BORCHERDS PRODUCTS WITH PRESCRIBED DIVISOR

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Abstract. Given an infinite set of special divisors satisfying a mild regularity condition, we prove the existence of a Borcherds product of non-zero weight whose divisor is supported on these special divisors. We also show that every meromorphic Borcherds product is the quotient of two holomorphic ones. The proofs of both results rely on the properties of vector valued Eisenstein series for the Weil representation.

1. Introduction and statement of results

Let (L, Q) be an even lattice of signature $(n, 2)$ with dual L'. We write $\mathcal{D}^+$ for the hermitian symmetric space associated with the connected component of the real points of the orthogonal group $\mathrm{O}(L)$ of L. Let $\Gamma \subset \mathrm{O}(L)$ be a congruence subgroup which preserves $\mathcal{D}^+$ and which acts trivially on the discriminant group L'/L. By the theory of Baily-Borel the quotient

$$X_\Gamma = \Gamma\backslash\mathcal{D}^+$$

has a structure as a quasi-projective algebraic variety over $\mathbb{C}$ of dimension n. For every $\mu \in L'/L$ and every positive $m \in \mathbb{Z}+Q(\mu)$ there exists a special divisor $Z(m, \mu)$ on X_Γ. In the projective model of $\mathcal{D}^+$ it is given by the orthogonal complements of vectors $\lambda \in L+\mu$ with $Q(\lambda) = m$, see [Bo2], [Ku1], [Br].

We briefly write L^- for the lattice $(L, -Q)$ of signature $(2, n)$. Recall that there is a Weil representation ρ_L of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[L'/L]$, see [Bo1], [Br]. By means of the standard $\mathbb{C}$-bilinear pairing on $\mathbb{C}[L'/L]$, the dual representation of ρ_L can be identified with ρ_{L^-}. In his celebrated paper [Bo1], R. Borcherds constructed a map from weakly holomorphic modular forms of weight $1-n/2$ for $\mathrm{Mp}_2(\mathbb{Z})$ with representation ρ_{L^-} to meromorphic modular forms on X_Γ whose divisors are supported on special divisors and which have particular infinite product expansions, see [Bo1, Theorem 13.3] and [Br, Theorem 3.22]. Since these Borcherds products give rise to explicit relations among special divisors in the Picard group of X_Γ, they are of great importance for algebraic and arithmetic applications, see e.g. [Bo2], [Ku2], [BHY]. In this note we prove two useful results about Borcherds products.

We call a set $\mathcal{S}$ of pairs $(m, \mu) \in \mathbb{Q}_{>0} \times L'/L$ *admissible*, if:

(1) For all $(m, \mu) \in \mathcal{S}$ there exists a $\lambda \in \mu + L$ with $Q(\lambda) = m$.
(2) There exists a positive integer A such that $\mathrm{ord}_p(m) \leq A$ for all $(m, \mu) \in \mathcal{S}$ and for all primes p dividing $2|L'/L|$.

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The first condition is equivalent to requiring that $Z(m,\mu)$ be a non-trivial divisor on X_Γ.

Theorem 1.1. *Assume that $n \geq 2$. Let $\mathcal{S}$ be an infinite admissible set of pairs $(m,\mu) \in \mathbb{Q}_{>0} \times L'/L$ Then there exists a Borcherds product Ψ of non-zero weight whose divisor is supported on divisors $Z(m,\mu)$ with $(m,\mu) \in \mathcal{S}$.*

Remark 1.2. It is much easier to see that there is also a (non-constant) Borcherds product of weight 0 whose divisor is supported on divisors $Z(m,\mu)$ with $(m,\mu) \in \mathcal{S}$.

This result can be used to construct sections of a *non-trivial* power of the tautological bundle over X_Γ whose divisor is supported on divisors $Z(m,\mu)$ with $(m,\mu) \in \mathcal{S}$. This is employed in the recent proof of the averaged Colmez conjecture by Andreatta, Goren, Howard, and Madapusi Pera [AGHM, Theorem 9.5.5].

