question, whether F is already the Borcherds lift of a *nearly holomorphic* modular form as in Theorem 3.22.

Recall that a hyperbolic plane is a lattice which is isomorphic to the lattice $\mathbb{Z}^2$ with the quadratic form $q((a,b)) = ab$.

We define two subspaces of $S_{\kappa,L}$ by

$$S^-_{\kappa,L} = \{f = \sum_{\gamma,n} c(\gamma,n)\mathfrak{e}_\gamma(n\tau) \in S_{\kappa,L}; \quad c(\lambda,-q(\lambda)) = 0 \text{ for all } \lambda \in L'\}, \tag{5.21}$$

$$S^+_{\kappa,L} = \left(S^-_{\kappa,L}\right)^\perp. \tag{5.22}$$

If the lattice L splits a hyperbolic plane over $\mathbb{Z}$, then $S^-_{\kappa,L} = 0$ and $S^+_{\kappa,L} = S_{\kappa,L}$. It is an immediate consequence of Theorem 5.9 (i) that the lifting ϑ vanishes identically on $S^-_{\kappa,L}$.

Theorem 5.11. *Assume that the restricted map*

$$\vartheta : S^+_{\kappa,L} \longrightarrow \widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$$

is injective. Then every meromorphic modular form F with respect to $\Gamma(L)$, whose divisor is a linear combination of Heegner divisors

$$(F) = \frac{1}{2}\sum_{\substack{\lambda\in L'\\ q(\lambda)<0}} c(\lambda)H(\lambda, q(\lambda)) \tag{5.23}$$

($c(\lambda) \in \mathbb{Z}$ with $c(\lambda) = c(-\lambda)$ and $c(\lambda) = 0$ for all but finitely many λ), is a Borcherds product; i.e. there exists a nearly holomorphic modular form $f \in M^!_{k,L}$ with principal part

$$\sum_{\substack{\lambda\in L'\\ q(\lambda)<0}} c(\lambda)\mathfrak{e}_\lambda(q(\lambda)\tau) \tag{5.24}$$

such that F is the Borcherds lift of f as in Theorem 3.22.

Proof. Let F be a meromorphic modular form for the group $\Gamma(L)$ with divisor (5.23). Then

$$\frac{1}{2}\sum_{\lambda\in L'} c(\lambda)c(H(\lambda, q(\lambda))) = 0$$

in $\widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$. According to Theorem 5.9 this implies that

$$\vartheta\left(\sum_{\lambda\in L'} c(\lambda)(-q(\lambda))^{\kappa-1}P_{\lambda,-q(\lambda)}\right) = 0$$

in $\widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$. Using the assumption on ϑ we find that

$$\sum_{\lambda\in L'} c(\lambda)(-q(\lambda))^{\kappa-1} P_{\lambda,-q(\lambda)}$$

is contained in $S^-_{\kappa,L}$. On the other hand, by Proposition 1.5 this linear combination of Poincaré series is orthogonal to $S^-_{\kappa,L}$. Therefore it vanishes identically. But this is equivalent to saying

$$\sum_{\lambda\in L'} c(\lambda)a_{\lambda,-q(\lambda)} = 0$$

in $S^*_{\kappa,L}$. By Theorem 1.17 there exists a nearly holomorphic modular form $f \in M^!_{k,L}$ with principal part (5.24). Let $B(f)$ be the Borcherds lift of f as in Theorem 3.22. Then $B(f)/F$ is a holomorphic modular form (with a multiplier system of finite order) without any zeros on $\mathbb{H}_l$. Hence it is constant. □

Theorem 5.12. *Let L be an even lattice of signature $(2,l)$, that splits two orthogonal hyperbolic planes over $\mathbb{Z}$. Then every meromorphic modular form for the group $\Gamma(L)$, whose divisor is a linear combination of Heegner divisors, is a Borcherds product as in Theorem 3.22.*

Proof. We write L as an orthogonal sum $L = D \oplus H_1 \oplus H_2$, where D is a negative definite lattice and H_1, H_2 are hyperbolic planes. Put $K = D \oplus H_1$. According to Theorem 5.11 it suffices to prove that the map $\vartheta : S^+_{\kappa,L} \to \widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$ is injective.

Let $f \in S^+_{\kappa,L}$ with $\vartheta(f) = 0$. Denote the Fourier coefficients of f by $c(\gamma, n)$. We have to show that $f \in S^-_{\kappa,L}$.

