CM-VALUES OF HILBERT MODULAR FUNCTIONS

JAN HENDRIK BRUINIER AND TONGHAI YANG

1. Introduction

Let X be an algebraic variety over $\mathbb{Q}$, and let Ψ be a rational function on X and $C = \sum n_P P$ be a rational 0-cycle on X, then $\Psi(C) = \prod \Psi(P)^{n_P}$ is a rational number. A natural question is to figure out its prime factorization. Of course, the question asked in such a generality is perhaps of no good since anything can happen. However, for some special functions and cycles, the answer could be very interesting, as first theoretically established by Gross and Zagier in 1985 [GZ1] for the classical j-function. More precisely, let d_1 and d_2 be two fundamental discriminants of imaginary quadratic fields such that $(d_1, d_2) = 1$, and $D = d_1 d_2$. Let $X = (\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^2$ be the self-product of the simplest modular curve $X_1 = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Let $\tau = (\tau_1, \tau_2)$ be a CM point in X such that τ_i is a CM point on X_1 of discriminant d_i, and denote by $\mathcal{CM}(d_1, d_2)$ the rational 0-cycle in X given by the sum of all such points. Gross and Zagier consider the value of the rational function $j(z_1) - j(z_2)$ on X at $\mathcal{CM}(d_1, d_2)$. They found a striking explicit formula for the prime factorization of

$$J(d_1, d_2) = \prod_{\substack{[\tau_1],[\tau_2] \\ \mathrm{disc}(\tau_i)=d_i}} \big(j(\tau_1) - j(\tau_2)\big)^{\frac{4}{w_1 w_2}},$$

which states that

$$J(d_1, d_2)^2 = \pm \prod_{\substack{x,n,n' \in \mathbb{Z}, nn'>0 \\ x^2+4nn'=D}} n^{\epsilon(n')}. \tag{1.1}$$

Here w_i is the number of roots of unity in $\mathbb{Q}(\sqrt{d_i})$, and ϵ is related to the genus character as follows: $\epsilon(n) = \prod \epsilon(l_i)^{a_i}$ if n has the prime factorization $n = \prod l_i^{a_i}$, and

$$\epsilon(l) = \begin{cases} (\frac{d_1}{l}) & \text{if } l \nmid d_1, \\ (\frac{d_2}{l}) & \text{if } l \nmid d_2, \end{cases}$$

for primes l with $(\frac{D}{l}) \neq -1$. The main purpose of this paper is to establish a similar formula for certain rational functions on a Hilbert modular surface and CM 0-cycles on the surface associated to a *non-biquadratic quartic CM field* containing the underlying real quadratic field of the Hilbert modular surface. One of the interesting features is that the

Date: April 26, 2004. Revision: February 23, 2005.

2000 *Mathematics Subject Classification.* 11G15, 11F41, 14K22.

T.Y. Yang is partially supported by a NSF grant DMS-0302043 and a NSA grant.

factorization is determined by the arithmetic of the reflex field. Now we describe the result in more detail.

Let $p \equiv 1 \pmod 4$ be a prime number and $F = \mathbb{Q}(\sqrt{p})$. We write $\mathcal{O}_F$ for the ring of integers of F, and $x \mapsto x'$ for the conjugation in F. Let $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ be the Hilbert modular group associated to F. The corresponding Hilbert modular surface $X = \Gamma\backslash\mathbb{H}^2$ is a normal quasi-projective algebraic variety defined over $\mathbb{Q}$.

Let $K = F(\sqrt{\Delta})$ be a non-biquadratic totally imaginary quadratic extension of F. We view both K and $F(\sqrt{\Delta'})$ as subfields of $\mathbb{C}$ with $\sqrt{\Delta}, \sqrt{\Delta'} \in \mathbb{H}$. Then $M = F(\sqrt{\Delta}, \sqrt{\Delta'})$ is Galois over $\mathbb{Q}$ and has an automorphism σ of order 4 such that

$$\sigma(\sqrt{\Delta}) = \sqrt{\Delta'}, \quad \sigma(\sqrt{\Delta'}) = -\sqrt{\Delta}. \tag{1.2}$$

Notice that K has four CM types: $\Phi = \{1, \sigma\}$, $\sigma\Phi = \{\sigma, \sigma^2\}$, $\sigma^2\Phi$, and $\sigma^3\Phi$. We write $(\tilde{K}, \tilde{\Phi})$ for the reflex of (K, Φ), and let $\tilde{F} = \mathbb{Q}(\sqrt{\Delta\Delta'})$ be the real quadratic subfield of $\tilde{K}$. We refer to [Sh] for details about CM types and reflex fields. For technical reasons (see Remark 9.2), we assume in this paper that

$$d_{K/F} \cap \mathbb{Z} = q\mathbb{Z}, \quad N_{F/\mathbb{Q}} d_{K/F} = q, \tag{1.3}$$

for a prime number $q \equiv 1 \pmod 4$. Here $d_{K/F}$ is the relative discriminant of K/F. This condition implies (Lemma 7.1) that $\tilde{F} = \mathbb{Q}(\sqrt{q})$, and

$$d_{\tilde{K}/\tilde{F}} = \tilde{\mathfrak{p}}, \quad N_{\tilde{F}/\mathbb{Q}} \tilde{\mathfrak{p}} = p, \tag{1.4}$$

for a prime ideal $\tilde{\mathfrak{p}}$ of $\tilde{F}$. For a nonzero element $t \in d_{\tilde{K}/\tilde{F}}^{-1}$ and a prime ideal $\mathfrak{l}$ of $\tilde{F}$, we define

$$B_t(\mathfrak{l}) = \begin{cases} 0 & \text{if } \mathfrak{l} \text{ is split in } \tilde{K}, \\ (\mathrm{ord}_{\mathfrak{l}}\, t + 1)\rho(t d_{\tilde{K}/\tilde{F}} \mathfrak{l}^{-1}) \log|\mathfrak{l}| & \text{if } \mathfrak{l} \text{ is non-split in } \tilde{K}, \end{cases} \tag{1.5}$$

and

$$B_t = \sum_{\mathfrak{l}} B_t(\mathfrak{l}). \tag{1.6}$$

Here $|\mathfrak{l}|$ is the norm of $\mathfrak{l}$, and $\rho(\mathfrak{a}) = \rho_{\tilde{K}/\tilde{F}}(\mathfrak{a})$ is defined as

$$\rho(\mathfrak{a}) = \#\{\mathfrak{A} \subset \mathcal{O}_{\tilde{K}} :\ N_{\tilde{K}/\tilde{F}} \mathfrak{A} = \mathfrak{a}\}. \tag{1.7}$$

We remark that $\rho(\mathfrak{a}) = 0$ for a non-integral ideal $\mathfrak{a}$, and that for every $t \neq 0$, there are at most finitely many prime ideals $\mathfrak{l}$ such that $B_t(\mathfrak{l}) \neq 0$. In fact, when $t > 0 > t'$, then $B_t = 0$ unless there is *exactly one* prime ideal $\mathfrak{l}$ such that $\chi_{\mathfrak{l}}(t) = -1$, in which case $B_t = B_t(\mathfrak{l})$ (see Remark 7.3). Here $\chi = \prod_{\mathfrak{l}} \chi_{\mathfrak{l}}$ is the quadratic Hecke character of $\tilde{F}$ associated to $\tilde{K}/\tilde{F}$.

Let $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ be the CM 0-cycle in X of CM abelian surfaces of CM type (K, Φ), i.e., the points on X with an $\mathcal{O}_K$ action via Φ (see Section 3). By the theory of complex multiplication [Sh], the field of moduli for $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ is the reflex field $\tilde{K}$ of (K, Φ). In fact, one can show that the field of moduli for $\mathcal{CM}(K) = \mathcal{CM}(K, \Phi, \mathcal{O}_F) + \mathcal{CM}(K, \sigma^3\Phi, \mathcal{O}_F)$ is $\mathbb{Q}$ (see Remark 3.5). Therefore, if Ψ is a rational function on X, i.e., a Hilbert modular function for Γ over $\mathbb{Q}$, then $\Psi(\mathcal{CM}(K))$ is a rational number, and it would be very interesting but highly nontrivial to find a factorization formula for this number. The main purpose of this paper is to find such a formula in some special cases.

Recall that for any positive integer m there is a distinguished divisor T_m on X, the so-called Hirzebruch-Zagier divisor of discriminant m (see Section 2). Here we consider Hilbert modular functions whose divisor is supported on Hirzebruch-Zagier divisors. The (multiplicative) group of these functions can be characterized as the image of the Borcherds lift [Bo1], [Br1], [BB], which can be viewed as a multiplicative analogue of the Doi-Naganuma lift [DN], [Na], [Za]. Observe that in the case of Gross and Zagier, one may regard $(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^2$ as a degenerate Hilbert modular surface associated to $\mathbb{Q}\oplus\mathbb{Q}$, and $j(z_1)-j(z_2)$ as the Borcherds lift of $j(\tau)-744$.

A holomorphic Hilbert modular form for the group Γ is called *normalized integral*, if all its Fourier coefficients at the cusp ∞ are rational integers with greatest common divisor 1. A meromorphic Hilbert modular form is called *normalized integral*, if it is the quotient of two normalized integral holomorphic Hilbert modular forms.

Theorem 1.1. *Let the notation and assumption be as in* (1.3)–(1.7). *Let* Ψ *be a normalized integral Hilbert modular function (of weight* 0*) for the group* Γ *such that*

$$\mathrm{div}(\Psi) = \sum_{m>0} \tilde{c}(-m) T_m,$$

with integral coefficients $\tilde{c}(-m) \in \mathbb{Z}$. *Then*

$$\log|\Psi(\mathcal{CM}(K))| = \frac{W_{\tilde{K}}}{4} \sum_{m>0} \tilde{c}(-m) b_m,$$

where $W_{\tilde{K}}$ *is the number of roots of unity in* $\tilde{K}$, *and*

$$b_m = \sum_{\substack{t=\frac{n+m\sqrt{q}}{2p}\in d_{\tilde{K}/\tilde{F}}^{-1} \\ |n|<m\sqrt{q}}} B_t. \tag{1.8}$$

We remark that $W_{\tilde{K}} = 2$ except when $K = \tilde{K} = \mathbb{Q}(\zeta_5)$, in which case $W_{\tilde{K}} = 10$. Notice that $\log|\Psi(\mathcal{CM}(K))|$ gives a bilinear pairing between the multiplicative group of Hilbert modular functions and the additive group of CM 0-cycles on X. Theorem 1.1 provides an explicit formula when the Hilbert modular function is normalized integral with divisor supported on Hirzebruch-Zagier divisors. This theorem can also be rephrased as a formula for the factorization of the rational number $\Psi(\mathcal{CM}(K))$.

Corollary 1.2. *Let the notation and assumption be as in Theorem 1.1. Then*

$$\Psi(\mathcal{CM}(K)) = \pm \prod_{l \text{ rational prime}} l^{e_l} \tag{1.9}$$

with

$$e_l = \frac{W_{\tilde{K}}}{4} \sum_{m>0} \tilde{c}(-m) b_m(l),$$

and

$$b_m(l)\log l = \sum_{\mathfrak{l}|l} \sum_{\substack{t=\frac{n+m\sqrt{q}}{2p}\in d_{\tilde{K}/\tilde{F}}^{-1} \\ |n|<m\sqrt{q}}} B_t(\mathfrak{l}).$$

Moreover, when $K/\mathbb{Q}$ is cyclic, the sign in (1.9) *is positive.*

Corollary 1.3. *Let the notation and assumption be as in Corollary 1.2. Then $e_l = 0$ unless $4pl|m^2q - n^2$ for some $m \in M := \{m \in \mathbb{Z}_{>0} : \tilde{c}(-m) \neq 0\}$ and some integer $|n| < m\sqrt{q}$. In particular, every prime factor of $\Psi(\mathcal{CM}(K))$ is less than or equal to $\frac{N^2q}{4p}$, where $N = \max(M)$.*

In short, the factorization of $\Psi(\mathcal{CM}(K))$ is determined by the arithmetic of the reflex field $\tilde{K}$. We illustrate these results with some examples for $p = 5$ and 13. For details we refer to Section 10. When $F = \mathbb{Q}(\sqrt{5})$, there are precisely three normalized integral holomorphic Hilbert modular forms of weight 10 for the group $\mathrm{SL}_2(\mathcal{O}_F)$ whose divisor is supported on Hirzebruch-Zagier divisors. They have the divisors $2T_1$, T_6, and T_{10}, respectively, and are denoted by Ψ_1^2, Ψ_6, and Ψ_{10}, accordingly. They are constructed explicitly as Borcherds lifts and as Doi-Naganuma lifts in Section 10. Table 1 gives the factorization of $\Psi(\mathcal{CM}(K))$ up to sign according Corollary 1.2, where q is the rational prime number such that $N_{F/\mathbb{Q}}d_{K/F} = q$.

TABLE 1. The case $F = \mathbb{Q}(\sqrt{5})$

q	$\frac{\Psi_6}{\Psi_1^2}(\mathcal{CM}(K))$	$\frac{\Psi_{10}}{\Psi_1^2}(\mathcal{CM}(K))$
5 (cyclic)	$2^{20}\cdot 3^{10}$	$2^{20}\cdot 5^{10}$
41	$2^{14}\cdot 3^{10}\cdot 61\cdot 73$	$2^{14}\cdot 5^{9}\cdot 37\cdot 41$
61	$2^{20}\cdot 3^{6}\cdot 13\cdot 97\cdot 109$	$2^{20}\cdot 5^{9}\cdot 61$
109	$2^{20}\cdot 3^{8}\cdot 61\cdot 157\cdot 193$	$2^{20}\cdot 5^{12}\cdot 73$
149	$2^{20}\cdot 3^{10}\cdot 31^2\cdot 37\cdot 229$	$2^{20}\cdot 5^{12}\cdot 17\cdot 113$
269	$2^{20}\cdot 3^{10}\cdot 13^{-2}\cdot 37^2\cdot 61\cdot 97\cdot 349\cdot 433$	$2^{20}\cdot 5^{14}\cdot 13^{-1}\cdot 53\cdot 73\cdot 233$

When $F = \mathbb{Q}(\sqrt{13})$, there are three normalized integral holomorphic Hilbert modular forms of weight 6 for $\mathrm{SL}_2(\mathcal{O}_F)$ whose divisor is supported on Hirzebruch-Zagier divisors. They have the divisors $6T_1$, T_{14}, and T_{26}, respectively, and are denoted by Ψ_1^6, Ψ_{14}, and Ψ_{26}, accordingly. They are also constructed as Borcherds lifts and as Doi-Naganuma lifts in Section 10. Table 2 gives the factorization of $\Psi(\mathcal{CM}(K))$ up to sign according Corollary 1.2.

These values were verified numerically using the representations of the involved Borcherds products as Doi-Naganuma lifts.

In the present paper we will in fact prove the following slightly more general result on meromorphic Hilbert modular forms of arbitrary weights, from which Theorem 1.1 is a special corollary. It is quite amusing to see how the weight affects the formula. It can be viewed as a period integral of the Borcherds product over a CM 0-cycle. The period

TABLE 2. The case $F = \mathbb{Q}(\sqrt{13})$

q	$\frac{\Psi_{14}}{\Psi_1^6}(\mathcal{CM}(K))$	$\frac{\Psi_{26}}{\Psi_1^6}(\mathcal{CM}(K))$
13 (cyclic)	$2^{12} \cdot 7^2$	$2^{12} \cdot 13^4$
17	$2^{13} \cdot 7^2 \cdot 53$	$2^{11} \cdot 13^5$
29	$2^{12} \cdot 5^2 \cdot 7^2 \cdot 109$	$2^{12} \cdot 13^4 \cdot 29$
113	$2^{20} \cdot 7^6 \cdot 11^2 \cdot 149 \cdot 277 \cdot 337 \cdot 401 \cdot 421$	$2^{22} \cdot 7^2 \cdot 11^2 \cdot 13^{11} \cdot 97 \cdot 109 \cdot 113$
157	$2^{12} \cdot 3^2 \cdot 7^2 \cdot 19^2 \cdot 37 \cdot 193$	$2^{12} \cdot 13^4 \cdot 17^2 \cdot 157$
269	$2^{36} \cdot 5^{-2} \cdot 7^6 \cdot 23^2 \cdot 37^3 \cdot 61^2 \cdot 67^2 \cdot 1009$	$2^{36} \cdot 5^{-2} \cdot 13^{12} \cdot 23^2 \cdot 73 \cdot 233^2 \cdot 269$

integral of the Borcherds product over the 2-cycle (i.e., the volume form) was studied by Kudla [Ku2] and Bruinier, Burgos, and Kühn [BBK, BK] independently.

Theorem 1.4. *Let the notation and assumption be as in Theorem 1.1, but let Ψ be a normalized integral meromorphic Hilbert modular form of weight $c(0)$ for the group Γ such that*

$$\operatorname{div}(\Psi) = \sum_{m>0} \tilde{c}(-m) T_m,$$

with integral coefficients $\tilde{c}(-m) \in \mathbb{Z}$. Then

$$\log \|\Psi(\mathcal{CM}(K))\|_{\mathrm{Pet}} = \frac{W_{\tilde{K}}}{4} \sum_{m>0} \tilde{c}(-m) b_m - \frac{W_{\tilde{K}}}{4} c(0) \alpha(\tilde{K}/\tilde{F})$$

with

$$\alpha(\tilde{K}/\tilde{F}) = \Lambda(0,\chi)\left(\Gamma'(1) + \frac{\Lambda'(0,\chi)}{\Lambda(0,\chi)} - \log 4\pi\right).$$

Here b_m and $W_{\tilde{K}}$ are given in Theorem 1.1, $\|\Psi\|_{\mathrm{Pet}}$ denotes the Petersson metric of Ψ normalized as in (2.27), χ is the quadratic Hecke character of $\tilde{F}$ associated to $\tilde{K}/\tilde{F}$, and $\Lambda(s,\chi)$ is the completed L-function of χ.

Although the proof of Theorem 1.4 is complicated and occupies the whole paper, the basic idea is clear. It roughly follows the analytic proof of (1.1) in [GZ1] although each step is more involved and needs new ideas. It consists of three main ingredients and essentially goes as follows.

By the converse theorem for the Borcherds lift the function Ψ is the Borcherds lift (see Theorem 2.4) of a weakly holomorphic modular form f of weight 0 for $\Gamma_0(p)$ with Nebentypus character ϵ_p. The logarithm of its Petersson metric can be viewed as a Green function for the divisor $\sum_{m>0} \tilde{c}(-m) T_m$. It turns out that T_m has a 'natural' *automorphic Green function* itself (see (2.24) and (2.25)). The first ingredient is to work out the exact relation between Ψ and the automorphic Green function (Theorem 2.8).

Next, using a CM point, one can relate the quadratic lattice defining the automorphic Green function of T_m with some ideal of the reflex field $\tilde{K}$ of (K, Φ) (Proposition 4.8). The condition (1.3) on K guarantees that every ideal of $\tilde{K}$ is 'hit' via some CM point. So the evaluation of the automorphic Green function at the CM 0-cycle is now related to the arithmetic of $\tilde{K}$ although the Green function itself is still involved. This part occupies Sections 3 to 5 with the main result as Theorem 5.1.

The third ingredient is to construct a holomorphic modular form of weight 2 for $\Gamma_0(p)$ with Nebentypus character ϵ_p, whose Fourier coefficients are the sum of two parts such that one part is exactly the evaluation of the automorphic Green function at the CM 0-cycle, and the other involves arithmetic of $\tilde{K}$ only and is basically a linear combination of the quantities b_m and $\alpha(\tilde{K}/\tilde{F})$ in Theorem 1.4. To construct such a modular form, we have to find a nice 'incoherent' Eisenstein series [Ku1] on $\tilde{F}$ associated to $\tilde{K}/\tilde{F}$, and compute its central derivative, restriction to $\mathbb{Q}$, and finally holomorphic projection as in [GZ1]. This part occupies Sections 6 to 8 with the main result recorded as Theorem 8.1.

Now by Serre duality (see Proposition 2.5), the existence of the weakly holomorphic form f of weight 0 (whose Borcherds lift is equal to Ψ) implies a certain relation among the Fourier coefficients of the weight 2 form which allows us to deduce the formula of Theorem 1.4. This is carried out in Section 9, where we also make some remarks about the relaxation of the condition (1.3), and a possible application to the Siegel modular three-fold.

We mention that J. Schofer [Sc] is obtaining a similar formula in his thesis for the evaluation of Borcherds products on a CM 0-cycle associated with a biquadratic CM field, using a different method.