Theorem 1.3. *Assume that $n \geq 1$. Every Borcherds product for Γ is the quotient of two Borcherds products for Γ which are holomorphic on X_Γ.*

This theorem is useful to reduce statements about Fourier expansions of Borcherds products to the holomorphic case. A slight variant (see Theorem 3.7), together with [HM, Theorem 6.3], can be employed to give a different proof of the converse theorem for Borcherds products [Br, Theorem 5.12] for lattices that split two hyperbolic planes over $\mathbb{Z}$. Similar results as Theorems 1.1 and 1.3 were obtained in [BBK, Section 4] for the special case of Hilbert modular surfaces for the full Hilbert modular group.

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2. Preliminaries

We begin by fixing some general notation. If $D \in \mathbb{Z} \setminus \{0\}$ is a discriminant, we write χ_D for the Dirichlet character $\chi_D(a) = \left(\frac{D}{a}\right)$. If a is a positive integer and χ is a Dirichlet character, we denote by $\sigma_s(a,\chi)$ the divisor sum

$$\sigma_s(a,\chi) = \sum_{d \mid a} \chi(d) d^s.$$

If $\chi = \chi_1$ is the trivial character modulo 1, we briefly write $\sigma_s(a) = \sigma_s(a,\chi_1)$. As usual the Moebius function is denoted by $a \mapsto \mu(a)$.

In this section we temporarily consider an even lattice (L,Q) of arbitrary signature (b^+,b^-). We write N for the level of L and $\det(L)$ for the Gram determinant of L. Recall that $|\det(L)| = |L'/L|$ and that N and $\det(L)$ have the same prime divisors. Moreover, we denote by $r(L)$ the Witt rank of L, i.e., the rank of a maximal totally isotropic sublattice.

As in [Bo1] we denote by $\mathrm{Mp}_2(\mathbb{Z})$ the metaplectic extension of $\mathrm{SL}_2(\mathbb{Z})$, realized by the two possible choices of a holomorphic square root $\sigma(\tau)$ of the automorphy factor $j(g,\tau) = c\tau + d$ of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ for τ in the upper complex half plane $\mathbb{H}$. If $k \in \frac{1}{2}\mathbb{Z}$, we write $M^!_k(\rho_L)$ for the space of $\mathbb{C}[L'/L]$-valued weakly holomorphic modular forms of weight k for the group $\mathrm{Mp}_2(\mathbb{Z})$ with representation ρ_L. The subspaces of holomorphic modular forms and cusp forms are denoted by $M_k(\rho_L)$ and $S_k(\rho_L)$, respectively.

2.1. **Eisenstein series.** Here we recall some facts about $\mathbb{C}[L'/L]$-valued Eisenstein series from [BK]. Let $\kappa \in \frac{1}{2}\mathbb{Z}$ with $2\kappa \equiv b^+ - b^- \pmod 4$. Assume that $\kappa > 2$.

Let $\Gamma'_\infty \subset \mathrm{Mp}_2(\mathbb{Z})$ be the stabilizer of the cusp ∞, that is, the subgroup of pairs $(g,\sigma) \in \mathrm{Mp}_2(\mathbb{Z})$ for which g is of the form $\pm \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. Let $(\chi_\mu)_{\mu \in L'/L}$ be the standard basis of $\mathbb{C}[L'/L]$. The element $\chi_0 \in \mathbb{C}[L'/L]$ transforms under Γ'_∞ with a character of the center. The corresponding Eisenstein series

$$E_{\kappa,L}(\tau) = \sum_{(g,\sigma)\in \Gamma'_\infty \setminus \mathrm{Mp}_2(\mathbb{Z})} \sigma(\tau)^{-2\kappa} \cdot \left(\rho_L(g,\sigma)^{-1}\chi_0\right)$$

defines a holomorphic function in $\tau \in \mathbb{H}$, satisfying the transformation law

$$E_{\kappa,L}(\gamma\tau) = \sigma(\tau)^{2\kappa}\rho_L(\gamma)E_{\kappa,L}(\tau)$$

for all $\gamma = (g,\sigma) \in \mathrm{Mp}_2(\mathbb{Z})$. In particular, $E_{\kappa,L}(\tau)$ belongs to $M_\kappa(\rho_L)$. It has a Fourier expansion

$$E_{\kappa,L}(\tau) = \sum_{\mu \in L'/L}\sum_{m\geq 0} e_{\kappa,L}(m,\mu)\cdot q^m \chi_\mu$$

with coefficients $e_{\kappa,L}(m,\mu)$, and $q = e^{2\pi i\tau}$. The constant term of $E_{\kappa,L}$ is given by

$$e_{\kappa,L}(0,\mu) = \begin{cases} 1, & \text{if } \mu = 0, \\ 0, & \text{if } \mu \neq 0. \end{cases}$$

This implies that $E_{\kappa,L}$ does not vanish identically.