For a suitable choice of the vectors z and z' the Fourier expansion of $\vartheta(f)$ is given by

$$\begin{aligned}\vartheta(f)(Z) = \vartheta_0(f)(Y) - 2^{-\kappa}\pi^{1/2-\kappa} \sum_{\substack{\lambda\in K'\\ q(\lambda)<0}} \sum_{n|\lambda} &|\lambda|^{2-2\kappa} n^{2\kappa-3} c(\lambda/n, -q(\lambda)/n^2)\\ &\times \partial\bar\partial \mathcal{V}_\kappa\left(\pi|\lambda||Y|, \pi(\lambda, Y)\right) e((\lambda, X)).\end{aligned}$$

The assumption $\vartheta(f) = 0$ implies that

$$|\lambda|^{2-2\kappa} \sum_{n|\lambda} n^{2\kappa-3} c(\lambda/n, -q(\lambda)/n^2) = 0$$

for all $\lambda \in K'$ with $q(\lambda) < 0$. By an easy inductive argument we find that $c(\lambda, -q(\lambda)) = 0$ for all $\lambda \in K'$ with $q(\lambda) < 0$.

Recall that there is an isomorphism between $H_1 \oplus H_2$ and the lattice $M_2(\mathbb{Z})$ of integral 2×2 matrices such that the quadratic form q corresponds to the determinant on $M_2(\mathbb{Z})$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $M_2(\mathbb{Z})$ by multiplication from both sides. This gives rise to a homomorphism

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{O}^+(H_1 \oplus H_2)$$

(cf. [FH] Lemma 4.4). Hence, by the theorem of elementary divisors for $\mathrm{SL}_2(\mathbb{Z})$, for any $\lambda \in L'$ there is a $\sigma \in \mathrm{O}^+(H_1 \oplus H_2) \subset \Gamma(L)$ such that $\sigma(\lambda) \in K'$.

Thereby we obtain $c(\lambda, -q(\lambda)) = 0$ for all $\lambda \in L'$ with $q(\lambda) < 0$. □

It seems likely that Theorem 5.11 can also be used to prove a more general version of Theorem 5.12. However, this seems to be a non-trivial problem, which might require additional arguments like some newform theory or Hecke theory for the space $S_{\kappa,L}$ and the lifting ϑ.

Example 5.13. Fix a positive integer t. Let L be the lattice $L = D \oplus H_1 \oplus H_2$, where H_1, H_2 are hyperbolic planes and D denotes the lattice $\mathbb{Z}$ with the negative definite quadratic form $q(a) = -ta^2$. Obviously L has signature $(2, 3)$, $k = -1/2$, and $\kappa = 5/2$. The discriminant group $L'/L \cong D'/D$ has order $2t$. In the same way as in [EZ] §5 it can be seen that the space $S_{5/2,L}$ is isomorphic to the space $\mathcal{J}_{3,t}^{\text{cusp}}$ of *skew-holomorphic* Jacobi cusp forms of weight 3 and index t (see [Sk], [Ko]).

The group $\Gamma(L)$ is isomorphic to the *paramodular group* of level t (cf. [GN] Lemma 1.9 and [GrHu]). Moreover, the quotient $\mathcal{X}_L = \mathbb{H}_l/\Gamma(L)$ is the moduli space of Abelian surfaces with a $(1, t)$-polarization. The Heegner divisors $H(\beta, m)$ are known as Humbert surfaces. Since L splits two hyperbolic planes over $\mathbb{Z}$, any $H(\beta, m)$ is a prime divisor in $D(\mathcal{X}_L)$ (cf. [FH] Lemma 4.4).

By Theorem 5.9 we have an injective map

$$\vartheta : \mathcal{J}_{3,t}^{\text{cusp}} \longrightarrow \mathcal{H}^{1,1}(\mathcal{X}_L)$$

from $\mathcal{J}_{3,t}^{\text{cusp}}$ into the space of square integrable harmonic $(1, 1)$-forms on $\mathcal{X}_L$. Observe that the 0-th Fourier coefficient $\vartheta_0(f)(Y)$ of the lift of $f \in \mathcal{J}_{3,t}^{\text{cusp}}$ can be identified with the Shimura lift of f. According to Theorem 5.12 any meromorphic modular form with respect to $\Gamma(L)$, whose only zeros and poles lie on Humbert surfaces, is a Borcherds product in the sense of Theorem 3.22.

