INFINITE PRODUCTS IN NUMBER THEORY AND GEOMETRY

JAN HENDRIK BRUINIER

ABSTRACT. We give an introduction to the theory of Borcherds products and to some number theoretic and geometric applications. In particular, we discuss how the theory can be used to study the geometry of Hilbert modular surfaces.

1. INTRODUCTION

Infinite products play an important role in many branches of mathematics. In number theory, they for instance provide an elegant way of encoding and manipulating combinatorial identities. The product expansion of the generating function of the partition function is a well known example. On the other hand, infinite products are a fundamental tool in complex analysis to construct meromorphic functions with prescribed zeros and poles, the Weierstrass product theorem being a prominent example. In that way, they become interesting for the study of geometric problems.

In the first part of the present paper we will present some examples of particularly interesting infinite products, called Borcherds products, which are characterized by a striking symmetry property: They are modular forms for the orthogonal group of a suitable rational quadratic space of signature $(2, \ell)$. Although some very classical modular forms appear here, as for instance certain Eisenstein series or the j-function, most of these product expansions were only discovered rather recently by R. Borcherds [Bo1, Bo4].

We will then consider more systematically the properties of Borcherds products on Hilbert modular surfaces. They can be used to recover important classical results on the geometry of such surfaces. In addition, they provide a new approach to various arithmetic questions.

Hilbert modular surfaces can be realized as arithmetic quotients associated to certain rational orthogonal groups of signature $(2, 2)$. In particular, they are just very special instances of those Shimura varieties that can be obtained as quotients from orthogonal groups of signature $(2, \ell)$, the general setting of Borcherds' theory. However, we feel that focusing on such a special case facilitates the presentation of central ideas. Moreover, the geometry of Hilbert modular surfaces is particularly beautiful. Most of the results stated in sections 2–7 actually hold in greater generality.

The present paper is not intended to be a survey on Borcherds products. It only covers a small part of the many interesting aspects of the theory. For further expository articles, in particular for the connection to generalized Kac-Moody algebras, we refer to [Bo2], [Bo3], [Bo5], [Ko].

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2. Infinite products and elliptic modular forms

Recall that an infinite product

$$(1 - a_1)(1 - a_2)(1 - a_3)\ldots$$

is said to converge absolutely, if the underlying series

$$a_1 + a_2 + a_3 + \ldots$$

converges absolutely. With this definition, an absolutely convergent infinite product vanishes, if and only if one of its factors vanishes. So for instance the product

$$h(q) = \prod_{n=1}^{\infty}(1 - q^n)$$

converges absolutely for any complex number q with $|q| < 1$ and does not vanish there. Its underlying series is the geometric series $\sum_{n\geq 1} q^n$.

This first example already has very interesting combinatorial and geometric properties. The function $1/h(q)$ is holomorphic for $|q| < 1$ and hence can be expanded in an infinite series

$$1/h(q) = \sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + \ldots.$$

It is easily verified that the function $p(n)$ is the so called partition function. It counts the number of ways a positive integer n can be written as a sum of positive integers. The number of summands is unrestricted, repetition is allowed, and the order of the summands is not taken into account. For instance there are 5 partitions of 4, namely 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$, $1 + 1 + 1 + 1$.

The arithmetic of partitions is quite involved and there are a number of unsolved questions about them (see e.g. [On]). One reason is that the partition function grows rather rapidly. This is not apparent from the first few values give above, but a quick computation (using e.g. Maple) shows that for instance $p(100) = 190\,569\,292$. In fact the celebrated Hardy-Rademacher-Ramanujan asymptotic states

(2.1)
$$p(n) \sim \frac{e^{K\sqrt{n}}}{4n\sqrt{3}}$$

as $n \to \infty$, where $K = \pi\sqrt{2/3}$ (c.f. [Ap] chapter 5). This follows from the fact that $1/h(q)$ is closely related to a (weakly holomorphic) modular form of weight $-1/2$. The Fourier coefficients of such modular forms satisfy similar asymptotics in general.

In a different direction, if we put $q = e^{2\pi i \tau}$ for τ in the upper complex half plane $\mathbb{H} = \{\tau \in \mathbb{C}; \Im(\tau) > 0\}$, we are lead to the discriminant function

$$(2.2) \qquad \Delta(\tau) = q \cdot h(q)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Since the product converges (locally uniformly) absolutely, it defines a nowhere vanishing holomorphic function on $\mathbb{H}$. Moreover, Δ has a striking symmetry property with respect to the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ by Moebius transformations $\tau \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = \frac{a\tau+b}{c\tau+d}$. It satisfies

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12}\Delta(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(for a simple proof see [Ko]).

To put this in a suitable context, we recall some basic notions of the theory of modular functions (see e.g. [Sh], [Ma], [Ap] for more details). Let k be an integer, Γ a subgroup of finite index of $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$, and χ an Abelian character of Γ. A function $f : \mathbb{H} \to \mathbb{C}$ is called a *weakly holomorphic modular form* (of weight k with respect to Γ and χ), if

(1) $f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(\gamma)(c\tau + d)^k f(\tau)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,
(2) f is holomorphic on $\mathbb{H}$,
(3) f is meromorphic at the cusps of Γ.

We do not want to explain the last condition in too much detail. If $\Gamma = \Gamma(1)$ (which is all we need in this section), then any function f satisfying the first two conditions has a Fourier expansion

$$f(\tau) = \sum_{n=-\infty}^{\infty} c(n)q^n,$$

because $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(1)$. Now the third condition means that only finitely many $c(n)$ with $n < 0$ are non-zero. If actually all $c(n)$ with $n < 0$ (respectively $n \leq 0$) vanish, then f is called a *holomorphic modular form* (respectively *cusp form*) for $\Gamma(1)$. If Γ has only finite index in $\Gamma(1)$, one has to require similar conditions for finitely many $\Gamma(1)$-translates of f corresponding to the cusps of Γ, i.e., the Γ-orbits of $P^1(\mathbb{Q}) = \mathbb{Q} \cup \infty$.

We write W_k (respectively M_k, S_k) for the space of weakly holomorphic modular forms (respectively holomorphic modular forms, cusp forms) of weight k for $\Gamma(1)$ with trivial character. One can show that $M_k = \{0\}$ for $k < 0$ or k odd, $M_0 = \mathbb{C}$, and $M_2 = 0$.

The properties of Δ stated above can be summarized by saying that Δ is a cusp form of weight 12 for $\Gamma(1)$ with trivial character, i.e., an element of S_{12}.

Further examples of modular forms are provided by Eisenstein series. If k is an even integer, $k > 2$, we define the Eisenstein series of weight k for $\Gamma(1)$ by

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} (c\tau + d)^{-k}.$$

By comparison with a suitable integral, one checks that E_k converges normally and defines a holomorphic function on $\mathbb{H}$. Consequently the transformation behavior of a modular form of weight k easily follows by reordering the summation. The Fourier expansion of E_k can be computed by means of the partial fraction expansion of the cotangens. One finds that

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where B_k denotes the k-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ the usual divisor sum function. In particular E_k is a (non-zero) holomorphic modular form of weight k for $\Gamma(1)$. As a consequence we get the decomposition $M_k = \mathbb{C}E_k \oplus S_k$ for $k > 2$.

The non-vanishing of Δ on $\mathbb{H}$ implies that we have an isomorphism

$$S_k \rightarrow M_{k-12}, \quad f \mapsto f/\Delta.$$

This observation is crucial. It can be deduced that the graded algebra $\bigoplus_{k \in \mathbb{Z}} M_k$ of modular forms for $\Gamma(1)$ is a polynomial ring in E_4 and E_6.

To get a more geometric interpretation of modular forms for $\Gamma \subset \Gamma(1)$, one considers the modular curve associated with Γ, that is, the quotient $Y_\Gamma = \Gamma \backslash \mathbb{H}$. The complex structure of $\mathbb{H}$ induces a structure as a non-compact Riemann surface on Y_Γ. One obtains a compact Riemann surface X_Γ by adding finitely many points, namely the cusps of Γ. Every compact Riemann surface X has a natural structure as a projective algebraic variety over $\mathbb{C}$. The Zariski topology is given by taking for the closed sets the finite subsets of X together with X itself. The structure sheaf $\mathcal{O}$ is given by taking for the regular functions $\mathcal{O}(U)$ on an open subset $U \subset X$ all meromorphic functions on X, which are holomorphic on U.

The modular curves Y_Γ and X_Γ associated to certain families of "congruence subgroups" Γ can actually be defined over algebraic number fields and even over their rings of integers (see [Sh], [DeRa], [DI]). This is due to their moduli interpretation. For instance, the modular curve $\Gamma(1)\backslash\mathbb{H}$ is the (coarse) moduli space of isomorphism classes of elliptic curves over $\mathbb{C}$. In fact, by the theory of the Weierstrass $\wp$-function, an elliptic curve over $\mathbb{C}$ is a torus, which in turn is given by a quotient $\mathbb{C}/L$, where $L \subset \mathbb{C}$ is a lattice. Two elliptic curves $\mathbb{C}/L$ and $\mathbb{C}/L'$ are isomorphic, if and only if $L = aL'$ for some non-zero $a \in \mathbb{C} - \{0\}$. In particular every isomorphism class has a representative $\mathbb{C}/L_\tau$, where $L_\tau = \mathbb{Z} + \tau\mathbb{Z}$ and $\tau \in \mathbb{H}$. It is easily checked that two elliptic curves $\mathbb{C}/L_\tau$ and $\mathbb{C}/L_{\tau'}$ with $\tau, \tau' \in \mathbb{H}$ are isomorphic, if and only if τ is equivalent to τ' with respect to the action of $\Gamma(1)$ on $\mathbb{H}$ (this corresponds to a change of basis of the lattice). Hence the assignment $\tau \mapsto \mathbb{C}/L_\tau$ induces the identification of $\Gamma(1)\backslash\mathbb{H}$ with the moduli space. The point is that the moduli problem makes sense not only over $\mathbb{C}$ but over any scheme over $\mathbb{Z}$, allowing to construct models over Dedekind rings.

We may regard Δ as a section of the line bundle of modular forms of weight 12 on $\Gamma(1)\backslash\mathbb{H}$. In view of the moduli interpretation one checks that Δ assigns to the elliptic curve $\mathbb{C}/L_\tau$ its discriminant $\Delta(\tau)$.

A further classical modular form is the j-function:

$$j(\tau) = E_4^3(\tau)/\Delta(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots.$$

Since Δ does not vanish on $\mathbb{H}$, the j-function is holomorphic on $\mathbb{H}$. However, because of the presence of the term q^{-1} in the Fourier expansion, it is only meromorphic at the cusp ∞. Hence $j \in W_0$. In terms of the moduli interpretation, j assigns to the elliptic curve $\mathbb{C}/L_\tau$ its j-invariant $j(\tau)$. In the theory of elliptic curves one shows that j classifies elliptic curves over an algebraically closed field up to isomorphism. Moreover, for every $z \in \mathbb{C}$ there is an elliptic curve with prescribed j-invariant z. In other words, j defines an analytic isomorphism

$$\Gamma(1)\backslash\mathbb{H} \longrightarrow \mathbb{C}.$$

It extends to an isomorphism $X_{\Gamma(1)} \to P^1(\mathbb{C})$ to the Riemann sphere $P^1(\mathbb{C})$.

What does j have to do with infinite products? There are different important infinite products involving j. Here we only present one of these. To this end it is convenient to define $J(\tau) = j(\tau) - 744 \in W_0$. We denote the Fourier expansion by $J(\tau) = \sum_{n \geq -1} c(n)q^n$ with coefficients $c(n) \in \mathbb{Z}$. In particular we have $c(-1) = 1$, and $c(0) = 0$. We consider the modular form of two variables

$$j(z_1) - j(z_2)$$

of weight 0 for the group $\Gamma(1) \times \Gamma(1)$. It vanishes at a point $(z_1, z_2) \in \mathbb{H} \times \mathbb{H}$, if and only if $z_1 = Mz_2$ for some $M \in \Gamma(1)$, because of the injectivity of j. In other words, the divisor of $j(z_1) - j(z_2)$ is given by quadratic equations

$$(2.3) \qquad\qquad cz_1z_2 + dz_1 - az_2 - b = 0$$

with integral coefficients and $ad - bc = 1$.

Theorem 2.1. *The modular form $j(z_1) - j(z_2)$ has the infinite product expansion*

$$(2.4) \qquad\qquad j(z_1) - j(z_2) = q_1^{-1} \prod_{\substack{m>0 \\ n\in\mathbb{Z}}} \left(1 - q_1^m q_2^n\right)^{c(mn)}.$$

Here $q_j = e^{2\pi i z_j}$, and $c(n)$ is the n-th Fourier coefficient of $J(\tau)$. The product converges normally for $\Im(z_1)\Im(z_2) > 1$.

This beautiful identity was found in the 80's independently by Borcherds, Koike, Norton, and Zagier. It is the denominator identity of the monster Lie algebra, which is a generalized Kac-Moody algebra with an action of the Monster simple group. It is crucial in Borcherds' proof of the moonshine conjecture (see [Bo5] for an introduction and further references).

One may wonder why Theorem 2.1 had not been discovered earlier. It only involves classical modular functions that were well understood already at the end of the 19th century. One reason might be that the product only converges on a sub-domain of $\mathbb{H}^2$. This is due to the fact that the $c(n)$, being the coefficients of a *weakly* holomorphic modular form with a pole at the cusp ∞, grow rather rapidly:

$$(2.5) \qquad\qquad c(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}},$$

as $n \to \infty$. This asymptotic is analogous to the one for the partition function (2.1). The restricted convergence of the product for $j(z_1) - j(z_2)$ implies that only part of the divisor

can be read off directly from the product. More precisely, it only tells us the vanishing along those divisors of type (2.3) with $c = 0$.