If $\kappa = 2$, the Eisenstein series $E_{\kappa,L}(\tau)$ can be defined similarly using the usual 'Hecke trick'. It has the same properties as in the case $\kappa > 2$ with the only difference that in the constant term an additional non-holomorphic contribution (a multiple of $\Im(\tau)^{-1}$) can occur. The coefficients with positive index are still constant (see e.g. [BrKü, Section 3]).

The Fourier expansion of this Eisenstein series was computed in [BK] and [KY]. (Note that in [BK] it was worked implicitly with the lattice $(L, -Q)$. Moreover, the Eisenstein series $2E_{\kappa,L}$ was considered.) A first result is the following.

Proposition 2.1. *For all $m \in \mathbb{Q}_{>0}$ and $\mu \in L'/L$ the coefficients $e_{\kappa,L}(m,\mu)$ are rational numbers. Moreover, the quantity*

$$(-1)^{(2\kappa - b^+ + b^-)/4} e_{\kappa,L}(m,\mu)$$

is non-negative.

Proof. The rationality of the coefficients is Corollary 8 in [BK]. The non-negativity follows from Theorem 7 in [BK] by means of standard bounds for Dirichlet L-functions in the region of convergence. □

Now we specialize to the case that $\kappa = (b^+ + b^-)/2$, still assuming that $\kappa \geq 2$. The condition that $2\kappa \equiv b^+ - b^- \pmod 4$ is then equivalent to requiring that b^- is even. For $\mu \in L'/L$ we let $d_\mu = \min\{b \in \mathbb{Z}_{>0} :\ b\mu = 0\}$ be the order of μ. If $m \in \mathbb{Z} + Q(\mu)$ we write

$$N_{m,\mu}(a) = \left|\{r \in L/aL :\ Q(r+\mu) \equiv m \pmod a\}\right|.$$

This is a multiplicative function in a. Theorem 11 of [BK] gives the following explicit formulas for the Fourier coefficients of $E_{\kappa,L}$.

Theorem 2.2. *Assume that $b^+ + b^-$ is even. Let $\mu \in L'/L$, and let $m \in \mathbb{Z} + Q(\mu)$ be positive. Then*

$$e_{\kappa,L}(m,\mu) = \frac{(2\pi)^\kappa m^{\kappa-1}(-1)^{b^-/2}}{\sqrt{|L'/L|}\Gamma(\kappa)} \cdot \frac{\sigma_{1-\kappa}(d_\mu^2 m, \chi_{4D})}{L(\kappa,\chi_{4D})} \prod_{\substack{p \text{ prime} \\ p|2N}} \frac{N_{m,\mu}(p^{w_p})}{p^{(2\kappa-1)w_p}},$$

where D denotes the discriminant $D = (-1)^{(b^+ + b^-)/2}\det(L)$, and

$$w_p = w_p(m,\mu) = 1 + 2\operatorname{ord}_p(2d_\mu m).$$

Theorem 2.3. *Assume that $b^+ + b^-$ is odd. Let $\mu \in L'/L$, and let $m \in \mathbb{Z} + Q(\mu)$ be positive. Write $md_\mu^2 = m_0 f^2$ for positive integers m_0, f with $(f, 2N) = 1$ and $\operatorname{ord}_p(m_0) \in \{0,1\}$ for all primes p coprime to $2N$. Then*

$$e_{\kappa,L}(m,\mu) = \frac{(2\pi)^\kappa m^{\kappa-1}(-1)^{b^-/2}}{\sqrt{|L'/L|}\Gamma(\kappa)} \cdot \frac{L(\kappa - 1/2, \chi_{D'})}{\zeta(2\kappa-1)}$$
$$\times \sum_{d|f} \mu(d)\chi_{D'}(d) d^{1/2-\kappa}\sigma_{2-2\kappa}(f/d) \prod_{\substack{p \text{ prime} \\ p|2N}} \frac{N_{m,\mu}(p^{w_p})}{(1-p^{1-2\kappa})p^{(2\kappa-1)w_p}},$$

where D' denotes the discriminant $D' = 2(-1)^{(b^+ + b^- + 1)/2} m_0 \det(L)$, and

$$w_p = w_p(m,\mu) = 1 + 2\operatorname{ord}_p(2d_\mu m).$$

From this result we obtain the following lower bound for the coefficients.