In this case the classical theta lift is essentially the Maass lift. It gives rise to an injective map

$$\iota : J_{3,t}^{\text{cusp}} \longrightarrow \mathcal{H}^{3,0}(\mathcal{X}_L)$$

from holomorphic Jacobi cusp forms of weight 3 and index t into $\mathcal{H}^{3,0}(\mathcal{X}_L)$. We obtain a partial answer to a question raised by W. Kohnen ([Ko] Problem (ii) in §2). Holomorphic Jacobi forms of weight 3 contribute via Maass lift to the middle cohomology of $\mathcal{X}_L$, whereas skew-holomorphic Jacobi forms contribute to the second cohomology. Here it is natural to ask, if there is a similar connection in higher weights.

References

[AbSt] *M. Abramowitz and I. Stegun*, Pocketbook of Mathematical Functions, Verlag Harri Deutsch (1984).

[Bo1] *R. E. Borcherds*, Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products, Invent. Math. **120** (1995), 161–213.

[Bo2] *R. E. Borcherds*, Automorphic forms with singularities on Grassmannians, Invent. Math. **132** (1998), 491–562.

[Bo3] *R. E. Borcherds*, The Gross-Kohnen-Zagier theorem in higher dimensions, Duke Math. J. **97** (1999), 219–233.

[Bo4] *R. E. Borcherds*, Sporadic groups and string theory, First E. C. M., Vol. I (Paris 1992), 411-421, Progr. Math. **119** (1994), Birkhäuser, Basel.

[Bl1] *A. Borel*, Introduction aux groupes arithmétiques, Hermann (1969).

[Bl2] *A. Borel*, Linear Algebraic Groups. In: Algebraic groups and discontinuous subgroups, Proceedings of Symposia in Pure Mathematics **9**, American Mathematical Society (1966), 3–19.

[Bl3] *A. Borel*, Cohomology of arithmetic groups, Proc. Int. Congr. of Mathematics, Vancouver 1974, Vol. 1 (1975), 435–442.

[Br1] *J. H. Bruinier*, Borcherds products and Chern classes of Hirzebruch-Zagier divisors, Invent. math. **138** (1999), 51–83.

[Br2] *J. H. Bruinier*, Borcherdsprodukte und Chernsche Klassen von Hirzebruch-Zagier-Zykeln, Dissertation, Universität Heidelberg (1998).

[BK] *J. H. Bruinier and M. Kuss*, Eisenstein series attached to lattices and modular forms on orthogonal groups, preprint (2000), to appear in Manuscr. Math.

[BKK] *J. I. Burgos, J. Kramer, and U. Kühn* , manuscript, in preparation (2001).

[DI] *F. Diamond and J. Im*, Modular forms and modular curves, Canadian Mathematical Society Conference Proceedings **17** (1995), 39–133.

[DN] *K. Doi and H. Naganuma*, On the functional equation of certain Dirichlet series, Invent. Math. **9** (1969), 1–14.

[E1] *A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi*, Higher Transcendental Functions, McGraw-Hill (1953).

[E2] *A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi*, Tables of Integral Transforms, vol. I, McGraw-Hill (1954).

[EZ] *M. Eichler and D. Zagier*, The Theory of Jacobi Forms, Progress in Math. **55** (1985), Birkhäuser.

[Fa] *J. Fay*, Fourier coefficients of the resolvent for a Fuchsian group, J. reine angew. Math. **294** (1977), 143–203.

[Fr1] *E. Freitag*, Siegelsche Modulfunktionen, Springer-Verlag (1983).

[Fr2] *E. Freitag*, Singular Modular Forms and Theta Relations, Lecture Notes in Mathematics **1487**, Springer-Verlag (1991).

[FH] *E. Freitag and C. F. Hermann*, Some modular varieties of low dimension, Adv. Math. **152** (2000), 203–287.

[Fu] *J. Funke*, Rational quadratic divisors and automorphic forms, PhD-thesis, University of Maryland (1999).

[GH] *P. Griffiths and J. Harris*, Principles of algebraic geometry, John Wiley (1978).

[GrHu] *V. Gritsenko and K. Hulek*, Minimal Siegel modular threefolds, Math. Proc. Camb. Phil. Soc. **123** (1998), 461–485.

[GN] *V. Gritsenko and V. Nikulin*, Automorphic forms and Lorentzian Kac-Moody algebras. Part II, Intern. J. of Math. **9** (1998), 201–275.

[GR] *R. C. Gunning and H. Rossi*, Analytic functions of several complex variables, Prentice-Hall (1965).

[GW] *R. E. Greene and H. Wu*, C^∞ approximations of convex, subharmonic, and plurisubharmonic functions, Ann. scient. Éc. Norm. Sup. **12** (1979), 47–84.