One might ask, whether the product expansions of the discriminant function $\Delta(\tau)$ and the function $j(z_1) - j(z_2)$ have anything in common. This is in fact true. Both are particular examples of *Borcherds products*. These are certain meromorphic modular forms in ℓ variables, which have a particular product expansion, and arise as lifts of weakly holomorphic modular forms of weight $1 - \ell/2$ for $\Gamma(1)$. Their zeros and poles are explicitly given in terms of so-called Heegner divisors (also referred to in the literature as "rational quadratic divisors" or "special divisors").

In the next section we will make this informal definition more precise. Let us just remark here that $j(z_1) - j(z_2)$ can be viewed as the Borcherds lift of the weight 0 modular form $J(\tau)$, and $\Delta(\tau)$ as the Borcherds lift of the classical weight 1/2 Jacobi theta function $12\theta(\tau) = 12 \sum_{n \in \mathbb{Z}} q^{n^2}$. It can be shown that the j-function and the Eisenstein series E_k, with $k = 4, 6, 8, 10, 14$ are also lifts of certain weakly holomorphic weight 1/2 modular forms. For instance

$$E_4 = \prod_{n=1}^{\infty} (1 - q^n)^{c(n^2)},$$

where $c(n)$ denote the coefficients of the weight 1/2 modular form (for $\Gamma_0(4)$ in the Kohnen plus-space)

$$f = q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 + 1707264q^8 - 4096240q^9 + 44330496q^{12} + \dots,$$

see [Bo1].

Finally, notice that Borcherds products naturally live on the hermitian symmetric space of type IV associated to the real orthogonal group $O(2, \ell)$ of signature $(2, \ell)$. This hermitian space has complex dimension ℓ. They are constructed using a regularized theta lift for the dual reductive pair $(SL_2(\mathbb{R}), O(2, \ell))$. In small dimensions however, there are exceptional isomorphisms relating $O(2, \ell)$ to other classical Lie groups. For instance $O(2, 1)$ is essentially isomorphic to $SL_2(\mathbb{R})$, and $O(2, 2)$ to $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, which is implicitly used in the construction of the examples above. Moreover, $O(2, 3)$ is essentially isomorphic to the symplectic group $Sp(2, \mathbb{R})$ of genus 2, and $O(2, 4)$ to the hermitian symplectic group of genus 2. In view of these isomorphisms, also the Heegner divisors alluded to above become classically well known objects. For instance in the $O(2, 1)$ case, one gets Heegner points on modular or Shimura curves, justifying the terminology. In the $O(2, 2)$ case, one can obtain Hirzebruch-Zagier divisors on Hilbert modular surfaces, in the $O(2, 3)$ case Humbert surfaces on Siegel modular threefolds.

3. Borcherds products on Hilbert modular surfaces

We now want to generalize the $j(z_1) - j(z_2)$ example of the the previous section and study Borcherds' construction of infinite automorphic products (in two variables) in a more systematic way. From a geometric point of view, the underlying modular variety $\Gamma(1)^2 \backslash \mathbb{H}^2$ in that example was not very exciting. By means of the j-function it is isomorphically mapped to the affine plane $\mathbb{C}^2$. If one wants to get more interesting varieties one has to

replace the discrete subgroup $\Gamma(1) \times \Gamma(1) \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ by more complicated groups. An important family of discrete subgroups is provided by Hilbert modular groups of real quadratic fields.

We first need to introduce some notation (see [Fr], [Ge2], [Go] for more details). Throughout we use $z = (z_1, z_2)$ as a variable on $\mathbb{H}^2$ and write (y_1, y_2) for its imaginary part. Let K be the real quadratic field of discriminant D. For simplicity we assume throughout that D is a prime (hence $D \equiv 1 \pmod 4$ and $K = \mathbb{Q}(\sqrt{D})$). We write $\mathcal{O}_K$ for the ring of integers and $x \mapsto x'$ for the conjugation in K. The Hilbert modular group $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$ associated with K can be viewed as a discrete subgroup of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ by means of the two embeddings of K into $\mathbb{R}$. In particular Γ_K acts on $\mathbb{H}^2$ by

$$(z_1, z_2) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right).$$

In the same way as with the modular curves Y_Γ of the previous section, we consider the quotient $Y_K = \Gamma_K \backslash \mathbb{H}^2$, which has a structure as a non-compact complex surface. It can be compactified by adding h_K points, namely the cusps of Γ_K, i.e., the Γ_K-orbits of $P^1(K)$. Here h_K is the class number of K. In contrast to the case of modular curves the resulting normal complex space X_K is not regular. There are quotient singularities at the elliptic fixed points, and furthermore, the cusps are highly singular points. By the theory of Hironaka the singularities can be resolved [Hi], there exists a desingularization $\pi : \widetilde{X}_K \to X_K$, such that $\mathbf{D}_K := \pi^{-1}(X_K^{sing})$ is a divisor with normal crossings. The minimal resolution of singularities was constructed by Hirzebruch [Hz].

According to the theory of Baily-Borel, the complex spaces Y_K, X_K, and $\widetilde{X}_K$ all have natural structures as quasi-projective algebraic varieties over $\mathbb{C}$. Moreover, Y_K has a moduli interpretation as the (coarse) moduli space of isomorphism classes of Abelian surfaces with multiplication by $\mathcal{O}_K$ and a certain class of polarizations (see section 8). This can be used to construct integral models. The surfaces Y_K, X_K, and $\widetilde{X}_K$ are all referred to as *Hilbert modular surfaces* associated with K.

On such surfaces there exist distinguished divisors, called Hirzebruch-Zagier divisors. (As already mentioned, they play the role of the Heegner divisors in the informal definition of Borcherds products of the previous section.) For every vector (a, b, λ) of positive norm $ab - \lambda\lambda'$ in the lattice $\mathbb{Z}^2 \oplus \mathfrak{d}^{-1}$, the subset

$$(a, b, \lambda)^\perp = \{(z_1, z_2) \in \mathbb{H}^2; \quad az_1z_2 + \lambda z_1 + \lambda' z_2 + b = 0\}$$

defines an analytic divisor on $\mathbb{H}^2$. Here $\mathfrak{d}^{-1} = \frac{1}{\sqrt{D}} \mathcal{O}_K$ is the inverse different of K. Observe the analogy with (2.3). Now let m be a positive integer. The sum

$$T(m) = \sum_{\substack{(a,b,\lambda) \in (\mathbb{Z}^2 \oplus \mathfrak{d}^{-1})/\{\pm 1\} \\ ab - \lambda\lambda' = m/D}} (a, b, \lambda)^\perp$$

is a Γ_K-invariant analytic divisor on $\mathbb{H}^2$. It descends to an algebraic divisor on the quasi-projective variety Y_K, the *Hirzebruch-Zagier divisor* of discriminant m. Moreover, we

obtain Hirzebruch-Zagier divisors on X_K by taking the closure of $T(m)$, and on $\widetilde{X}_K$ by taking the pullback with respect to the desingularization morphism.

One easily sees that $T(m) = \emptyset$, if and only if $\chi_D(m) = -1$, where χ_D is the quadratic character corresponding to K given by the Legendre symbol $\chi_D(x) = \left(\frac{D}{x}\right)$. If m is square-free and a norm of $\mathcal{O}_K$, then the normalization of $T(m)$ on X_K is isomorphic to the modular curve $X_0(m)$. If m is square-free and not a norm of $\mathcal{O}_K$ (but $\chi_D(m) = 1$), then the normalization of $T(m)$ is isomorphic to a Shimura curve associated to a certain order in the indefinite quaternion algebra $Q_m = \left(\frac{D, -m/D}{\mathbb{Q}}\right)$.

One can show that the Hirzebruch-Zagier divisors are actually defined over $\mathbb{Q}$. Moreover, they often have a moduli interpretation, which can be used to extend them to integral models. For instance, in the latter case, where the normalization of $T(m)$ is a Shimura curve, the points on $T(m)$ correspond to Abelian surfaces with quaternionic multiplication by a certain order of Q_m.

Let k be an integer. Recall that a meromorphic (respectively holomorphic) *Hilbert modular form* of weight k for the group Γ_K is a meromorphic (respectively holomorphic) function f on $\mathbb{H}^2$ satisfying the transformation law

$$(3.1) \qquad f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z_1, z_2)\right) = (cz_1 + d)^k (c'z_2 + d')^k f(z_1, z_2)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_K$. Notice that in contrast to the definition of modular forms for subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in the previous section, we do not have to require that f be meromorphic (respectively holomorphic) at the cusps. This is automatic by the Koecher principle.

We are now ready to explain Borcherds' lift in the case of Hilbert modular surfaces. Let us first describe the "input data" which is used to feed the lift. We denote by $W_k(D, \chi_D)$ the space of weakly holomorphic modular forms of weight k for the Hecke group

$$(3.2) \qquad \Gamma_0(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}); \quad c \equiv 0 \pmod{D} \right\}$$

with character χ_D (where $\chi_D(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \chi_D(d)$). Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(D)$, any such modular form f has a Fourier expansion of the form $f = \sum_{n \gg -\infty} c(n)q^n$. We let $W_k^+(D, \chi_D)$ be the subspace of those $f \in W_k(D, \chi_D)$, whose Fourier coefficients $c(n)$ satisfy the so-called *plus space condition*, i.e., $c(n) = 0$ whenever $\chi_D(n) = -1$. Moreover, we write $M_k^+(D, \chi_D)$ (respectively $S_k^+(D, \chi_D)$) for the subspace of holomorphic modular forms (respectively cusp forms) in $W_k^+(D, \chi_D)$. For even $k \geq 2$, Hecke proved that there is a unique normalized Eisenstein series $E_k(\tau)$ in $M_k^+(D, \chi_D)$, and

$$(3.3) \qquad M_k^+(D, \chi_D) = S_k^+(D, \chi_D) \oplus \mathbb{C} E_k(\tau),$$

see [He], and section 7 here.

If $f = \sum_{n \in \mathbb{Z}} c(n)q^n \in \mathbb{C}((q))$ is a formal Laurent series, we put

$$\tilde{c}(n) = \begin{cases} c(n), & \text{if } n \not\equiv 0 \pmod{D}, \\ 2c(n), & \text{if } n \equiv 0 \pmod{D}. \end{cases}$$

Modular forms in the plus space $W_k^+(D, \chi_D)$ can also be realized as vector valued modular forms for the full modular group $\Gamma(1)$ transforming with a certain D-dimensional unitary representation (see [BB]). As a consequence, there is a bilinear pairing assigning to two modular forms $f \in W_k^+(D, \chi_D)$ and $g \in W_{k'}^+(D, \chi_D)$ a modular form $\langle f, g \rangle \in W_{k+k'}$ for the group $\Gamma(1)$. If $f = \sum_n c(n)q^n$ and $g = \sum_n b(n)q^n$, it can be described in terms of the Fourier expansions as follows:

$$(3.4) \qquad \langle f, g \rangle = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \tilde{c}(m) b(Dn - m) q^n.$$

Notice that this pairing only depends on the transformation properties of modular forms in the plus space and naturally extends to non-holomorphic forms.

Theorem 3.1 (Borcherds). *Let $f = \sum_{n \gg -\infty} c(n)q^n$ be a weakly holomorphic modular form in $W_0^+(D, \chi_D)$ and assume that $\tilde{c}(n) \in \mathbb{Z}$ for all $n < 0$. Then there exists a meromorphic Hilbert modular form $\Psi(z_1, z_2, f)$ for Γ_K (with some multiplier system of finite order) such that:*

(i) *The weight of Ψ is equal to the constant term $c(0)$ of f.*

(ii) *The divisor $Z(f)$ of Ψ is determined by the principal part of f at the cusp ∞. It equals*

$$Z(f) = \sum_{n < 0} \tilde{c}(n) T(-n).$$

(iii) *Let $W \subset \mathbb{H}^2$ be a Weyl chamber attached to f and put $N = \min\{n; \ c(n) \neq 0\}$. The function Ψ has the Borcherds product expansion*

$$\Psi(z_1, z_2, f) = q_1^\rho q_2^{\rho'} \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ (\nu, W) > 0}} \left(1 - q_1^\nu q_2^{\nu'}\right)^{\tilde{c}(D\nu\nu')},$$

which converges normally for all (z_1, z_2) with $y_1 y_2 > |N|/D$ outside the set of poles. Here $\rho \in K$ is the Weyl vector corresponding to W and f, and $q_j^\nu = e^{2\pi i \nu z_j}$ for $\nu \in K$.

A few additional explanations regarding Weyl chambers are in order. For each $\lambda \in \mathfrak{d}^{-1}$ of negative norm the subset

$$\lambda^\perp = \{(z_1, z_2) \in \mathbb{H}^2; \quad \lambda y_1 + \lambda' y_2 = 0\}$$

is a hyperplane of real codimension 1 in $\mathbb{H}^2$. Since f is meromorphic at the cusps, and by reduction theory, the union

$$S(f) = \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \lambda\lambda' < 0}} c(D\lambda\lambda')\lambda^\perp$$

has only finitely many Γ_K-orbits. It is invariant under the stabilizer in Γ_K of the cusp ∞. Its complement $\mathbb{H}^2 - S(f)$ decomposes into connected components, which are called the Weyl chambers attached to f. To such a Weyl chamber W (and f) one can associate the so-called Weyl vector $\rho \in K$, which we do not want to define here (see [BB] for its

explicit computation in the present case, and [Bo1, Bo4] for more general facts). Moreover, if $\nu \in K$, then one writes $(\nu, W) > 0$, if $\lambda y_1 + \lambda' y_2 > 0$ for all $(z_1, z_2) \in W$.

If $\operatorname{div}(\Psi)$ is compact in Y_K, i.e., if the normalizations of the irreducible components are Shimura curves, then $S(f)$ is empty and there is just the one Weyl chamber $W = \mathbb{H}^2$. In this case $\rho = 0$, and the condition $(\nu, W) > 0$ becomes just the condition that ν be totally positive.