Gross and Zagier also gave an algebraic arithmetic proof of their result, using the moduli interpretation of $(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^2$ and of the CM points on it. In this direction, Goren and Lauter have recently obtained very interesting results on the CM-values of Igusa genus two invariants (some Siegel modular functions of genus two) using arithmetic ideas [GL]. Let $\mathcal{X}$ be the moduli stack over $\mathbb{Z}$ of abelian surfaces with real multiplication by $\mathcal{O}_F$ and fixed polarization $(\partial_F^{-1}, \partial_F^{-1,+})$, so that $\mathcal{X} \otimes \mathbb{Q} = X$. Let $\mathcal{T}_m$ and $\mathcal{CM}(K)$ be the Zariski closures of T_m and $\mathcal{CM}(K)$ in $\mathcal{X}$. Then Theorem 1.4 suggests the following conjectural intersection formula:

$$\mathcal{T}_m \cdot \mathcal{CM}(K) = \frac{W_{\tilde{K}}}{4} b_m. \tag{1.10}$$

The left hand side makes sense as the generic fibers do not intersect at all. A CM point over $\mathbb{C}$ associated to a non-biquadratic quartic CM field cannot lie on a Hirzebruch-Zagier divisor T_m. The conjectured formula (1.10) is true in the degenerated case $F = \mathbb{Q} \oplus \mathbb{Q}$ by the work of Gross and Zagier [GZ2]. For $p = 5, 13, 17$, our work implies that (1.10) is true for every integer $m \geq 1$ if it is true for one single m. We hope to come back to this formula in the near future.

We would like to thank S. Kudla and D. Zagier for many stimulating discussions and valuable suggestions. We thank N. Boston for help on MAGMA and H. Hippauf for her help on computations with PARI/GP. The second author's interest in this project was first inspired by numerous discussions with K. Lauter on her conjectures on Igusa invariants [La], and he thanks her for the inspiration. The second author also thanks the Max-Planck Institut für Mathematik at Bonn for providing the excellent working environment where part of this project was done during his visit in summer 2003. Finally, we thank the referee

for his/her careful reading our manuscript and for the very insightful questions and detailed comments.

2. Borcherds products and automorphic Green functions

In this section, we review some facts about Borcherds products on Hilbert modular surfaces. By [Br1] they are related to certain automorphic Green functions associated with Hirzebruch-Zagier divisors. We state this relationship in a precise form which is convenient for the purposes of the present paper (Theorem 2.8). The constant term of the automorphic Green function associated to a Hirzebruch-Zagier divisor T_m is dictated by the m-th Fourier coefficient of a certain Eisenstein series of weight 2, whose definition and Fourier expansion we review as well. This mainly serves to fix normalizations and notation.

We use $\tau = u + iv$ as a standard variable on the upper complex half plane $\mathbb{H}$ and put $q = e^{2\pi i\tau}$ as usual. If $w \in \mathbb{C}$ we briefly write $e(w) = e^{2\pi i w}$. Let $p \equiv 1 \pmod 4$ be a prime. Moreover, let k be an integer, and denote by $W_k(p, \epsilon_p)$ the space of weakly holomorphic modular forms of weight k for the group

$$\Gamma_0(p) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \; c \equiv 0 \pmod p \}$$

with Nebentypus character $\epsilon_p = \left(\frac{\cdot}{p}\right)$. Recall that a *weakly holomorphic modular form* is a meromorphic modular form which is holomorphic outside the cusps. We let $W_k^+(p, \epsilon_p)$ be the subspace of those $f \in W_k(p, \epsilon_p)$, whose Fourier coefficients $c(n)$ satisfy the so-called *plus space condition*, i.e., $c(n) = 0$ whenever $\epsilon_p(n) = -1$. If $f \in W_k^+(p, \epsilon_p)$ with Fourier expansion $f(\tau) = \sum_{n \gg -\infty} c(n) q^n$, then the polynomial

$$P(f) = \sum_{n<0} c(n) q^n$$

is called the *principal part* of f. Moreover, we write $M_k^+(p, \epsilon_p)$ (respectively $S_k^+(p, \epsilon_p)$) for the subspace of holomorphic modular forms (respectively cusp forms) in $W_k^+(p, \epsilon_p)$. If k is odd, then $W_k(p, \epsilon_p) = \{0\}$. Throughout we therefore assume that k is even. The space $M_k^+(p, \epsilon_p)$ decomposes as a direct sum

$$M_k^+(p, \epsilon_p) = S_k^+(p, \epsilon_p) \oplus \mathbb{C} E_k^+(\tau, 0),$$

where $E_k^+(\tau, 0)$ is the value at $s = 0$ of a non-holomorphic Eisenstein series. Since we will need this Eisenstein series at several places in this section, we briefly discuss its construction and its Fourier expansion.

There are two non-holomorphic Eisenstein series

$$E_k^\infty(\tau, s) = \sum_{\substack{c,d \in \mathbb{Z} \\ c \equiv 0 \ (p)}} \epsilon_p(d) \frac{1}{(c\tau + d)^k} \frac{v^s}{|c\tau + d|^{2s}}, \tag{2.1}$$

$$E_k^0(\tau, s) = \sum_{c,d \in \mathbb{Z}} \epsilon_p(c) \frac{1}{(c\tau + d)^k} \frac{v^s}{|c\tau + d|^{2s}} \tag{2.2}$$

of weight k for $\Gamma_0(p)$ with character ϵ_p. The former series corresponds to the cusp ∞ of $\Gamma_0(p)$, the latter to the cusp 0. It is easily seen that these series converge normally on $\mathbb{H}$

if $\mathrm{Re}(s) > 1 - k/2$, and it is well known that they have a meromorphic continuation to all $s \in \mathbb{C}$, which is holomorphic at $s = 0$. By means of Lemma 3 of [BB] one can check that the linear combination

$$E_k^+(\tau, s) = \frac{1}{2L(k+2s, \epsilon_p)} \left(p^s E_k^\infty(\tau, s) + p^{1/2-k-s} E_k^0(\tau, s) \right) \tag{2.3}$$

satisfies the plus space condition (extended to non-holomorphic modular forms in the analogous way). Here $L(s, \epsilon_p)$ denotes the usual Dirichlet series associated with the Dirichlet character ϵ_p.

Let $W_{\nu,\mu}(z)$ be the usual W-Whittaker function as in [AbSt] Chapter 13, given by (when $\mathrm{Re}(\frac{1}{2} + \mu - \nu) > 0$, $\mathrm{Re}\, z > 0$, see [AbSt] (13.2.5))

$$W_{\nu,\mu}(z) = \frac{e^{-\frac{z}{2}} z^{\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2} + \mu - \nu)} \int_0^\infty e^{-tz} t^{-\frac{1}{2}+\mu-\nu} (1+t)^{-\frac{1}{2}+\mu+\nu} dt. \tag{2.4}$$

To lighten the notation we put for $s \in \mathbb{C}$ and $v \in \mathbb{R} \setminus \{0\}$:

$$\mathcal{W}_s(v) = |v|^{-k/2} W_{\mathrm{sgn}(v)k/2, (1-k)/2-s}(|v|). \tag{2.5}$$

Notice that

$$\mathcal{W}_0(v) = \begin{cases} e^{-v/2}, & \text{if } v > 0, \\ e^{-v/2}\Gamma(1-k, |v|), & \text{if } v < 0, \end{cases} \tag{2.6}$$

where $\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt$ denotes the incomplete Gamma function as in [AbSt] p. 81. We have suppressed the dependency of k from the notation $\mathcal{W}_s(v)$, since it will be fixed (equal to 2) later.

We define a function $h(\tau, s)$ on $\mathbb{H} \times \mathbb{C}$ by

$$h(\tau, s) = \sum_{d \in \mathbb{Z}} \frac{1}{(\tau + d)^k} \frac{1}{|\tau + d|^{2s}}. \tag{2.7}$$

It converges for $\mathrm{Re}(s) > (1-k)/2$ and is clearly periodic in τ.

Lemma 2.1. *The function $h(\tau, s)$ has the Fourier expansion*

$$\begin{aligned} h(\tau, s) = {} & 2^{2-k-2s} \pi i^k \frac{\Gamma(2s+k-1)}{\Gamma(s+k)\Gamma(s)} v^{1-k-2s} \\ & + \frac{(2\pi i)^k \pi^s}{\Gamma(s) v^s} \sum_{n<0} |n|^{s+k-1} \mathcal{W}_s(4\pi n v) e(nu) \\ & + \frac{(2\pi i)^k \pi^s}{\Gamma(s+k) v^s} \sum_{n>0} |n|^{s+k-1} \mathcal{W}_s(4\pi n v) e(nu). \end{aligned}$$

Here the Whittaker function $\mathcal{W}_s(v)$ is defined by (2.5)*.*

Proof. We write the Fourier expansion of $h(\tau, s)$ in the form

$$h(\tau, s) = \sum_{n \in \mathbb{Z}} c_n(v, s) e(nu),$$

with

$$c_n(v,s) = \int_{u=0}^{1} h(\tau,s)e(-nu)\,du.$$

By Poisson summation

$$c_n(v,s) = \int_{u=-\infty}^{\infty} \frac{e(-nu)}{\tau^{k+s}\bar{\tau}^s}\,du,$$

and this integral is computed in (3.11), (3.12) of [BK]. We find that

$$c_n(v,s) = \begin{cases} 2^k\pi^{s+k}i^k|n|^{s+k-1}v^{-s}\mathcal{W}_s(4\pi nv)\Gamma(s)^{-1}, & n<0 \\ 2^{2-k-2s}\pi i^k\frac{\Gamma(k+2s-1)}{\Gamma(k+s)\Gamma(s)}v^{1-k-2s}, & n=0, \\ 2^k\pi^{s+k}i^k|n|^{s+k-1}v^{-s}\mathcal{W}_s(4\pi nv)\Gamma(k+s)^{-1}, & n>0. \end{cases}$$

This implies the assertion. □

Theorem 2.2. *The Eisenstein series $E_k^+(\tau,s)$ defined in* (2.3) *has the Fourier expansion*

$$E_k^+(\tau,s) = c(0,s,v) + \sum_{n\in\mathbb{Z}\setminus\{0\}} C(n,s)\mathcal{W}_s(4\pi nv)e(nu),$$

where

$$c(0,s,v) = (pv)^s + 2^{2-k-2s}\pi i^k p^{1/2-k-s} v^{1-k-s}\frac{\Gamma(2s+k-1)}{\Gamma(s+k)\Gamma(s)}\frac{L(2s+k-1,\epsilon_p)}{L(2s+k,\epsilon_p)}, \tag{2.8}$$

$$C(n,s) = 2\left(\frac{p}{4\pi}\right)^s\frac{\cos(\pi s)\Gamma(2s+k)\sigma_{|n|}(1-k-2s)}{\Gamma(s)L(1-k-2s,\epsilon_p)}, \quad \textit{if } n<0, \tag{2.9}$$

$$C(n,s) = 2\left(\frac{p}{4\pi}\right)^s\frac{\cos(\pi s)\Gamma(2s+k)\sigma_n(1-k-2s)}{\Gamma(s+k)L(1-k-2s,\epsilon_p)}, \quad \textit{if } n>0, \tag{2.10}$$

and $\sigma_n(s)$ denotes the generalized divisor sum

$$\sigma_n(s) = n^{(k-1-s)/2}\sum_{d\mid n} d^s\left(\epsilon_p(d)+\epsilon_p(n/d)\right). \tag{2.11}$$

Proof. It is easily seen that

$$E_k^\infty(\tau,s) = 2v^sL(2s+k,\epsilon_p) + 2v^sp^{-k-2s}\sum_{d' \bmod p}\epsilon_p(d')\sum_{c\in\mathbb{N}}h(c\tau+d'/p,s),$$

$$E_k^0(\tau,s) = 2v^s\sum_{c\in\mathbb{N}}\epsilon_p(c)h(c\tau,s).$$

The assertion can be deduced from (2.3) by means of the Fourier expansion of $h(\tau,s)$ and the functional equation of $L(s,\epsilon_p)$ as in (3.30) of [BK]. □

Observe that $\sigma_n(s)=\sigma_n(-s)$, and $\sigma_n(k-1) = \sum_{d\mid n} d^{k-1}(\epsilon_p(d)+\epsilon_p(n/d))$.

Corollary 2.3. *For $k \geq 2$, the special value $E_k^+(\tau, 0)$ is an element of $M_k^+(p, \epsilon_p)$ with Fourier expansion*

$$E_k^+(\tau, 0) = 1 + \frac{2}{L(1-k, \epsilon_p)} \sum_{n=1}^{\infty} \sigma_n(k-1) q^n. \tag{2.12}$$

In particular, $C(n, 0) = \frac{2}{L(1-k,\epsilon_p)} \sigma_n(k-1)$.

We now define *Hirzebruch-Zagier divisors* on the Hilbert modular surface X. We use $z = (z_1, z_2)$ as a standard variable on $\mathbb{H}^2$ and denote its imaginary part by (y_1, y_2). It is well known that the Hilbert modular group can also be viewed as a discrete subgroup of the orthogonal group of the rational quadratic space

$$V = \{A \in M_2(F) : \quad {}^tA = A'\} = \{A = \left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) : \quad a, b \in \mathbb{Q},\ \lambda \in F\}, \tag{2.13}$$

equipped with the quadratic form $Q(A) = \det(A)$. Here ${}^tA$ is the transpose of A. The Hilbert modular group $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ acts on V via

$$\gamma.A = \gamma' A\, {}^t\gamma. \tag{2.14}$$

We consider the even integral lattice

$$L^0 = \{A = \left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) : \quad a, b \in \mathbb{Z},\ \lambda \in \mathcal{O}_F\} \tag{2.15}$$

in V. It has level p and its dual is given by

$$L = \{A = \left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) : \quad a, b \in \mathbb{Z},\ \lambda \in \partial_F^{-1}\}, \tag{2.16}$$

where $\partial_F = \sqrt{p}\mathcal{O}_F$ denotes the different of F. The discriminant group L/L^0 is simply $\mathbb{Z}/p\mathbb{Z}$. For a positive integer m we denote

$$L_m = \{A \in L : \quad Q(A) = m/p\}. \tag{2.17}$$

The subset

$$T_m = \bigcup_{\left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) \in L_m/\{\pm 1\}} \{(z_1, z_2) \in \mathbb{H}^2 : \quad a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b = 0\} \tag{2.18}$$

defines an Γ-invariant analytic divisor on $\mathbb{H}^2$, which by Chow's lemma descends to an algebraic divisor on X. This is the Hirzebruch-Zagier divisor of discriminant m, which we also denote by T_m. The multiplicities of all irreducible components of T_m are equal to 1. Moreover, $T_m = 0$ if $\epsilon_p(m) = -1$.

We are now ready to state Borcherds' Theorem (see [Bo1] Theorem 13.3). In the present form it was obtained in [BB], Theorem 9. We refer the reader to [BB] for details, in particular on Weyl chambers and on the computation of the Weyl vector. If $f = \sum_{n \in \mathbb{Z}} c(n) q^n \in \mathbb{C}((q))$ is a formal Laurent series, we put

$$\tilde{c}(n) = \begin{cases} c(n), & \text{if } n \not\equiv 0 \pmod{p}, \\ 2c(n), & \text{if } n \equiv 0 \pmod{p}. \end{cases}$$

Theorem 2.4. *Let $f = \sum_{n\gg-\infty} c(n)q^n \in W_0^+(p,\epsilon_p)$ be a weakly holomorphic modular form and assume that $\tilde{c}(n) \in \mathbb{Z}$ for all $n<0$. Then there exists a meromorphic Hilbert modular form $\Psi(z,f)$ for $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ (with some multiplier system of finite order) such that:*

(i) *The weight of $\Psi(z,f)$ is equal to the constant term $c(0)$ of f.*

(ii) *The divisor T_f of $\Psi(z,f)$ is determined by the principal part of f at the cusp ∞. It equals*

$$T_f = \sum_{n<0} \tilde{c}(n) T_{-n}.$$

(iii) *Let $W \subset \mathbb{H}^2$ be a Weyl chamber associated to f and put $N = \max\{n :\ c(-n) \neq 0\}$. The function $\Psi(z,f)$ has the Borcherds product expansion*

$$\Psi(z,f) = q_1^{\rho} q_2^{\rho'} \prod_{\substack{\nu \in \partial_F^{-1} \\ (\nu,W)>0}} \left(1 - q_1^{\nu} q_2^{\nu'}\right)^{\tilde{c}(p\nu\nu')},$$

which converges normally for all $z=(z_1,z_2)$ with $y_1 y_2 > N/p$ outside the set of poles. Here $\rho \in F$ is the Weyl vector corresponding to W and f, and $q_j^{\nu} = e^{2\pi i \nu z_j}$ for $\nu \in F$.

(iv) *A sufficiently large power of Ψ is a normalized integral Hilbert modular form for Γ.*

Hilbert modular forms that arise as lifts via Theorem 2.4 are called *Borcherds products*. They provide a vast supply of Hilbert modular forms for which our main results Theorems 1.1 and 1.4 apply.

The existence of weakly holomorphic forms in $W_0^+(p,\epsilon_p)$ with prescribed principal part is dictated by the Fourier coefficients of holomorphic modular forms in $M_2^+(p,\epsilon_p)$.

Proposition 2.5. *Let $P = \sum_{n<0} c(n)q^n \in \mathbb{C}[q^{-1}]$ be a polynomial with $c(n)=0$ if $\epsilon_p(n) = -1$. The following statements are equivalent:*

(i) *There is a weakly holomorphic form $f \in W_{2-k}^+(p,\epsilon_p)$ with principal part P.*

(ii) *For every $g = \sum_{m>0} b(m)q^m \in S_k^+(p,\epsilon_p)$ we have $\sum_{n<0} \tilde{c}(n)b(-n) = 0$.*

(iii) *For every $g = \sum_{m>0} b(m)q^m \in S_k(p,\epsilon_p)$ we have $\sum_{n<0} \tilde{c}(n)b(-n) = 0$.*

This is proved for certain vector valued modular forms in [Bo2] and in the present form in [BB], Theorem 6. In the same way one also finds:

Proposition 2.6. *Let $f = \sum_{n\gg-\infty} c(n)q^n \in W_{2-k}^+(p,\epsilon_p)$ be a weakly holomorphic modular form. Then the constant term can be computed in terms of the principal part by*

$$c(0) = -\frac{1}{2} \sum_{n<0} \tilde{c}(n) C(-n,0).$$

Here $C(m,0)$ is the m-th coefficient of the Eisenstein series $E_k^+(\tau,0)$, see Corollary 2.3.

We now recall from [Br1] and [BBK] how Borcherds products are related to automorphic Green functions in a form which is convenient for our purposes. This relationship is the first step in the proof of our main results.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $(z_1, z_2) \in \mathbb{H}^2 \backslash T_m$ we define the *automorphic Green function* for T_m by

$$\Phi_m(z_1, z_2, s) = \sum_{\left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) \in L_m} Q_{s-1}\left(1 + \frac{|az_1z_2 + \lambda z_2 + \lambda' z_1 + b|^2}{2y_1y_2 m/p}\right), \tag{2.19}$$

where $Q_{s-1}(t)$ denotes the Legendre function of the second kind (cf. [AbSt] §8) which is the unique solution to the second order Legendre (ordinary differential) equation

$$(1-t^2)Q''(t) - 2tQ'(t) + s(s-1)Q(t) = 0 \tag{2.20}$$

satisfying $Q_{s-1}(t) = O(t^{-s})$ when t goes to ∞, and

$$Q_{s-1}(t) = -\frac{1}{2}\log(t-1) + O(1) \tag{2.21}$$

when t goes to 1. In fact, $Q_{s-1}(t)$ can be given by

$$Q_{s-1}(t) = \int_0^\infty (t + \sqrt{t^2-1}\cosh v)^{-s} dv,$$

and

$$Q_{s-1}\left(\frac{1+t}{1-t}\right) = \frac{\Gamma(s)^2}{2\Gamma(2s)}(1-t)^s F(s,s,2s;1-t) \quad (0 < t < 1).$$

It is easily seen that $\Phi_m(z_1, z_2, s)$ converges normally for $(z_1, z_2) \in \mathbb{H}^2 \backslash T_m$ and $\mathrm{Re}(s) > 1$ and therefore defines a Γ-invariant function, which has a logarithmic singularity along T_m. According to [Br1] it has a meromorphic continuation to a neighborhood of $s = 1$, and a simple pole at $s = 1$. The residue is equal to $-\frac{1}{2}C(m, 0)$, where $C(m, 0)$ is the coefficient of $E_2^+(\tau, 0)$. This follows from [BK] (in particular Example 2) or from [BBK] Theorem 2.11 combined with the observation that the function $\varphi_m(s)$ of [BBK] satisfies

$$\varphi_m(s) = (\pi/p)^{s-1}\frac{s\Gamma(s-1/2)}{(2-4s)\Gamma(1/2)}C(m, s-1), \tag{2.22}$$

$$\varphi_m(1) = -\frac{1}{2}C(m, 0). \tag{2.23}$$

So $C(m, s-1)$ occurs in the constant term of the Fourier expansion of $\Phi_m(z_1, z_2, s)$ and dictates the residue at $s = 1$.

