IDENTITIES AND CONGRUENCES FOR RAMANUJAN'S $\omega(q)$

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For George E. Andrews on his 70th birthday

Abstract. Recently, the authors [3] constructed *generalized Borcherds products* where modular forms are given as infinite products arising from weight 1/2 harmonic Maass forms. Here we illustrate the utility of these results in the special case of Ramanujan's mock theta function $\omega(q)$. We obtain identities and congruences modulo 512 involving the coefficients of $\omega(q)$.

1. Introduction and statement of results

In a recent paper, the authors [3] obtained results concerning *generalized Borcherds products*. Loosely speaking, these are modular forms which are infinite products whose exponents are coefficients of weight 1/2 harmonic Maass forms (see [6] for a survey on harmonic Maass forms in number theory). The authors then employed these results to study the vanishing of derivatives of modular L-functions.

Here we illustrate the implications of these results for partitions and q-series. We consider the special case of Ramanujan's mock theta function $\omega(q)$

$$\omega(q)=\sum_{n=0}^{\infty} a_{\omega}(n)q^n := \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} = 1+2q+3q^2+4q^3+6q^4+8q^5+\cdots. \tag{1.1}$$

As usual, we use the customary notation

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

Thanks to Fine's identity (see (26.84) of [4])[1]

$$q\omega(q)=\sum_{n=0}^{\infty}\frac{q^{n+1}}{(q;q^2)_{n+1}}=\sum_{n=1}^{\infty}\frac{q^n}{(1-q^{1+0})(1-q^{2+1})\cdots(1-q^{n+(n-1)})}, \tag{1.2}$$

we find that $\omega(q)$ is a generating function for an elegant partition function. The coefficient $a_{\omega}(n)$ denotes the number of partitions of $n-1$ whose summands, apart from one of maximal size, form pairs of consecutive non-negative integers.

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[1] The reader is also encouraged to see Andrews's recent paper [1] for more on the combinatorial interpretation of $\omega(q)$.

Example. Here are the partitions of 6:

$$6,\ 5+1,\ 4+2,\ 4+1+1,\ 3+3,\ 3+2+1,\ 3+1+1+1,$$
$$2+2+2,\ 2+2+1+1,\ 2+1+1+1+1,\ 1+1+1+1+1+1.$$

Eight of these partitions correspond to partitions who summands, apart from one of the largest summands, occur in pairs of consecutive non-negative integers:

$$\begin{aligned}&6,\ 5+(1+0),\ 4+(1+0)+(1+0),\ 3+(2+1),\ 3+(1+0)+(1+0)+(1+0),\\ &2+(2+1)+(1+0),\ 2+(1+0)+(1+0)+(1+0)+(1+0),\\ &1+(1+0)+(1+0)+(1+0)+(1+0)+(1+0).\end{aligned}$$

This corresponds to our observation that $a_\omega(5)=8$.

Here we investigate the arithmetic properties of the partition function $a_\omega(n)$. We shall relate this function to the classical divisor functions

$$\sigma_\nu(n) := \sum_{1\leq d\mid n} d^\nu \tag{1.3}$$

which play central roles in the theory of modular forms. To this end we define a "strange" divisor function using the coefficients $a_\omega(n)$, the Legendre symbol $\left(\frac{\bullet}{3}\right)$, and the classical Jacobi-symbol character $\chi(m) := \left(\frac{-8}{m}\right)$. We define $\widehat{\sigma}_\omega(n)$ by

$$\widehat{\sigma}_\omega(n) := \sum_{1\leq d\mid n} \left(\frac{d}{3}\right)\chi(n/d)d\cdot a_\omega\left(\frac{2d^2-2}{3}\right), \tag{1.4}$$

and we consider the following two generating functions:

$$L_\omega(q) := \sum_{n\geq 1}\widehat{\sigma}_\omega(n)q^n = q-6q^2+q^3+116q^4-506q^5-6q^6+\cdots \tag{1.5}$$

$$\widetilde{L}_\omega(q) := \sum_{\substack{n\geq 1\\ \gcd(n,6)=1}} \widehat{\sigma}_\omega(n)q^n = q-506q^5+9736q^7-3638260q^{11}+\cdots. \tag{1.6}$$

We prove the following curious theorem.

Theorem 1.1. *The q-series $L_\omega(q)$ (resp. $\widetilde{L}_\omega(q)$) is the Fourier expansion of a weight 2 meromorphic modular form on $\Gamma_0(6)$ (resp. $\Gamma_0(216)$), where $q:=e^{2\pi iz}$.*

An explicit form of this result (see Section 2) gives the following congruences.