Theorem 3.1 is contained (in a slightly different formulation) in Theorem 13.3 of [Bo4] (to obtain the above form see [BB]). The idea of the proof is as follows.

First, we notice that by an elementary argument the Fourier coefficients of f are bounded by

$$(3.5) \qquad c(n) = O\left(e^{4\pi \sqrt{|Nn|}}\right), \qquad n \to \infty,$$

see [BF] section 3. This implies the convergence of the product in the stated region. The estimate (3.5) is also a consequence of the (much more precise) Hardy-Rademacher-Ramanujan asymptotic for the coefficients of weakly holomorphic modular forms, the general theorem behind (2.1) and (2.5).

As already mentioned, the group $(\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}))/\{\pm 1\}$ is isomorphic to the connected component of real orthogonal group $\mathrm{O}(2, 2)$. The latter group and $\mathrm{SL}_2(\mathbb{R})$ form a dual reductive pair in the sense of Howe [Ho]. Thus we can construct Hilbert modular forms for Γ_K from modular forms on $\mathrm{SL}_2(\mathbb{R})$ by integrating against a certain kernel function, the Siegel theta function $\Theta_S(\tau; z_1, z_2)$ associated to the lattice $L = \mathbb{Z}^2 \oplus \mathcal{O}_K$ of signature $(2, 2)$. Here the Siegel theta function is a certain non-holomorphic modular form for $\Gamma_0(D)$ satisfying the plus-space condition in the variable τ, and a Γ_K-invariant function in the variable (z_1, z_2). This suggests that we look at the integral

$$(3.6) \qquad \int_{\mathcal{F}} \langle f(\tau), \Theta_S(\tau; z_1, z_2) \rangle \, d\mu.$$

Here $\mathcal{F} = \{\tau \in \mathbb{H};\ |\tau| \geq 1,\ |u| \leq 1/2\}$ denotes the standard fundamental domain for the action of $\Gamma(1)$ on $\mathbb{H}$, and $d\mu = \frac{du\,dv}{v^2}$ is the invariant measure on $\mathbb{H}$ (with $\tau = u + iv$). Since f and the Siegel theta function are in weight 0, the integrand is $\Gamma(1)$-invariant, so that the integral makes formally sense. Unfortunately, because f grows exponentially as τ approaches the cusp ∞, it diverges wildly. However, Harvey and Moore discovered that it is possible to regularize the integral [HM]. Following their idea, Borcherds defines the regularized theta lift $\Phi(z_1, z_2, f)$ of f to be the constant term in the Laurent expansion at $s = 0$ of the meromorphic continuation in s of

$$(3.7) \qquad \lim_{t \to \infty} \int_{\mathcal{F}_t} \langle f(\tau), \Theta_S(\tau; z_1, z_2) \rangle \, v^{-s} \, d\mu.$$

Here $\mathcal{F}_t = \{\tau \in \mathcal{F};\ |v| \leq t\}$ is the truncated fundamental domain. One can show that this regularized integral still makes sense even though (3.6) does not. It defines a Γ_K-invariant

real analytic function on $\mathbb{H}^2 - \text{supp}(Z(f))$ with a logarithmic singularity[1] along the divisor $-4Z(f)$.

Moreover, it can be shown that the limit in (3.7) exists and is holomorphic at $s = 0$, if the constant term $c(0)$ of f vanishes. It follows that $\Phi(z_1, z_2, f)$ is equal to

$$\lim_{t \to \infty} \int_{\mathcal{F}_t} \left(\langle f(\tau), \Theta_S(\tau; z_1, z_2) \rangle - \tilde{c}(0)v \right) d\mu + A\tilde{c}(0),$$

where A is the constant term in the Laurent expansion at $s = 0$ of $\lim_{t \to \infty} \int_{\mathcal{F}_t} v^{1-s} d\mu$. This could be taken as an alternative definition.

The Fourier expansion of $\Phi(z_1, z_2, f)$ can be computed explicitly by applying some partial Poisson summation on the theta kernel. It turns out that

$$\Phi(z_1, z_2, f) = -2 \log \left| \Psi(z_1, z_2, f)^2 (16\pi^2 y_1 y_2)^{c(0)} \right| + 2c(0) \left(\log(8\pi) - \Gamma'(1) \right),$$

giving the meromorphic continuation, the divisor, and the transformation behavior of the infinite product $\Psi(z_1, z_2, f)$.

Notice that a weakly holomorphic modular form $f = \sum_n c(n)q^n$ in $W_k^+(D, \chi_D)$ of weight $k \le 0$ is uniquely determined by its *principal part*

$$\sum_{n<0} c(n)q^n \in \mathbb{C}[q^{-1}].$$

For the difference of two elements of $W_k^+(D, \chi_D)$ with the same principal part is a holomorphic modular form of weight $k \le 0$ with Nebentypus, hence vanishes identically. Moreover, the Galois group $\text{Gal}(\mathbb{C}/\mathbb{Q})$ acts on weakly holomorphic modular forms by mapping f to

$$f = \sum_n c(n)^\sigma q^n, \qquad \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}).$$

Here $c(n)^\sigma$ denotes the Galois conjugate of $c(n)$. This follows from the fact that the spaces $W_k(D, \chi_D)$ have bases of modular forms with integral rational Fourier coefficients (see [DeRa], [DI]). Consequently, if f has principal part in $\mathbb{Q}[q^{-1}]$, then all its Fourier coefficients are rational with bounded denominators. We may conclude that some power of any holomorphic Borcherds product has coprime integral rational Fourier coefficients and trivial multiplier system. This observation is crucial. By the q-expansion principle (see [Ge2], chapter X.3 Theorem 3.3), such a modular form corresponds to a section of the line bundle of Hilbert modular forms over $\mathbb{Z}$ on the moduli stack over $\mathbb{Z}$ representing the moduli problem "isomorphism classes of Abelian schemes with real multiplication by $\mathcal{O}_K$". Hence Borcherds products can be viewed as modular forms over $\mathbb{Z}$. In fact, they provide a powerful tool to study arithmetic intersection numbers on Hilbert modular surfaces, see [BBK] and section 9 here.

We conclude this section with an example for Theorem 3.1. We consider the Hilbert modular group Γ_K of the real quadratic field $K = \mathbb{Q}(\sqrt{5})$. The fundamental unit of K is

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[1]If X is a normal complex space, $D \subset X$ a Cartier divisor, and f a smooth function on $X - \text{supp}(D)$, then f has a logarithmic singularity along D, if for any local equation g for D on an open subset $U \subset X$, the function $f - \log|g|$ is smooth on U.

$\varepsilon_0 = \frac{1}{2}(1 + \sqrt{5}) \in \mathcal{O}_K$. Gundlach constructed a particular holomorphic Hilbert modular form $\Theta(z_1, z_2)$ of weight 5 for Γ_K as a product of 10 theta functions of weight $1/2$ [Gu]. He showed that the divisor of Θ is equal to $T(1)$ and used this fact to determine the graded algebra of Hilbert modular forms for Γ_K. From the construction one also finds that Θ has Fourier coefficients in $\mathbb{Z}$ with greatest common divisor 64.

One can recover Gundlach's function using the Borcherds lift: We need to look at the "input space" $W_0^+(5, \chi_5)$. Using some basic facts on modular forms for $\Gamma_0(5)$ due to Hecke one finds that there is a modular form $f_1 \in W_0^+(5, \chi_5)$ with Fourier expansion

$$f_1 = q^{-1} + 5 + 11\,q - 54\,q^4 + 55\,q^5 + 44\,q^6 - 395\,q^9 + 340\,q^{10} + 296\,q^{11} - 1836\,q^{14} + \dots.$$

If we plug this into the Borcherds lift, we get a Hilbert modular form Ψ for Γ_K of weight 5 with divisor $T(1)$. Hence Ψ must be a constant multiple of Θ. From the Borcherds product expansion it follows that Ψ has coprime Fourier coefficients in $\mathbb{Z}$. Consequently the constant factor is $1/64$. If we compute the Weyl vector as in [BB], we obtain the product expansion

$$(3.8) \qquad \frac{1}{64}\Theta(z_1, z_2) = q_1^{\varepsilon_0/\sqrt{5}} q_2^{(\varepsilon_0/\sqrt{5})'} \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \varepsilon_0 \nu' - \varepsilon_0' \nu > 0}} \left(1 - q_1^\nu q_1^{\nu'}\right)^{\tilde{c}(5\nu\nu')},$$

where the $c(n)$ denote the Fourier coefficients of f_1.

4. Obstructions for constructing Borcherds products

Borcherds' theorem of the previous section provides a way of constructing many Hilbert modular forms with known divisor supported on Hirzebruch-Zagier divisors $T(m)$. It is natural to seek for a precise description of those linear combinations of Hirzebruch-Zagier divisors, which are divisors of Borcherds products. Since the divisor of a Borcherds product $\Psi(z_1, z_2, f)$ is determined by the principal part of the weakly holomorphic modular form f, which is used to construct it, it suffices to understand which Fourier polynomials $\sum_{n<0} c(n)q^n \in \mathbb{C}[q^{-1}]$ can occur as principal parts of elements of $W_0^+(D, \chi_D)$. A necessary condition is easily obtained. If $f \in W_k^+(D, \chi_D)$ with Fourier coefficients $c(n)$, and $g \in M_{2-k}^+(D, \chi_D)$ with Fourier coefficients $b(n)$, then the pairing $\langle f, g \rangle$ is a weakly holomorphic modular form of weight 2 for $\Gamma(1)$. Thus

$$\langle f, g \rangle d\tau$$

is a meromorphic differential on the Riemann sphere whose only pole is at the cusp ∞. By the residue theorem its residue has to vanish. But this residue is just the constant term in the Fourier expansion of $\langle f, g \rangle$. In view of (3.4) we find that

$$(4.1) \qquad \sum_{n \le 0} \tilde{c}(n)b(-n) = 0.$$

Using Serre duality for vector bundles on Riemann surfaces, Borcherds showed that this condition is essentially also sufficient (see [Bo6] and [BB] Theorem 6).

Theorem 4.1. *There exists a weakly holomorphic modular form $f \in W_k^+(D, \chi_D)$ with prescribed principal part $\sum_{n<0} c(n)q^n$ (where $c(n) = 0$ if $\chi_D(n) = -1$), if and only if*

$$\sum_{n<0} \tilde{c}(n)b(-n) = 0$$

for every cusp form $g = \sum_{m>0} b(m)q^m$ in $S_{2-k}^+(D, \chi_D)$.

This result shows that $S_2^+(D, \chi_D)$ is precisely the space of obstructions for constructing Borcherds products on Y_K with prescribed divisor. (In the same way $M_2^+(D, \chi_D)$ can be viewed as the space of obstructions for constructing Borcherds products with prescribed divisor and weight.) The dimension of this space can be computed by means of the Riemann-Roch theorem or the Selberg trace formula. In our case, where D is prime, this was already done by Hecke [He]. One finds that

$$\dim S_2^+(D, \chi_D) = \dim M_2^+(D, \chi_D) - 1 = \left[\frac{D-5}{24}\right].$$

In particular $S_2^+(D, \chi_D) = \{0\}$ for the primes $D = 5, 13, 17$. In these cases there are no obstructions, and for any $T(m)$ there is a Borcherds products product with divisor $T(m)$. For all other prime discriminants D there are obstructions. Then for instance $T(1)$ is not the divisor of a Borcherds product, since there is a normalized Hecke eigenform g in $S_2(D, \chi_D)$. Its first Fourier coefficient is 1, and the "real part" of g is an element of $S_2^+(D, \chi_D)$, whose first Fourier coefficient is equal to 1 as well (see [Ge2], chapter VI.4). Hence q^{-1} cannot be the principal part of a weakly holomorphic modular form in $W_0^+(D, \chi_D)$.

Let us look at Theorem 4.1 from a little different angle. Let $\mathbb{C}[q^{-1}]^+$ (respectively $\mathbb{C}[[q]]^+$) be the space of Fourier polynomials (respectively formal power series), whose coefficients satisfy the plus space condition. We define a non-degenerate bilinear pairing between these spaces by putting

$$\{f, g\} = \sum_{n \leq 0} \tilde{c}(n)b(-n)$$

for $f = \sum_{n \leq 0} c(n)q^n \in \mathbb{C}[q^{-1}]^+$ and $g = \sum_{m \geq 0} b(m)q^m \in \mathbb{C}[[q]]^+$. For $k \leq 0$ the space $W_k^+(D, \chi_D)$ is identified with a subspace of $\mathbb{C}[q^{-1}]^+$ by mapping a weakly holomorphic modular form with Fourier coefficients $c(n)$ to $\sum_{n \leq 0} c(n)q^n$. The space $M_{2-k}^+(D, \chi_D)$ is identified with a subspace of $\mathbb{C}[[q]]^+$ by mapping a modular form to its q-expansion. Now Theorem 4.1 implies that the orthogonal complement of $S_{2-k}^+(D, \chi_D)$ with respect to the pairing $\{\cdot, \cdot\}$ is equal to $W_k^+(D, \chi_D) \oplus \mathbb{C}$. Using the splitting (3.3) of $M_{2-k}^+(D, \chi_D)$ one concludes that the orthogonal complement of $M_{2-k}^+(D, \chi_D)$ is precisely $W_k^+(D, \chi_D)$. Since the pairing is non-degenerate, and since $M_{2-k}^+(D, \chi_D)$ has finite dimension, it follows by linear algebra that conversely $M_{2-k}^+(D, \chi_D)$ is the orthogonal complement of $W_k^+(D, \chi_D)$. In other words:

Corollary 4.2. *A formal power series $\sum_{m \geq 0} b(m)q^m \in \mathbb{C}[[q]]^+$ is the q-expansion of a modular form in $M_{2-k}^+(D, \chi_D)$, if and only if*

$$\sum_{n \leq 0} \tilde{c}(n)b(-n) = 0$$

for every $f = \sum_n c(n)q^n$ in $W_k^+(D, \chi_D)$.