We define the regularized Green function $\Phi_m(z_1, z_2)$ for the divisor T_m as the constant term in the Laurent expansion of $\Phi_m(z_1, z_2, s)$ at $s = 1$, that is

$$\Phi_m(z_1, z_2) = \lim_{s \to 1}\left(\Phi_m(z_1, z_2, s) + \frac{C(m, 0)}{2(s-1)}\right). \tag{2.24}$$

For questions regarding the arithmetic of Borcherds products, it is convenient to renormalize the Green function $\Phi_m(z_1, z_2)$ as in [BBK], Definition 2.13:

Definition 2.7. If m is a positive integer we put

$$G_m(z_1, z_2) = \frac{1}{2}\left(\Phi_m(z_1, z_2) - \mathcal{L}_m\right), \tag{2.25}$$

where

$$\mathcal{L}_m = \frac{C(m,0)}{2}\left(1+\Gamma'(1)-\log(4\pi)-\frac{C'(m,0)}{C(m,0)}\right). \tag{2.26}$$

Recall that the Petersson metric of a Hilbert modular form F of weight k is defined as

$$\|F(z_1,z_2)\|_{\mathrm{Pet}} = |F(z_1,z_2)|\big(16\pi^2 y_1 y_2\big)^{k/2}. \tag{2.27}$$

Theorem 2.8. *Let F be a meromorphic Hilbert modular form of weight k for the group Γ such that*

$$\operatorname{div}(F) = \sum_{m>0} \tilde{c}(-m) T_m,$$

with integral coefficients $\tilde{c}(-m) \in \mathbb{Z}$.

(i) *Then there exists a weakly holomorphic form $f \in W_0^+(p,\epsilon_p)$ with principal part $\sum_{m<0} c(m)q^m$ and a non-zero constant C such that $F(z) = C\Psi(z,f)$, that is, F is a constant multiple of the Borcherds lift of f.*

(ii) *The Petersson metric of F is given by*

$$\log\|F(z_1,z_2)\|_{\mathrm{Pet}} = \log|C| - \sum_{m>0} \tilde{c}(-m) G_m(z_1,z_2).$$

Here C is the same constant as in (i).

(iii) *If F is in addition normalized integral, then $C = \pm 1$ in (i) and (ii).*

Proof. The first assertion is the converse Theorem (see [Br1] Theorem 9), see also [Br3], Section 5, for background information.

To prove (iii), we notice that, by Theorem 2.4 (iv), there is a positive integer N such that $\Psi(z,f)^N$ is a normalized integral Hilbert modular form for Γ. So $F(z)^N/\Psi(z,f)^N = C^N$ is a normalized integral modular form which is constant. This implies $C^N = \pm 1$. Now we notice that the Fourier coefficient of $\Psi(z,f)$ with index given by the Weyl vector ρ is equal to 1. On the other hand the corresponding coefficient of F is rational by assumption. This implies that C is rational, and hence $C = \pm 1$.

Finally, the assertion (ii) follows from [BBK] Theorem 4.3 (iv). (Notice that we use the same normalization of G_m as in [BBK].) □

3. Hilbert modular varieties and CM 0-cycles

In this section, we review some basic facts about CM 0-cycles on a Hilbert modular variety, which are known to experts but not in the literature. For that reason, we make it more general than it perhaps should be and hope it will be useful for non-experts in this field. We first recall some facts on modular varieties and refer to [Ge] and [Go] for details.

Let F be a totally real number field of degree g with ring of integers $\mathcal{O}_F$. Let $\{\sigma_1,\dots,\sigma_g\}$ be the real embeddings of F. For a subset S of F, we denote S^+ for the subset of S of elements which are totally positive.

For a fractional ideal $\mathfrak{f}_0$ of F, let

$$\Gamma = \Gamma(\mathfrak{f}_0) = \mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{f}_0) = \{g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(F) :\ a,d \in \mathcal{O}_F, b \in \mathfrak{f}_0, c \in \mathfrak{f}_0^{-1}\}. \tag{3.1}$$

It acts on $\mathbb{H}^g$ via

$$\gamma(z_1, \dots, z_g) = (\sigma_1(\gamma)z_1, \dots, \sigma_g(\gamma)z_g).$$

The quotient space $X = X(\mathfrak{f}_0) = \Gamma\backslash\mathbb{H}^g$ is the so-called open Hilbert modular variety (associated to $\mathfrak{f}_0$). Clearly, for $a \in F^+$ one has

$$X(\mathfrak{f}_0) \xrightarrow{\sim} X(a\mathfrak{f}_0), \quad z = (z_i) \mapsto az = (\sigma_i(a)z_i).$$

More generally, for any ideal $\mathfrak{a}_0$ of F, there is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \cap \begin{pmatrix} \mathfrak{a}_0 & \mathfrak{a}_0\mathfrak{f}_0 \\ (\mathfrak{a}_0\mathfrak{f}_0)^{-1} & (\mathfrak{a}_0)^{-1} \end{pmatrix}.$$

Moreover,

$$\phi : X(\mathfrak{f}_0) \longrightarrow X(\mathfrak{a}_0^2\mathfrak{f}_0), \quad z \mapsto Az = (\sigma_1(A)z_1, \dots, \sigma_g(A)z_g) \tag{3.2}$$

is an isomorphism of varieties.

It is known ([Go], Theorem 2.17) that $X(\mathfrak{f}_0)$ parameterizes the isomorphism classes of the triples $(A, \imath, m)$, where $(A, \imath)$ is an abelian variety with real multiplication $\imath : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$, and

$$m : (\mathfrak{M}_A, \mathfrak{M}_A^+) \longrightarrow \left((\partial_F\mathfrak{f}_0)^{-1}, (\partial_F\mathfrak{f}_0)^{-1,+}\right)$$

is an $\mathcal{O}_F$-isomorphism between $\mathfrak{M}_A$ and $(\partial_F\mathfrak{f}_0)^{-1}$ which preserves the 'direction', i.e., maps $\mathfrak{M}_A^+$ onto $(\partial_F\mathfrak{f}_0)^{-1,+}$. Here

$$\mathfrak{M}_A = \{\lambda : A \to A^\vee :\ \lambda \text{ is a symmetric } \mathcal{O}_F\text{-linear homomorphism}\}$$

is the polarization module of A and

$$\mathfrak{M}_A^+ = \{\lambda \in \mathfrak{M}_A :\ \lambda \text{ is a polarization}\}$$

is its positive cone. In terms of lattices, $X(\mathfrak{f}_0)$ parameterizes isomorphism classes of tuples $(\Lambda, (\bigwedge^2_{\mathcal{O}_F}\Lambda)^{*,+}, i, m)$, where Λ is a projective $\mathcal{O}_F$-module of rank 2,

$$i : \Lambda \otimes_{\mathcal{O}_F} \mathbb{R} \xrightarrow{\sim} \mathbb{C}^2$$

is a complex structure on $\Lambda \otimes_{\mathcal{O}_F} \mathbb{R}$, and

$$\left(\bigwedge{}^2_{\mathcal{O}_F}\Lambda\right)^* = \mathrm{Hom}\left(\bigwedge{}^2_{\mathcal{O}_F}\Lambda, \mathbb{Z}\right),$$

and $(\bigwedge^2_{\mathcal{O}_F}\Lambda)^{*,+}$ is a sub-semigroup of $(\bigwedge^2_{\mathcal{O}_F}\Lambda)^*$ (direction), and

$$m : \left((\bigwedge{}^2_{\mathcal{O}_F}\Lambda)^*, (\bigwedge{}^2_{\mathcal{O}_F}\Lambda)^{*,+}\right) \xrightarrow{\sim} \left((\partial_F\mathfrak{f}_0)^{-1}, (\partial_F\mathfrak{f}_0)^{-1,+}\right).$$

Let K be a totally imaginary quadratic extension of F and let $\Phi = (\sigma_1, \dots, \sigma_g)$ be a CM type of K. Then a point $z = (A, \imath, m) \in X(\mathfrak{f}_0)$ is said to be a CM point of type (K, Φ) if one of the following equivalent conditions holds:

(1) As a point $z \in \mathbb{H}^g$, there is $\tau \in K$ such that $\Phi(\tau) = (\sigma_1(\tau), \dots, \sigma_g(\tau)) = z$ and such that $\Lambda_\tau = \mathcal{O}_F\tau + \mathfrak{f}_0$ is a fractional ideal of K.

(2) $(A, \imath')$ is a CM abelian variety of type (K, Φ) by $\mathcal{O}_K$, $\imath' : \mathcal{O}_K \hookrightarrow \mathrm{End}A$, such that $\imath = \imath'|_{\mathcal{O}_F}$.

To relate CM points with ideals of K, it is convenient to fix a $\xi_0 \in K^*$ such that $\bar{\xi}_0 = -\xi_0$ and that $\Phi(\xi_0) = (\sigma_1(\xi_0), \dots, \sigma_g(\xi_0)) \in \mathbb{H}^g$. We will say an element $z \in K$ is in $\mathbb{H}^g$ if $\Phi(z) \in \mathbb{H}^g$ and will identify z with $\Phi(z)$ to lighten up the notation sometimes. We also denote $K^+ = K \cap \mathbb{H}^g$ via the above identification (it depends on Φ).

Lemma 3.1. *Let $\mathfrak{a}$ be a fractional ideal of K. Then the ideal class of $\mathfrak{f}_\mathfrak{a} = \xi_0 \partial_{K/F} \mathfrak{a}\bar{\mathfrak{a}} \cap F$ is the Steinitz class of $\mathfrak{a}$ as a projective $\mathcal{O}_F$-module. More precisely, we have an $\mathcal{O}_F$-linear isomorphism*

$$\mathfrak{f}_\mathfrak{a}^{-1} \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_F}\left(\bigwedge\nolimits^2_{\mathcal{O}_F} \mathfrak{a}, \mathcal{O}_F\right), \quad a \mapsto E_{a\xi_0},$$

where

$$E_\xi : K \times K \longrightarrow F, \quad E_\xi(x, y) = \operatorname{tr}_{K/F} \xi \bar{x} y.$$

Moreover, a is totally positive if and only if $\operatorname{tr}_{F/\mathbb{Q}} E_{a\xi_0}$ is a non-degenerate Riemann form.

Proof. Since $K = F + F\xi_0$, $a \mapsto E_{a\xi_0}$ is an F-isomorphism between F and F-symplectic forms on K. We have $E_{a\xi_0}(x, y) \in \mathcal{O}_F$ for all $x, y \in \mathfrak{a}$ if and only if $a \in \mathfrak{f}_\mathfrak{a}^{-1}$. □

From this lemma, it is easy to see that the CM abelian variety $(A_\mathfrak{a} = \mathbb{C}^g/\Phi(\mathfrak{a}), \imath)$ has the following polarization module

$$(\mathfrak{M}_A, \mathfrak{M}_A^+) = \left((\partial_F \mathfrak{f}_\mathfrak{a})^{-1}, (\partial_F \mathfrak{f}_\mathfrak{a})^{-1,+}\right).$$

To give an $\mathcal{O}_F$-isomorphism between this pair with $((\partial_F \mathfrak{f}_0)^{-1}, (\partial_F \mathfrak{f}_0)^{-1,+})$, is the same as to give an $r \in F^+$ such that $\mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0$. Therefore, to give a CM point $(A, \imath, m) \in X(\mathfrak{f}_0)$ is the same as to give a pair $(\mathfrak{a}, r)$ where $\mathfrak{a}$ is a fractional ideal K and $\mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0$ with $r \in F^+$. Two such pairs are equivalent if they give the same CM point, that is, there is $\alpha \in K^*$ such that

$$\mathfrak{a}_2 = \alpha \mathfrak{a}_1, \quad r_2 = r_1 \alpha \bar{\alpha}.$$

We write $[\mathfrak{a}, r]$ for the equivalence class of $(\mathfrak{a}, r)$ and identify it with its associated CM point $(A_\mathfrak{a}, \imath, m) \in X(\mathfrak{f}_0)$. In terms of its coordinates in $\mathbb{H}^g$, we have the following

Lemma 3.2. *Given a CM point $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$, there is a decomposition*

$$\mathfrak{a} = \mathcal{O}_F \alpha + \mathfrak{f}_0 \beta \tag{3.3}$$

with $z = \frac{\alpha}{\beta} \in K^* \cap \mathbb{H}^g$. Moreover z represents the point $[\mathfrak{a}, r]$ in $X(\mathfrak{f}_0)$.

Proof. Lemma 3.1 implies that the Steinitz class of $\mathfrak{a}$ in $\mathcal{CL}(F)$ is $\mathfrak{f}_0$. So we can decompose $\mathfrak{a}$ as in (3.3) with $\alpha, \beta \in K^*$. This implies that

$$(\bar{\alpha}\beta - \alpha\bar{\beta})\mathfrak{f}_0 \mathcal{O}_K = \partial_{K/F} \mathfrak{a}\bar{\mathfrak{a}}, \text{ and } \xi_0(\bar{\alpha}\beta - \alpha\bar{\beta})\mathfrak{f}_0 = \mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0.$$

So $\xi_0(\bar{\alpha}\beta - \alpha\bar{\beta}) = r\epsilon$ for some unit $\epsilon \in O_F^*$. Replacing β by $\beta\epsilon^{-1}$ if necessary, we may assume $\epsilon = 1$. This implies

$$\xi_0(\bar{z} - z) = \frac{r}{\beta\bar{\beta}} \in F^+$$

and thus $z \in K^+ = \{z \in K^* : \Phi(z) \in \mathbb{H}^g\}$. Now notice that

$$\mathfrak{a} = \beta\Lambda_z, \quad A_\mathfrak{a} \cong A_z,$$

where $\Lambda_z = \mathcal{O}_F z + \mathfrak{f}_0$ and $A_z = \mathbb{C}^g/\Lambda_z$ is the abelian variety associated to z. So z represents $[\mathfrak{a}, r]$ in $X(\mathfrak{f}_0)$. $\square$

Lemma 3.3. *Let $\mathfrak{a}_0$ be an ideal of F. Then, under the isomorphism $\phi : X(\mathfrak{f}_0) \longrightarrow X(\mathfrak{a}_0^2\mathfrak{f}_0)$ given by (3.2), one has $\phi([\mathfrak{a}, r]) = [\mathfrak{a}\mathfrak{a}_0, r]$.*

Proof. We may assume $\mathfrak{a} = \mathcal{O}_F z + \mathfrak{f}_0$ with $z = [\mathfrak{a}, r] \in X(\mathfrak{f}_0)$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \cap \begin{pmatrix} \mathfrak{a}_0 & \mathfrak{a}_0\mathfrak{f}_0 \\ (\mathfrak{a}_0\mathfrak{f}_0)^{-1} & (\mathfrak{a}_0)^{-1} \end{pmatrix}$$

be the matrix defining ϕ in (3.2). Then

$$\phi(z) = Az = \frac{az+b}{cz+d},$$

and

$$(cz+d)\Lambda_{\phi(z)} = \mathcal{O}_F\alpha + \mathfrak{a}_0^2\mathfrak{f}_0\beta \subset \mathfrak{a}\mathfrak{a}_0$$

with

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = A\begin{pmatrix} z \\ 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} d\alpha - b\beta \\ -c\alpha + a\beta \end{pmatrix}.$$

Now $z = d\alpha - b\beta$ implies

$$\mathfrak{a}_0 z \subset \mathfrak{a}_0 d\alpha - \mathfrak{a}_0 b\beta \subset \mathcal{O}_F\alpha + \mathfrak{a}_0^2\mathfrak{f}_0\beta = (cz+d)\Lambda_{\phi(z)},$$

and $1 = -c\alpha + a\beta$ implies

$$\mathfrak{a}_0\mathfrak{f}_0 \subset \mathfrak{a}_0\mathfrak{f}_0 c\alpha + \mathfrak{a}_0\mathfrak{f}_0 a\beta \subset \mathcal{O}_F\alpha + \mathfrak{a}_0^2\mathfrak{f}_0\beta = (cz+d)\Lambda_{\phi(z)}.$$

So

$$\mathfrak{a}\mathfrak{a}_0 = (cz+d)\Lambda_{\phi(z)}$$

and $\phi(z) = [\mathfrak{a}\mathfrak{a}_0, r] \in X(\mathfrak{a}_0^2\mathfrak{f}_0)$. $\square$

We will denote $\mathcal{CM}(K, \Phi, \mathfrak{f}_0)$ for the set of CM points $[\mathfrak{a}, r]$ in $X(\mathfrak{f}_0)$, and view it as a 0-cycle in $X(\mathfrak{f}_0)$. We also denote

$$\mathcal{CM}(K, \Phi) = \sum_{\mathfrak{f}_0 \in \mathcal{CL}^+(F)} \mathcal{CM}(K, \Phi, \mathfrak{f}_0).$$

Notice that the forgetful map

$$\mathcal{CM}(K, \Phi) \longrightarrow \mathcal{CL}(K), \quad [\mathfrak{a}, r] \mapsto [\mathfrak{a}]$$

is surjective. All the fibers are indexed by $\epsilon \in \mathcal{O}_F^{*,+}/N_{K/F}\mathcal{O}_K^*$, since every element in the fiber of $\mathfrak{a}$ is of the form $[\mathfrak{a}, r\epsilon]$ with r fixed and $\epsilon \in \mathcal{O}_F^{*,+}$ a totally positive unit. It is the same as $[\mathfrak{a}, r]$ if and only if $\epsilon \in N_{K/F}\mathcal{O}_K^*$. In particular, all the fibers have the same cardinality of $\mathcal{O}_F^{*,+}/N_{K/F}\mathcal{O}_K^*$ which is 1 or 2.

Let $(\tilde{K}, \tilde{\Phi})$ be the reflex type of (K, Φ). Then there is a type norm map N_Φ both on elements and on ideals,

$$N_\Phi : K^* \longrightarrow \tilde{K}^*, \quad \alpha \mapsto N_\Phi(\alpha) = \prod_{\sigma \in \Phi} \sigma(\alpha), \tag{3.4}$$

and

$$N_\Phi : I(K) \longrightarrow I(\tilde{K}), \quad N_\Phi(\mathfrak{a}) = \prod_{\sigma \in \Phi} \sigma(\mathfrak{a})\mathcal{O}_M \cap \tilde{K}, \tag{3.5}$$

where M is a Galois extension of $\mathbb{Q}$ containing both K and $\tilde{K}$, and $I(K)$ is the group of all fractional ideals of K. Notice that

$$N_\Phi(\mathfrak{a})\overline{N_\Phi(\mathfrak{a})} = N\mathfrak{a}\mathcal{O}_{\tilde{F}}.$$

Here $N\mathfrak{a} = \#\mathcal{O}_K/\mathfrak{a}$. Let $H(K)$ be the subgroup of $I(K)$ of ideals $\mathfrak{a}$ such that

$$N_\Phi\mathfrak{a} = \mu\mathcal{O}_{\tilde{K}}, \quad N\mathfrak{a} = \mu\bar{\mu} \quad \text{for some } \mu \in \tilde{K}^*. \tag{3.6}$$

We call the quotient $\mathcal{CC}(K) = \mathcal{CC}(K, \Phi) = I(K)/H(K)$ the CM ideal class group of K. According to [Sh], page 112, Main Theorem 1, the class field of $\tilde{K}$ associated to the CM ideal class group $\mathcal{CC}(\tilde{K}, \tilde{\Phi})$ is the composite of $\tilde{K}$ with the field of the moduli of any polarized CM abelian variety of type (K, Φ) by $\mathcal{O}_K$. This CM ideal class group acts on $\mathcal{CM}(K, \Phi, \mathfrak{f}_0)$ via

$$[\mathfrak{a}, r]^{\sigma_{\tilde{\mathfrak{b}}}} = [\mathfrak{a}N_{\tilde{\Phi}}\tilde{\mathfrak{b}}, rN\tilde{\mathfrak{b}}]. \tag{3.7}$$

Therefore, $\mathcal{CM}(K, \Phi, \mathfrak{f}_0)$, as a 0-cycle, is defined over $\tilde{K}$. Let $\bar{\Phi}$ be the complement of Φ, consisting of $\sigma \circ \rho$ for all $\sigma \in \Phi$, where ρ is the complex conjugation in K.