[Ha] *Harish-Chandra*, Automorphic Forms on Semisimple Lie Groups, Lecture Notes in Mathematics **62**, Springer-Verlag (1968).

[He] *D. A. Hejhal*, The Selberg Trace Formula for $\mathrm{PSL}(2,\mathbb{R})$, Lecture Notes in Mathematics **1001**, Springer-Verlag (1983).

[Hel] *S. Helgason*, Differential geometry and symmetric spaces, Academic Press (1962).

[HM] *J. Harvey and G. Moore*, Algebras, BPS states, and strings, Nuclear Phys. B **463** (1996), no. 2-3, 315–368.

[HZ] *F. Hirzebruch und D. Zagier*, Intersection Numbers of Curves on Hilbert Modular Surfaces and Modular Forms of Nebentypus, Invent. Math. **36** (1976), 57–113.

[Ho] *R. Howe*, θ-series and invariant theory, Proceedings of Symposia in Pure Mathematics **33**, part 1, American Mathematical Society (1979), 275–285.

[Ka] *N. Katz*, p-adic properties of modular schemes and modular forms. In: Modular Functions of One Variable III, Lecture Notes in Math. **350**, Springer-Verlag (1973), 69–190.

[Ko] *W. Kohnen*, Jacobi forms and Siegel modular forms: Recent results and problems, L'Ens. Math. **39** (1993), 121–136.

[Kon] *M. Kontsevich*, Product formulas for modular forms on $O(2,n)$, Séminaire Bourbaki **821** (1996).

[KM1] *S. Kudla and J. Millson*, The theta correspondence and harmonic forms I, Math. Ann. **274**, (1986), 353–378.

[KM2] *S. Kudla and J. Millson*, The theta correspondence and harmonic forms II, Math. Ann. **277**, (1987), 267–314.

[KM3] *S. Kudla and J. Millson*, Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables, IHES Publi. Math. **71** (1990), 121–172.

[Ma1] *H. Maass*, Modular functions of one complex variable, Spinger-Verlag (1983).

[Ma2] *H. Maass*, Automorphe Funktionen in mehreren Veränderlichen und Dirichletsche Reihen, Abh. Math. Sem. Univ. Hamburg **16** (1949), 72–100.

[Ma3] *H. Maass*, Über ein Analogon zur Vermutung von Saito-Kurokawa, Invent. Math. **60** (1980), 85–104.

[Mar] *G. A. Margulis*, Discrete Subgroups of Semisimple Lie Groups, Springer-Verlag (1991).

[Me] *O. Meyer*, Über die Satakekompaktifizierung arithmetischer Quotienten von beschränkten hermiteschen Bereichen des Types IV_n, Diplomarbeit, Universität Heidelberg (1998).

[Na] *H. Naganuma*, On the coincidence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hilbert modular forms over a real quadratic field, J. Math. Soc. Japan **25** (1973), 547–555.

[Nak] *S. Nakajima*, On Invariant Differential Operators on Bounded Symmetric Domains of Type IV, Proc. Japan Acad. **58** Ser. A (1982), 235–238.

[Ni] *D. Niebur*, A class of nonanalytic automorphic functions, Nagoya Math. J. **52** (1973), 133–145.

[No] *A. Nobs*, Die irreduziblen Darstellungen der Gruppen $\mathrm{SL}_2(\mathbb{Z}_p)$, insbesondere $\mathrm{SL}_2(\mathbb{Z}_2)$. I. Teil, Comment Math. Helvetici **51** (1976), 465–489.

[Od] *T. Oda*, On Modular Forms Associated with Indefinite Quadratic Forms of Signature $(2, n-2)$, Math. Ann. **231** (1977), 97–144.

[PS1] *I. Piateskii-Shapiro*, On the Saito-Kurokawa lifting, Invent. Math. **71** (1983), 309–338.

[PS2] *I. Piateskii-Shapiro*, Cuspidal automorphic representations associated to parabolic subgroups and Ramanujan conjecture. In: Number theory related to Fermat's last theorem, Prog. Math. **26**, Birkhäuser (1982), 143–151.

[RS] *S. Rallis and G. Schiffmann*, On a relation between SL_2 cusp forms and cusp forms on tube domains associated to orthogonal groups, Trans. AMS **263** (1981), 1–58.

[Rh] *G. de Rham*, Differentiable Manifolds, Springer-Verlag (1984).

[Ro] *W. Roelcke*, Über den Laplace-Operator auf Riemannschen Manigfaltigkeiten mit diskontinuierlichen Gruppen, Math. Nachr. **21** (1960), 131–149.