Since the pairing $\{\cdot, \cdot\}$ is defined over $\mathbb{Q}$ with respect to the natural rational structures $\mathbb{Q}[q^{-1}]^+$ and $\mathbb{Q}[[q]]^+$, and since $M_{2-k}^+(D, \chi_D)$ and $W_k^+(D, \chi_D)$ have bases of modular forms with integral coefficients, an analogous assertion holds for modular forms over $\mathbb{Q}$. Moreover, it suffices to check the condition in this corollary for every $f \in W_k^+(D, \chi_D)$ with integral coefficients.

If X is a regular projective algebraic variety, we write $\mathrm{CH}^1(X)$ for its first Chow group, i.e., the group of algebraic divisors on X modulo rational equivalence. Furthermore, we put $\mathrm{CH}^1(X)_{\mathbb{Q}} = \mathrm{CH}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Recall that $\mathrm{CH}^1(X)$ is isomorphic to the Picard group of X, the group of isomorphism classes of algebraic line bundles on X. The isomorphism is given by mapping a line bundle $\mathcal{L}$ to the class $\mathrm{c}_1(\mathcal{L})$ of the divisor of a rational section of $\mathcal{L}$. The Chow group $\mathrm{CH}^1(X)$ is an important invariant of X. It is finitely generated.

Meromorphic (respectively holomorphic) Hilbert modular forms can be interpreted as rational (respectively regular) sections of the sheaf $\mathcal{M}_k(\mathbb{C})$ of modular forms, which can be defined as follows: If we write $p : \mathbb{H}^2 \to Y_K$ for the canonical projection, then the sections over an open subset $U \subset \Gamma \backslash \mathbb{H}^2$ are holomorphic functions on $p^{-1}(U)$, satisfying the transformation law (3.1). By the Koecher principle, this sheaf on Y_K extends to X_K. Moreover, we obtain a sheaf on $\widetilde{X}_K$, also denoted by $\mathcal{M}_k(\mathbb{C})$, by taking the pullback with respect to the desingularization morphism. By the theory of Baily-Borel, there is a positive integer n_K such that $\mathcal{M}_k(\mathbb{C})$ is an algebraic line bundle if $n_K | k$, and therefore defines an element of $\mathrm{Pic}(\widetilde{X}_K)$. Notice that $\mathcal{M}_{nk}(\mathbb{C}) = \mathcal{M}_k(\mathbb{C})^{\otimes n}$ for any positive integer n. If k is any rational number, we chose an integer n such that nk is a positive integer divisible by n_K and put $\mathrm{c}_1(\mathcal{M}_k(\mathbb{C})) = \frac{1}{n} \mathrm{c}_1(\mathcal{M}_{nk}(\mathbb{C})) \in \mathrm{CH}^1(\widetilde{X}_K)_{\mathbb{Q}}$.

It is natural to study the positions of the Hirzebruch-Zagier divisors in $\mathrm{CH}^1(\widetilde{X}_K)$. To this end we consider the generating series

$$(4.2) \qquad A(\tau) = \mathrm{c}_1(\mathcal{M}_{-1/2}(\mathbb{C})) + \sum_{m > 0} T(m)q^m \in \mathbb{Q}[[q]]^+ \otimes_{\mathbb{Q}} \mathrm{CH}^1(\widetilde{X}_K)_{\mathbb{Q}}.$$

Combining Theorem 3.1 and Corollary 4.2 one obtains the following striking application.

Theorem 4.3. *The Hirzebruch-Zagier divisors generate a subspace of $\mathrm{CH}^1(\widetilde{X}_K)_{\mathbb{Q}}$ of dimension $\leq \dim(M_2^+(D, \chi_D))$. The generating series $A(\tau)$ is a modular form in $M_2^+(D, \chi_D)$ with values in $\mathrm{CH}^1(\widetilde{X}_K)_{\mathbb{Q}}$, i.e., an element of $M_2^+(D, \chi_D) \otimes_{\mathbb{Q}} \mathrm{CH}^1(\widetilde{X}_K)_{\mathbb{Q}}$.*

In other words, if λ is a linear functional on $\mathrm{CH}^1(\widetilde{X}_K)_{\mathbb{Q}}$, then

$$\lambda\left(\mathrm{c}_1(\mathcal{M}_{-1/2}(\mathbb{C}))\right) + \sum_{m > 0} \lambda(T(m))q^m \in M_2^+(D, \chi_D).$$

A typical linear functional, one can take for λ, is given by the intersection pairing with a divisor on $\widetilde{X}_K$. Theorem 4.3 was first proved by Hirzebruch and Zagier by computing intersection numbers on $\widetilde{X}_K$ of Hirzebruch-Zagier divisors with other such divisors and with the exceptional divisors coming from the resolution of the cusp singularities [HZ]. Their discovery triggered important investigations by several people, showing that more generally periods of certain special cycles in arithmetic quotients of orthogonal or unitary type can be viewed as the coefficients of Siegel modular forms. For instance, Oda considered cycles on quotients of $O(2, \ell)$ given by embedded quotients of $O(1, \ell)$ [Od1], and Kudla-Millson studied more general cycles on quotients of $O(p, q)$ and $U(p, q)$ using the Weil representation and theta functions with values in closed differential forms [KM1, KM2, KM3], see also [Fu] for the case of non-compact quotients. The relationship of the Kudla-Millson lift and Borcherds' regularized theta lift is clarified in [BF].

Using Borcherds products, Theorem 4.3 can be proved as follows (see [Bo6]). In view of Corollary 4.2 it suffices to show that

$$\tilde{c}(0)\, c_1(\mathcal{M}_{-1/2}(\mathbb{C})) + \sum_{n<0} \tilde{c}(n) T(-n) = 0 \in \mathrm{CH}^1(\widetilde{X}_K)_{\mathbb{Q}}$$

for every $f = \sum_n c(n) q^n$ in $W_0^+(D, \chi_D)$ with integral Fourier coefficients. But this is an immediate consequence of Theorem 3.1: The Borcherds lift of f is a rational section of $\mathcal{M}_{c(0)}(\mathbb{C})$ with divisor $\sum_{n<0} \tilde{c}(n) T(-n)$. Notice that we have only used (i) and (ii) of Theorem 3.1. The product expansion (iii) is not required. Finally, we mention that this argument generalizes to Heegner divisors on quotients of $O(2, \ell)$.

5. CONVERSE THEOREMS

By Theorem 4.1 of the previous section we know precisely which linear combinations of Hirzebruch-Zagier divisors occur as divisors of Borcherds products on Y_K. Here it is natural to ask, whether every Hilbert modular form on Y_K, whose divisor is a linear combination of Hirzebruch-Zagier divisors, is a Borcherds product, i.e., in the image of the lift of Theorem 3.1. In this section we discuss this question in some detail. To answer it, we first simplify the problem. We extend the Borcherds lift to a larger space of "input modular forms", given by certain Maass wave forms, and answer the question for this extended lift. In that way we are led to automorphic Green functions associated with Hirzebruch-Zagier divisors.

Let k be an integer, Γ a subgroup of finite index of $\Gamma(1)$, and χ a character of Γ. A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}$ is called a *weak Maass form* (of weight k with respect to Γ and χ), if

(1) $f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(\gamma)(c\tau+d)^k f(\tau)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,
(2) f has polynomial growth at the cusps of Γ (in terms of local parameters),
(3) $\Delta_k f(\tau) = 0$.

Here

$$(5.1) \qquad \Delta_k = -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right)$$

denotes the usual hyperbolic Laplace operator in weight k and $\tau = u + iv$.

So if we compare this with the definition of a weakly holomorphic modular form, we see that we simply replaced the condition that f be holomorphic on $\mathbb{H}$ by the weaker condition that f be annihilated by Δ_k, and the meromorphicity at the cusps by the corresponding growth condition. The third condition implies that f is actually real analytic. Because of the transformation behavior, it has a Fourier expansion involving besides the exponential function a second type of Whittaker function. (See [BF] section 3 for more details.)

There are two fundamental differential operators on modular forms for Γ, the Maass raising and lowering operators

$$R_k = 2i\frac{\partial}{\partial\tau} + kv^{-1} \qquad \text{and} \qquad L_k = -2iv^2\frac{\partial}{\partial\bar{\tau}}.$$

If f is a differentiable function on $\mathbb{H}$ satisfying the transformation law (1) above in weight k, then $L_k f$ transforms in weight $k - 2$, and $R_k f$ in weight $k + 2$. It can be shown that the assignment

$$f(\tau) \mapsto \xi_k(f)(\tau) := v^{k-2}\overline{L_k f(\tau)} = R_{-k}v^k\overline{f(\tau)}$$

defines an antilinear map ξ_k from weak Maass forms of weight k to weakly holomorphic modular forms of weight $2 - k$. Its kernel is precisely the space of weakly holomorphic modular forms in weight k.

We write $N_k(D, \chi_D)$ for the space of weak Maass forms of weight k with respect to $\Gamma_0(D)$ and χ_D. Let us have a closer look at map $\xi_k : N_k(D, \chi_D) \to W_{2-k}(D, \chi_D)$. We denote by $\mathcal{N}_k(D, \chi_D)$ the inverse image of $S_{2-k}(D, \chi_D)$ under ξ_k, and its plus subspace by $\mathcal{N}_k^+(D, \chi_D)$. (Note that our notation is not consistent with the notation of [BF].)

Theorem 5.1. *We have the following exact sequence:*

$$0 \longrightarrow W_k^+(D, \chi_D) \longrightarrow \mathcal{N}_k^+(D, \chi_D) \xrightarrow{\xi_k} S_{2-k}^+(D, \chi_D) \longrightarrow 0.$$

This can be proved using Serre duality for the Dolbeault resolution of the structure sheaf on a modular curve (see [BF] Theorem 3.7) or by means of Hejhal-Poincaré series (see [Br2] chapter 1).

For every weak Maass form $f \in \mathcal{N}_k^+(D, \chi_D)$ there is a unique Fourier polynomial $P(f) = \sum_{n<0} c(n)q^n$ in $\mathbb{C}[q^{-1}]^+$ such that $f(\tau) - P(f)(\tau)$ is bounded as $v \to \infty$, uniformly in u. It is called the principal part of f. This generalizes the notion of the principal part of a weakly holomorphic modular form. One can show that every prescribed Fourier polynomial as above occurs as the principal part of a unique $f \in \mathcal{N}_k^+(D, \chi_D)$. This is a key fact, which suggests to study the Borcherds lift of weak Maass forms.

If $f \in \mathcal{N}_0^+(D, \chi_D)$, then we define its regularized theta lift $\Phi(z_1, z_2, f)$ by (3.7), in the same way as for weakly holomorphic modular forms. One can show that the regularized theta integral defines a Γ_K-invariant function on $\mathbb{H}^2$ with a logarithmic singularity along $-4Z(f)$, where

$$Z(f) = \sum_{n<0} \tilde{c}(n)T(-n),$$

and $\sum_{n<0} c(n)q^n$ denotes the principal part of f [Br2], [BF]. It is almost harmonic (outside the singularities) in the following sense. If $\Delta^{(1)}$ and $\Delta^{(2)}$ denote the $\mathrm{SL}_2(\mathbb{R})$-invariant hyperbolic Laplace operators on $\mathbb{H}^2$ in the first and second variable, then $\Delta^{(j)}\Phi(z_1, z_2, f) =$ constant.

The Fourier expansion of $\Phi(z_1, z_2, f)$ can be computed explicitly. It can be used to determine the growth behavior at the boundary of Y_K in $\widetilde{X}_K$. It turns out that the boundary singularities are of log and log-log type. More precisely, one can view $\frac{1}{4}\Phi(z_1, z_2, f)$ as a pre-log-log Green function for the divisor $Z(f)$ on $\widetilde{X}_K$ in the sense of [BKK1] (see section 8 here, and [BBK] Proposition 2.15).

Moreover, one finds that $\Phi(z_1, z_2, f)$ can be split into a sum

$$(5.2) \qquad \Phi(z_1, z_2, f) = -2\log|\Psi(z_1, z_2, f)|^2 + \xi(z_1, z_2, f),$$

where $\xi(z_1, z_2, f)$ is real analytic on the whole domain $\mathbb{H}^2$ and $\Psi(z_1, z_2, f)$ is a meromorphic function on $\mathbb{H}^2$ whose divisor equals $Z(f)$. If f is weakly holomorphic, the function $\xi(z_1, z_2, f)$ is simply equal to $2c(0)\left(\log(8\pi) - \Gamma'(1) - \log(16\pi^2 y_1 y_2)\right)$ and we are back in the case of Borcherds' original lift. However, if f is an honest weak Maass form, then ξ is a complicated function and Ψ far from being modular.

In any case, via the usual Poincaré-Lelong argument, the above splitting implies that the $(1,1)$ form

$$(5.3) \qquad \Lambda_B(f) := \frac{1}{4}dd^c\Phi(z_1, z_2, f) = \frac{1}{4}dd^c\xi(z_1, z_2, f)$$

represents the Chern class of the divisor $Z(f)$ in the second cohomology $H^2(Y_K)$. One can further show that it is a square integrable harmonic representative. (It can also be regarded as a pre-log-log form on $\widetilde{X}_K$, representing the class of $T(m)$ on $\widetilde{X}_K$ in $H^2(\widetilde{X}_K)$.)

Using suitable L^p-estimates for the functions $\Phi(z_1, z_2, f)$, and results of Andreotti-Vesentini and Yau on (sub-) harmonic functions on complete Riemann manifolds that satisfy such integrability conditions, the following *weak converse theorem* for the Borcherds lift can be proved (see [Br2] chapter 5).