Lemma 3.4. *One has*

$$\mathcal{CM}(K, \Phi, \mathfrak{f}_0) = \mathcal{CM}(K, \bar{\Phi}, \mathfrak{f}_0).$$

More generally, if ϕ is an automorphism of K over $\mathbb{Q}$ such that $\phi(\mathfrak{f}_0) = \mathfrak{f}_0$, then

$$\mathcal{CM}(K, \Phi \circ \phi^{-1}, \mathfrak{f}_0) = \mathcal{CM}(K, \Phi, \mathfrak{f}_0).$$

In particular, if $K/\mathbb{Q}$ is Galois, then $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ is rational over $\mathbb{Q}$.

Proof. If $[\mathfrak{a}, r] \in \mathcal{CM}(K, \Phi, \mathfrak{f}_0)$, then Lemma 3.2 implies that $\mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0$ and

$$\mathfrak{a} = \mathcal{O}_F\alpha + \mathfrak{f}_0\beta$$

with $z = \Phi(\frac{\alpha}{\beta}) \in \mathbb{H}^g$. In this case, z is the associated CM point in $X(\mathfrak{f}_0)$. Notice that

$$\phi(\mathfrak{a}) = \mathcal{O}_F\phi(\alpha) + \mathfrak{f}_0\phi(\beta)$$

and $\Phi \circ \phi^{-1}(\phi(\alpha)/\phi(\beta)) = \Phi(\alpha/\beta) = z$. On the other hand, it is easy to see that $\phi(\partial_{K/F}) = \partial_{K/F}$, and

$$\phi(\mathfrak{f}_\mathfrak{a}) = \phi(\xi_0)\partial_{K/F}\phi(\mathfrak{a})\overline{\phi(\mathfrak{a})} \cap F = \phi(r)\phi(\mathfrak{f}_0) = \phi(r)\mathfrak{f}_0.$$

So $\phi(\mathfrak{f}_{\mathfrak{a}})$ can be viewed as $\mathfrak{f}_{\phi(\mathfrak{a})}$ with respect to $\phi(\xi_0)$. Therefore, $[\mathfrak{a}, r]$ (with respect to Φ) and $[\phi(\mathfrak{a}), \phi(r)]$ (with respect to $\Phi \circ \phi^{-1}$) give rise to the same point z in $X(\mathfrak{f}_0)$. Since $\mathfrak{a} \mapsto \phi(\mathfrak{a})$ is an automorphism of $\mathcal{CL}(K)$, one has thus

$$\mathcal{CM}(K, \Phi \circ \phi^{-1}, \mathfrak{f}_0) = \mathcal{CM}(K, \Phi, \mathfrak{f}_0).$$

□

Remark 3.5. Now we are in a position to explain the claims in the introduction about the rationality of the CM 0-cycles and prove that Corollary 1.2 follows from Theorem 1.1. Let (K, Φ) be as in the introduction. By (3.7), $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ is defined over $\tilde{K}$. So

$$\mathcal{CM}(K, \Phi, \mathcal{O}_F) + \mathcal{CM}(K, \sigma\Phi, \mathcal{O}_F) + \mathcal{CM}(K, \sigma^2\Phi, \mathcal{O}_F) + \mathcal{CM}(K, \sigma^3\Phi, \mathcal{O}_F)$$

is defined over $\mathbb{Q}$. Since σ^2 is the complex conjugation, Lemma 3.4 implies that

$$\mathcal{CM}(K, \Phi, \mathcal{O}_F) = \mathcal{CM}(K, \sigma^2\Phi, \mathcal{O}_F), \quad \mathcal{CM}(K, \sigma\Phi, \mathcal{O}_F) = \mathcal{CM}(K, \sigma^3\Phi, \mathcal{O}_F).$$

So $\mathcal{CM}(K)$ is rational over $\mathbb{Q}$, and $\Psi(\mathcal{CM}(K))$ is a rational number. Now Corollary 1.2 follows from Theorem 1.1. When $K/\mathbb{Q}$ is cyclic, the four CM 0-cycles above are all the same by Lemma 3.4. So $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ is defined over $\mathbb{Q}$, and

$$\Psi(\mathcal{CM}(K)) = \Psi(\mathcal{CM}(K, \Phi, \mathcal{O}_F))^2 > 0.$$

4. CM number fields of degree 4

The automorphic Green function $\Phi_m(z_1, z_2, s)$ associated to T_m is by definition an infinite sum over L_m, the vectors of norm m/p in the lattice L of rank 4. In this section, we show (Proposition 4.8) that when (z_1, z_2) is a CM point associated to (K, Φ), the lattice L can be replaced by an ideal of $\tilde{K}$ via some isometry ρ defined in (4.11).

Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, and let $K = F(\sqrt{\Delta})$ be a totally imaginary quadratic extension of F, and let Φ be a CM type of K. Let $\tilde{F} = \mathbb{Q}(\sqrt{\Delta\Delta'})$, where $a + b\sqrt{D} \mapsto (a + b\sqrt{D})' = a - b\sqrt{D}$ is the non-trivial automorphism of F over $\mathbb{Q}$. We first record an easy lemma (see [Sh], page 64).

Lemma 4.1. *Let the notation be as above.*

(1) *K/F is biquadratic if and only if $\tilde{F} = \mathbb{Q}$.*
(2) *K/F is cyclic if and only if $\tilde{F} = F$.*
(3) *K/F is non-Galois if and only if $\tilde{F} \neq F$ is a real quadratic field.*
(4) *(K, Φ) is a primitive CM type if and only if K is not biquadratic.*

From now on, we assume that K is non-biquadratic, i.e., $\tilde{F}$ is a real quadratic field. We view both $\sqrt{\Delta}$ and $\sqrt{\Delta'}$ as complex numbers with positive imaginary parts. Then $M = \mathbb{Q}(\sqrt{\Delta}, \sqrt{\Delta'})$ is a Galois closure of K over $\mathbb{Q}$. There are two cases:

(a) **Cyclic Case:** In this case, $M = K$, and $\mathrm{Gal}(M/\mathbb{Q}) =<\sigma>$ with

$$\sigma(\sqrt{\Delta}) = \sqrt{\Delta'},\ \sigma(\sqrt{\Delta'}) = -\sqrt{\Delta}. \tag{4.1}$$

(b) **Non-Galois case:** In this case, $\mathrm{Gal}(M/\mathbb{Q}) =<\sigma, \tau>$ is D_4, with σ as above and

$$\tau(\sqrt{\Delta}) = \sqrt{\Delta'},\ \tau(\sqrt{\Delta'}) = \sqrt{\Delta}. \tag{4.2}$$

It is clear that $\sigma^2 = $ '$-$' is the complex conjugation. Let $\Phi = \{1, \sigma\}$ be a CM type of K, which is primitive since K is non-biquadratic. Let $(\tilde{K}, \tilde{\Phi})$ be its reflex. Then $\tilde{F}$ is the maximal totally real subfield of $\tilde{K}$. Recall

$$K^+ = K^+(\Phi) = \{z \in K : \Phi(z) = (z, \sigma(z)) \in \mathbb{H}^2\}.$$

Define $\tilde{D} = N_{F/\mathbb{Q}} d_{K/F}$. Then

$$\Delta\Delta' = \tilde{D} a^2, \tilde{D} = d_{\tilde{F}} b^2 \tag{4.3}$$

for some rational numbers $a, b \in \frac{1}{2}\mathbb{Z}$.

Lemma 4.2. *Let $\mathfrak{f}_0$ be an integral ideal of F, and let $\mathfrak{a} = \mathcal{O}_F \alpha + \mathfrak{f}_0 \beta$ be a fractional ideal of K. Then*

$$\begin{aligned}\sqrt{\tilde{D}} N\mathfrak{a} &= \pm 4 \operatorname{Im}(\alpha\bar{\beta}) \operatorname{Im}(\sigma(\alpha)\overline{\sigma(\beta)}) N\mathfrak{f}_0 \\ &= \mp(\alpha\bar{\beta} - \bar{\alpha}\beta)(\sigma(\alpha)\overline{\sigma(\beta)} - \overline{\sigma(\alpha)}\sigma(\beta)) N\mathfrak{f}_0.\end{aligned}$$

Here N means the absolute norm from a number field to $\mathbb{Q}$. When $z = \frac{\alpha}{\beta} \in K^+$, the sign $\pm$ becomes $+1$.

Proof. First we recall a general fact. Let K/F be a finite extension of number fields of degree d, and let $\mathfrak{a}$ be a fractional ideal of K generated by $x_1, \dots, x_d$ over $\mathcal{O}_F$. Then

$$d_{K/F}\{x_1, \dots, x_d\} = N_{K/F}\mathfrak{a}^2 d_{K/F}, \tag{4.4}$$

where

$$d_{K/F}\{x_1, \dots, x_d\} = \det(\operatorname{tr} x_i x_j) = \det((\sigma_i(x_j)))^2$$

with σ_i the different embeddings of K into $\bar{F}$ fixing F. In our case, let $\{e_1, e_2\}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_F$ and $\{f_1, f_2\}$ be a $\mathbb{Z}$-basis of $\mathfrak{f}_0$, then $\{e_1\alpha, e_2\alpha, f_1\beta, f_2\beta\}$ is a $\mathbb{Z}$-basic for $\mathfrak{a}$. Then a simple calculation gives

$$\begin{aligned}d_{K/\mathbb{Q}} \cdot (N\mathfrak{a})^2 &= d_{K/\mathbb{Q}}\{e_1\alpha, e_2\alpha, f_1\beta, f_2\beta\} \\ &= (\alpha\bar{\beta} - \bar{\alpha}\beta)(\sigma(\alpha)\overline{\sigma(\beta)} - \overline{\sigma(\alpha)}\sigma(\beta)) d_{F/\mathbb{Q}}\{e_1, e_2\} d_{F/\mathbb{Q}}\{f_1, f_2\} \\ &= (\alpha\bar{\beta} - \bar{\alpha}\beta)(\sigma(\alpha)\overline{\sigma(\beta)} - \overline{\sigma(\alpha)}\sigma(\beta)) d_F^2 \cdot (N\mathfrak{f}_0)^2.\end{aligned}$$

Notice that $d_K = d_F^2 \tilde{D}$, one proves the lemma by taking the square root of the above identity. ◻

Let

$$V = \{A \in M_2(F) : {}^t A = A'\} = \{A = \left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) : a, b \in \mathbb{Q}, \lambda \in F\} \tag{4.5}$$

with quadratic form $Q(A) = \det A = ab - \lambda\lambda'$ as in Section 2. For an ideal $\mathfrak{f}_0$ of F, let $f_0 = N\mathfrak{f}_0 \in \mathbb{Q}^+$. The Hilbert modular group $\Gamma = \Gamma(\mathfrak{f}_0)$ acts on V via

$$\gamma.A = \gamma' A\, {}^t\gamma. \tag{4.6}$$

Let

$$L^0(\mathfrak{f}_0) = \{A = \left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) \in V : a \in \mathbb{Z}, b \in f_0\mathbb{Z}, \lambda \in \mathfrak{f}_0\} \tag{4.7}$$

be an $\mathfrak{f}_0$-integral lattice of V (so $Q(L^0) \subset f_0\mathbb{Z}$) which is preserved by the action of Γ. Its dual is $\frac{1}{f_0}L(\mathfrak{f}_0)$, where

$$L(\mathfrak{f}_0) = \{A = \left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) :\ a \in \mathbb{Z}, b \in f_0\mathbb{Z}, \lambda \in \mathfrak{f}_0\partial_F^{-1}\}. \tag{4.8}$$

Notice that $Q(L(\mathfrak{f}_0)) \subset \frac{f_0}{D}\mathbb{Z}$, and that L is the same lattice as in (2.16) when $\mathfrak{f}_0 = \mathcal{O}_F$. For a positive integer $m \geq 1$, let

$$L_m = L_m(\mathfrak{f}_0) = \{A \in L :\ \det A = m\frac{f_0}{D}\}, \quad L_m^0 = \{A \in L^0 :\ \det A = mf_0\}. \tag{4.9}$$

It can be proved that

$$L_m^0 = L_{mD}. \tag{4.10}$$

For any $(\alpha, \beta) \in K^2$, we define a map

$$\rho_{(\alpha,\beta)} : V \longrightarrow \tilde{K}, \quad \rho_{(\alpha,\beta)}(A) = (\sigma(\alpha), \sigma(\beta))A\begin{pmatrix}\alpha \\ \beta\end{pmatrix}. \tag{4.11}$$

Explicitly,

$$\rho_{\alpha,\beta}(A) = a\alpha\sigma(\alpha) + \alpha\sigma(\lambda\beta) + \sigma(\alpha)\lambda\beta + b\beta\sigma(\beta) \quad \text{if } A = \begin{pmatrix} a & \lambda \\ \lambda' & b\end{pmatrix}. \tag{4.12}$$

We define

$$\gamma.(\alpha, \beta) = (\alpha, \beta)\gamma.$$

Then for $\gamma \in \Gamma(\mathfrak{f}_0)$ and $a \in K^*$, one has

$$\rho_{\gamma.(\alpha,\beta)}(A) = \rho_{(\alpha,\beta)}(\gamma.A), \quad \rho_{(a\alpha,a\beta)}(A) = a\bar{a}\rho_{(\alpha,\beta)}(A). \tag{4.13}$$

For a CM point $[\mathfrak{a}, r] \in \mathcal{CM}(K, \Phi, \mathfrak{f}_0)$, we write

$$\mathfrak{a} = \mathcal{O}_F\alpha + \mathfrak{f}_0\beta, \quad z = \frac{\alpha}{\beta} \in K^+ \tag{4.14}$$

as in (3.3).

We define two $\mathbb{Q}$-quadratic forms on $\tilde{K}$ via

$$\begin{aligned} Q^-(\rho) &= \operatorname{tr}_{\tilde{F}/\mathbb{Q}} \frac{1}{\sqrt{\tilde{D}}}\rho\bar{\rho} = \frac{1}{\sqrt{\tilde{D}}}(\rho\bar{\rho} - \sigma(\rho)\overline{\sigma(\rho)}), \\ Q^+(\rho) &= \operatorname{tr}_{\tilde{F}/\mathbb{Q}} \rho\bar{\rho} = \rho\bar{\rho} + \sigma(\rho)\overline{\sigma(\rho)}. \end{aligned} \tag{4.15}$$

Then

$$\rho\bar{\rho} = \frac{1}{2}(Q^+(\rho) + \sqrt{\tilde{D}}Q^-(\rho)). \tag{4.16}$$

Proposition 4.3. *Let $[\mathfrak{a}, r] \in \mathcal{CM}(K, \Phi, \mathfrak{f}_0)$, and write $\mathfrak{a} = \mathcal{O}_F\alpha + \mathfrak{f}_0\beta$ as in (4.14). Then the map $\rho_{(\alpha,\beta)}$ is a $\mathbb{Q}$-linear isometry between quadratic spaces $(V, \det)$ and $(\tilde{K}, -\frac{N\mathfrak{f}_0}{N\mathfrak{a}}Q^-)$. That is,*

$$Q^-(\rho_{\alpha,\beta}(A)) = -\frac{N\mathfrak{a}}{N\mathfrak{f}_0}\det(A).$$

Proof. We may assume $\beta = 1$ and write $z = \alpha$ and

$$\rho_z(A) = \rho_{z,1}(A) = az\sigma(z) + z\lambda' + \sigma(z)\lambda' + b. \tag{4.17}$$

When K is cyclic over $\mathbb{Q}$, $\tilde{K} = K$, ρ_z is clearly a $\mathbb{Q}$-linear map. When K is non-Galois, $\tilde{K}$ is the subfield of M fixed by τ, and $\rho_z(A)$ is fixed by τ and thus belongs to $\tilde{K}$. So ρ_z is again a $\mathbb{Q}$-linear map. If $\rho_z(A) = 0$, so is $\sigma(\rho_z(A))$, and thus

$$0 = \rho_z(A) - \sigma(\rho_z(A)) = (a\sigma(z) + \lambda')(z - \bar{z}),$$

which is impossible since $z \notin F$. So ρ_z is injective and thus an isomorphism since $\dim V = \dim \tilde{K} = 4$. To check the isometry, set $\rho = \rho_z(A)$. It is easy to check that

$$\begin{aligned} \rho - \sigma(\rho) &= (a\sigma(z) + \lambda')(z - \bar{z}), \\ \rho - \overline{\sigma(\rho)} &= (az + \lambda)(\sigma(z) - \overline{\sigma(z)}), \\ \bar{\rho}\sigma(z) - \sigma(\rho)\overline{\sigma(z)} &= (\lambda'\bar{z} + b)(\sigma(z) - \overline{\sigma(z)}). \end{aligned} \tag{4.18}$$

So

$$\begin{aligned} \rho\bar{\rho} - \sigma(\rho)\overline{\sigma(\rho)} &= \bar{\rho}(\rho - \sigma(\rho)) + \sigma(\rho)(\bar{\rho} - \overline{\sigma(\rho)}) \\ &= (z - \bar{z})\left(a(\bar{\rho}\sigma(z) - \sigma(\rho)\overline{\sigma(z)}) + \lambda'(\bar{\rho} - \sigma(\rho))\right) \\ &= (z - \bar{z})(\sigma(z) - \overline{\sigma(z)}) \det A. \end{aligned}$$

By Lemma 4.2, we have then

$$Q^-(\rho_z(A)) = -\frac{N\mathfrak{a}}{N\mathfrak{f}_0}\det(A).$$

□

To determine the image of the latices L^0 and L, we need some more preparation. Without loss of generality, we may write

$$\Delta = c\frac{-a + b\sqrt{D}}{2}, \quad a, b, c \in \mathbb{Z} \tag{4.19}$$

such that $\frac{-a+b\sqrt{D}}{2} \in \mathcal{O}_F$ is *primitive* in the sense that it does not have any rational prime factor, and that c is square-free. For each prime p, we write $\mathfrak{a}^{(p)}$ for the p-part of an ideal $\mathfrak{a}$, and $\mathfrak{a}^{\mathrm{odd}}$ for the prime to 2-part of $\mathfrak{a}$.

Lemma 4.4. *Let the notation be as above. Then*

$$d_{K/F}^{\mathrm{odd}} \cap \mathbb{Z} = d_{\tilde{F}}^{\mathrm{odd}} \prod_{p|c, p\nmid 2Dd_{\tilde{F}}} p\mathbb{Z},$$

and

$$N_{F/\mathbb{Q}} d_{K/F}^{\mathrm{odd}} = d_{\tilde{F}}^{\mathrm{odd}} \prod_{p|c, p\nmid 2Dd_{\tilde{F}}} p^2\mathbb{Z}.$$

Proof. By looking at each finite prime of F locally ($\mathfrak{p}$-adic completion), one sees $d_{K/F}^{\mathrm{odd}}|\Delta\mathcal{O}_F$. One also has $d_{\tilde F}^{\mathrm{odd}}|\Delta\Delta'$. Write $\Delta = c\Delta_1$ with $\Delta_1 = \frac{-a+b\sqrt{D}}{2}$ primitive. Let $p|\Delta\Delta' = c^2\Delta_1\Delta_1'$ be an odd prime number. We will prove the formulas prime-by-prime.

If p is inert in F, then $p|c$ and $p \nmid \Delta_1\Delta_1'$. So $\tilde F$ is unramified at p, and K/F is ramified at p ($\mathrm{ord}_p \Delta = 1$), i.e., $d_{K/F}^{(p)} = p\mathcal{O}_F$. The p-parts of the formulas hold.

If $p = \mathfrak{p}^2$ is ramified in F, then

$$\mathrm{ord}_{\mathfrak{p}} \Delta = 2\,\mathrm{ord}_p c + \mathrm{ord}_{\mathfrak{p}} \Delta_1 = 2\,\mathrm{ord}_p c + \mathrm{ord}_p \Delta_1\Delta_1'.$$

So K/F is ramified at $\mathfrak{p}$ if and only if $\tilde F$ is ramified at p, and thus

$$d_{K/F}^{(p)} = \mathfrak{p}^f \quad \text{if } d_{\tilde F}^{(p)} = p^f.$$

The p-parts of the formulas hold again in this case.