Proposition 2.4. *Assume that $\kappa = \frac{b^+ + b^-}{2} > 2$, and let $A \geq 0$. There exists a constant $C > 0$ (depending only on A and L) such that for all $(m,\mu) \in \mathbb{Q}_{>0} \times L'/L$ satisfying*

(i) *m is represented by $L + \mu$,*
(ii) *$\operatorname{ord}_p(m) \leq A$ for all primes p dividing $2N$,*

we have

$$(-1)^{b^-/2} e_{\kappa,L}(m,\mu) > C \cdot m^{\kappa-1}.$$

Proof. This is a direct consequence of Theorem 2.2 and Theorem 2.3, combined with elementary estimates for L-functions of quadratic Dirichlet characters.

For instance, in the case when n is odd, we have

$$L(\kappa - 1/2, \chi_{D'}) > \frac{\zeta(2\kappa-1)}{\zeta(\kappa - 1/2)}.$$

This follows from the Euler product expansion which converges absolutely since $\kappa \geq 5/2$. In the sum over the divisors of f, the term $d = 1$ is dominating. Using again the fact that

$\kappa \geq 5/2$, we obtain

$$\begin{aligned}\sum_{d|f} \mu(d)\chi_{D'}(d) d^{1/2-\kappa}\sigma_{2-2\kappa}(f/d) &\geq 1 - \sum_{\substack{d|f\\ d>1}} d^{1/2-\kappa}\sigma_{2-2\kappa}(f/d)\\ &\geq 1-(\zeta(2)-1)\zeta(3)\\ &\geq 1/5.\end{aligned}$$

Finally, the condition (i) implies that the representation numbers $N_{m,\mu}(a)$ modulo a are all at least 1. Condition (ii) implies that for primes p dividing $2N$ the quantities w_p are bounded by $1 + 2(\mathrm{ord}_p(2N) + A)$. Hence

$$\prod_{\substack{p \text{ prime}\\ p|2N}} \frac{N_{m,\mu}(p^{w_p})}{p^{(2\kappa-1)w_p}}$$

is greater than a positive constant. This concludes the proof of the proposition. □

Remark 2.5. If $\kappa = 2$ then the assertion of Proposition 2.4 is still true with the slightly weaker lower bound

$$(-1)^{b^-/2} e_{\kappa,L}(m,\mu) > C \cdot m^{\kappa-1-\varepsilon}$$

for any $\varepsilon > 0$, and a constant C depending in addition on ε. Here the extra $m^{-\varepsilon}$ term comes from bounding $\sigma_{1-\kappa}(d_\mu^2 m, \chi_{4D})$ in this case.

3. Proofs

Here we turn to the proofs of the theorems stated in the introduction.

3.1. **Weakly holomorphic modular forms.**

Throughout this subsection we assume that L has signature $(n, 2)$ with $n \geq 2$ and put $\kappa = 1 + n/2$.

Let $\mathcal{S}$ be an infinite admissible set of pairs $(m,\mu) \in \mathbb{Q}_{>0} \times L'/L$ as in the introduction. In view of of [Bo1, Theorem 13.3], Theorem 1.1 of the introduction is a consequence of the following proposition.

Proposition 3.1. *There exists a weakly holomorphic modular form $f \in M^!_{2-\kappa}(\rho_{L^-})$ with integral Fourier coefficients $c_f(l,\nu)$ with the properties:*

(i) *if $(m,\mu) \in \mathbb{Q}_{>0} \times L'/L$ with $c_f(-m,\mu) \neq 0$, then $(m,\mu) \in \mathcal{S}$,*
(ii) *$c_f(0,0) \neq 0$.*

To prove this proposition, we recall the following result from [Bo2] (see also Corollary 3.9 in [BF]).

Proposition 3.2. *There exists a weakly holomorphic modular form $f \in M^!_{2-\kappa}(\rho_{L^-})$ with Γ'_∞-invariant prescribed principal part*

$$\sum_{\nu\in L'/L} \sum_{l<0} c(l,\nu)\, q^l \chi_\nu \in \mathbb{C}[L'/L][q^{-1/N}]$$

at the cusp ∞ if and only if

$$\sum_{\nu\in L'/L}\sum_{l>0} c(-l,\nu)\, b(l,\nu) = 0 \tag{3.1}$$

for every $g = \sum_\nu \sum_l b(l,\nu) q^l \chi_\nu \in S_\kappa(\rho_L)$.