[Sc] *B. Schoeneberg*, Elliptic Modular Functions, Springer-Verlag (1974).

[SeSt] *J.-P. Serre and H. M. Stark*, Modular forms of weight 1/2. In: Modular Functions of One Variable VI, Lecture Notes in Math. **627**, Springer-Verlag (1977), 27–67.

[Sh] *G. Shimura*, Introduction to the Arithmetic Theory of Automorphic Functions, Princeton University Press (1971).

[Sk] *N.-P. Skoruppa*, Developments in the theory of Jacobi forms. In: Proceedings of the conference on automorphic funtions and their applications, Chabarovsk (eds.: N. Kuznetsov and V. Bykovsky), The USSR Academy of Science (1990), 167–185. (see also MPI-preprint 89-40, Bonn (1989).)

[Sn] *T. Shintani*, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J. **58** (1975), 83–126.

[SABK] *C. Soulé, D. Abramovich, J.-F. Burnol, and J. Kramer*, Lectures on Arakelov Geometry, Cambridge Studies in Advanced Mathematics **33**, Cambridge University Press (1992).

[Wei] *R. Weissauer*, The Picard group of Siegel modular threefolds, J. reine angew. Math. **430** (1992), 179–211.

[Wel] *R. O. Wells*, Differential analysis on complex manifolds, Springer-Verlag (1980).

[Yau] *S.-T. Yau*, Some function-theoretic properties of complete Riemannian Manifolds and their applications to geometry, Indiana Univ. Math. J. **25** (1976), 659–670.

[Za] *D. Zagier*, Modular Forms Associated to Real Quadratic Fields, Invent. Math. **30** (1975), 1–46.

Notation

$\sqrt{\cdot}$	The principal branch of the square root
$f(\cdot) \ll g(\cdot)$	$\lvert f(\cdot)\rvert \leq C\lvert g(\cdot)\rvert$
$f(\cdot) \ll_\varepsilon g(\cdot)$	$\lvert f(\cdot)\rvert \leq C(\varepsilon)\lvert g(\cdot)\rvert$
$f \approx g$	See p. 51
$\vert_\kappa$	The Petersson slash operator of weight κ with representation ϱ_L (p. 17)
$\vert^*_\kappa$	The Petersson slash operator of weight κ with representation ϱ^*_L (p. 17)
$\langle a, b\rangle$	The standard scalar product on $\mathbb{C}[L'/L]$ for $a, b \in \mathbb{C}[L'/L]$
(a, b)	The bilinear form on L for $a, b \in L$
(f, g)	The Petersson scalar product of $f, g \in S_{\kappa,L}$ (p. 22)
$\uparrow^L_K$	See p. 129
$\downarrow^L_K$	See p. 129
λ^2	$= (\lambda, \lambda)$ for $\lambda \in V$
$\lvert\lambda\rvert$	$= \lvert(\lambda, \lambda)\rvert^{1/2}$ for $\lambda \in V$
$\lambda^\perp$	The orthogonal complement of λ
λ_v	The orthogonal projection of $\lambda \in V$ onto v
$a_{\gamma,n}$	A certain functional in $S^*_{\kappa,L}$ (p. 36) or $\mathcal{S}^*_{\kappa,L}(\mathbb{Z})$ (p. 120)
$\mathcal{A}_{\kappa,L}(\mathbb{Z})$	See p. 120
$b(\gamma, n, s)$	The Fourier coefficients of $F_{\beta,m}(\tau, s)$ (p. 30)
$b_{\beta,m}(\gamma, n)$	$= b(\gamma, n) = b(\gamma, n, 1 - k/2)$
$B_r(x)$	The r-th Bernoulli polynomial (p. 61)
$\mathbb{B}_r(x)$	See p. 62
$c(D)$	The Chern class of the divisor D in the second cohomology
$c(\gamma, n; y, s)$	The Fourier coefficients of $F_{\beta,m}(\tau, s)$ (p. 57)
C	The positive cone in $K \otimes \mathbb{R}$ (p. 77)
$C_{\beta,m}$	A certain constant (p. 72)
$\mathrm{Cl}(X/\Gamma)$	The divisor class group of X/Γ (p. 117)
$\widetilde{\mathrm{Cl}}(X/\Gamma)$	The modified divisor class group of X/Γ (p. 117)
$\widetilde{\mathrm{Cl}}_{\mathrm{H}}(\mathcal{X}_L)$	The subgroup of $\widetilde{\mathrm{Cl}}(\mathcal{X}_L)$ generated by the Heegner divisors $H(\beta, m)$
$\mathbb{C}$	The complex numbers
$\mathbb{C}[L'/L]$	The group algebra of the discriminant group L'/L
$\mathcal{C}_{s=a}[f(s)]$	The constant term of the Laurent expansion of f at $s = a$ (p. 50)
χ	A multiplier system or character
d	Often a primitive isotropic vector in K