Theorem 5.2. *Let F be a meromorphic Hilbert modular form of weight r for Γ_K, whose divisor $\mathrm{div}(F) = \sum_{n<0} \tilde{c}(n)T(-n)$ is a linear combination of Hirzebruch-Zagier divisors. Then*

$$-2\log|F(z_1, z_2)^2 (16\pi^2 y_1 y_2)^r| = \Phi(z_1, z_2, f) + constant,$$

where f is the unique weak Maass form in $\mathcal{N}_0^+(D, \chi_D)$ with principal part $\sum_{n<0} c(n)q^n$.

Corollary 5.3. *The assignment $Z(f) \mapsto \Lambda_B(f)$ defines a linear map from the subgroup $\mathrm{CH}_{HZ}^1(\widetilde{X}_K)$ of $\mathrm{CH}^1(\widetilde{X}_K)$, generated by the Hirzebruch-Zagier divisors, to $\mathcal{H}^{1,1}(Y_K)$, the space of square integrable harmonic $(1,1)$-forms on Y_K.*

Summing up, we get the following commutative diagram:

$$(5.4) \qquad \begin{array}{ccc} \mathcal{N}_0^+(D,\chi_D) \longrightarrow \mathcal{N}_0^+(D,\chi_D)/W_0^+(D,\chi_D) \xrightarrow[\xi_0]{\sim} S_2^+(D,\chi_D) \,. \\ \downarrow \qquad\qquad\qquad\qquad \downarrow \\ \mathrm{Z}_{HZ}^1(\widetilde{X}_K) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathrm{CH}_{HZ}^1(\widetilde{X}_K) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathcal{H}^{1,1}(Y_K) \end{array}$$

Here $\mathrm{Z}_{HZ}^1(\widetilde{X}_K)$ denotes the subgroup of the divisor group $\mathrm{Z}^1(\widetilde{X}_K)$ generated by the $T(m)$ ($m \in \mathbb{N}$). The left vertical arrow is defined by $f \mapsto Z(f) - \frac{c(0)}{k_G} \operatorname{div} G$, where $c(0)$ denotes the constant term of f, and G a fixed meromorphic Borcherds product of weight k_G. The vertical arrow in the middle is given by $f \mapsto Z(f) - c(0)\,\mathrm{c}_1(\mathcal{M}_1(\mathbb{C}))$.

In particular, the above diagram gives rise to a linear map $S_2^+(D,\chi_D) \to \mathcal{H}^{1,1}(Y_K)$. It can be explicitly described in terms of the Fourier expansions. One finds that the image is in the subspace of $\mathcal{H}^{1,1}(Y_K)$ given by forms which are symmetric with respect to the interchange of the coordinates z_1, z_2. It is known that this subspace is isomorphic to $\mathbb{C}\Omega \oplus S_2(\Gamma_K)$, where

$$(5.5) \qquad\qquad\qquad \Omega = \frac{dx_1\,dy_1}{4\pi y_1^2} + \frac{dx_2\,dy_2}{4\pi y_2^2}$$

is the symmetric invariant Kähler form on $\mathbb{H}^2$, and $S_2(\Gamma_K)$ denotes the space of Hilbert cusp forms of weight 2 for Γ_K. Consequently, we get a linear map $S_2^+(D,\chi_D) \to S_2(\Gamma_K)$.

To answer the surjectivity question for the Borcherds lift raised at the beginning of this section, it now suffices to show that this map is injective. This can for instance be deduced by means of the Fourier expansion of the image. We obtain the following *strong converse theorem* for the Borcherds lift (see [Br1], [Br2] chapter 5).

Theorem 5.4. *Let F be a meromorphic Hilbert modular form for Γ_K, whose divisor $\operatorname{div}(F) = \sum_{n<0} \tilde{c}(n)T(-n)$ is given by Hirzebruch-Zagier divisors. Then there is a weakly holomorphic modular form $f \in W_0^+(D,\chi_D)$ with principal part $\sum_{n<0} c(n)q^n$, and, up to a constant multiple, F is equal to the Borcherds lift of f in the sense of Theorem 3.1.*

As a corollary it can be deduced that the dimension of $\mathrm{CH}_{HZ}^1(\widetilde{X}_K)_{\mathbb{Q}}$ is equal to the dimension of $M_2^+(D,\chi_D)$ complementing Theorem 4.3. It is not hard to see that our map $S_2^+(D,\chi_D) \to S_2(\Gamma_K)$ coincides with the celebrated Doi-Naganuma lift [DN], [Na], [Za]. The above construction can be viewed as a new approach to it.

The weak converse theorem is proved in much greater generality in [Br2]. Combining the argument of [Br2] with techniques of [BF] it could probably be extended to hold in full generality for $\mathrm{O}(2,\ell)$. However, for the strong converse theorem the situation seems far more complicated. It is proved in [Br2] for modular forms on $\Gamma(L) \subset \mathrm{O}(2,\ell)$, where $\Gamma(L)$ denotes the discriminant kernel of the orthogonal group of an even lattice L of signature $(2,\ell)$ that splits two hyperbolic planes over $\mathbb{Z}$. For example, if we go to congruence subgroups of the Hilbert modular group Γ_K, it is not clear whether the analogue of Theorem 5.4 holds or not.

6. Automorphic Green functions

In this section we look at the regularized theta lifts of weak Maass forms from a different perspective. By the discussion of the previous section, for every positive integer m there exists a unique weak Maass form $f_m \in \mathcal{N}_0^+(D, \chi_D)$, whose principal part is equal to q^{-m} if $m \not\equiv 0 \pmod{D}$, and equal to $\frac{1}{2} q^{-m}$, if $m \equiv 0 \pmod{D}$. The lift

$$\phi_m(z_1, z_2) = \frac{1}{2} \Phi(z_1, z_2, f_m)$$

of f_m is a real analytic function on Y_K with a logarithmic singularity along $-2T(m)$.

Here we present a different, more naive, construction of $\phi_m(z_1, z_2)$. For details see [Br1]. The idea is to construct $\phi_m(z_1, z_2)$ directly as a Poincaré series by summing over the logarithms of the defining equations of $T(m)$. We consider the sum

$$(6.1) \qquad \sum_{\substack{(a,b,\lambda) \in \mathbb{Z}^2 \oplus \mathfrak{d}^{-1} \\ ab - \lambda\lambda' = m/D}} \log \left| \frac{a z_1 \bar{z}_2 + \lambda z_1 + \lambda' \bar{z}_2 + b}{a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b} \right|.$$

The denominators of the summands ensure that this function has a logarithmic singularity along $-2T(m)$ in the same way as $\phi_m(z_1, z_2)$. The enumerators are smooth on the whole $\mathbb{H}^2$. They are included to make the sum formally Γ_K-invariant. Unfortunately, the sum diverges. However, it can be regularized in the following way. If we put $Q_0(z) = \frac{1}{2} \log\left(\frac{z+1}{z-1} \right)$, we may rewrite the summands as

$$\log \left| \frac{a z_1 \bar{z}_2 + \lambda z_1 + \lambda' \bar{z}_2 + b}{a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b} \right| = Q_0 \left(1 + \frac{|a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b|^2}{2 y_1 y_2 m/D} \right).$$

Now we replace Q_0 by the 1-parameter family Q_{s-1} of Legendre functions of the second kind (cf. [AbSt] §8), defined by

$$(6.2) \qquad Q_{s-1}(z) = \int_0^\infty (z + \sqrt{z^2 - 1} \cosh u)^{-s} du.$$

Here $z > 1$ and $s \in \mathbb{C}$ with $\Re(s) > 0$. If we insert $s = 1$, we get back the above Q_0. Hence we consider

$$(6.3) \qquad \Phi_m(z_1, z_2, s) = \sum_{\substack{a,b \in \mathbb{Z} \\ \lambda \in \mathfrak{d}^{-1} \\ ab - \mathrm{N}(\lambda) = m/D}} Q_{s-1} \left(1 + \frac{|a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b|^2}{2 y_1 y_2 m/D} \right).$$

It is easily seen that this series converges normally for $(z_1, z_2) \in \mathbb{H}^2 - T(m)$ and $\Re(s) > 1$ and therefore defines a Γ_K-invariant function, which has logarithmic growth along $-2T(m)$. It is an eigenfunction of the hyperbolic Laplacians $\Delta^{(j)}$ with eigenvalue $s(s-1)$, because of the differential equation satisfied by Q_{s-1}. Notice that for $D = m = 1$ the function $\Phi_m(z_1, z_2, s)$ is simply the classical resolvent kernel for $\mathrm{SL}_2(\mathbb{Z})$ (cf. [Hej], [Ni]). One can compute the Fourier expansion of $\Phi_m(z_1, z_2, s)$ explicitly and use it to obtain a meromorphic continuation to $s \in \mathbb{C}$. At $s = 1$ there is a simple pole, reflecting the divergence of the

formal sum (6.1). We define the regularization $\Phi_m(z_1, z_2)$ of (6.1) to be the constant term in the Laurent expansion of $\Phi_m(z_1, z_2, s)$ at $s = 1$.

It turns out that $\Phi_m(z_1, z_2)$ is, up to an additive constant L_m, equal to the function $\phi_m(z_1, z_2)$ above (see [Br2] Proposition 2.11 and Theorem 2.14). Here the constant L_m is quite interesting, since it is given by the derivative of the m-th coefficient of a certain Eisenstein series $E_2(\tau, s)$ of weight 2 for $\Gamma_0(D)$ and χ_D [BK], [BBK]. We will come back to this in section 7.

One may use the Fourier expansion of $\Phi_m(z_1, z_2)$ and identities for certain finite exponential sums of [Za] to obtain a different independent proof of Theorem 3.1.

The following integral formula is fundamental (see [BK] Theorem 4.7, [BBK]). It justifies why $\Phi_m(z_1, z_2, s)$ (and also $\Phi_m(z_1, z_2)$) is called an *automorphic Green function* for the divisor $T(m)$.

Theorem 6.1. *Let $h : Y_K \to \mathbb{C}$ be a bounded eigenfunction of the Laplacian $\Delta^{(1)}$ (or $\Delta^{(2)}$) with eigenvalue λ. Then for $s \in \mathbb{C}$ with $\Re(s) > 1$ we have*

$$\int\limits_{\Gamma_K \backslash \mathbb{H}^2} \Phi_m(z_1, z_2, s) h(z_1, z_2) \, \Omega^2 = \frac{1}{s(s-1) - \lambda} \int\limits_{T(m)} h(z_1, z_2) \, \Omega.$$

Here Ω is defined by (5.5) so that Ω^2 is an invariant volume form on Y_K.

Such automorphic Green functions are constructed in greater generality for $O(2, \ell)$ in [Br2] using the regularized theta lift of Hejhal-Poincaré series, and independently in [OT] from the point of view of spherical functions on real Lie groups.

7. INTEGRALS OF AUTOMORPHIC GREEN FUNCTIONS

It is well known that the volume of Y_K is given by $\mathrm{vol}_\Omega(Y_K) = \int_{Y_K} \Omega^2 = \zeta_K(-1)$, where $\zeta_K(s)$ denotes the Dedekind zeta function of K. The volume of a divisor C on Y_K is defined as the integral $\mathrm{vol}_\Omega(C) = \int_C \Omega$. One can show that the integral is finite, see e.g. [Br3]. If C is effective, then its volume is positive.

It is a well known fact that the volumes of Hirzebruch-Zagier divisors are given by the Fourier coefficients of the unique normalized Eisenstein series in $M_2^+(D, \chi_D)$ (see [Fra], [Ha], and [Ge2] chapter V.5). Let us recall the definition of that Eisenstein series. In weight k there are the two non-holomorphic Eisenstein series

$$E_k^\infty(\tau, s) = \sum_{\substack{c,d \in \mathbb{Z} \\ c \equiv 0 \, (D)}} \chi_D(d) \frac{1}{(c\tau + d)^k} \frac{y^s}{|c\tau + d|^{2s}},$$

$$E_k^0(\tau, s) = \sum_{c,d \in \mathbb{Z}} \chi_D(c) \frac{1}{(c\tau + d)^k} \frac{y^s}{|c\tau + d|^{2s}}$$

for $\Gamma_0(D)$ with character χ_D, the former corresponding to the cusp ∞ of $\Gamma_0(D)$, the latter to the cusp 0. (By our assumption that D be prime, these are the only cusps of $\Gamma_0(D)$.) They converge for $\Re(s) > 1 - k/2$ and have a meromorphic continuation in s to the full

complex plane. If $k \geq 2$, the special values $E_k^\infty(\tau, 0)$ and $E_k^0(\tau, 0)$ are holomorphic in τ and define elements of $M_k(D, \chi_D)$. One can show that the linear combination

$$E_k(\tau, s) = \frac{1}{2L(k + 2s, \chi_D)} \left(D^s E_k^\infty(\tau, s) + D^{1/2 - k - s} E_k^0(\tau, s) \right)$$

satisfies the plus space condition. (This follows most easily from Lemma 3 of [BB].) Here $L(s, \chi_D)$ denotes the L-series associated with the Dirichlet character χ_D. In particular we have $E_k(\tau, 0) \in M_k^+(D, \chi_D)$. The Fourier expansion of $E_k(\tau, s)$ has the form

$$(7.1) \qquad E_k(\tau, s) = \sum_{n \in \mathbb{Z}} C(n, s) \mathcal{W}_s(4\pi n v) e^{2\pi i n u},$$

where the $C(n, s)$ are complex coefficients independent of v; and $\mathcal{W}_s(v)$ is a certain Whittaker function, which we normalize as in [BK] (3.2). The precise normalization is not important for our purposes here, we only need that it is a universal function for all n of the same sign. The coefficients $C(n, s)$ are computed for instance in [BK] section 5, Example 2. Here we only state the special value

$$(7.2) \ \ E_k(\tau, 0) = 1 + \sum_{n \geq 1} C(n, 0) q^n = 1 + \frac{2}{L(1 - k, \chi_D)} \sum_{n \geq 1} \sum_{d | n} d^{k-1} \left(\chi_D(d) + \chi_D(n/d) \right) q^n,$$

which is obtained in the standard way (see [He], Werke p. 818) using the functional equation of $L(s, \chi_D)$.