If $n > 2$ or $n = 2 > r(L)$, the constant term $c(0,0)$ of such an f is given in terms of the coefficients of the Eisenstein series $E_{\kappa,L} \in M_\kappa(\rho_L)$ by

$$c(0,0) = -\sum_{\nu\in L'/L}\sum_{l>0} c(-l,\nu)\, e_{\kappa,L}(l,\nu). \tag{3.2}$$

Remark 3.3. The condition on the Witt rank $r(L)$ in the second part of the proposition implies that the Eisenstein series $E_{\kappa,L}$ is holomorphic even in the case $n = 2$.

Proof of Proposition 3.1. We generalize the argument of [BBK, Lemma 4.11]. According to [McG], the space $M^!_{2-\kappa}(\rho_{L^-})$ has a basis of weakly holomorphic modular forms with integral coefficients. Hence, it suffices to show the existence of an $f \in M^!_{2-\kappa}(\rho_{L^-})$ with *rational* coefficients satisfying (i) and (ii). Let us first assume that $n > 2$ or $n = 2 > r(L)$, so that (3.2) holds.

To lighten the notation, throughout the proof we write M_κ for the $\mathbb{Q}$-vector space of holomorphic modular forms in $M_\kappa(\rho_L)$ with rational coefficients. We write S_κ for the subspace of cusp forms with rational coefficients. The $\mathbb{Q}$-dual spaces are denoted by $M_\kappa^\vee$ and $S_\kappa^\vee$, respectively. The natural inclusion $S_\kappa \to M_\kappa$ induces a surjective linear map

$$\mathrm{pr} : M_\kappa^\vee \to S_\kappa^\vee, \quad a \mapsto \mathrm{pr}(a).$$

For $\mu \in L'/L$ and $m \in \mathbb{Z} + Q(\mu)$, we write $a_{m,\mu}$ for the element in $M_\kappa^\vee$ taking an element $g = \sum_\nu \sum_l b(l,\nu) q^l \chi_\nu \in M_\kappa$ to the Fourier coefficient

$$a_{m,\mu}(g) = b(m,\mu).$$

We let $M^\vee_{\kappa,\mathcal{S}} \subset M_\kappa^\vee$ be the subspace generated by the functionals $a_{m,\mu}$ with $(m,\mu) \in \mathcal{S}$. According to Proposition 3.2, it suffices to show that there exists an $a \in M^\vee_{\kappa,\mathcal{S}}$ with $\mathrm{pr}(a) = 0$ and $a(E_{\kappa,L}) \neq 0$.

Let $a_1, \dots, a_d \in M^\vee_{\kappa,\mathcal{S}}$ such that $\mathrm{pr}(a_1), \dots, \mathrm{pr}(a_d)$ is a basis of $\mathrm{pr}(M^\vee_{\kappa,\mathcal{S}}) \subset S_\kappa^\vee$. Then for every $(m,\mu) \in \mathcal{S}$ there exists a unique vector $r(m,\mu) = (r_1(m,\mu), \dots, r_d(m,\mu)) \in \mathbb{Q}^d$ such that

$$\mathrm{pr}(a_{m,\mu}) = r_1(m,\mu)\cdot \mathrm{pr}(a_1) + \ldots + r_d(m,\mu)\cdot \mathrm{pr}(a_d).$$

The linear combination

$$\tilde a_{m,\mu} := a_{m,\mu} - r_1(m,\mu)\cdot a_1 - \ldots - r_d(m,\mu)\cdot a_d \in M^\vee_{\kappa,\mathcal{S}} \tag{3.3}$$

is in the kernel of pr. Evaluating $\tilde a_{m,\mu}$ at the Eisenstein series, we obtain

$$\tilde a_{m,\mu}(E_{\kappa,L}) = e_{\kappa,L}(m,\mu) - r_1(m,\mu)\cdot a_1(E_{\kappa,L}) - \ldots - r_d(m,\mu)\cdot a_d(E_{\kappa,L}). \tag{3.4}$$

In view of (3.2), it suffices to show that there is an $(m,\mu) \in \mathcal{S}$ such that $\tilde a_{m,\mu}(E_{\kappa,L})$ is non-zero.