If $p = \mathfrak{p}\mathfrak{p}'$ is split in F, then there are two sub-cases. When $p|d_{\tilde F}$, $\mathrm{ord}_p \Delta\Delta'$ is odd. But

$$\mathrm{ord}_p \Delta\Delta' = \mathrm{ord}_{\mathfrak{p}} \Delta + \mathrm{ord}_{\mathfrak{p}} \Delta' = \mathrm{ord}_{\mathfrak{p}} \delta + \mathrm{ord}_{\mathfrak{p}'} \Delta,$$

exactly one of $\mathrm{ord}_{\mathfrak{p}} \Delta$ or $\mathrm{ord}_{\mathfrak{p}} \Delta'$ is odd. So

$$d_{K/F}^{(p)} = \mathfrak{p} \text{ or } \mathfrak{p}', \quad d_{\tilde F}^{(p)} = p,$$

and the p-parts of the formulas hold again in this case. When $p \nmid d_{\tilde F}$, $\tilde F$ is unramified at p. In this case,

$$\mathrm{ord}_p \Delta_1\Delta_1' = \mathrm{ord}_{\mathfrak{p}} \Delta_1 + \mathrm{ord}_{\mathfrak{p}'} \Delta_1,$$

is even, and one of $\mathrm{ord}_{\mathfrak{p}} \Delta_1$ or $\mathrm{ord}_{\mathfrak{p}'} \Delta_1$ is zero since Δ_1 is primitive. So both are even, and

$$\mathrm{ord}_{\mathfrak{p}} \Delta \equiv \mathrm{ord}_{\mathfrak{p}'} \Delta \equiv \mathrm{ord}_p c \mod 2.$$

So

$$d_{K/F}^{(p)} = \begin{cases} p\mathcal{O}_F & \text{if } p|c, \\ \mathcal{O}_F & \text{if } p \nmid c. \end{cases}$$

The p-parts of the formulas hold in this case too. □

Recall that $(\tilde K, \tilde\Phi)$ is the reflex type of (K, Φ), and $\tilde K = \tilde F(\sqrt{\tilde\Delta})$ with

$$\tilde\Delta = (\sqrt{\Delta} + \sqrt{\Delta'})^2 = c(-a - 2\sqrt{\Delta_1\Delta_1'}).$$

It is easy to check that $-a - 2\sqrt{\Delta_1\Delta_1'}$ is prime-to-2 primitive. So the above lemma gives,

Corollary 4.5. *Let the notation be as above. Then*

$$d_{\tilde K/\tilde F}^{\mathrm{odd}} \cap \mathbb{Z} = D^{\mathrm{odd}} \prod_{p|c, p\nmid 2Dd_{\tilde F}} p\mathbb{Z},$$

and

$$N_{\tilde F/\mathbb{Q}} d_{\tilde K/\tilde F}^{\mathrm{odd}} = D^{\mathrm{odd}} \prod_{p|c, p\nmid 2Dd_{\tilde F}} p^2\mathbb{Z}.$$

Here D^{odd} is as before the odd part of D.

Corollary 4.6. *Let the notation be as above, and assume that Δ is primitive. Then*

$$(\partial_F \mathcal{O}_M)^{\mathrm{odd}} = (\partial_{\tilde{K}/\tilde{F}} \sigma(\partial_{\tilde{K}/\tilde{F}}) \mathcal{O}_M)^{\mathrm{odd}},$$

and

$$(\partial_{\tilde{F}} \mathcal{O}_M)^{\mathrm{odd}} = (\partial_{K/F} \sigma(\partial_{K/F}) \mathcal{O}_M)^{\mathrm{odd}}.$$

Proof. Let $\mathfrak{A}_M = \partial_{\tilde{K}/\tilde{F}} \sigma(\partial_{\tilde{K}/\tilde{F}}) \mathcal{O}_M$. Recall that $\mathrm{Gal}(M/F) =< \tau\sigma, \sigma^2 >$ and that σ^2 is the complex conjugation. Clearly,

$$\sigma^2(\mathfrak{A}_M) = \mathfrak{A}_M,$$

and (since $\tau\sigma = \sigma^{-1}\tau$ and τ fixes $\tilde{K}$)

$$\begin{aligned} \tau\sigma(\mathfrak{A}_M) &= \tau\sigma\tau^{-1}(\partial_{\tilde{K}/\tilde{F}})\tau\sigma^2(\partial_{\tilde{K}/\tilde{F}}) \\ &= \sigma^{-1}(\partial_{\tilde{K}/\tilde{F}})\sigma^2\tau(\partial_{\tilde{K}/\tilde{F}}) \\ &= \sigma^2(\mathfrak{A}_M) = \mathfrak{A}_M. \end{aligned}$$

So $\mathfrak{A}_M$ is invariant under $\mathrm{Gal}(M/F)$, and there is thus an ideal $\mathfrak{a}_F$ such that $\mathfrak{A}_M = \mathfrak{a}_F \mathcal{O}_M$. On the other hand,

$$\mathfrak{A}_M^2 = \mathfrak{A}_M \overline{\mathfrak{A}_M} = N_{\tilde{F}}(d_{\tilde{K}/\tilde{F}}) \mathcal{O}_M.$$

So Proposition 4.4 implies

$$(\mathfrak{a}_F^2)^{\mathrm{odd}} = d_F^{\mathrm{odd}} \mathcal{O}_F$$

and thus $\mathfrak{a}_F^{\mathrm{odd}} = \partial_F^{\mathrm{odd}}$. This proves the first formula of this corollary. The second formula is the same. $\square$

For the rest of this section, we assume that

$$\Delta \text{ is primitive and } d_K \text{ is odd.} \tag{4.20}$$

According to a theorem of Hilbert ([Co], Theorem 17.20, see also [Ya2], Appendix A), d_K is odd if D and Δ are odd, and Δ is a square modulo 4. Under our assumption, d_F^{odd} becomes d_F and so on in the above lemma and corollaries.

Proposition 4.7. *Let the notation be as in Proposition 4.3, and assume that the condition (4.20) holds. Then*

$$\rho_{\alpha,\beta}(L^0) = N_\Phi \mathfrak{a}.$$

Proof. We may assume that $\alpha = z$ and $\beta = 1$. Notice first that for any $A = \left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) \in L^0(\mathfrak{f}_0)$, $a \in \mathbb{Z}$, $b \in f_0\mathbb{Z} \subset \mathfrak{f}_0\sigma(\mathfrak{f}_0) \subset \mathfrak{a}\sigma(\mathfrak{a})$, and $\lambda \in \mathfrak{f}_0 \subset \mathfrak{a}$. So

$$\rho_z(A) = az\sigma(z) + z\lambda' + \sigma(z)\lambda' + b \in N_\Phi(\mathfrak{a}).$$

Conversely, if $\rho = \rho_z(A) \in N_\Phi(\mathfrak{a})$, we first assume that $\mathfrak{f}_0 = \mathcal{O}_F$. Then $\mathcal{O}_F \subset \mathfrak{a}$, and thus $\rho, \sigma(\rho) \in \mathfrak{a}\bar{\mathfrak{a}}\sigma(\mathfrak{a})\mathcal{O}_M$. By (4.18), one has then

$$\rho - \sigma(\rho) = (a\sigma(z) + \lambda')(z - \bar{z}) \in \mathfrak{a}\bar{\mathfrak{a}}\sigma(\mathfrak{a})\mathcal{O}_M.$$

On the other hand, $\mathfrak{a} = \mathcal{O}_F z + \mathcal{O}_F$ implies that

$$(z - \bar{z})\mathcal{O}_K = \partial_{K/F}\mathfrak{a}\bar{\mathfrak{a}}. \tag{4.21}$$

Hence

$$a\sigma(z) + \lambda' \in \sigma(\mathfrak{a})\partial_{K/F}^{-1}\mathcal{O}_M,$$

and

$$a\overline{\sigma(z)} + \lambda' \in \overline{\sigma(\mathfrak{a})}\partial_{K/F}^{-1}\mathcal{O}_M.$$

Thus

$$a(\sigma(z) - \overline{\sigma(z)}) \in \sigma(\mathfrak{a})\overline{\sigma(\mathfrak{a})}\partial_{K/F}^{-1}\mathcal{O}_M.$$

Now (4.21) and Corollary 4.6 implies

$$a \in \partial_{K/F}^{-1}\sigma(\partial_{K/F}^{-1})\mathcal{O}_M \cap \mathbb{Q} = \partial_{\tilde{F}}^{-1} \cap \mathbb{Q} = \mathbb{Z}.$$

This implies

$$\lambda' \in \sigma(\mathfrak{a})\partial_{K/F}^{-1}\mathcal{O}_M \cap F = \mathcal{O}_F.$$

It is easy now to verify that $b \in \mathbb{Z}$, i.e., $A \in L^0$. This proves the proposition in the case $\mathfrak{f}_0 = \mathcal{O}_F$. Shifting by a generator of $\mathfrak{f}_0$, we see that this is also true if $\mathfrak{f}_0$ is principal. For a general ideal $\mathfrak{f}_0$, we localize as every prime ideal of $\mathcal{O}_F$, and the above argument shows that $A \in L(\mathfrak{f}_0)_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of F, and so $A \in L(\mathfrak{f}_0)$. □

For the lattice L, it is clear from the definition

$$\rho_{\alpha,\beta}(A) = a\alpha\sigma(\alpha) + \alpha\sigma(\lambda\beta) + \sigma(\alpha)\lambda\beta + b\beta\sigma(\beta) \in \partial_F^{-1}\mathfrak{a}\sigma(\mathfrak{a})\mathcal{O}_M \cap K.$$

However, the converse is not true. In fact, one of the key technical results in this paper is the following proposition, which connects the quadratic lattice L related to the Hirzebruch-Zagier curves with some ideal of the reflex field $\tilde{K}$ via the CM point involved.

Proposition 4.8. *Let the notation be as in Proposition 4.3, and assume that the condition (4.20) holds. Then*

$$\rho_{\alpha,\beta}(L) = \partial_{\tilde{K}/\tilde{F}}^{-1}N_\Phi\mathfrak{a}.$$

In particular, for any integer $m \geq 1$, one has

$$\rho_{(\alpha,\beta)}(L_m) = \{\rho \in \partial_{\tilde{K}/\tilde{F}}^{-1}N_\Phi(\mathfrak{a}) :\ \rho\bar{\rho} = \frac{n - m\sqrt{\tilde{D}}}{2D}N\mathfrak{a} \in N\mathfrak{a}d_{\tilde{K}/\tilde{F}}{}^{-1}\}. \tag{4.22}$$

Proof. **The cyclic case:** We first assume that K is cyclic over $\mathbb{Q}$ so $M = K = \tilde{K}$ and $F = \tilde{F}$. We may also assume that $\alpha = z$, $\beta = 1$, $\mathfrak{f}_0 = \mathcal{O}_F$, and that $(N\mathfrak{a}, d_K) = 1$. If $A = \left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) \in L$ with $a, b \in \mathbb{Z}$ and $\lambda \in \partial_F^{-1} = \frac{1}{\sqrt{D}}\mathcal{O}_F$, one has by definition

$$\rho_z(A) = az\sigma(z) + b + \lambda' z + \lambda\sigma(z) \in \partial_F^{-1}N_\Phi\mathfrak{a}.$$

Clearly $az\sigma(z) + b \in N_\Phi\mathfrak{a}$. Write $\lambda = \frac{\lambda_1}{\sqrt{D}}$ with $\lambda_1 \in \mathcal{O}_F$, then

$$\lambda' z + \lambda\sigma(z) = \frac{\sigma(\lambda_1' z) - \lambda_1' z}{\sqrt{D}}.$$

Since $\sqrt{D}\mathcal{O}_K = \partial_{K/F}^2$, it suffices to prove that for any $z \in \mathcal{O}_K$,

$$\sigma(z) - z \in \partial_{K/F}.$$

For any prime P of K dividing $\partial_{K/F}$, P is totally ramified in $K/\mathbb{Q}$ and so $\mathcal{O}_K/P = \mathbb{Z}/p$ where p is the prime below P. So

$$\sigma(z) - z \equiv 0 \mod P$$

and thus

$$\sigma(z) - z \in \partial_{K/F}.$$

This proves one inclusion. Conversely, if $\rho \in \partial_{K/F}^{-1} N_\Phi \mathfrak{a}$, write $\rho = \rho_z(A)$ for some $A \in V$ by Proposition 4.3. By our assumption at the beginning of the proof, we know $\mathfrak{a} \supset \mathcal{O}_F$. Hence

$$\rho \in \partial_{K/F}^{-1} \mathfrak{a}\sigma(\mathfrak{a})\bar{\mathfrak{a}},$$
$$\sigma(\rho) \in \partial_{K/F}^{-1} \mathfrak{a}\sigma(\mathfrak{a})\bar{\mathfrak{a}},$$

since $\sigma(\partial_{K/F}) = \partial_{K/F}$. So

$$\rho - \sigma(\rho) = (az + \lambda')(z - \bar{z}) \in \partial_{K/F}^{-1} \mathfrak{a}\sigma(\mathfrak{a})\bar{\mathfrak{a}}.$$

Now (4.21) implies

$$az + \lambda' \in \partial_{K/F}^{-2} \sigma(\mathfrak{a})$$

and the same argument as in the proof of the previous proposition gives

$$a \in \partial_{K/F}^{-3} \cap \mathbb{Q} = \mathbb{Z}.$$

This implies

$$\lambda \in \partial_{K/F}^{-2} \sigma(\mathfrak{a}) \cap F = \frac{1}{\sqrt{D}} \mathcal{O}_F.$$

Now it easy to see by definition $b \in \mathbb{Z}$, i.e., $A \in L$. This proves the cyclic case.

Easy non-Galois case: Now we assume that $K/\mathbb{Q}$ is non-Galois and a further condition that $(d_F, d_{\tilde{F}}) = 1$. By Corollary 4.6, one has

$$\frac{1}{\sqrt{D}} \mathcal{O}_M = \partial_{\tilde{K}/\tilde{F}}^{-1} \sigma(\partial_{\tilde{K}/\tilde{F}})^{-1} \mathcal{O}_M.$$

Claim: $\sigma(\partial_{\tilde{K}/\tilde{F}}^{-1}) \mathcal{O}_M \cap \tilde{K} = \mathcal{O}_{\tilde{K}}$.

Indeed, by Lemma 4.4, we only need to check primes $p | d_F$. By the same lemma, p splits into two primes $\mathfrak{p}_{\tilde{F}}$ and $\mathfrak{p}'_{\tilde{F}}$, such that $\mathfrak{p}_{\tilde{F}}$ is ramified in $\tilde{K}$ while $\mathfrak{p}'_{\tilde{F}}$ is unramified in $\tilde{K}$. On the other hand, the following diagram and Lemma 4.4 imply that the prime ideal of F above p is unramified in M (since $(d_F, d_{\tilde{F}}) = 1$).

$$\begin{tikzcd}
 & & & M \arrow[dlll, dash] \arrow[dl, dash] \arrow[dr, dash] \arrow[drrr, dash] & & & \\
\sigma(K) \arrow[dr, dash] & & K \arrow[dl, dash] & & \tilde{K} \arrow[dr, dash] & & \sigma(\tilde{K}) \arrow[dl, dash] \\
 & F \arrow[drr, dash] & & & & \tilde{F} \arrow[dll, dash] & \\
 & & & \mathbb{Q} & & &
\end{tikzcd}$$

So the diagram and Corollary 4.6 imply that the prime ideal(s) of $\tilde{K}$ above $\mathfrak{p}'_{\tilde{F}}$ is ramified in M while $\mathfrak{p}'_{\tilde{F}}$ is ramified in $\sigma(\tilde{K})$ and becomes unramified in M. Therefore for a prime P of M above $\mathfrak{p}'_{\tilde{F}}$ (i.e., above $\sigma(\partial_{\tilde{K}/\tilde{F}})$), one has

$$\operatorname{ord}_P \sigma(\partial_{\tilde{K}/\tilde{F}})\mathcal{O}_M = 1, \quad e(P/P_{\tilde{K}}) = 2$$

where $P_{\tilde{K}} = P \cap \tilde{K}$, and $e(P/P_{\tilde{K}})$ is the ramification index. So

$$\sigma(\partial_{\tilde{K}/\tilde{F}})^{-1}\mathcal{O}_M \cap \tilde{K} = \mathcal{O}_{\tilde{K}}.$$

Next, the claim and Corollary 4.6 imply

$$\begin{aligned}\frac{1}{\sqrt{D}}\mathfrak{a}\sigma(\mathfrak{a})\mathcal{O}_M \cap \tilde{K} &= (\partial_{\tilde{K}/\tilde{F}})^{-1}\mathfrak{a}\sigma(\mathfrak{a})\sigma(\partial_{\tilde{K}/\tilde{F}})^{-1}\mathcal{O}_M \cap \tilde{K} \\ &= (\partial_{\tilde{K}/\tilde{F}})^{-1}\mathfrak{a}\sigma(\mathfrak{a})\mathcal{O}_M \cap \tilde{K} = (\partial_{\tilde{K}/\tilde{F}})^{-1}N_\Phi\mathfrak{a}.\end{aligned}$$

So $\rho(L) \subset (\partial_{\tilde{K}/\tilde{F}})^{-1}N_\Phi\mathfrak{a}$. The other direction is the same as the cyclic case with following modification:

$$\rho \in \partial_{\tilde{K}/\tilde{F}}^{-1}\mathfrak{a}\sigma(\mathfrak{a})\bar{\mathfrak{a}}\mathcal{O}_M, \quad \sigma(\rho) \in \sigma(\partial_{\tilde{K}/\tilde{F}}^{-1})\mathfrak{a}\sigma(\mathfrak{a})\bar{\mathfrak{a}}\mathcal{O}_M.$$

So

$$\rho - \rho(z) = (az + \lambda')(z - \bar{z}) \in \partial_{\tilde{K}/\tilde{F}}^{-1}\sigma(\partial_{\tilde{K}/\tilde{F}}^{-1})\mathfrak{a}\sigma(\mathfrak{a})\bar{\mathfrak{a}}\mathcal{O}_M = \partial_F^{-1}\mathfrak{a}\sigma(\mathfrak{a})\bar{\mathfrak{a}}\mathcal{O}_M,$$

and thus

$$az + \lambda' \in \partial_{K/F}^{-1}\partial_F^{-1}\sigma(\mathfrak{a})\mathcal{O}_M.$$

The same argument as in the cyclic case shows

$$a \in \partial_F^{-1}\partial_{\tilde{F}}^{-1}\mathcal{O}_M \cap \mathbb{Q} = \mathbb{Z}$$

since $(d_F, d_{\tilde{F}}) = 1$. This proves the easy non-Galois case. The cyclic and easy non-Galois case are all we need for the proof of Theorem 1.4.

The general non-Galois case: The proposition is local in nature, we can prove it one prime at a time. For $p \nmid d_F d_{\tilde{F}}$, there is nothing to prove. When p divides exactly one of the d_F and $d_{\tilde{F}}$. The same argument as in the easy non-Galois case applies. When $p|d_F$ and $p|d_{\tilde{F}}$, the argument is similar to that of the cyclic case. Indeed, to prove $\rho_z(L) \subset \partial_{\tilde{K}/\tilde{F}}^{-1} N_\Phi(\mathfrak{a})$, it suffices to prove, by Corollary 4.6, that for any $z \in \mathcal{O}_K$,

$$\sigma(z) - z \in \sigma(\partial_{\tilde{K}/\tilde{F}})\mathcal{O}_M.$$

Since $p|d_F$ and $p|d_{\tilde{F}}$, Lemma 4.4 implies that p is totally ramified in K and $\tilde{K}$ and the prime ideal P_K (resp. $P_{\tilde{K}}$) of K (resp. $\tilde{K}$) above p becomes unramified in M. Write

$$z = a + b\delta, \quad a, b \in \mathcal{O}_{F_p},$$

where $\delta \in K_p$ is a uniformizer with $\bar{\delta} = -\delta$. Then

$$\sigma(z) - z = a - a' + b\delta - b'\delta'.$$

Clearly $a - a' \in \sqrt{D}\mathcal{O}_{F_p} \subset \sigma(\partial_{\tilde{K}/\tilde{F}})\mathcal{O}_{M_p}$. On the other hand,

$$b\delta - b'\delta' = b\delta - \sigma(b\delta) \in \sigma(\partial_{\tilde{K}/\tilde{F}}).$$

So

$$\sigma(z) - z \in \sigma(\partial_{\tilde{K}/\tilde{F}})\mathcal{O}_{M_p}.$$

For the other direction, we notice that both $P_{\tilde{K}}$ and $\sigma(P_{\tilde{K}})$ are unramified in M, and

$$P_{\tilde{K}}\mathcal{O}_M = \sigma(P_{\tilde{K}})\mathcal{O}_M = P_K\mathcal{O}_M = \sigma(O_K)\mathcal{O}_M.$$

This implies

$$\sigma(\partial_{\tilde{K}_p/\tilde{F}_p}\mathcal{O}_{M_p}) = \partial_{\tilde{K}_p/\tilde{F}_p}\mathcal{O}_{M_p} = P_{\tilde{K}}\mathcal{O}_{M_p}.$$

So the same argument as the cyclic case gives

$$\rho, \sigma(\rho) \in (\partial_{\tilde{K}/\tilde{F}})^{-1}\mathfrak{a}\sigma(\mathfrak{a})\bar{\mathfrak{a}}\mathcal{O}_{M_p}.$$

The same argument as in the cyclic case then shows

$$az + \lambda' \in \partial_{K/F}^{-1}(\partial_{\tilde{K}/\tilde{F}})^{-1}\sigma(a)\mathcal{O}_{M_p} = \partial_{K/F}^{-2}\sigma(a)\mathcal{O}_{M_p}$$

and

$$a \in \partial_{K/F}^{-3}\mathcal{O}_{M_p} \cap \mathbb{Q}_p = \mathbb{Z}_p.$$

So the p-part of a is integral. The same argument as in the cyclic case implies then that the p-part of λ is in ∂_F^{-1}. Since this holds for every p, we finally proved the proposition. $\square$

5. CM values of automorphic Green functions

Now we come back to our situation in the introduction and provide the second ingredient for the proof of Theorem 1.4. The theorem itself should be of independent interest.