Theorem 1.2. *The following are true:*

(1) *We have that*

$$L_\omega(q) \equiv \sum_{n=0}^{\infty}\left(q^{(2n+1)^2}+q^{3(2n+1)^2}\right) \pmod 2.$$

(2) *We have that*

$$\widetilde{L}_\omega(q) \equiv \sum_{\substack{n\geq 1\\ \gcd(n,6)=1}} \sigma_1(n)q^n \pmod{512}.$$

In particular, if $p \geq 5$ is prime, then

$$a_\omega\left(\frac{2p^2-2}{3}\right) \equiv \begin{cases} \left(\frac{p}{3}\right) \pmod{512} & \text{if } p \equiv 1,3 \pmod 8,\\ \left(\frac{p}{3}\right)\left(1+2p^{255}\right) \pmod{512} & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Example. If $p=7$, then Theorem 1.2 (2) implies that

$$a_\omega(32) = 1391 \equiv 367 \equiv \left(\frac{7}{3}\right)\left(1+2\cdot 7^{255}\right) \pmod{512}.$$

Three Remarks.

(1) It is natural to ask whether there is a combinatorial explanation for the fact that $a_\omega\left(\frac{2p^2-2}{3}\right) \equiv \pm 1 \pmod{512}$ for the "half" of the primes which satisfy the congruence $p \equiv 1,3 \pmod 8$.

(2) The results presented here are examples of a general theory in the case of a single generalized Borcherds product for $\omega(q)$. There are infinitely many such Borcherds products for $\omega(q)$. For any given product, one may obtain congruences modulo arbitrary powers of infinitely many primes (for example, see [2]). For $L_\omega(q)$, these are the primes p for which $\left(\frac{-2}{p}\right) \in \{0,-1\}$. For these p we have that $L_\omega(q)$ is a p-adic modular form in the sense of Serre [7] (for example, see [2]). In the present paper we are content with $p=2$ and the p-power modulus $2^9 = 512$.

(3) More generally, one may construct such generalized Borcherds products for all of Ramanujan's mock theta functions using Theorems 6.1 and 6.2 of [3]. These modular forms will have twisted Heegner divisors, as well as logarithmic derivatives which are meromorphic weight 2 modular forms, which for certain primes p will turn out to be p-adic modular forms.

In Section 2 we prove Theorems 1.1 and 1.2 using the results of [3] combined with various standard arguments from the theory of modular forms.

2. Proofs

Our results follow from a generalized Borcherds product obtained in [3]. Using the coefficients of $\omega(q)$, we define the formal power series

$$\begin{aligned} B_\omega(z) &:= \prod_{m=1}^{\infty}\left(\frac{1+\sqrt{-2}q^m-q^{2m}}{1-\sqrt{-2}q^m-q^{2m}}\right)^{-4\left(\frac{m}{3}\right)a_\omega\left(\frac{2m^2-2}{3}\right)} \\ &= \left(\frac{1+\sqrt{-2}q-q^2}{1-\sqrt{-2}q-q^2}\right)^{-4}\cdot\left(\frac{1+\sqrt{-2}q^2-q^4}{1-\sqrt{-2}q^2-q^4}\right)^{12}\cdots \\ &= 1-8\sqrt{-2}q-(64-24\sqrt{-2})q^2+(384+168\sqrt{-2})q^3\cdots. \end{aligned} \tag{2.1}$$

This formal power series, where $q := e^{2\pi i z}$ for $z \in \mathbb{H}$, is discussed in Example 8.2 of [3]. Thanks to work of Zwegers [8], Theorems 6.1 and 6.2 of [3] implies the following theorem.

Theorem 2.1. *The q-series $B_\omega(z)$ is the Fourier expansion of a weight 0 modular form on the congruence subgroup $\Gamma_0(6)$.*

Proof of Theorem 1.1. That $\widetilde{L}_\omega(q)$ is a meromorphic modular form on $\Gamma_0(216)$ will follow from the assertion that $L_\omega(q)$ is the Fourier expansion of a weight 2 meromorphic modular form on $\Gamma_0(6)$. One simply uses the standard U and V operators (for example, see §2.4 of [5]).

Let $\Theta := q \cdot \frac{q}{dq} = \frac{1}{2\pi i} \cdot \frac{d}{dz}$. If $f(z)$ is a meromorphic modular form (for example, see §2.3 of [5]), it is a standard fact that $\Theta(f)/f$ is a weight 2 meromorphic modular form. Therefore, it follows that

$$\begin{aligned}\frac{\Theta(B_\omega(z))}{B_\omega(z)} &= -8\sqrt{-2}q + 48\sqrt{-2}q^2 - 8\sqrt{-2}q^3 - 928\sqrt{-2}q^4 + 4048\sqrt{-2}q^5 + \cdots \\ &= -8\sqrt{-2}\left(q - 6q^2 + q^3 + 116q^4 - 506q^5 - 6q^6 + 9736q^7 - \cdots\right) \\ &= -8\sqrt{-2} \cdot G_\omega(q)\end{aligned}$$

is a weight 2 meromorphic modular form on $\Gamma_0(6)$. It suffices to prove that $G_\omega(q) = L_\omega(q)$.