To see this, we assume on the contrary that $\tilde{a}_{m,\mu}(E_{\kappa,L}) = 0$ for all $(m,\mu) \in \mathcal{S}$. We let $\|r\|$ be the euclidian norm of a vector $r \in \mathbb{R}^d$. Moreover, we also denote by $\|\cdot\|$ a norm on $S_\kappa^\vee \otimes \mathbb{R}$, say the operator norm. Since $\operatorname{pr}(a_1), \dots, \operatorname{pr}(a_d)$ are linearly independent, there exists an $\varepsilon > 0$ such that

$$\|r_1 \operatorname{pr}(a_1) + \ldots + r_d \operatorname{pr}(a_d)\| \geq \varepsilon \|r\|$$

for all $r = (r_1, \dots, r_d) \in \mathbb{R}^d$. By means of (3.3) we obtain

$$\|\operatorname{pr}(a_{m,\mu})\| \geq \varepsilon \cdot \|r(m,\mu)\|. \tag{3.5}$$

On the other hand, our assumption $\tilde{a}_{m,\mu}(E_{\kappa,L}) = 0$ and (3.4) imply that there is a constant $C' > 0$ such that

$$|e_{\kappa,L}(m,\mu)| \leq C' \cdot \|r(m,\mu)\|. \tag{3.6}$$

Together, (3.5) and (3.6) imply that

$$|e_{\kappa,L}(m,\mu)| \leq \frac{C'}{\varepsilon} \cdot \|\operatorname{pr}(a_{m,\mu})\| \tag{3.7}$$

for all $(m,\mu) \in \mathcal{S}$. The Weil bound for the coefficients of (scalar valued) cusp forms of weight κ for $\Gamma(N)$ implies that $\|\operatorname{pr}(a_{m,\mu})\| = O(m^{\kappa/2-1/4+\delta})$ as $m \to \infty$ for any $\delta > 0$. Combining this with (3.7) we obtain

$$|e_{\kappa,L}(m,\mu)| = O(m^{\kappa/2-1/4+\delta})$$

for $(m,\mu) \in \mathcal{S}$ and $m \to \infty$, contradicting Proposition 2.4 and Remark 2.5.

We finally consider the remaining case $n = 2 = r(L)$. From (3.1) and the fact that $\mathcal{S}$ is infinite, we easily deduce the existence of an f satisfying condition (i) of Proposition 3.1, but possibly violating condition (ii). The fact that $r(L) = 2$ implies that there is an even overlattice $M \supset L$ which is isomorphic to the even unimodular lattice $II_{2,2}$ of signature $(2,2)$. This in turn implies that $\mathbb{C}[L'/L]$ contains a rational vector f_0 which is invariant under the Weil representation ρ_{L^-} and which has non-zero χ_0-component. In other words, f_0 is a non-zero element of $M_0(\rho_{L^-})$. A suitable linear combination of f and f_0 satisfies both conditions of Proposition 3.1. □

3.2. **Proof of Theorem 1.3.** Throughout this subsection we assume that (L,Q) has signature $(n,2)$ with $n \geq 1$. We briefly write L^- for the lattice $(L,-Q)$ of signature $(2,n)$.

Lemma 3.4. *Let $b \in \mathbb{Z}$ such that $k := 1-n/2+12b$ is greater than 2. The Eisenstein series $E_{k,L^-} \in M_k(\rho_{L^-})$ has non-negative Fourier coefficients $e_{k,L^-}(l,\mu)$. When $l \in -Q(\mu+L)$ is positive, the coefficient $e_{k,L^-}(l,\mu)$ is strictly positive.*

Proof. The non-negativity of the coefficients is a direct consequence of Proposition 2.1. When $l \in -Q(\mu+L)$, then the congruence representation numbers $N_{l,\mu}(p^\nu)$ for the lattice L^- are all positive, since there even exists a global solution. Therefore the claimed positivity follows from Theorem 2.2 and Theorem 2.3. □

For $\mu \in L'/L$ we define

$$t_\mu = \min\{-Q(\lambda) \mid \ \lambda \in \mu + L \text{ and } -Q(\lambda) > 0\},$$
$$T = \max\{t_\mu \mid \ \mu \in L'/L\}.$$

Since L is indefinite, t_μ has a finite value in $\frac{1}{N}\mathbb{Z}_{>0}$. The coefficient $e_{k,L^-}(t_\mu, \mu)$ of the Eisenstein series E_{k,L^-} of Lemma 3.4 is positive (for any choice of b).