Theorem 5.1. *Let $F = \mathbb{Q}(\sqrt{p})$ be a real quadratic field with $p \equiv 1 \pmod 4$ being prime. Let $K = F(\sqrt{\Delta})$ be a totally imaginary quadratic extension of F such that*

$$d_{K/F} \cap \mathbb{Z} = q\mathbb{Z}, \quad \textit{and} \quad N_{F/\mathbb{Q}} d_{K/F} = q \tag{5.1}$$

for some prime number $q \equiv 1 \pmod 4$, as in the introduction. Let $\Phi = \{1, \sigma\}$ be the CM type of K defined in (1.2), and let $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ be the CM 0-cycle in X defined in Section 3. Let $\Phi_m(z, s)$ be the automorphic Green function for T_m as in (2.19). Then[1]

$$\Phi_m(\mathcal{CM}(K, \Phi, \mathcal{O}_F), s) = W_{\tilde{K}} \sum_{\mu = \frac{n - m\sqrt{q}}{2p} \in d_{\tilde{K}/\tilde{F}}^{-1,+}} Q_{s-1}\left(\frac{n}{m\sqrt{q}}\right) \rho_{\tilde{K}/\tilde{F}}(\mu d_{\tilde{K}/\tilde{F}}).$$

Here $(\tilde{K}, \tilde{\Phi})$ is the reflex type of (K, Φ), and $\tilde{F} = \mathbb{Q}(\sqrt{q})$. Finally, $W_{\tilde{K}}$ is the number of roots of unity in $\tilde{K}$, and $\rho_{\tilde{K}/\tilde{F}}$ is defined as in (1.7).

Proof. For $A = \left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) \in V$, set $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and

$$d_A(z_1, z_2) = 1 + \frac{|az_1 z_2 + \lambda' z_1 + \lambda z_2 + b|^2}{2 y_1 y_2 \det A}. \tag{5.2}$$

Then for a CM point $[\mathfrak{a}, r] \in \mathcal{CM}(K, \Phi, \mathcal{O}_F)$, we write as in (4.14)

$$\mathfrak{a} = \mathcal{O}_F \alpha + \mathcal{O}_F \beta, \quad z = \frac{\alpha}{\beta} \in K^+.$$

Proposition 4.3 and Lemma 4.2, imply ($\rho = \rho_{(\alpha,\beta)}(A)$)

$$d_A(z, \sigma(z)) = 1 + \frac{2\rho\bar{\rho}}{4 \operatorname{Im}(z) \operatorname{Im}(\sigma(z)) \det A} = \frac{Q^+(\rho)}{\sqrt{q}} \frac{1}{N\mathfrak{a} \det A}.$$

Set

$$\mu(z, A) = \frac{\rho_{(\alpha,\beta)(A)} \overline{\rho_{(\alpha,\beta)(A)}}}{N\mathfrak{a}} = \frac{Q^+(\rho) + \sqrt{\tilde{D}} Q^-(\rho)}{2N\mathfrak{a}}. \tag{5.3}$$

The identity (4.13) implies

$$\mu(\gamma z, A) = \mu(z, {}^t\gamma.A). \tag{5.4}$$

So $\mu(z, A)$ depends only on the CM point $z = [\mathfrak{a}, r] \in \mathcal{CM}(K, \Phi, \mathcal{O}_F)$, up to a $\mathrm{SL}_2(\mathcal{O}_F)$ action. Now $A \in L_m$ implies $\det A = \frac{m}{p}$. Proposition 4.8 implies $\mu(z, A) \in d_{\tilde{K}/\tilde{F}}^{-1,+} \subset \frac{1}{p}\mathcal{O}_{\tilde{F}}$, so we can write

$$\mu(z, A) = \frac{n - m\sqrt{q}}{2p} \in d_{\tilde{K}/\tilde{F}}^{-1,+}.$$

Then $d_A(z, \sigma(z)) = \frac{n}{m\sqrt{q}}$, and

$$\begin{aligned} \Phi_m(z, s) &= \sum_{A \in L_m} Q_{s-1}(d_A(z, \sigma(z))) \\ &= \sum_{\mu = \frac{n - m\sqrt{q}}{2p} \in d_{\tilde{K}/\tilde{F}}^{-1}} Q_{s-1}\left(\frac{n}{m\sqrt{q}}\right) \sum_{\substack{A \in L_m \\ \mu(z,A) = \mu}} 1. \end{aligned} \tag{5.5}$$

[1]Throughout, for the Green functions G_m and Φ_m the evaluation on a 0-cycle is additive.

So Proposition 4.8 implies

$$\Phi_m(\mathcal{CM}(K,\Phi,\mathcal{O}_F),s)=\sum_{\mu=\frac{n-m\sqrt{q}}{2p}\in d_{\tilde K/\tilde F}^{-1}} Q_{s-1}\left(\frac{n}{m\sqrt{q}}\right)C(\mu) \tag{5.6}$$

with

$$C(\mu)=\#\{[\mathfrak{a},r]\in\mathcal{CM}(K,\Phi,\mathcal{O}_F),\rho\in\partial_{\tilde K/\tilde F}^{-1}N_\Phi\mathfrak{a}:\ \rho\bar\rho=\mu N\mathfrak{a}\}. \tag{5.7}$$

Fix a $\xi_0\in K^*$ with $\bar\xi_0=-\xi_0$, then $\xi_0\partial_{K/F}=\mathfrak{f}_0\mathcal{O}_K$ for some ideal $\mathfrak{f}_0$ of F. Since $p\equiv 1 \bmod 4$ is prime, the narrow class number of F is odd, and thus $\mathfrak{f}_0=\lambda_0\mathfrak{g}_0^2$ for some ideal $\mathfrak{g}_0$ of F and $\lambda_0\gg 0$. Recall that $\mathcal{CM}(K,\Phi,\mathcal{O}_F)$ is the equivalence classes of $[\mathfrak{a},r]$ where $\mathfrak{a}$ is an ideal of K and $r\in F^+$ with

$$\mathfrak{f}_{\mathfrak{a}}:=\xi_0\partial_{K/F}\mathfrak{a}\bar{\mathfrak{a}}\cap F=\lambda_0\mathfrak{g}_0^2N_{K/F}\mathfrak{a}=r\mathcal{O}_F. \tag{5.8}$$

To continue the proof, we need two lemmas.

Lemma 5.2. *Let the notation and assumption be as in Theorem 5.1. Let $\mathcal{CL}_0(K)=\{[\mathfrak{a}]\in\mathcal{CL}(K):\ N_{K/F}\mathfrak{a}=\mu\mathcal{O}_F$ for some $\mu\gg 0\}$. Then $[\mathfrak{a},r]\mapsto[\mathfrak{a}\mathfrak{g}_0]$ gives an bijection between $\mathcal{CM}(K,\Phi,\mathcal{O}_F)$ and $\mathcal{CL}_0(K)$. Moreover, for $[\mathfrak{a}\mathfrak{g}_0^{-1},r]\in\mathcal{CM}(K,\Phi,\mathcal{O}_F)$ with $[\mathfrak{a}]\in\mathcal{CL}_0(K)$ and $\mu\in d_{\tilde K/\tilde F}^{-1}$, one has*

$$\begin{aligned}&\#\{\rho\in\partial_{\tilde K/\tilde F}^{-1}N_\Phi(\mathfrak{a}\mathfrak{g}_0^{-1}):\ \rho\bar\rho=\mu N_{K/\mathbb{Q}}(\mathfrak{a}\mathfrak{g}_0^{-1})\}\\ &=W_{\tilde K}\#\{\tilde{\mathfrak{b}}\in[\partial_{\tilde K/\tilde F}][N_\Phi(\mathfrak{a})]^{-1}\ \textit{integral}:\ N_{\tilde K/\tilde F}\tilde{\mathfrak{b}}=\mu d_{\tilde K/\tilde F}\}.\end{aligned}$$

Proof. Since every totally positive unit of F is a square, the comment after Lemma 3.3 implies that $[\mathfrak{a}\mathfrak{g}_0^{-1},r]\mapsto[\mathfrak{a}]$ gives a bijection between $\mathcal{CM}(K,\Phi,\mathcal{O}_F)$ and $\mathcal{CL}_0(K)$.

Secondly, $N_\Phi\mathfrak{g}_0=N_{F/\mathbb{Q}}\mathfrak{g}_0=g_0\in\mathbb{Q}^*$, $\rho\mapsto\rho g_0$ gives a bijection between

$$S_1=\{\rho\in\partial_{\tilde K/\tilde F}^{-1}N_\Phi(\mathfrak{a}\mathfrak{g}_0^{-1}):\ \rho\bar\rho=\mu N_{K/\mathbb{Q}}(\mathfrak{a}\mathfrak{g}_0^{-1})\}$$

and

$$S_2=\{\rho\in\partial_{\tilde K/\tilde F}^{-1}N_\Phi(\mathfrak{a}):\ \rho\bar\rho=\mu N_{K/\mathbb{Q}}(\mathfrak{a})\}.$$

Thirdly, $\rho\mapsto\tilde{\mathfrak{b}}=\rho\partial_{\tilde K/\tilde F}N_\Phi\mathfrak{a}^{-1}$ is a map from S_2 to

$$S_3=\{\tilde{\mathfrak{b}}\in[\partial_{\tilde K/\tilde F}][N_\Phi(\mathfrak{a})]^{-1}\ \text{integral}:\ N_{\tilde K/\tilde F}\tilde{\mathfrak{b}}=\mu d_{\tilde K/\tilde F}\}.$$

Conversely, if $\tilde{\mathfrak{b}}\in S_3$, then

$$\tilde{\mathfrak{b}}\partial_{\tilde K/\tilde F}^{-1}N_\Phi(\mathfrak{a})=\rho\mathcal{O}_K,\quad \rho\bar\rho\mathcal{O}_F=\mu N\mathfrak{a}\mathcal{O}_F$$

for some $\rho\in K^*$. So $\rho\in\partial_{\tilde K/\tilde F}^{-1}N_\Phi(\mathfrak{a})$, and $\rho\bar\rho=\mu N\mathfrak{a}\epsilon$ for some $\epsilon\in O_F^*$ totally positive. Again, one has $\epsilon=\epsilon_1^2$ for some unit $\epsilon_1\in\mathcal{O}_F^*$ and $\rho\epsilon_1^{-1}$ maps to $\tilde{\mathfrak{b}}$ under our map. Finally, if ρ_1 and ρ_2 map to the same ideal $\tilde{\mathfrak{b}}$, then

$$\rho_1=\epsilon\rho_2,\quad \rho_1\bar\rho_1=\rho_1\bar\rho_1=\mu N\mathfrak{a}$$

for some unit $\epsilon \in \mathcal{O}_{\tilde{K}}^*$. So $\epsilon\bar{\epsilon} = 1$, and therefore ϵ is a root of unity in $\tilde{K}$. So the map from S_2 to S_3 is surjective and $W_{\tilde{K}}$ to one, and thus

$$\#S_1 = \#S_2 = W_{\tilde{K}} \#S_3.$$

□

Lemma 5.3. *Under the assumption of Theorem 5.1, the type norm map*

$$N_\Phi : \mathcal{CL}(K) \longrightarrow \mathcal{CL}(\tilde{K}), \quad [\mathfrak{a}] \mapsto [\mathfrak{a}\sigma(\mathfrak{a})\mathcal{O}_M \cap \tilde{K}]$$

induces an isomorphism between $\mathcal{CL}_0(K)$ *and* $\mathcal{CL}_0(\tilde{K})$.

Proof. Since $N_\Phi(\mathfrak{a})\overline{N_\Phi(\mathfrak{a})} = N\mathfrak{a}\mathcal{O}_{\tilde{K}}$, one sees $[N_\Phi \mathfrak{a}] \in \mathcal{CL}_0(\tilde{K})$ always. Next, for an ideal class $[\mathfrak{a}] \in \mathcal{CL}_0(K)$, we may choose a representative $\mathfrak{a}$ so that $\mathfrak{a}$ and $\sigma(\mathfrak{a})$ are relatively prime in the Galois closure M of K. Then one has

$$\begin{aligned} N_{\tilde{\Phi}} \circ N_\Phi(\mathfrak{a}) &= N_\Phi(\mathfrak{a})\sigma^{-1}(N_\Phi(\mathfrak{a}))\mathcal{O}_M \cap K \\ &= \mathfrak{a}^2\sigma(\mathfrak{a}\bar{\mathfrak{a}}). \end{aligned}$$

Since $[\mathfrak{a}] \in \mathcal{CL}_0(K)$, $\mathfrak{a}\bar{\mathfrak{a}} = \lambda\mathcal{O}_K$ for some $\lambda \in F^+$. So

$$[N_{\tilde{\Phi}} \circ N_\Phi(\mathfrak{a})] = [\mathfrak{a}]^2$$

is the square map. By [CH], Corollary 13.9, $\mathcal{CL}(K)$ and thus $\mathcal{CL}_0(K)$ is odd. So the square map and thus N_Φ and $N_{\tilde{\Phi}}$ are isomorphisms. □

Now we return to the proof of Theorem 5.1. By the above two lemmas, one has for $\mu \in d_{\tilde{K}/\tilde{F}}^{-1,+}$:

$$\begin{aligned} C(\mu) &= W_{\tilde{K}}\#\{[\mathfrak{a}] \in \mathcal{CL}_0(K),\, \tilde{\mathfrak{b}} \in [\partial_{\tilde{K}/\tilde{F}}][N_\Phi\mathfrak{a}]^{-1} \text{ integral} : \ N_{\tilde{K}/\tilde{F}}\tilde{\mathfrak{b}} = \mu d_{\tilde{K}/\tilde{F}}\} \\ &= W_{\tilde{K}}\#\{[\tilde{\mathfrak{a}}] \in \mathcal{CL}_0(\tilde{K}),\, \tilde{\mathfrak{b}} \in [\partial_{\tilde{K}/\tilde{F}}][\tilde{\mathfrak{a}}] \text{ integral} : \ N_{\tilde{K}/\tilde{F}}\tilde{\mathfrak{b}} = \mu d_{\tilde{K}/\tilde{F}}\} \\ &= W_{\tilde{K}}\#\{\tilde{\mathfrak{b}} \subset \mathcal{O}_{\tilde{K}} : \ N_{\tilde{K}/\tilde{F}}\tilde{\mathfrak{b}} = \mu d_{\tilde{K}/\tilde{F}}\} \\ &= W_{\tilde{K}}\rho_{\tilde{K}/\tilde{F}}(\mu d_{\tilde{K}/\tilde{F}}). \end{aligned}$$

The last identity is due to the fact that if an integral ideal $\tilde{\mathfrak{b}}$ of $\tilde{K}$ satisfies $N_{\tilde{K}/\tilde{F}}\tilde{\mathfrak{b}} = \mu d_{\tilde{K}/\tilde{F}}$, then $\tilde{\mathfrak{a}} = \tilde{\mathfrak{b}}\partial_{\tilde{K}/\tilde{F}}^{-1} \in \mathcal{CL}_0(\tilde{K})$. Combining this formula for $C(\mu)$ with (5.6), one proves the theorem. □

Let $\Phi' = \{1, \sigma^{-1}\} = \sigma^3\Phi$ be another CM type of K, and let $(\tilde{K}', \tilde{\Phi}')$ be the reflex of (K, Φ'). Then $\sigma(\tilde{K}') = \tilde{K}$ and $\sigma(\partial'_{\tilde{K}/\tilde{F}}) = \partial_{\tilde{K}/\tilde{F}}$, where $\partial_{\tilde{K}/\tilde{F}}$ is the ideal of $\tilde{K}'$ defined the same manner as that of $\partial_{\tilde{K}/\tilde{F}}$. So Theorem 5.1 implies:

Corollary 5.4. *Let the notation be as above. Then*

$$\begin{aligned}\Phi_m(\mathcal{CM}(K,\Phi',\mathcal{O}_F)) &= W_{\tilde{K}'} \sum_{\mu=\frac{n-m\sqrt{q}}{2p}\in d'^{-1,+}_{\tilde{K}/\tilde{F}}} Q_{s-1}\left(\frac{n}{m\sqrt{q}}\right)\rho_{\tilde{K}'/\tilde{F}}(\mu d'_{\tilde{K}/\tilde{F}}) \\ &= W_{\tilde{K}} \sum_{\mu=\frac{n+m\sqrt{q}}{2p}\in d^{-1,+}_{\tilde{K}/\tilde{F}}} Q_{s-1}\left(\frac{n}{m\sqrt{q}}\right)\rho_{\tilde{K}/\tilde{F}}(\mu d_{\tilde{K}/\tilde{F}}).\end{aligned}$$

6. Incoherent Eisenstein series

In [Ya1], one of the authors constructed holomorphic modular forms by restricting a 'coherent' Eisenstein series over a totally real number field to $\mathbb{Q}$. The purpose of next three sections is to construct a holomorphic cuspidal modular form of weight 2 from an 'incoherent' Eisenstein series over a real quadratic field by means of central derivative, diagonal restriction, and holomorphic projection. This provides the third and last ingredient we need for the proof of Theorem 1.4. We start with a more general setting with new notations. The quadratic field F here will be $\tilde{F}$ in Theorem 1.4 in the end.

Let $F=\mathbb{Q}(\sqrt{D})$ be a real quadratic field with fundamental discriminant $D>0$, and let $K=F(\sqrt{\Delta})$ be a totally imaginary quadratic extension with $\Delta \ll 0$ totally negative. Let $\chi=\chi_{K/F}$ be the quadratic Hecke character of F associated to K/F.

Let $\psi_{\mathbb{Q}}$ be the canonical unramified additive character of $\mathbb{Q}_{\mathbb{A}}$ with $\psi_{\mathbb{Q}_\infty}(x)=e(x)=e^{2\pi i x}$. Let ψ be the unramified additive character of $F_{\mathbb{A}}$ given by

$$\psi(x)=\psi\left(\operatorname{tr}_{F/\mathbb{Q}}\frac{x}{\sqrt{D}}\right). \tag{6.1}$$

We fix one embedding F into $\mathbb{R}$ such that $\sqrt{D}>0$, and denote the other embedding by $x=a+b\sqrt{D}\mapsto x'=a-b\sqrt{D}$. In particular, $(\sqrt{D})'=-\sqrt{D}<0$. We denote the corresponding two infinite places by ∞ and ∞'. We also fix a CM type $\Phi=\{1,\sigma\}$ of K so that $\sqrt{\Delta}\in i\mathbb{R}_{>0}$ and $\sigma(\sqrt{\Delta})=\sqrt{\Delta'}\in i\mathbb{R}_{>0}$. Under these identifications, one has

$$\psi_\infty(x)=e\left(\frac{x}{\sqrt{D}}\right),\quad \psi_{\infty'}(x)=e\left(-\frac{x}{\sqrt{D}}\right). \tag{6.2}$$

Let $I(s,\chi)=\otimes' I(s,\chi_v)$ be the induced representation of $\mathrm{SL}_2(F_{\mathbb{A}})$, consisting of Schwartz functions $\Phi(g,s)$ on $\mathrm{SL}_2(F_{\mathbb{A}})$ such that

$$\Phi(n(b)m(a)g,s)=\chi(a)|a|^{s+1}\Phi(g,s),\quad n(b)=\begin{pmatrix}1&b\\0&1\end{pmatrix},\quad m(a)=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}. \tag{6.3}$$

Recall first that for a factorizable standard section $\Phi=\prod_v\Phi_v\in I(s,\chi)$, the Eisenstein series

$$E(g,s,\Phi)=\sum_{\gamma\in B\backslash\mathrm{SL}_2(F)}\Phi(\gamma g,s) \tag{6.4}$$

is absolutely convergent when $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to the whole complex s-plane with finitely many poles and is holomorphic on the unitary line $\mathrm{Re}(s) = 0$. Here B is the usual Borel subgroup of $\mathrm{SL}_2(F)$.