Theorem 7.1. *We have*

$$E_2(\tau, 0) = 1 - \frac{2}{\mathrm{vol}_\Omega(Y_K)} \sum_{m \geq 1} \mathrm{vol}_\Omega(T(m)) q^m.$$

Similar identities hold in much greater generality for special cycles on arithmetic quotients of $O(p, q)$ and $U(p, q)$, see e.g. [Ge1], [Ku4], [Ku2], [Od2]. (Observe that our normalization of $\mathrm{vol}_\Omega(T(m))$ equals twice the volume of $T(m)$ in [BBK].)

Let us briefly indicate, how Theorem 7.1 can be deduced from the properties of the automorphic Green functions $\Phi_m(z_1, z_2, s)$. For instance, from the description as a regularized theta lift it follows that the residue at $s = 1$ of $\Phi_m(z_1, z_2, s)$ is equal to the constant coefficient $a_m(0)$ of the weak Maass form $f_m \in \mathcal{N}_0^+(D, \chi_D)$ defined at the beginning of section 6. By means of the relationship of the spaces $\mathcal{N}_k^+(D, \chi_D)$ and $M_{2-k}^+(D, \chi_D)$, which is also implicit in (5.1), one finds that $a_m(0) = -\frac{1}{2} C(m, 0)$ (see [BF] Proposition 3.5). Therefore we have

$$\Phi_m(z_1, z_2) = \lim_{s \to 1} \left(\Phi_m(z_1, z_2, s) + \frac{C(m, 0)}{2(s - 1)} \right).$$

Using growth estimates for $\Phi_m(z_1, z_2, s)$, which can be deduced from the constant coefficients of the Fourier expansions, we obtain:

Proposition 7.2. *The function* $\Phi_m(z_1, z_2)$ *belongs to* $L^p(Y_K, \Omega^2)$ *for any* $p < 2$, *and*

$$\int_{Y_K} \Phi_m(z_1, z_2) \, \Omega^2 = \lim_{s \to 1} \int_{Y_K} \left(\Phi_m(z_1, z_2, s) + \frac{C(m, 0)}{2(s - 1)} \right) \Omega^2.$$

If we apply Theorem 6.1 for the constant function $h = 1$, we may compute the integral. It is equal to

$$(7.3) \quad \frac{1}{s(s-1)} \int_{T(m)} \Omega + \frac{C(m,0)}{2(s-1)} \int_{Y_K} \Omega^2 = \frac{1}{(s-1)} \left(\frac{\text{vol}_\Omega(T(m))}{s} + \frac{C(m,0)}{2} \text{vol}_\Omega(Y_K) \right).$$

Since the limit $s \to 1$ exists, the quantity in parenthesis on the right hand side has to vanish at $s = 1$. This yields the assertion of Theorem 7.1.

So far we have essentially exploited the existence of the integral $\int_{Y_K} \Phi_m(z_1, z_2) \Omega^2$, which means that the residue in the Laurent expansion of (7.3) at $s = 1$ vanishes. We may actually compute the constant term of that expansion, that is, the value of the integral. It is equal to $-\text{vol}_\Omega(T(m))$.

One can further improve this result by observing that the full coefficient $C(m, s)$ as a function of s occurs in the constant term of $\Phi_m(z_1, z_2, s)$. More precisely, if we define

$$(7.4) \qquad G_m(z_1, z_2) = \frac{1}{2} \lim_{s \to 1} \left(\Phi_m(z_1, z_2, s) + B(s)\zeta(2s-1)C(m, s-1) \right),$$

with

$$B(s) = \frac{(16\pi)^{s-1}\Gamma(s-1/2)s}{\Gamma(1/2)(2s-1)},$$

then one can show that $G_m(z_1, z_2) = \frac{1}{4}\Phi(z_1, z_2, f_m) + \frac{1}{2}a_m(0)(\Gamma'(1) - \log(8\pi))$ (which is essentially the calculation of the constant L_m on page 20). This means in particular that if F is the Borcherds lift of a weakly holomorphic modular form f with coefficients $a(n)$, then its *Petersson metric* is given by

$$(7.5) \qquad \log \|F(z_1, z_2)\|_{Pet} := \log\left(|F(z_1, z_2)|(16\pi^2 y_1 y_2)^{a(0)/2} \right) = -\sum_{n<0} \tilde{a}(n)G_{-n}(z_1, z_2).$$

The latter identity can be viewed as a generalization of the Kronecker limit formula expressing the logarithm of the absolute value of the discriminant function (2.2) as the constant term in the Laurent expansion at $s = 1$ of the non-holomorphic Eisenstein series of weight 0 for $\text{SL}_2(\mathbb{Z})$ (see [BK] (4.14)). Notice that the constant $B(s)$ in (7.4) does not depend on m. It changes if the normalization of the Whittaker function $\mathcal{W}_s(v)$ is varied. Arguing as above we find that (see [BK] Theorem 4.10)

$$\int_{Y_K} G_m(z_1, z_2) \Omega^2 = -\frac{\text{vol}_\Omega(T(m))}{2} \left(C'(m,0)/C(m,0) + \log(4\pi) - \Gamma'(1) \right).$$

If we insert the explicit formula for $C(m, s)$, we get

$$(7.6) \qquad \int_{Y_K} G_m(z_1, z_2) \Omega^2 = -\text{vol}_\Omega(T(m)) \left(\frac{L'(-1, \chi_D)}{L(-1, \chi_D)} + \frac{1}{2} - \frac{\sigma'_m(-1)}{\sigma_m(-1)} + \frac{1}{2}\log(D) \right),$$

where

$$(7.7) \qquad \sigma_m(s) = m^{(1-s)/2} \sum_{d|m} d^s \left(\chi_D(d) + \chi_D(m/d) \right).$$

In particular, in view of (7.5), the integral over the logarithm of the Petersson metric of any Borcherds product can be computed explicitly (see also [Ku4]). For example, if $K = \mathbb{Q}(\sqrt{5})$, we obtain for the Gundlach theta function

$$\int_{Y_K} \log\left(2^{-6}|\Theta(z_1, z_2)|(16\pi^2 y_1 y_2)^{5/2}\right) \Omega^2 = -\zeta(-1)\left(2\frac{L'(-1, \chi_D)}{L(-1, \chi_D)} + 1 + \log(5)\right).$$

Such integrals play a fundamental role in the Arakelov intersection theory of Hirzebruch-Zagier divisors. We will come back to that in the section 9.

The integral of the logarithm of the Petersson metric of a Borcherds product was first calculated by Kudla in [Ku4] using a different approach based on the Siegel-Weil formula. We recall that the quantity $\Phi(z_1, z_2, f)$ we want to integrate is given by the theta integral (3.7) of a weakly holomorphic modular form f. Now the idea is to interchange the (z_1, z_2)-integration with the regularized integration over τ and to compute

$$(7.8) \qquad \int_{\mathcal{F}} \left\langle f(\tau), \int_{Y_K} \Theta_S(\tau; z_1, z_2)\, d\mu_z \right\rangle d\mu_\tau,$$

where $d\mu_z$ denotes the invariant measure on Y_K. (Notice that this needs a careful justification.) The inner integral over the Siegel theta function can be determined by means of the Siegel-Weil formula. It yields an Eisenstein series of weight 0 for $\Gamma_0(D)$, which can be written in terms of the lowering operator and our Eisenstein series (7.1) of weight 2 as $\frac{1}{s-1}L_2 E_2(\tau, s)$. The integrand for the remaining regularized integral over τ is now essentially $\frac{1}{s-1}d(\langle f(\tau), E_2(\tau, s)\rangle d\tau)$ at $s = 1$, so that we may use Stoke's theorem to compute it. The derivative of $E_2(\tau, s)$ occurs because of the factor $\frac{1}{s-1}$.

We conclude this section by giving a characterization of the automorphic Green function $\Phi_m(z_1, z_2)$. It can be proved in a similar way as Theorem 5.2.

Proposition 7.3. *Let f be a smooth function on $Y_K - T(m)$ with the properties:*

 (i) *f has a logarithmic singularity along $T(m)$,*
 (ii) *$(\Delta^{(1)} + \Delta^{(2)})f = constant$,*
 (iii) *$f \in L^{1+\varepsilon}(Y_K, \Omega^2)$ for some $\varepsilon > 0$,*
 (iv) *$\int_{Y_K} f(z_1, z_2)\, \Omega^2 = \frac{1}{2}\,\mathrm{vol}_\Omega(T(m))$.*

Then $f(z_1, z_2) = -\frac{1}{2}\Phi_m(z_1, z_2)$.

8. ARITHMETIC OF HIRZEBRUCH-ZAGIER DIVISORS

In their paper on the intersection of modular correspondences, Gross and Keating interpreted classical results of Hurwitz and Kronecker by the observation that the intersection number of two modular correspondences on $Y_{\mathbb{Q}\oplus\mathbb{Q}} = \Gamma(1)^2\backslash\mathbb{H}^2$ is given by the coefficients of the classical Siegel Eisenstein series $E^{(2)}(Z, s)$ of weight 2 and genus 2 at $s = 0$. Their main result was that the arithmetic intersection numbers of three such modular correspondences on the regular model $\mathrm{Spec}\,\mathbb{Z}[j, j']$ of $Y_{\mathbb{Q}\oplus\mathbb{Q}}$ is given by the coefficients of the derivative of the Siegel Eisenstein series $E^{(3)}(Z, s)$ of weight 2 and genus 3 at $s = 0$ [GK]. Observe that

$Y_{\mathbb{Q}\oplus\mathbb{Q}}$ can be viewed as the "degenerate" Hilbert modular surface with discriminant $D = 1$ and the modular correspondences as Hirzebruch-Zagier divisors in this case.

Kudla proved that the arithmetic intersection numbers in the sense of Arakelov geometry of certain arithmetic special divisors on a regular model of a Shimura curve are dictated by the coefficients of the derivative of a Siegel Eisenstein series of weight 2 and genus 2 at $s = 0$ [Ku1]. (So Kudla considers an arithmetic surfaces, rather than an arithmetic 3-fold as in the case of Gross and Keating. This explains the different genus.) Here the arithmetic divisors are pairs consisting of a special divisor on the regular model and a certain Green function for the induced divisor on the corresponding complex variety, fitting in the setup of arithmetic intersection theory as in [SABK].

In further works Kudla, Rapoport, and Yang developed an extensive program relating arithmetic special divisors on Shimura varieties of type $O(2, \ell)$ and their arithmetic intersection theory to automorphic forms, in particular to the coefficients of the derivatives of Siegel Eisenstein series. Most of this is conjectural, but in important special cases these conjectures are meanwhile proved. (See e.g. [Ku6] for the $O(2, 1)$ case of Shimura curves, [KRY] for the $O(2, 0)$ case of CM elliptic curves, [KR] for partial results in the $O(2, 3)$ case of Siegel modular threefolds, and [Ku5] for an overview.) Notice that $Y_{\mathbb{Q}\oplus\mathbb{Q}}$ can be described in terms of $O(2, 2)$.

One conclusion of this general picture is that the geometric results over $\mathbb{C}$ of Hirzebruch and Zagier (as e.g. Theorems 4.3 and 7.1) and their generalizations to $O(2, \ell)$ should have arithmetic analogues over $\mathbb{Z}$. Here the classical intersection theory has to be replaced by Arakelov intersection theory.

In this section we discuss, how Borcherds products can be used to obtain new results in that direction. We begin by recalling some facts on the arithmetic of Hilbert modular surfaces.

In section 2 we briefly discussed that modular curves have a moduli interpretation as a moduli space for isomorphism classes of elliptic curves with additional structure. The same is true for Hilbert modular surfaces, which is the starting point for arithmetic investigations. More precisely, Y_K parametrizes isomorphism classes of triples (A, ι, ψ), where A is an abelian surface over $\mathbb{C}$, ι is an $\mathcal{O}_K$-multiplication, that is, a ring homomorphism $\mathcal{O}_K \to \operatorname{End}(A)$, and ψ is a $\mathfrak{d}^{-1}$-polarization, that is, an isomorphism of $\mathcal{O}_K$-modules $\mathfrak{d}^{-1} \to \operatorname{Hom}_{\mathcal{O}_K}(A, A^\vee)^{\operatorname{sym}}$ from the inverse different $\mathfrak{d}^{-1} = \frac{1}{\sqrt{D}}\mathcal{O}_K$ to the module of $\mathcal{O}_K$-linear symmetric homomorphisms, taking the totally positive elements of $\mathfrak{d}^{-1}$ to $\mathcal{O}_K$-linear polarizations (see [Go] Chapter 2).

The moduli description now makes sense over any scheme S over $\mathbb{Z}$. (Here one has to require that ψ fulfill an extra technical condition called the Deligne-Pappas condition, see [DePa]. That condition is automatically fulfilled in characteristic 0.) Due to the work of Rapoport, Deligne, and Pappas it is known that the moduli problem "Abelian surfaces over S with $\mathcal{O}_K$-multiplication and $\mathfrak{d}^{-1}$-polarization with Deligne-Pappas condition" is represented by a regular algebraic stack $\mathcal{H}$, which is flat and of relative dimension two over $\operatorname{Spec}\mathbb{Z}$. It is smooth over $\operatorname{Spec}\mathbb{Z}[1/D]$, and the fiber of $\mathcal{H}$ above D is smooth outside a closed subset of codimension 2.