Lemma 3.5. *Let $b \in \mathbb{Z}_{>0}$ such that $k := 1 - n/2 + 12b$ is greater than 2. There exists an element $h \in M^!_{1-n/2}(\rho_{L^-})$ with non-negative rational Fourier coefficients $c_h(l, \mu)$ such that*

$$c_h(l, \mu) > 0$$

for all $\mu \in L'/L$ and all $l \in \mathbb{Z} - Q(\mu)$ with $l \geq T - b$.

Proof. Let $E_{k,L^-} \in M_k(\rho_{L^-})$ be the Eisenstein series of weight k of Lemma 3.4. Then

$$h(\tau) = \Delta(\tau)^{-b} E_{k,L^-}(\tau)$$

belongs to $M^!_{1-n/2}(\rho_{L^-})$. The product expansion $\Delta = q \prod_{j\geq 1}(1 - q^j)^{24}$ of the discriminant function implies that the Fourier coefficients $c_{\Delta^{-1}}(j)$ of Δ^{-1} with index $j \geq -1$ are all positive. Consequently, the coefficients of $c_{\Delta^{-b}}(j)$ of Δ^{-b} with index $j \geq -b$ are all positive. By Lemma 3.4 we obtain that the coefficients of h are all non-negative.

If $\mu \in L'/L$ and $l \in \mathbb{Z} - Q(\mu)$ with $l \geq T - b$, we have that

$$\begin{aligned} c_h(l, \mu) &= \sum_{j \in \mathbb{Z}} c_{\Delta^{-b}}(j) \cdot e_{k,L^-}(l - j, \mu) \\ &= c_{\Delta^{-b}}(l - t_\mu) \cdot e_{k,L^-}(t_\mu, \mu) + \sum_{\substack{j \in \mathbb{Z} \\ j \neq l - t_\mu}} c_{\Delta^{-b}}(j) \cdot e_{k,L^-}(l - j, \lambda). \end{aligned}$$

The hypothesis $l \geq T - b$ implies that $l - t_\lambda \geq -b$, and therefore, by Lemma 3.4, the first quantity on the right hand side of the latter equation is positive. Since the second quantity is non-negative, we obtain the assertion. □

We say that a weakly holomorphic modular form $f \in M^!_k(\rho_{L^-})$ with Fourier coefficients $c_f(l, \mu)$ has *non-negative principal part* if

$$c_f(l, \mu) \geq 0$$

for all $\mu \in L'/L$ and all $l < 0$. Note that the Borcherds lift $\Psi(z, f)$ of any $f \in M^!_{1-n/2}(\rho_{L^-})$ with integral and non-negative principal part is holomorphic on X_Γ. Theorem 1.3 is a direct consequence of the following proposition.

Proposition 3.6. *Let $f \in M^!_{1-n/2}(\rho_{L^-})$. There exist $f_1, f_2 \in M^!_{1-n/2}(\rho_{L^-})$ with non-negative principal part such that $f = f_1 - f_2$. If f has integral principal part, we may also choose f_1 and f_2 with integral principal part.*

Proof. Let $b \in \mathbb{Z}_{>0}$ such that $b-T$ is greater than the order of the pole of f at ∞. Let h be the corresponding element of $M^!_{1-n/2}(\rho_{L^-})$ as in Lemma 3.5. Then there exists a positive integer c such that

$$f_1 = f + c \cdot h$$

has non-negative principal part. Setting in addition $f_2 = c \cdot h$ gives the desired representation of f. □

We close this section with a variant of Theorem 1.3, which can be proved similarly.

Theorem 3.7. *For every $\mu \in L'/L$ and every positive $m \in Q(\mu + L)$ there exists a (non-zero) holomorphic Borcherds product for Γ which vanishes along $Z(m, \mu)$.*

In view of [HM, Theorem 6.3] and [HM, Remark 7.2] this result can be employed to prove the converse theorem for Borcherds products for lattices that split two hyperbolic planes over $\mathbb{Z}$ (see [Br, Theorem 5.12]) in a completely different way.

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