In this paper, we choose $\Phi^K = \prod \Phi_v \in I(s, \chi)$ as follows. Let $V = K$ with the quadratic form $Q(z) = z\bar{z}$. Then the reductive dual pair $(O(V), \mathrm{SL}_2)$ gives rise to the Weil representation $\omega = \omega_\psi$ of $\mathrm{SL}_2(F_\mathbb{A})$ on the space of Schwartz functions $S(V_\mathbb{A}) = S(K_\mathbb{A})$, and there is $\mathrm{SL}_2(F_\mathbb{A})$-equivariant map

$$\lambda = \lambda_V : S(V_\mathbb{A}) \longrightarrow I(0, \chi), \quad \phi \mapsto \omega(g)\phi(0). \tag{6.5}$$

For any finite prime v of F, let $\Phi_v \in I(s, \chi_v)$ be the standard section such that

$$\Phi_v(g, 0) = \lambda_v(\mathrm{char}(\mathcal{O}_{K_v}))(g).$$

That is, Φ_v is Φ_v^+ associated to ψ_v in the notation of [Ya1]. For an infinite v, we take $\Phi_v = \Phi_\mathbb{R}^1 \in I(s, \chi_v)$ to be the unique eigenfunction of $\mathrm{SO}_2(\mathbb{R})$ of weight 1, i.e.,

$$\Phi_v(gk_\theta, s) = \Phi_v(g, s)e^{i\theta}, \quad \Phi(1, s) = 1, \quad k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right).$$

We remark that due to our choice of ψ, every Φ_v except for $v = \infty'$ comes from the quadratic space V_v while $\Phi_{\infty'}$ actually comes from $-V_{\infty'}$ (the same space with the negative quadratic form), thus 'incoherent' according to Kudla ([Ku1]).

We normalize

$$E^*(\tau, \tau', s, \Phi^K) = (vv')^{-\frac{1}{2}} E(g_\tau g_{\tau'}, s, \Phi^K)\Lambda(s + 1, \chi). \tag{6.6}$$

Here $(\tau, \tau') = (u + iv, u' + iv') \in \mathbb{H}^2$, $A = DN_{F/\mathbb{Q}}d_{K/F}$, and

$$\Lambda(s, \chi) = A^{\frac{s}{2}} L(s, \chi)\Gamma_\mathbb{R}(s + 1) = A^{\frac{s}{2}} \prod_v L(s, \chi_v), \quad \Gamma_\mathbb{R}(s) = \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2}). \tag{6.7}$$

The same proof as that of [Ya1], Theorem 1.2, implies:

Theorem 6.1. *The Eisenstein series $E^*(\tau, \tau', s, \Phi^K)$ has the following properties.*

(1) *It is a (non-holomorphic) Hilbert modular form for the group $\Gamma_0(d_{K/F})$ of weight $(1, 1)$ and character χ. That is,*

$$E^*(\gamma\tau, \gamma'\tau', s, \Phi^K) = \chi(\gamma)(c\tau + d)(c'\tau' + d')E^*(\tau, \tau', s, \Phi^K),$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(d_{K/F})$, i.e., $c \in d_{K/F}$. Here $\chi(\gamma)$ is defined as

$$\chi(\gamma) := \prod_{v | d_{K/F}} \chi_v(d). \tag{6.8}$$

(2) *It satisfies the functional equation*

$$E^*(\tau, \tau', s, \Phi^K) = -E^*(\tau, \tau', -s, \Phi^K).$$

(3) *The constant term is given by*

$$E_0^*(\tau, \tau', s, \Phi^K) = (vv')^{\frac{s}{2}}\Lambda(-s, \chi) - (vv')^{-\frac{s}{2}}\Lambda(s, \chi).$$

Here $\Gamma_\mathbb{R}(s) = \pi^{-\frac{s}{2}}\Gamma(s)$, and

$$\Lambda(s, \chi) = A^{\frac{s}{2}}\Gamma_\mathbb{R}(s + 1)^2 L(s, \chi), \quad A = DN_{F/\mathbb{Q}}d_{K/F}$$

is the complete L-function of χ.

(4) $E_t^*(\tau, \tau', s, \Phi^K) = 0$ *unless* $t \in \mathcal{O}_F$.

In particular, $E^*(\tau, \tau', 0, \Phi^K) = 0$ automatically, and it is interesting to compute the central derivative $E^{*,\prime}(\tau, \tau', 0, \Phi^K)$. In fact, we need a slightly more general result, which we now describe. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)$, it is easy to check that

$$\begin{aligned} E^*(\tau, \tau', s, \Phi)|_1\gamma &:= (c\tau + d)^{-1}(c'\tau' + d')^{-1} E^*(\gamma\tau, \gamma'\tau', s, \Phi) \\ &= E^*(\tau, \tau', s, \gamma_f^{-1}\Phi), \end{aligned} \tag{6.9}$$

where γ_f is the image of γ in $\mathrm{SL}_2(\mathbb{A}_f)$ under the diagonal embedding.

We decompose $d_{K/F} = \mathfrak{d}_c \mathfrak{d}_c'$ according to c such that $\mathfrak{d}_c'$ is relatively prime to $c\mathcal{O}_F$ and every prime factor of $\mathfrak{d}_c$ divides $c\mathcal{O}_F$. Define

$$C_t(\gamma) = \prod_{v|\mathfrak{d}_c} \chi_v(d) \prod_{v|\mathfrak{d}_c'} \chi_v(c)\epsilon(\chi_v, \psi_v)|d_{K/F}|^{\frac{1}{2}} \psi(-\tfrac{d}{c}t). \tag{6.10}$$

In particular, $C_t(1) = 1$ and $C_t(w) = (N_{F/\mathbb{Q}} d_{K/F})^{-\frac{1}{2}}$. Finally, we define

$$\delta(t) = \prod_{v|d_{K/F}} (1 + \chi_v(t)), \tag{6.11}$$

and for an ideal $\mathfrak{a}$ of F (as in (1.7))

$$\rho(\mathfrak{a}) = \#\{\mathfrak{A} \subset \mathcal{O}_K : N_{K/F}(\mathfrak{A}) = \mathfrak{a}\}. \tag{6.12}$$

Then $\rho(\mathfrak{a}) = 0$ for a non-integral integral $\mathfrak{a}$. If $\mathfrak{a}$ is integral, one has

$$\rho(\mathfrak{a}) = \prod_{v<\infty} \rho_v(\mathfrak{a}), \tag{6.13}$$

with

$$\rho_v(\mathfrak{a}) = \begin{cases} 1 & \text{if } v|d_{K/F}, \\ \frac{1+(-1)^{\mathrm{ord}_v \mathfrak{a}}}{2} & \text{if } v \text{ inert in} K/F, \\ 1 + \mathrm{ord}_v \mathfrak{a} & \text{if } v \text{ split in} K/F. \end{cases} \tag{6.14}$$

The rest of this section is to prove the following theorem.

Theorem 6.2. *Let the notation be as above. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)$ and assume that the 2-part of $d_{K/F}$ either divides $c\mathcal{O}_F$ or is prime to $c\mathcal{O}_F$. Then*

$$E^{*,\prime}(\tau, \tau', 0, \gamma_f^{-1}\Phi^K) = \sum_{t \in (\mathfrak{d}_c')^{-1}} C_t(\gamma) A_t(v, v') e\left(\mathrm{tr}\, \frac{t\tau}{\sqrt{D}}\right),$$

where $\tau = u + iv$, $\tau' = u' + iv'$, and $A_t(v, v')$ are given as follows.

(1) *The constant term is*

$$A_0(v, v') = \log(vv')\Lambda(0, \chi) + 2\Lambda'(0, \chi).$$

(2) *When $t, t' > 0$, one has*

$$A_t(v, v') = -2\delta(t)\rho(td_{K/F})\beta_1\left(\frac{4\pi|t'|v'}{\sqrt{D}}\right).$$

(3) *When $t, t' < 0$, one has*

$$A_t(v, v') = -2\delta(t)\rho(td_{K/F})\beta_1\left(\frac{4\pi|t|v}{\sqrt{D}}\right)\prod_{v|\mathfrak{d}_c}\chi_v(a).$$

(4) *When $t > 0 > t'$, and there is a ramified finite prime v_0 such that $\chi_{v_0}(t) = -1$, one has*

$$A_t(v, v') = A_t = -4\,\mathrm{ord}_{v_0}(td_{K/F})\rho(td_{K/F})\prod_{v_0\neq v|d_{K/F}}(1 + \chi_v(t))\log|\mathfrak{p}_{v_0}|$$

is independent of τ or τ'. Here $\mathfrak{p}_{v_0}$ is the prime ideal of F associated to v_0, for an integral ideal $\mathfrak{a}$, we write $|\mathfrak{a}|$ for the order of $\mathcal{O}_F/\mathfrak{a}$.

(5) *When $t > 0 > t'$ and there is an inert finite prime v_0 such that $\chi_{v_0}(t) = -1$, one has*

$$A_t(v, v') = A_t = -2\delta(t)\rho(td_{K/F}\mathfrak{p}_{v_0}^{-1})(\mathrm{ord}_{v_0} t + 1)\log|\mathfrak{p}_{v_0}|$$

is independent of τ or τ'.

(6) *In all other cases, one has*

$$A_t(v, v') = 0.$$

The proof is local in nature. We first recall that for any factorizable section $\Phi = \prod \Phi_v$, the Eisenstein series $E(g, s, \Phi)$ has the Fourier expansion

$$E(g, s, \Phi) = E_0(g, s, \Phi) + \sum_{t\in F^*} E_t(g, s, \Phi), \tag{6.15}$$

where, for $t \in F^*$,

$$E_t(g, s, \Phi) = \prod_v W_{t,v}(g, s, \Phi_v) \tag{6.16}$$

with

$$W_{t,v}(g, s, \Phi_v) = \int_{F_v} \Phi_v(wn(b)g, s)\psi_v(-tb)db. \tag{6.17}$$

Here db is the Haar measure on F_v with respect to the character ψ_v. The constant term is

$$E_0(g, s, \Phi) = \Phi(g, s) + W_0(g, s, \Phi) = \Phi(g, s) + M(s)\Phi(g, s). \tag{6.18}$$

We normalize

$$W^*_{t,v}(g, s, \Phi) = L(s + 1, \chi_v)W_{t,v}(g, s, \Phi). \tag{6.19}$$

Then our normalized Eisenstein series is

$$E^*(\tau, \tau', s, \Phi) = E^*_0(\tau, \tau', s, \Phi) + \sum_{t\in F^*} E^*_t(\tau, \tau', s, \Phi)$$

with $(t \ne 0)$

$$E^*_t(\tau, \tau', s, \Phi) = A^{\frac{s+1}{2}} W^*_{t,\infty}(\tau, s, \Phi_\infty) W^*_{t',\infty'}(\tau', s, \Phi_\infty) \prod_{v<\infty} W^*_{t,v}(1, s, \Phi_v).$$

The constant term E^*_0 is similar. We now recall the local results in [Ya1].

Lemma 6.3. *([Ya1], Proposition 2.1) Assume $v \nmid d_{K/F}\infty\infty'$.*

(1) *Φ_v is the unique eigenfunction of $K = \mathrm{SL}_2(\mathcal{O}_v)$ with trivial eigencharacter such that $\Phi_v(1, s) = 1$.*
(2) *Its Whittaker function satisfies $W^*_{t,v}(1, s, \Phi_v) = 0$ unless $t \in \mathcal{O}_v$. When $t \in \mathcal{O}_v$, $W^*_{t,v}(1, 0, \Phi_v) = \rho_v(t\mathcal{O}_v)$.*
(3) *$W^*_{t,v}(1, 0, \Phi_v) = 0$ if and only if v is inert in K/F and $\chi_v(t) = -1$. In such a case,*

$$W^{*,\prime}_{t,v}(1, 0, \Phi_v) = \frac{1}{2}(\mathrm{ord}_v\, t + 1) \log |\mathfrak{p}_v|.$$

(4) *$M^*_v(s)\Phi_v(g, s) = L(s, \chi_v)\Phi_v(g, -s)$.*

Lemma 6.4. *([Ya1], Proposition 2.3) Assume that $v | d_{K/F}$. Then*

(1) *Φ_v is an eigenfunction of $K_0(d_{K/F})$ with eigencharacter χ_v, where*

$$K_0(d_{K/F}) = \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_v) : c \equiv 0 \mod d_{K/F}\}, \quad \chi_v(\gamma) = \chi_v(d).$$

Moreover,

$$\Phi_v(1) = 1, \quad \Phi_v(w) = \chi_v(-1)\epsilon(\chi_v, \psi_v)|d_{K/F}|_v^{\frac{1}{2}}, \quad \Phi_v(n^-(c)) = 0$$

for $n^-(c) = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ with $0 < \mathrm{ord}_v\, c < \mathrm{ord}_v\, d_{K/F}$.

(2) *Its Whittaker function with respect to ψ_v satisfies*

$$W^*_t(1, s, \Phi_v) = \Phi_v(w)(1 + \chi_v(t)|td_{K/F}|_v^s)\mathrm{char}(\mathcal{O}_v)(t),$$
$$W^*_t(w, s, \Phi_v) = \Phi_v(w)^2(1 + \chi_v(t)|td_{K/F}|_v^s)\mathrm{char}(d_{K/F}^{-1}\mathcal{O}_v)(t).$$

(3) *For $t \in \mathcal{O}_F$, $W^*_t(1, 0, \Phi_v) = 0$ if and only if $\chi_v(t) = -1$. In such a case,*

$$W^{*,\prime}_{t,v}(1, 0, \Phi_v) = \Phi_v(w)(\mathrm{ord}_v\, td_{K/F}) \log |\mathfrak{p}_v|.$$

(4) *For $t \in d_{K/F}^{-1}$, $W^*_{t,v}(w, 0, \Phi_v) = 0$ if and only if $\chi_v(t) = -1$. In such a case,*

$$W^{*,\prime}_{t,v}(w, 0, \Phi_v) = \Phi_v(w)^2(\mathrm{ord}_v\, td_{K/F}) \log |\mathfrak{p}_v|.$$

(5) *One has the functional equation*

$$M^*_v(s)\Phi_v(g, s) = \Phi_v(g, -s)\Phi_v(w).$$

Next, we consider the infinite primes ∞ and ∞'. Notice first that $\psi_\infty(x) = \psi_{\mathbb{Q}_\infty}(\frac{x}{\sqrt{D}})$, so the self-dual Haar measure $d_{\psi_\infty} b$ with respect to ψ_∞ is

$$d_{\psi_\infty} b = D^{-\frac{1}{4}} db,$$

where db is the usual Haar measure on $\mathbb{R}$, self-dual respect to $\phi_{\mathbb{Q}_\infty}$. Let Φ_∞ be the weight one section. Then its t-th Whittaker function with respect to ψ_∞ is

$$\begin{aligned} W_{t,\infty}(\tau, s, \Phi_\infty) &= v^{-\frac{1}{2}} \int_{\mathbb{R}} \Phi_{\mathbb{R}}^1(wn(b)g_\tau, s)\psi_{\mathbb{Q}_\infty}(-\tfrac{tb}{\sqrt{D}})d_{\psi_\infty}b \\ &= -v^{-\frac{1}{2}} \int_{\mathbb{R}} \Phi_{\mathbb{R}}^1(w^{-1}n(b)g_\tau, s)\psi_{\mathbb{Q}_\infty}(-\tfrac{tb}{\sqrt{D}})d_{\psi_\infty}b \\ &= -v^{-\frac{1}{2}} D^{-\frac{1}{4}} W_{\frac{t}{\sqrt{D}},\mathbb{R}}(g_\tau, s, \Phi_{\mathbb{R}}^1). \end{aligned}$$

Here $W_{t,\mathbb{R}}(g_\tau, s, \Phi_{\mathbb{R}}^1)$ is the Whittaker function of $\Phi_{\mathbb{R}}^1$ with respect to $\psi_{\mathbb{R}}$, which is computed in [KRY1]. Notice that [KRY1] uses w^{-1} to define local Whittaker functions. Similarly, one has

$$W_{t,\infty'}(\tau', s, \Phi_{\infty'}) = -D^{-\frac{1}{4}} v'^{,-\frac{1}{2}} W_{-\frac{t}{\sqrt{D}},\mathbb{R}}(\tau', s, \Phi_{\mathbb{R}}^1).$$

We set

$$\begin{aligned} W_{t,\infty}^*(\tau, \tau', s) &= W_{t,\infty}^*(\tau, s, \Phi_\infty) W_{t',\infty'}^*(\tau', s, \Phi_{\infty'}) \\ &= D^{-\frac{1}{2}} W_{\frac{t}{\sqrt{D}},\mathbb{R}}^*(\tau, s, \Phi_{\mathbb{R}}^1) W_{-\frac{t'}{\sqrt{D}},\mathbb{R}}^*(\tau', s, \Phi_{\mathbb{R}}^1). \end{aligned} \tag{6.20}$$

Here $W_{t,\mathbb{R}}^*(\tau, s, \Phi_{\mathbb{R}}^1)$ is the $v^{-\frac{1}{2}}$ times of the function $W_{t,\infty}^*(\tau, s)$ in [KRY1]. The lemma and Proposition 2.6 of [KRY1] imply

Lemma 6.5. *Let $t \in F$.*

(1) *When $t, t' > 0$, i.e., $t \gg 0$, one has $W_{t,\infty}^*(\tau, \tau', 0) = 0$, and*

$$W_{t,\infty}^{*,\prime}(\tau, \tau', 0) = -\frac{2}{\sqrt{D}} e\left(\frac{t\tau - t'\tau'}{\sqrt{D}}\right) \beta_1\left(\frac{4\pi|t'|v'}{\sqrt{D}}\right).$$

Here

$$\beta_1(x) = -Ei(-t) = \int_1^\infty e^{-ux}u^{-1}du, \quad x > 0.$$

(2) *When $t \ll 0$ is totally negative, one has $W_{t,\infty}^*(\tau, \tau', 0) = 0$, and*

$$W_{t,\infty}^{*,\prime}(\tau, \tau', 0) = -\frac{2}{\sqrt{D}} e\left(\frac{t\tau - t'\tau'}{\sqrt{D}}\right) \beta_1\left(\frac{4\pi|t|v}{\sqrt{D}}\right).$$

(3) *When $t < 0 < t'$, one has*

$$W_{t,\infty}^{*,\prime}(\tau, \tau', 0) = W_{t,\infty}^*(\tau, \tau', 0) = 0.$$

(4) *When $t > 0 > t'$, one has*

$$W_{t,\infty}^*(\tau, \tau', 0) = -\frac{4}{\sqrt{D}} e\left(\frac{t\tau - t'\tau'}{\sqrt{D}}\right).$$

(5) *When $t = 0$, one has*

$$W_{0,\infty}^*(\tau, \tau', s) = -(vv')^{-\frac{s}{2}} D^{-\frac{1}{2}} L(s, \chi_\infty) L(s, \chi_{\infty'}).$$