The corresponding complex variety $\mathcal{H}(\mathbb{C})$ is isomorphic to Y_K. The isomorphism is obtained by associating to $z = (z_1, z_2) \in \mathbb{H}^2$ the abelian surface $A_z = \mathbb{C}^2/\Lambda_z$ over $\mathbb{C}$ given by the lattice

$$\Lambda_z = \left\{ \begin{pmatrix} \alpha z_1 + \beta \\ \alpha' z_2 + \beta' \end{pmatrix} \in \mathbb{C}^2; \quad \alpha, \beta \in \mathcal{O}_K \right\} \subset \mathbb{C}^2,$$

together with the $\mathcal{O}_K$-multiplication ι induced by the natural action $\iota(\nu) = \left(\begin{smallmatrix} \nu & 0 \\ 0 & \nu' \end{smallmatrix} \right)$ of $\mathcal{O}_K$ on $\mathbb{C}^2$, and a certain $\mathfrak{d}^{-1}$-polarization.

For $k \in \mathbb{Z}$ sufficiently divisible there exists a line bundle $\mathcal{M}_k$ on $\mathcal{H}$ (the k-th power of the pull-back along the zero section of the determinant of the relative cotangent bundle of the universal family over $\mathcal{H}$) such that the induced bundle on $\mathcal{H}(\mathbb{C})$ can be identified with the line bundle $\mathcal{M}_k(\mathbb{C})$ of Hilbert modular forms of weight k for Γ_K of the previous sections. By the q-expansion principle and the Koecher principle, the global sections of $\mathcal{M}_k$ can be identified with Hilbert modular forms of weight k for Γ_K with integral rational Fourier coefficients.

There exists an arithmetic Baily-Borel compactification $\overline{\mathcal{H}}$ of the coarse moduli space corresponding to $\mathcal{H}$, which can be described as

(8.1) $$\overline{\mathcal{H}} = \mathrm{Proj}\left(\bigoplus_k H^0(\mathcal{H}, \mathcal{M}_k) \right).$$

The scheme $\overline{\mathcal{H}}$ is normal, projective, and flat over $\mathrm{Spec}\,\mathbb{Z}$ (see [Ch], p. 549), and $\overline{\mathcal{H}}(\mathbb{C}) \cong X_K$. Furthermore, its fibers over $\mathrm{Spec}\,\mathbb{Z}$ are irreducible (see [DePa], p. 65). By construction, the bundle $\mathcal{M}_k$ extends to $\overline{\mathcal{H}}$.

Throughout the rest of this paper we will make the following

Assumption 8.1. *There exists a desingularization $\pi : \widetilde{\mathcal{X}}_K \to \overline{\mathcal{H}}$ by a regular scheme $\widetilde{\mathcal{X}}_K$, which is projective and flat over $\mathbb{Z}$, such that the regular locus $\overline{\mathcal{H}}^{r\acute{e}g}$ is fiber-wise dense in $\widetilde{\mathcal{X}}_K$, and such that the induced morphism $\widetilde{\mathcal{X}}_K(\mathbb{C}) \to X_K$ is a desingularization as in the previous sections $\widetilde{X}_K$.*

This assumption simplifies the exposition (it might actually be too optimistic). Notice that the singularities of $\overline{\mathcal{H}}$ at the boundary can be resolved by considering a suitable toroidal compactification of $\mathcal{H}$. So only the singularities corresponding to elliptic fixed points would need to be resolved. If one wants to obtain unconditional results one can impose an additional level structure in order to get a fine moduli problem and work with a suitable toroidal compactification of the corresponding moduli scheme (as is done in [BBK]). Unfortunately, in that way one only gets a regular scheme $\widetilde{\mathcal{H}}(N)$ which is projective and flat over $\mathbb{Z}[\zeta_N, 1/N]$, where ζ_N denotes a primitive N-th root of unity and $N \geq 3$ the level.

We define the line bundle of modular forms of weight k on $\widetilde{\mathcal{X}}_K$ as the pullback $\pi^*(\mathcal{M}_k)$. For simplicity we will also denote it by $\mathcal{M}_k$.

It can be shown that the Hirzebruch-Zagier divisors on X_K are defined over $\mathbb{Q}$, that is, $T(m)$ is obtained by base change from a divisor on the generic fiber $\overline{\mathcal{H}} \times_{\mathbb{Z}} \mathbb{Q}$ of $\overline{\mathcal{H}}$. We define the Hirzebruch-Zagier divisor $T(m)$ on the generic fiber $\widetilde{\mathcal{X}}_K \times_{\mathbb{Z}} \mathbb{Q}$ as the pullback of

$T(m)$ on $\overline{\mathcal{H}} \times_{\mathbb{Z}} \mathbb{Q}$. Moreover, we define the Hirzebruch-Zagier divisor $\mathcal{T}(m)$ on $\widetilde{\mathcal{X}}_K$ as the Zariski closure of $T(m)$.

We now briefly recall some basic properties of arithmetic Chow rings (see e.g. [SABK]). Since Y_K is non-compact, the natural metrics on automorphic vector bundles have singularities at the boundary [Mu], [BKK2]. Therefore we need to work with the extended arithmetic Chow ring $\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{pre}})$ constructed in [BKK1]. In this ring the Green objects satisfy beside the usual logarithmic additional log-log growth conditions.

Let $\mathcal{X}$ be an arithmetic variety over $\mathbb{Z}$, i.e., a regular scheme, which is projective and flat over $\mathbb{Z}$. Moreover, let $\mathbf{D}$ be a fixed normal crossing divisor on the complex variety $\mathcal{X}(\mathbb{C})$, which is stable under complex conjugation. An arithmetic divisor on $\mathcal{X}$ (in the sense of [BKK1]) is a pair

$$(y, \mathfrak{g}_y),$$

where y is a divisor on the scheme $\mathcal{X}$ and $\mathfrak{g}_y$ is a pre-log-log Green object for the induced divisor $y(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$. In particular, a pair (y, g_y) where g_y is a *pre-log-log Green function* for y, determines an arithmetic divisor. This essentially means that g_y is a smooth function on $\mathcal{X}(\mathbb{C}) - (y(\mathbb{C}) \cup \mathbf{D})$, invariant under complex conjugation, with logarithmic singularities along the irreducible components of $y(\mathbb{C})$ and pre-log-log singularities along $\mathbf{D}$ such that the $\partial\bar{\partial}$-equation of currents holds:

$$-2\partial\bar{\partial}[g] = [-2\partial\bar{\partial}g] - \delta_y.$$

Here $[\cdot]$ denotes the current associated to a differential form and δ_y the Dirac current for y normalized as in [BKK1] and [BBK]. A differential form α is called pre-log singular (pre-log-log singular), if α, $\partial\alpha$, $\bar{\partial}\alpha$, and $\partial\bar{\partial}\alpha$ have only logarithmic growth (respectively log-log growth). We write $\widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{D}_{\mathrm{pre}})$ for the free abelian group generated by the arithmetic divisors on $\mathcal{X}$. Here $\mathcal{D}_{\mathrm{pre}}$ stands for the Deligne algebra with pre-log-log forms along $\mathbf{D}$, which is needed for the precise description of Green objects in [BKK1]. Moreover, we write $\widehat{\mathrm{Rat}}^1(\mathcal{X})$ for the subgroup of $\widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{D}_{\mathrm{pre}})$ given by arithmetic divisors of the form $(\mathrm{div}(f), -\log|f|)$, where f is a rational function on $\mathcal{X}$ and $|f|$ the absolute value of the induced function on $\mathcal{X}(\mathbb{C})$. The first *arithmetic Chow group* of $\mathcal{X}$ with log-log growth along $\mathbf{D}$ is defined by

$$\widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) = \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) / \widehat{\mathrm{Rat}}^1(\mathcal{X}).$$

More generally, in [BKK1] arithmetic Chow groups $\widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{D}_{\mathrm{pre}})$ of codimension p arithmetic cycles with log-log growth along $\mathbf{D}$ are defined. There exists an arithmetic intersection product

$$\widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \otimes \widehat{\mathrm{CH}}^q(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \longrightarrow \widehat{\mathrm{CH}}^{p+q}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}},$$

and

$$\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}} = \bigoplus_{p \geq 0} \widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

equipped with this product has the structure of a commutative associative ring.

For instance, if $\mathcal{X} = \operatorname{Spec} \mathbb{Z}$, then the closed points of $\operatorname{Spec} \mathbb{Z}$ can be identified with the primes of $\mathbb{Z}$. An arithmetic divisor is a pair $(\sum_p n_p p, \, g)$ consisting of a finite formal $\mathbb{Z}$-linear combination of primes p and a real number g. The elements of $\widehat{\operatorname{Rat}}^1(\operatorname{Spec} \mathbb{Z})$ are the pairs of the form $(\sum_{p|N} \operatorname{ord}_p(N)p, \, -\log |N|)$ for $N \in \mathbb{Q}$. This implies that

$$\widehat{\deg} : \widehat{\operatorname{CH}}^1(\operatorname{Spec} \mathbb{Z}, \mathcal{D}_{\mathrm{pre}}) \longrightarrow \mathbb{R}, \quad \left(\sum n_p p, \, g \right) \mapsto g + \sum n_p \log(p)$$

is an isomorphism. It is common to identify $\widehat{\operatorname{CH}}^1(\operatorname{Spec} \mathbb{Z}, \mathcal{D}_{\mathrm{pre}})$ with $\mathbb{R}$.

There also is an arithmetic analogue of the Picard group: The *arithmetic Picard group* $\widehat{\operatorname{Pic}}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}})$ is the group of isomorphism classes of pre-log singular hermitian line bundles on $\mathcal{X}$. Here a pre-log singular hermitian line bundle is a pair $\overline{\mathcal{L}} = (\mathcal{L}, \| \cdot \|)$ consisting of a line bundle $\mathcal{L}$ on $\mathcal{X}$, and a smooth hermitian metric $\| \cdot \|$ on the induced complex line bundle on $\mathcal{X}(\mathbb{C}) - \mathbf{D}$, invariant under complex conjugation, and such that $-\log \|s\|$ has logarithmic singularities along $\operatorname{div}(s)(\mathbb{C})$ and pre-log-log singularities along $\mathbf{D}$ for any rational section s of $\mathcal{L}$. If $\overline{\mathcal{L}}$ is a pre-log singular hermitian line bundle and s a rational section of $\mathcal{L}$, then, essentially by the Poincaré-Lelong lemma,

$$\widehat{c}_1(\overline{\mathcal{L}}) = (\operatorname{div}(s), -\log \|s\|)$$

defines a class in $\widehat{\operatorname{CH}}^1(\mathcal{X}, \mathcal{D}_{\mathrm{pre}})$, which is independent of the choice of s. It is called the first arithmetic Chern class of $\overline{\mathcal{L}}$. The assignment $\overline{\mathcal{L}} \mapsto \widehat{c}_1(\overline{\mathcal{L}})$ actually induces an isomorphism

$$\widehat{c}_1 : \widehat{\operatorname{Pic}}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \longrightarrow \widehat{\operatorname{CH}}^1(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}).$$

We now consider the arithmetic Chow ring $\widehat{\operatorname{CH}}^*(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$ of the model $\widetilde{\mathcal{X}}_K$ of our Hilbert modular surface, where we take for $\mathbf{D}$ the normal crossing divisor $\mathbf{D}_K = \pi^{-1}(X_K^{sing})$. For details we refer to [BBK]. The Green functions of section 6 turn out to be particularly nice, because they fit into the arithmetic Chow theory of $\widetilde{\mathcal{X}}_K$.

Theorem 8.2. *The pair*

$$\widehat{\mathcal{T}}(m) = (\mathcal{T}(m), G_m)$$

defines an element of $\widehat{\operatorname{CH}}^1(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})$, called the arithmetic Hirzebruch-Zagier divisor of discriminant m. Here G_m is the automorphic Green function defined by (7.4).

Notice that G_m always has log-log singularities along $\mathbf{D}_K$, even if $T(m)$ is disjoint to $\mathbf{D}_K$. So G_m does not define a Green function for $T(m)$ in the classical arithmetic Chow theory due to Gillet and Soulé. We therefore really need the extension of [BKK1]. Observe that the arithmetic divisors of Theorem 8.2 slightly differ from those considered by Kudla, Rapoport, and Yang. For instance, they often contain boundary components (which is possible since Y_K is non-compact), and are built with different Green functions.

Moreover, we obtain an element of $\widehat{\operatorname{Pic}}(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})$, by equipping the line bundle of modular forms with the *Petersson metric*. Recall that if $F \in \mathcal{M}_k(\mathbb{C})(U)$ is a rational section

over an open subset $U \subset Y_K$, then its Petersson metric is given by

$$\|F(z_1, z_2)\|_{\mathrm{Pet}}^2 = |F(z_1, z_2)|^2 (16\pi^2 y_1 y_2)^k.$$

This defines a pre-log singular hermitian metric on $\mathcal{M}_k(\mathbb{C})$ (with respect to $\mathbf{D}_K$). We denote the corresponding pre-log singular hermitian line bundle by $\overline{\mathcal{M}}_k = (\mathcal{M}_k, \|\cdot\|_{\mathrm{Pet}}^2)$. (That the Petersson metric has singularities at the boundary is easily seen: For instance, if $z = (z_1, z_2)$ approaches the cusp ∞ of Y_K, then $y_1 y_2 \to \infty$ by construction of the Baily-Borel topology. At the elliptic fixed points it is continuous, but the derivatives do have singularities.)

A central idea in [BBK] is to connect the arithmetic of Borcherds products and the properties of the automorphic Green functions G_m to derive information on $\widehat{\mathrm{CH}}_{HZ}^1(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$, the subspace of $\widehat{\mathrm{CH}}^1(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$ spanned by the arithmetic Hirzebruch-Zagier divisors.