d'	An element of K' with $(d, d') = 1$
$\tilde{d}$	Usually $d' - q(d')d$
D	$= K \cap d^\perp \cap d'^\perp$, or a divisor
$\delta_{*,*}$	The Kronecker-delta
Δ_k	The Laplace operator of weight k (p. 28)
$e(z)$	$= e^{2\pi i z}$ for $z \in \mathbb{C}$
$\mathfrak{e}_\gamma$	An element of the standard basis $(\mathfrak{e}_\gamma)_{\gamma \in L'/L}$ of $\mathbb{C}[L'/L]$
$\mathfrak{e}_\gamma(\tau)$	$= \mathfrak{e}_\gamma e(\tau)$
$E_\beta^L(\tau)$	An Eisenstein series in $M_{\kappa,L}$ (p. 23)
$E(\tau)$	$= E_0(\tau) = E_0^L(\tau)$
$E(u,v)$	An Eichler transformation (p. 105)
η	See p. 120
$F_{\beta,m}^L(\tau,s)$	A non-holomorphic Poincaré series of weight k (p. 29)
$\widetilde{F}_{\beta,m}(\tau,s)$	See p. 30
$F(a,b,c;z)$	The Gauss hypergeometric function as in [AbSt]
$F_{\beta,m}^K(\tau,s;r,t)$	See p. 45
$\mathcal{F}$	The standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$
$\mathcal{F}_u$	The truncated fundamental domain (p. 47)
$\mathrm{Gal}(E/F)$	The Galois group of a field extension E/F
$\mathrm{Gr}(L)$	The Grassmannian of L
Γ	Usually a subgroup of finite index of $\Gamma(L)$
$\Gamma(N)$	The principal congruence subgroup of level N
$\Gamma(y)$	The Gamma function
$\Gamma(a,x)$	The incomplete Gamma function as in [AbSt] (p. 28)
Γ_1	$= \mathrm{SL}_2(\mathbb{Z})$
Γ_∞	$= \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right); \quad n \in \mathbb{Z}\} \leq \mathrm{SL}_2(\mathbb{Z})$
$\tilde{\Gamma}_\infty$	$= \{(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), 1); \quad n \in \mathbb{Z}\} \leq \mathrm{Mp}_2(\mathbb{Z})$
$\Gamma(L)$	$= \mathrm{O}^+(V) \cap \mathrm{O}_d(L)$
$H(\beta,m)$	The Heegner divisor of index (β,m) (p. 47 and p. 117)
$h_{\beta,m}(Z)$	The representative $\frac{1}{4}\partial\bar{\partial}\xi_{\beta,m}^L(Z)$ of the Chern class of $H(\beta,m)$
$H_c^*(\beta,m,\gamma,n)$	A generalized Kloosterman sum (p. 19)
$H_c(\beta,m,\gamma,n)$	A generalized Kloosterman sum (p. 30)
$H^2(\mathcal{X}_L,\mathbb{C})$	The second cohomology of $\mathcal{X}_L$ with coefficients in $\mathbb{C}$
$\tilde{H}^2(\mathcal{X}_L,\mathbb{C})$	See p. 122
$\mathcal{H}^{1,1}(\mathcal{X}_L)$	The space of square integrable harmonic $(1,1)$-forms on $\mathcal{X}_L$ (p. 123)
$\widetilde{\mathcal{H}}^{1,1}(\mathcal{X}_L)$	See p. 124
$\mathbb{H}$	The complex upper half plane $\{\tau \in \mathbb{C};\ \Im(\tau) > 0\}$
$\mathbb{H}_l$	The generalized upper half plane (p. 77)
$\mathbb{H}_l(\Lambda)$	The generalized upper half plane attached to Λ (p. 108)
$\mathcal{H}$	The upper half space model of hyperbolic space (p. 65)
$I_\nu(y)$	The I-Bessel function as in [AbSt] or [E1]
$\Im(\cdot)$	The imaginary part
$J_\nu(y)$	The J-Bessel function as in [AbSt] or [E1]
$j(\sigma,Z)$	The automorphic factor $(\sigma(Z_L), z)$ (p. 82)