To prove this assertion, we let

$$P(X) := \frac{1 + \sqrt{-2}X - X^2}{1 - \sqrt{-2}X - X^2}.$$

If m is a positive integer, then a straightforward calculation reveals that

$$\frac{\Theta(P(q^m))}{P(q^m)} = 2m\sqrt{-2}\sum_{n=1}^{\infty} \chi(n) q^{mn}.$$

Using this result, it follows that

$$\frac{\Theta(B_\omega(z))}{B_\omega(z)} = -8\sqrt{-2}\sum_{m=1}^{\infty} m \left(\frac{m}{3}\right) a_\omega\left(\frac{2m^2-2}{3}\right)\sum_{n=1}^{\infty}\chi(n)q^{mn}.$$

That $G_\omega(q) = L_\omega(q)$ now follows immediately, and so $L_\omega(q)$ is a meromorphic modular form on $\Gamma_0(6)$. □

Now we turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. We recall the explicit description of the meromorphic modular form $B_\omega(z)$ given in Example 8.2 of [3]. Let $j_6^*(z)$ be the usual Hauptmodul for $\Gamma_0^*(6)$, the extension of $\Gamma_0(6)$ by all the Atkin-Lehner involutions. It is not difficult to verify that

$$j_6^*(z) := \left(\frac{\eta(z)\eta(2z)}{\eta(3z)\eta(6z)}\right)^4 + 4 + 3^4\left(\frac{\eta(3z)\eta(6z)}{\eta(z)\eta(2z)}\right)^4 = q^{-1} + 79q + 352q^2 + 1431q^3 + \cdots,$$

where $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta-function. Let α_1 and α_2 be the Heegner points

$$\alpha_1 := \frac{-2+\sqrt{-2}}{6} \quad \text{and} \quad \alpha_2 := \frac{2+\sqrt{-2}}{6}.$$

We have that $j_6^*(\alpha_1) = j_6^*(\alpha_2) = -10$. Therefore, it follows that $j_6^*(z) + 10$ is a rational modular function on $X_0(6)$ whose divisor consists of the 4 cusps with multiplicity -1 and the points α_1 and α_2 with multiplicity 2.

Let $E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n$ be the standard weight 4 Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, and let

$$\delta(z) := \eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2 = q - 2q^2 - 3q^3 + 4q^4 + \cdots .$$

Using $E_4(z)$ and $\delta(z)$, we define the weight 4 holomorphic $\Gamma_0(6)$-modular form $\phi(z)$ by

$$\begin{aligned}450\phi(z) : &= (3360 - 1920\sqrt{-2})\delta(z) + (1 - 7\sqrt{-2})E_4(z)\\ &\quad + (4 - 28\sqrt{-2})E_4(2z) + (89 + 7\sqrt{-2})E_4(3z) + (356 + 28\sqrt{-2})E_4(6z).\end{aligned}$$

It turns out that $\phi(z)$ has divisor $4(\alpha_1)$. In terms of $\phi(z), j_6^*(z)$ and $\delta(z)$, it turns out that

$$B_\omega(z) = \frac{\phi(z)}{(j_6^*(z) + 10)\delta(z)}.$$

By Theorem 1 of [2], generalized to $\Gamma_0(6)$ and $B_\omega(z)$ in the obvious way, we have that $-8\sqrt{-2}L_\omega(q) = \Theta(B_\omega(z))/B_\omega(z)$ is a 2-adic modular form of weight 2. This then implies that $L_\omega(z) \pmod{2^k}$, for every positive integer k, is the reduction of a holomorphic modular form.

To obtain Theorem 1.2, we now employ the identity

$$\mathcal{E}(z) := \frac{\eta(4z)^8}{\eta(2z)^4} = \sum_{n=0}^{\infty} \sigma_1(2n+1)q^{2n+1} = q + 4q^3 + 6q^5 + \cdots . \tag{2.2}$$

Congruence (1) is equivalent to the assertion that

$$L_\omega(q) \equiv \mathcal{E}(z) + \mathcal{E}(3z) \pmod{2},$$

while (2) is equivalent to the assertion that

$$\widetilde{L}_\omega(q) \equiv \mathcal{E}(z) - \mathcal{E}(z)|U(3)|V(3) \pmod{512}.$$

These congruences are easily confirmed using the constructive proof of Theorem 1 of [2], combined with Sturm's Theorem (see Theorem 2.58 of [5]). That

$$a_\omega\left(\frac{2p^2-2}{3}\right) \equiv \begin{cases} \left(\frac{p}{3}\right) \pmod{512} & \text{if } p \equiv 1, 3 \pmod{8},\\ \left(\frac{p}{3}\right)(1 + 2p^{255}) \pmod{512} & \text{if } p \equiv 5, 7 \pmod{8}\end{cases}$$

follows easily from (2), namely that

$$\sigma_1(p) \equiv \widehat{\sigma}_\omega(p) \pmod{512},$$

and the definition of $\widehat{\sigma}_\omega(p)$. ∎

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