Theorem 8.3. *Recall Assumption 8.1. The homomorphism* $\mathrm{Z}_{HZ}^1(\widetilde{X}_K) \to \widehat{\mathrm{Z}}_{HZ}^1(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})$ *defined by* $T(m) \mapsto \widehat{\mathcal{T}}(m)$ *induces an isomorphism*

$$\mathrm{CH}_{HZ}^1(\widetilde{X}_K)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{CH}}_{HZ}^1(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}},$$

taking $\mathrm{c}_1(\mathcal{M}_k(\mathbb{C}))$ *to* $\widehat{\mathrm{c}}_1(\mathcal{M}_k)$.

Sketch of the proof. We have to show that if there is a relation in $\mathrm{Z}^1(\widetilde{X}_K)$ among the $T(m)$, we can lift it to a relation in $\widehat{\mathrm{Z}}^1(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})$, and that every relation among arithmetic Hirzebruch-Zagier divisors arises in that way (up to torsion).

So suppose that F is a rational function on $\widetilde{X}_K$ with divisor $\sum_{n<0} \tilde{c}(n) T(-n)$. Then by the strong converse theorem (Theorem 5.4), we may assume that F is a Borcherds product, that is, the lift of a weakly holomorphic modular form $f \in W_0^+(D, \chi_D)$ with Fourier expansion $\sum_n c(n) q^n$ as in Theorem 3.1. It can be shown that any meromorphic Borcherds product is the quotient of two holomorphic ones ([BBK] Proposition 4.5). Therefore we may write $F = F_1/F_2$, where F_1, F_2 are holomorphic Borcherds products of the same weight. But then the Borcherds product expansion (Theorem 3.1 (iii)) implies that a positive power of F_j has integral rational Fourier coefficients. Without loss of generality we may assume that already the F_j have integral rational Fourier coefficients. According to the q-expansion principle F_j defines a section $\mathcal{F}_j$ of $\mathcal{M}_k$ on the model $\widetilde{\mathcal{X}}_K$. Hence the quotient of these sections is a rational function on $\widetilde{\mathcal{X}}_K$ that specializes to F on the generic fiber.

We claim that the divisor of $\mathcal{F}_j$ on $\widetilde{\mathcal{X}}_K$ is horizontal. To see this, we notice that by work of Rapoport, Deligne, and Pappas, the geometric fibers of $\mathcal{H}$ are irreducible (see [Ra], [DePa]). It follows by Assumption 8.1, that the same holds for the geometric fibers of $\widetilde{\mathcal{X}}_K$. Suppose that $\mathrm{div}(\mathcal{F}_j)$ contains a vertical component above a prime p. Then, because of the irreducibility of the fibers, $\mathrm{div}(\mathcal{F}_j)$ contains the full fiber above p. By the q-expansion principle, this implies that all Fourier coefficients of F_j are divisible by p. But the Borcherds product expansion of F_j shows that the coefficients are coprime (in fact, the coefficient corresponding to the Weyl vector $\rho(F_j)$ is 1), and therefore a contradiction.

Thus the divisor of $\mathcal{F}_1/\mathcal{F}_2$ is horizontal and equal to $\sum_{n<0} \tilde{c}(n)\mathcal{T}(-n)$. In view of (7.5) we may conclude that

$$\sum_{n<0} \tilde{c}(n)\widehat{\mathcal{T}}(-n) = (\mathrm{div}(\mathcal{F}_1/\mathcal{F}_2), -\log|F|) = 0 \in \widehat{\mathrm{CH}}^1(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}.$$

Conversely, every relation among arithmetic Hirzebruch-Zagier divisors obviously specializes to a relation on the generic fiber. $\qquad\square$

As a corollary we see that $\dim(\widehat{\mathrm{CH}}^1_{HZ}(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}) = \dim(\mathrm{CH}^1_{HZ}(\widetilde{X}_K)_{\mathbb{Q}}) = [\frac{D+19}{24}]$. Moreover, diagram (5.4) has an arithmetic analogue, where one has to replace $\mathrm{Z}^1_{HZ}(\widetilde{X}_K)$ by $\widehat{\mathrm{Z}}^1_{HZ}(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})$ and $\mathrm{CH}^1_{HZ}(\widetilde{X}_K)_{\mathbb{Q}}$ by $\widehat{\mathrm{CH}}^1_{HZ}(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$. Finally, in view of Theorem 4.3, one obtains the following arithmetic Hirzebruch-Zagier theorem (cf. [BBK] Theorem 6.2):

Theorem 8.4. *The arithmetic generating series*

$$(8.2) \qquad \widehat{A}(\tau) = \widehat{c}_1(\overline{\mathcal{M}}_{-1/2}) + \sum_{m>0} \widehat{\mathcal{T}}(m)q^m$$

is a holomorphic modular form in $M_2^+(D, \chi_D)$ with values in $\widehat{\mathrm{CH}}^1(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$, i.e., an element of $M_2^+(D, \chi_D) \otimes_{\mathbb{Q}} \widehat{\mathrm{CH}}^1(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$.

9. ARITHMETIC INTERSECTION NUMBERS

The first Chern form of the line bundle $\mathcal{M}_k(\mathbb{C})$ equipped with the Petersson metric is equal to

$$c_1(\mathcal{M}_k(\mathbb{C}), \|\cdot\|_{\mathrm{Pet}}) = 2\pi i k \cdot \Omega,$$

where Ω denotes the Kähler form (5.5). Consequently, $\mathrm{vol}_\Omega(Y_K) = \zeta_K(-1)$ can also be regarded as the geometric self intersection number $\mathcal{M}_1(\mathbb{C})^2$ of the line bundle of modular forms of weight 1. Moreover, Theorem 7.1 can be rephrased by saying that the intersection of the geometric generating series (4.2) and $c_1(\mathcal{M}_k(\mathbb{C}))$ is given by

$$A(\tau) \cdot c_1(\mathcal{M}_k(\mathbb{C})) = -\frac{k}{2}\zeta_K(-1) \cdot E_2(\tau, 0),$$

where $E_2(\tau, 0) \in M_2^+(D, \chi_D)$ is the Eisenstein series (7.2).

In view of this result it is natural to ask, what the intersection of the arithmetic generating series (8.2) with the class $\widehat{c}_1(\overline{\mathcal{M}}_k)^2 \in \widehat{\mathrm{CH}}^2(\widetilde{\mathcal{X}}_K, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$ is.

Theorem 9.1. *Recall Assumption 8.1. We have the following identities of arithmetic intersection numbers:*

$$(9.1) \qquad \widehat{A}(\tau) \cdot \widehat{c}_1(\overline{\mathcal{M}}_k)^2 = \frac{k^2}{2}\zeta_K(-1)\left(\frac{\zeta_K'(-1)}{\zeta_K(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2}\log(D)\right) \cdot E_2(\tau, 0),$$

where $E_2(\tau, 0)$ denotes the Eisenstein series defined in (7.2). In particular, the arithmetic self intersection number of $\overline{\mathcal{M}}_k$ is given by:

$$(9.2) \qquad \overline{\mathcal{M}}_k^3 = -k^3 \zeta_K(-1) \left(\frac{\zeta_K'(-1)}{\zeta_K(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2} \log(D) \right).$$

Let us briefly indicate how Theorem 9.1 can be proved (see [BBK] Theorem 6.4 for details). For simplicity we assume that $M_2^+(D, \chi_D) = \mathbb{C} E_2(\tau, 0)$ (that is $D = 5, 13$, or 17). In this case, regarding Theorem 8.4, we only have to determine the constant term of $\widehat{A}(\tau) \cdot \widehat{c}_1(\overline{\mathcal{M}}_k)^2$, that is, essentially the arithmetic self intersection number of $\overline{\mathcal{M}}_k$. The hypothesis on $M_2^+(D, \chi_D)$ implies in particular that G_m is the logarithm of the Petersson metric of a holomorphic Borcherds product with divisor $T(m)$ for any m.

Let p be any prime that is split in $\mathcal{O}_K$ (that is $\chi_D(p) = 1$). It can be shown that there exist infinitely many m_2 and infinitely many m_3, such that $\chi_D(m_j) = 1$, $T(m_2)$ is disjoint to the boundary, and such that all possible intersections of $T(p)$, $T(m_2)$, $T(m_3)$ on X_K are proper.

Let F_1, F_2, F_3 be the Borcherds products on X_K with divisors $T(p)$, $T(m_2)$, $T(m_3)$, respectively. By the bilinearity of the arithmetic intersection pairing we may assume that these Borcherds products are integral, i.e., have trivial multiplier system and integral rational Fourier coefficients. We may further assume that they all have the same (sufficiently divisible) weight k. Thus they define sections of $\mathcal{M}_k$. The definition of the arithmetic self intersection number then says:

$$\overline{\mathcal{M}}_k^3 = \widehat{\deg} \left(h_* \left(\operatorname{div}(F_1) \cdot \operatorname{div}(F_2) \cdot \operatorname{div}(F_3) \right) \right)$$

$$(9.3) \qquad + \frac{1}{(2\pi i)^2} \int\limits_{\widetilde{\mathcal{X}}_K(\mathbb{C})} (-\log \|F_1\|_{\mathrm{Pet}}) * (-\log \|F_2\|_{\mathrm{Pet}}) * (-\log \|F_3\|_{\mathrm{Pet}}).$$

Here the integral is over the star product of the Green functions corresponding to the sections F_j of $\overline{\mathcal{M}}_k$. It describes the intersection at the Archimedian place. Moreover, $h : \widetilde{\mathcal{X}}_K \to \operatorname{Spec}(\mathbb{Z})$ denotes the structure morphism. The first summand is the intersection at the finite places.

Using growth estimates for certain boundary terms, one finds that the integral in (9.3) is equal to

$$k^2 \int\limits_{\widetilde{X}_K} G_p \, \Omega^2 + k \int\limits_{T(p)'} -\log \|F_2\|_{\mathrm{Pet}} \, \Omega + \int\limits_{T(p)' \cap \operatorname{div}(F_2)'} -\log \|F_3\|_{\mathrm{Pet}},$$

where $T(p)'$ denotes the strict transform of the divisor $T(p)$ in $\widetilde{X}_K$ ([BBK] Theorem 3.13). The integral of G_p was computed in (7.6).

There is a birational morphism φ from the modular curve $X_0(p)$ onto $T(p)$ (which extends to integral models over $\mathbb{Z}[1/p]$). This fact can be used to interpret the sum of the latter two integrals as a star product on the modular curve $X_0(p)$, where it can be evaluated by means of the results of [Kü2] or [Bost].

The finite intersection can also be reduced to a finite intersection on the minimal regular model of $X_0(p)$ by applying the projection formula for the morphism φ.

It turns out that the finite contribution and the Archimedian contribution fit together rather nicely and yield the desired result up to contributions from the fiber above p. But now we can vary p, that is, take different Borcherds products for the F_j, to get the precise formula for $\overline{\mathcal{M}}_k^3$.

In the general case, one can argue similarly, since it can be proved that $\mathrm{CH}^1_{HZ}(\widetilde{X}_K)_{\mathbb{Q}}$ is already generated by Hirzebruch-Zagier divisors $T(p)$ of prime discriminant p (with $\chi_D(p) = 1$), see [BBK] section 4.2.

Formula (9.2) provides evidence for a conjecture of Kramer, based on results obtained in [Kr] and [Kü1], saying that the arithmetic volume of an arithmetic variety as $\widetilde{\mathcal{X}}_K$ is essentially the derivative of the zeta value for the geometric volume of $\widetilde{\mathcal{X}}_K(\mathbb{C})$. In the same way, it provides further evidence for the conjecture of Kudla on the constant term of the derivative of certain Eisenstein series [Ku2], [Ku3], [Ku5], and the conjecture of Maillot and Roessler on special values of logarithmic derivatives of Artin L-functions [MaRo].

It would be very interesting to find a more conceptual explanation for the fact that the geometric intersection $A(\tau) \cdot \mathrm{c}_1(\mathcal{M}_k(\mathbb{C}))$ is proportional to $\widehat{A}(\tau) \cdot \widehat{\mathrm{c}}_1(\overline{\mathcal{M}}_k)^2$.

We may apply Theorem 9.1 and (7.6) to compute the Faltings height of $\mathcal{T}(m)$ with respect to $\overline{\mathcal{M}}_k$ (as defined in [BKK1] and [BBK] section 1). We find:

Theorem 9.2. *Recall Assumption 8.1. If $T(m)$ is a Hirzebruch-Zagier divisor which is disjoint to the boundary of X_K, then the Faltings height of its model $\mathcal{T}(m) \in \mathrm{Z}^1(\widetilde{\mathcal{X}}_K)$ is given by*

$$\mathrm{ht}_{\overline{\mathcal{M}}_k}(\mathcal{T}(m)) = -2k^2 \, \mathrm{vol}_\Omega(T(m)) \left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} + \frac{1}{2}\frac{\sigma'_m(-1)}{\sigma_m(-1)} \right).$$

Here $\sigma_m(s)$ is the generalized divisor sum defined in (7.7).

We conclude by noticing that Assumption 8.1 can be avoided in the above theorems by introducing a level structure to rigidify the moduli problem. For instance, in [BBK] the full level N-structure is used (where N is an arbitrary integer ≥ 3). Then the moduli problem is represented by an arithmetic variety over $\mathbb{Z}[\zeta_N, 1/N]$. However, since N is inverted in the base, one only gets arithmetic intersection numbers in $\mathbb{R}_N = \mathbb{R}/\langle \sum_{p|N} \mathbb{Q} \cdot \log(p) \rangle$.

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MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, WEYERTAL 86–90, D-50931 KÖLN, GERMANY
E-mail address: bruinier@math.uni-koeln.de