$J_{k,t}$	The space of holomorphic Jacobi forms of weight k and index t
$\mathcal{J}_{k,t}$	The space of skew-holomorphic Jacobi forms of weight k and index t
$K_\nu(y)$	The K-Bessel function as in [AbSt] or [E1]
k	Usually $1 - l/2$
κ	Usually $1 + l/2$
K	$= L \cap z^\perp \cap z'^\perp$
$\mathcal{K}$	See p. 76
$\mathcal{K}^+$	A component of $\mathcal{K}$
$\widetilde{\mathcal{K}}^+$	The cone over $\mathcal{K}^+$ (p. 82)
l	An integer ≥ 3
L	An even lattice of signature (b^+, b^-)
L'	The dual lattice of L
L_0'	$= \{\lambda \in L';\quad (\lambda, z) \equiv 0 \pmod N\}$
L_k	A Maass differential operator (p. 95)
$L^p(\mathbb{H}_l/\Gamma)$	See p. 101
Λ	Usually an admissible index-tuple (p. 107)
$\tilde{M}$	$= (M, \sqrt{c\tau + d})$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$
$M_{\nu,\,\mu}(y)$	The M-Whittaker function as in [AbSt] (p. 27)
$\mathcal{M}_s(y)$	$= y^{-k/2} M_{-k/2,\, s-1/2}(y)$
$M_{\kappa,L}$	The space of modular forms of weight κ with respect to ϱ_L^* and $\mathrm{Mp}_2(\mathbb{Z})$
$M^!_{k,L}$	The space of nearly holomorphic modular forms of weight k (p. 34)
$\mathrm{Mp}_2(\mathbb{R})$	The metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$ (p. 15)
$\mathrm{Mp}_2(\mathbb{Z})$	The integral metaplectic group (p. 16)
$\mathcal{M}_{\kappa,L}$	$= \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \cdot M_{\kappa,L}$ (p. 120)
μ	Usually a certain vector in $V \cap z^\perp$ (p. 42)
N	The unique positive integer with $(z, L) = N\mathbb{Z}$, or the level of L
$\mathcal{N}$	The zero-quadric in $P(V(\mathbb{C}))$ (p. 76)
$\mathbb{N}$	$= \{1, 2, 3, \dots\}$
$\mathbb{N}_0$	$= \mathbb{N} \cup \{0\}$
$\mathrm{O}(V)$	The (special) orthogonal group of V (p. 40)
$\mathrm{O}^+(V)$	The connected component of the identity of $\mathrm{O}(V)$
$\mathrm{O}(L)$	$= \{g \in \mathrm{O}(V);\quad gL = L\}$
$\mathrm{O}_d(L)$	The discriminant kernel of $\mathrm{O}(L)$, see p. 40
$\mathrm{O}^+_{\mathbb{Q}}(L)$	The rational orthogonal group of L (p. 101)
Ω	The invariant Laplace operator on $\mathbb{H}_l$ (p. 94)
$\mathit{\Omega}(\tau)$	A kernel function, see p. 123
$\mathit{\Omega}^L(Z, \tau)$	The kernel function of ϑ, see p. 128
$\mathit{\Omega}^K(Y, \tau)$	A kernel function, see p. 130
p	Usually a certain projection $L_0' \to K'$ (p. 41)
$P^L_{\beta,m}(\tau)$	The Poincaré series of index (β, m) in $S_{\kappa,L}$ (p. 19)
$p_{\beta,m}(\gamma, n)$	The Fourier coefficients of $P^L_{\beta,m}$
$P(V(\mathbb{C}))$	The projective space of $V(\mathbb{C})$
$\Phi^L_{\beta,m}(v, s)$	The regularized theta lift of $F_{\beta,m}(\tau, s)$ (p. 47)
$\Phi_{\beta,m}(v)$	$= \mathcal{C}_{s=1-k/2}\,[\Phi_{\beta,m}(v, s)]$, the regularized theta lift of $F_{\beta,m}(\tau, 1 - k/2)$

$\psi^L_{\beta,m}$	See p. 66 and p. 79
$\Psi_{\beta,m}(Z)$	See p. 80
$q(a)$	$= \frac{1}{2}(a,a)$, the quadratic form on L
$q_\beta(\gamma,n)$	The Fourier coefficients of $E^L_\beta(\tau)$
$q(\gamma,n)$	The Fourier coefficients of $E(\tau)$
$\mathbb{Q}$	The rational numbers
R_k	A Maass differential operator (p. 95)
$\mathbb{R}$	The real numbers
$\mathcal{R}_t$	The set of Y defined by (4.18b)–(4.18d)
$\Re(\cdot)$	The real part
ϱ_L	The Weil representation attached to $(L'/L,q)$ (p. 16)
$\varrho_{\gamma\delta}(M,\phi)$	A coefficient of the representation ϱ_L (p. 17)
ϱ^*_L	The dual representation of ϱ_L
$\varrho_{\beta,m}(W)$	See p. 67
$\varrho_f(W)$	See p. 86
s	A complex variable
S	$= \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right) \in \mathrm{Mp}_2(\mathbb{Z})$
$\mathrm{SL}_2(\mathbb{R})$	The group of real 2×2-matrices of determinant 1
$\mathrm{SL}_2(\mathbb{Z})$	The group of integral 2×2-matrices of determinant 1
$S_{\kappa,L}$	The subspace of cusp forms in $M_{\kappa,L}$
$S^*_{\kappa,L}$	The dual space of $S_{\kappa,L}$
$S^-_{\kappa,L}$	A certain subspace of $S_{\kappa,L}$ (p. 136)
$S^+_{\kappa,L}$	A certain subspace of $S_{\kappa,L}$ (p. 136)
$\mathrm{supp}(D)$	The support of the divisor D
$\mathcal{S}_t$	A Siegel domain in $\mathbb{H}_l$ (p. 101)
$\tilde{\mathcal{S}}_t(\Lambda)$	A Siegel domain in $\mathcal{K}^+$ (p. 108)
$\mathcal{S}(\beta,m,U)$	See p. 47 and p. 81
$\mathcal{S}_{\kappa,L}$	$= \mathrm{Gal}(\mathbb{C}/\mathbb{Q})\cdot S_{\kappa,L}$ (p. 119)
$\mathcal{S}_{\kappa,L}(\mathbb{Z})$	The module of cusp forms in $\mathcal{S}_{\kappa,L}$ whose coefficients all lie in $\mathbb{Z}$
$\mathcal{S}^*_{\kappa,L}(\mathbb{Z})$	The dual module of $\mathcal{S}_{\kappa,L}(\mathbb{Z})$
σ	Often the real part of s
T	$= \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right) \in \mathrm{Mp}_2(\mathbb{Z})$
τ	A variable in $\mathbb{H}$
$\Theta_L(\tau,v;r,t)$	The Siegel theta function attached to L (p. 39)
$\theta_\gamma(\tau,v;r,t)$	A component of $\Theta_L(\tau,v;r,t)$
ϑ	A lifting, see p. 128
$\vartheta_0(f)(Y)$	The 0-th Fourier coefficient of $\vartheta(f)$ (p. 133)
$\Theta^D(Y_D,\tau)$	A theta series, see p. 130
v	Usually an element of $\mathrm{Gr}(L)$
V	$L\otimes\mathbb{R}$
$V(\mathbb{C})$	$= V\otimes\mathbb{C}$
$\mathcal{V}_\kappa(A,B)$	A special function, see p. 71
w	The orthogonal complement of z_v in v
W	A Weyl chamber (p. 61), or a complex variable

$W_{\nu,\,\mu}(y)$	The W-Whittaker function as in [AbSt] (p. 27)
$\mathcal{W}_s(y)$	$= \lvert y\rvert^{-k/2} W_{k/2\,\mathrm{sgn}(y),\,s-1/2}(\lvert y\rvert)$
x	The real part of τ
X	The real part of $Z \in \mathbb{H}_l$, or a normal irreducible complex space
$\mathcal{X}_L$	$= \mathbb{H}_l/\Gamma(L)$
$\xi^L_{\beta,m}$	See p. 66 and p. 79
y	The imaginary part of τ
Y	Usually the imaginary part of $Z \in \mathbb{H}_l$
z	Usually a primitive isotropic vector in L
z'	An element of L' with $(z, z') = 1$
$\tilde{z}$	Usually $z' - q(z')z$
Z	$= \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right) \in \mathrm{Mp}_2(\mathbb{Z})$, or a variable in $\mathbb{H}_l$
Z_L	Usually $(Z, 1, -q(Z) - q(z')) \in \widetilde{\mathcal{K}}^+$ for $Z \in \mathbb{H}_l$
$\mathbb{Z}$	The integers
ζ	A vector in L with $(\zeta, z) = N$
$\zeta(s)$	The Riemann zeta function
ζ_N	A primitive N-th root of unity

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