(6) *One has the functional equation for* $v = \infty, \infty'$,

$$M_v^*(s)\Phi_v(g, s) = -iD^{-\frac{1}{4}} L(s, \chi_v)\Phi_v(g, -s).$$

Proof of Theorem 6.2. Now we are ready to prove Theorem 6.2. We check the case $t \neq 0$ and leave the constant term to the reader. By Lemma 6.3, one has $\gamma_v^{-1}\Phi_v = \Phi_v$ for $v \nmid d_{K/F}\infty\infty'$. For $v|\mathfrak{d}_c$, the condition in Theorem 6.2 implies $\gamma_v \in K_0(d_{K/F})$, and thus

$$\gamma_v^{-1}\Phi_v = \chi_v(d)\Phi_v.$$

For $v|\mathfrak{d}'_c$, one has $c \in \mathcal{O}_v^*$ and

$$\gamma_v^{-1} = -m(c^{-1})n(-c^{-1}d)wn(-c^{-1}a)$$

and thus

$$\begin{aligned} W_{t,v}^*(1, s, \gamma_v^{-1}\Phi_v) &= \int_{F_v} \Phi_v(wn(b)\gamma_v^{-1})\psi_v(-tb)db \\ &= \chi_v(-c)|c|_v^{s+1} \int_{F_v} \Phi_v(wn(c^2b - cd)w)\psi_v(-tb)db \\ &= \chi_v(-c)\psi_v(-\frac{d}{c}t)W_{\frac{t}{c^2},v}^*(w, s, \Phi_v). \end{aligned}$$

Since $c \in \mathcal{O}_v^*$, Lemma 6.4(2) implies then

$$W_{t,v}^*(1, s, \gamma_v^{-1}\Phi_v) = \chi_v(-c)\psi_v(-\frac{d}{c}t)W_{t,v}^*(w, s, \Phi_v). \tag{6.21}$$

So

$$\begin{aligned} E_t^*(\tau, \tau', s, \gamma_f^{-1}\Phi^K) &= A^{\frac{s+1}{2}} W_{t,\infty}^*(\tau, \tau', s) \prod_{v \nmid d_{K/F}\infty\infty'} W_{t,v}^*(1, s, \Phi_v) \\ &\quad \cdot \prod_{v|\mathfrak{d}_c} \chi_v(d)W_{t,v}^*(1, s, \Phi_v) \prod_{v|\mathfrak{d}'_c} \chi_v(-c)\psi_v(-\frac{d}{c}t)W_{t,v}^*(w, s, \Phi_v). \end{aligned} \tag{6.22}$$

So Lemmas 6.3 and 6.4 imply that $E_t^*(\tau, \tau, 0, \gamma_f^{-1}\Phi^K) = 0$ unless $t \in \mathfrak{d}_c'^{-1}$. We assume $t \in \mathfrak{d}_c'^{-1}$ from now on. These lemmas also imply

$$\prod_{v \nmid d_{K/F}\infty\infty'} W_{t,v}^*(1, 0, \Phi_v) = \rho(td_{K/F}) \tag{6.23}$$

and

$$\begin{aligned} &\prod_{v|\mathfrak{d}_c} \chi_v(d)W_{t,v}^*(1, 0, \Phi_v) \prod_{v|\mathfrak{d}'_c} \chi_v(-c)\psi_v(-\frac{d}{c}t)W_{t,v}^*(w, 0, \Phi_v) \\ &= \delta(t) \prod_{v|d_{K/F}} \Phi_v(w) \prod_{v|\mathfrak{d}_c} \chi_v(d) \prod_{v|\mathfrak{d}'_c} \chi_v(-c)\psi_v(-\frac{d}{c}t)\Phi_v(w) \\ &= \frac{\delta(t)}{\sqrt{N_{F/\mathbb{Q}}d_K/F}} C_t(\gamma). \end{aligned} \tag{6.24}$$

If $t \gg 0$ is totally positive, $W^*_{t,\infty}(\tau, \tau', 0) = 0$ by Lemma 6.5. Moreover, Lemmas 6.3–6.5 and the above calculation imply (recall $A = DN_{F/\mathbb{Q}}d_{K/F}$)

$$\begin{aligned}
&E^{*,\prime}_t(\tau, \tau', 0, \gamma_f^{-1}\Phi^K)\\
&= A^{\frac{1}{2}} W^{*,\prime}_{t,\infty}(\tau, \tau', 0) \prod_{v \nmid d_{K/F}\infty\infty'} W^*_{t,v}(1, 0, \Phi_v)\\
&\quad \cdot \prod_{v|\mathfrak{d}_c} \chi_v(d) W^*_{t,v}(1, 0, \Phi_v) \prod_{v|\mathfrak{d}'_c} \chi_v(-c)\psi_v(-\frac{d}{c}t) W^*_{t,v}(w, 0, \Phi_v)\\
&= -2\delta(t)\rho(td_{K/F})C_t(\gamma)\beta_1\left(\frac{4\pi|t'|v'}{\sqrt{D}}\right) e\left(\operatorname{tr}\frac{t\tau}{\sqrt{D}}\right)
\end{aligned}$$

as claimed. The case $t \ll 0$ is the same.

Next, if $t > 0 > t'$ but there is a prime v_0 of F inert in K/F such that $\chi_{v_0}(t) = -1$, then $W^*_{t,v_0}(1, 0, \Phi_v) = 0$ and $\operatorname{ord}_{v_0} t > 0$ is odd. This implies $\rho_{v_0}(t\mathfrak{p}_{v_0}^{-1}) = 1$, and

$$\prod_{v \nmid v_0 d_{K/F}\infty\infty'} W^*_{t,v}(1, 0, \Phi_v) = \rho(td_{K/F}\mathfrak{p}_{v_0}^{-1}).$$

Now Lemmas 6.3–6.5 and the above calculation give again

$$\begin{aligned}
&E^{*,\prime}_t(\tau, \tau', 0, \gamma_f^{-1}\Phi^K)\\
&= A^{\frac{1}{2}} W^{*,\prime}_{t,v_0}(1, 0, \Phi_{v_0}) \cdot \text{other values}\\
&= -2\delta(t)\rho(td_{K/F}\mathfrak{p}_{v_0}^{-1})(\operatorname{ord}_{v_0} t + 1)C_t(\gamma) e\left(\operatorname{tr}\frac{t\tau}{\sqrt{D}}\right)
\end{aligned}$$

as claimed. The same calculation also verifies the case when $t > 0 > t'$ and there is a $v_0|d_{K/F}$ such that $\chi_{v_0}(t) = -1$. In all other cases, $E^*_t(\tau, \tau', s, \gamma_f^{-1}\Phi^K)$ has at least a double zero at $s = 0$ and thus the central derivative is zero. □

7. Diagonal restriction

In this section we study the restriction to the diagonal of $E^{*,\prime}(\tau, \tau', 0, \Phi^K)$. We first determine the level and the Nebentypus character precisely, and then compute the Fourier expansion and the image under the Fricke involution.

Setting $\tau' = \tau$, one sees from Theorem 6.2 that $E^{*,\prime}(\tau, \tau, 0, \Phi^K)$ is a non-holomorphic modular form of weight 2 for the group $\Gamma_0(N)$ with Nebentypus character $\tilde{\chi}$. Here $N\mathbb{Z} = d_{K/F} \cap \mathbb{Z}$, and $\tilde{\chi}$ is the composition of χ and the embedding $(\mathbb{Z}/N)^* \hookrightarrow (\mathcal{O}_F/d_{K/F})^*$. The following lemma ([Ya1], Lemma 5.6) makes N and $\tilde{\chi}$ more explicit in some cases.

Lemma 7.1. *Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, and let $\Delta \in \mathcal{O}_F$ be an odd totally negative primitive element (without any rational prime factor) such that Δ is a square modulo 4. Let $K = F(\sqrt{\Delta})$ and $\tilde{F} = \mathbb{Q}(\sqrt{\Delta\Delta'})$. Then*

(1) *$d_{K/F} \cap \mathbb{Z} = d_{\tilde{F}}\mathbb{Z}$, and the character $\tilde{\chi}$ just defined above is the quadratic Dirichlet character associated to $\tilde{F}/\mathbb{Q}$.*

(2) *$N_{F/\mathbb{Q}}d_{K/F} = d_{\tilde{F}}$ is odd.*

In particular, under the condition of the lemma, every prime factor of $d_{\tilde{F}}$ is split in F, and the diagonal restriction $E^{*,\prime}(\tau,\tau,0,\Phi^K)$ is a non-holomorphic modular form of weight 2, level $N = d_{\tilde{F}}$, and Nebentypus character $(\frac{\cdot}{N})$.

From now on, we assume in addition that:

$$d_{K/F} = \mathfrak{p}, \text{ and } d_{\tilde{F}} = p = N_{F/\mathbb{Q}}\mathfrak{p} \text{ are prime.} \tag{7.1}$$

So the situation is more general than in Theorem 1.4, where we also assume that $F/\mathbb{Q}$ is only ramified at one prime. We hope that this will be useful for possible generalizations (see Remark 9.2).

For the proof of Theorem 1.4, it is more convenient to consider the modular form

$$\tilde{f}(\tau) = \frac{1}{\sqrt{p}} E^{*,\prime}(\tau,\tau,0,\Phi^K)|_2 W_p, \tag{7.2}$$

where $W_p = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right) = w\mathrm{diag}(p,1)$. Since W_p normalizes $\Gamma_0(p)$, $\tilde{f}(\tau)$ is also a non-holomorphic modular form of weight 2, level p, and Nebentypus character $(\frac{\cdot}{p})$. It is easy to check

$$\begin{aligned} \tilde{f}(\tau) &= p^{\frac{1}{2}} E^{*,\prime}(p\tau, p\tau, 0, \Phi^K)|_2 w \\ &= p^{\frac{1}{2}} E^{*,\prime}(p\tau, p\tau, 0, w_f^{-1}\Phi^K). \end{aligned}$$

Theorem 7.2. *Let $F = \mathbb{Q}(\sqrt{D})$ and let $K = F(\sqrt{\Delta})$ be a CM quadratic extension of F satisfying the condition in Lemma 7.1 and the equation (7.1). Then one has for $\tau = u + iv \in \mathbb{H}$*

$$\tilde{f}(\tau) = \sum_{m\in\mathbb{Z}} a_m(v) e(m\tau),$$

where $a_m(v)$ are given as follows.

(1) *The constant term is*

$$a_0(v) = 2\Lambda(0,\chi)\log pv + 2\Lambda'(0,\chi) - 8\sum_{n=1}^{\infty} \rho(n\mathcal{O}_F)\beta_1(\frac{4\pi npv}{\sqrt{D}}).$$

(2) *For $m \neq 0$, one has $a_m(v) = -4b_m - 4c_m(v)$, where*

$$b_m = \sum_{\substack{t\in d_{K/F}^{-1} \\ t>0>t' \\ t-t'=\frac{m}{p}\sqrt{D}}} B_t = \sum_{\substack{t=\frac{n+m\sqrt{D}}{2p}\in d_{K/F}^{-1} \\ |n|<m\sqrt{D}}} B_t \tag{7.3}$$

with

$$B_t = (\mathrm{ord}_{\mathfrak{l}}\, t + 1)\rho(td_{K/F}\mathfrak{l}^{-1})\log|\mathfrak{l}|, \tag{7.4}$$

where $\mathfrak{l}$ is a prime ideal of F non-split in K such that $\chi_{\mathfrak{l}}(t) = -1$. Moreover,

$$c_m(v) = \sum_{t=\frac{n+m\sqrt{D}}{2p}\in d_{K/F}^{-1,+}} \rho(td_{K/F})\beta_1(\frac{4\pi t'pv}{\sqrt{D}}) + \sum_{t=\frac{n+m\sqrt{D}}{2p}\in (d'_{K/F})^{-1,+}} \rho(t'd_{K/F})\beta_1(\frac{4\pi t'pv}{\sqrt{D}}). \tag{7.5}$$

Remark 7.3. Before the proof, we remark that B_t and b_m are well-defined and the same as in (1.6) and (1.8) except that they are with respect to K and F here. Indeed, since $t > 0 > t'$, the identity

$$1 = \chi(t) = -\prod_{\mathfrak{l}<\infty} \chi_{\mathfrak{l}}(t) = -\prod_{\substack{\mathfrak{l}<\infty \\ \mathfrak{l} \text{ non-split}}} \chi_{\mathfrak{l}}(t)$$

implies that $\chi_{\mathfrak{l}}(t) = -1$ for an odd number of non-split prime ideals $\mathfrak{l}$ of F. However, $\rho(td_{K/F}\mathfrak{l}^{-1}) = 0$ if there is another $\mathfrak{l}' \neq \mathfrak{l}$ such that $\chi_{\mathfrak{l}'}(t) = -1$ and $\mathfrak{l}'$ is inert in K/F by (6.13) and (6.14) (recall that K/F is only ramified at one prime). So B_t is well-defined and is zero unless there is a unique non-split prime ideal $\mathfrak{l}$ of F such that $\chi_{\mathfrak{l}}(t) = -1$. This also shows that B_t is the same as the one defined in the introduction.

Proof of Theorem 7.2. By Theorem 6.2, one has

$$a_m(v) = \sum_{t \in d_{K/F}^{-1}, t-t' = \frac{m}{p}\sqrt{D}} A_t(pv, pv).$$

First notice that since $d_{K/F} = \mathfrak{p}$ is a prime, one has

$$\delta(t)\rho(td_{K/F}) = 2\rho(td_{K/F}) \quad \text{for } t \gg 0. \tag{7.6}$$

For $m = 0$, the sum is then over $t \in \mathbb{Q} \cap d_{K/F}^{-1} = \mathbb{Z}$. So

$$\begin{aligned} a_0(v) &= \log(pv)^2\Lambda(0,\chi) + 2\Lambda'(0,\chi) + \sum_{n=1}^{\infty} A_n(pv,pv) + \sum_{n=1}^{\infty} A_{-n}(pv,pv) \\ &= 2\Lambda(0,\chi)\log pv + 2\Lambda'(0,\chi) - 8\sum_{n=1}^{\infty} \rho(n\mathcal{O}_F)\beta_1\left(\frac{4\pi npv}{\sqrt{D}}\right). \end{aligned}$$

Here we have used the fact that $\rho(\mathfrak{a}d_{K/F}) = \rho(\mathfrak{a})$ for any integral ideal $\mathfrak{a}$ of F.

For $m \neq 0$, Theorem 6.2 and (7.6) imply

$$\begin{aligned} a_m(v) = -4 &\sum_{\substack{t \in d_{K/F}^{-1,+} \\ t-t' = \frac{m}{p}\sqrt{D}}} \rho(td_{K/F})\beta_1\left(\frac{4\pi t'pv}{\sqrt{D}}\right) \\ &- 4\sum_{\substack{t \in d_{K/F}^{-1}, t \ll 0 \\ t-t' = \frac{m}{p}\sqrt{D}}} \rho(td_{K/F})\beta_1\left(\frac{4\pi|t|pv}{\sqrt{D}}\right) \\ &- 4\sum_{\substack{t \in d_{K/F}^{-1}, t>0>t' \\ t-t' = \frac{m}{p}\sqrt{D}}} A_t(pv,pv). \end{aligned}$$

A substitution $t \mapsto -t'$ in the second sum implies that the first two sums give $-4c_m(v)$ in (7.5). On the other hand, Theorem 6.2(4)(5) imply $A_t(pv,pv) = b_m$ in the third sum.

Here we remark that

$$(\mathrm{ord}_{\mathfrak{p}} t + 1)\rho(td_{K/F}) \log p = (\mathrm{ord}_{\mathfrak{p}} t + 1)\rho(td_{K/F}\mathfrak{p}^{-1}) \log |\mathfrak{p}|.$$

So $a_m(v) = -4b_m - 4c_m(v)$ as claimed. This proves the theorem. □

To carry out the holomorphic projection, we also need to know the behavior of $\tilde{f}$ at the cusp $0 = w(\infty)$. Since $W_p w = \mathrm{diag}(-1, -p)$, one has

$$\begin{aligned}\tilde{f}|_2 w(\tau) &= \frac{1}{\sqrt{p}} E^{*,\prime}(\tau, \tau, 0, \Phi^K)|_2 \mathrm{diag}(-1, -p) \\ &= p^{-\frac{3}{2}} E^{*,\prime}\left(\frac{\tau}{p}, \frac{\tau}{p}, 0, \Phi^K\right).\end{aligned}$$

So we have by Theorem 6.2:

Theorem 7.4. *Under the assumption (7.1), one has*

$$\tilde{f}|_2 w(\tau) = p^{-\frac{3}{2}} \sum_{m \in \mathbb{Z}} a_m^0(v) e\left(\frac{m\tau}{p}\right),$$

where $a_m^0(v)$ are given as follows.

(1) *The constant term is*

$$a_0^0(v) = 2\Lambda(0, \chi) \log \frac{v}{p} + 2\Lambda'(0, \chi) - 8 \sum_{n=1}^{\infty} \rho(n\mathcal{O}_F)\beta_1\left(\frac{4\pi n v}{p\sqrt{D}}\right).$$

(2) *For $m \neq 0$, one has $a_m^0(v) = -4b_m^0 - 4c_m^0(v)$, where*

$$b_m^0 = \sum_{\substack{t \in \mathcal{O}_F \\ t > 0 > t' \\ t - t' = m\sqrt{D}}} B_t \tag{7.7}$$

and

$$c_m^0(v) = \sum_{\substack{t \in \mathcal{O}_F^+ \\ t - t' = m\sqrt{D}}} (\rho(t\mathcal{O}_F) + \rho(t'\mathcal{O}_F))\beta_1\left(\frac{4\pi t' v}{p\sqrt{D}}\right). \tag{7.8}$$

8. Holomorphic projection

Let the notation be as in Section 7, and retain the assumption (7.1). Following Sturm [St], and Gross and Zagier [GZ2], one obtains a holomorphic modular form of weight 2, level p, and Nebentypus character $\epsilon_p = (\frac{\cdot}{p})$ by computing the holomorphic projection of $\tilde{f}$. This is the unique holomorphic cusp form f of the same type as $\tilde{f}$ with the property that

$$\langle f, g \rangle = \langle \tilde{f}, g \rangle \tag{8.1}$$

for every holomorphic cusp form g of the same type. This provides the third and last ingredient in the proof of Theorem 1.4: An explicit holomorphic cusp form which connects the CM value of the automorphic Green function of T_m and the arithmetic of $\tilde{K}$ (Theorem 8.1).

Theorem 8.1. *Let $\tilde{f}$ be the non-holomorphic form of weight 2, level p, and Nebentypus character ϵ_p defined in (7.2), and let $f = \sum_{m\ge1} a_m q^m$ be the holomorphic cusp form of weight 2, level p, and Nebentypus character ϵ_p defined by the holomorphic projection of $\tilde{f}$. Then*

$$a_m = -4b_m - 4c_m - 2C(m,0)\Lambda(0,\chi_{K/F})\alpha(K/F),$$

where $b_m = b_m(K/F)$ is given in Theorem 7.2, $C(m,0)$ and $\mathcal{L}_m$ are given in Section 2,

$$\alpha(K/F) = \left(\Gamma'(1) + \frac{\Lambda'(0,\chi_{K/F})}{\Lambda(0,\chi_{K/F})} - \log 4\pi\right), \tag{8.2}$$

and $c_m = c_m(K/F)$ is given by

$$\begin{aligned} c_m = \lim_{s\to1}\{2 \sum_{t=\frac{n\pm m\sqrt{D}}{2p}\in d_{K/F}^{-1,+}} \rho(td_{K/F})Q_{s-1}(\frac{n}{m\sqrt{D}}) \\ + \frac{\Lambda(0,\chi_{K/F})C(m,0)}{2(s-1)} - \Lambda(0,\chi_{K/F})\mathcal{L}_m\}. \end{aligned}$$

Proof. It is easy to see from Theorem 2.2 that

$$E_2^+(\tau,s) = 1 + s\log(pv) + O(v^{-1}\log v), \tag{8.3}$$

$$E_2^+|_2 w(\tau,s) = p^{-\frac{3}{2}}(1 + s\log\tfrac{v}{p} + O(v^{-1}\log v)) \tag{8.4}$$

as $v\to\infty$ and $\operatorname{Re} s > 0$. Set

$$\mathcal{E}(\tau,s) = 2\Lambda(s,\chi_{K/F})E_2^+(\tau,s) = \tilde{C}(0,s,v) + \sum_{m\neq1}\tilde{C}(m,s)\mathcal{W}_s(4\pi m v)e(mu)$$

with $\tilde{C}(m,s) = 2\Lambda(s,\chi_{K/F})C(m,s)$. Then

$$\mathcal{E}'(\tau,0) = 2\Lambda'(0,\chi_{K/F}) + 2\Lambda(0,\chi_{K/F})\log pv + O(v^{-1}\log v),$$
$$\mathcal{E}'|_2 w(\tau,0) = p^{-\frac{3}{2}}(2\Lambda'(0,\chi_{K/F}) + 2\Lambda(0,\chi_{K/F})\log\tfrac{v}{p}) + O(v^{-1}\log v).$$

So Theorems 7.2 and 7.4 imply that

$$\tilde{f}_1(\tau) = \tilde{f} - \mathcal{E}'(\tau,0)$$

has the following property:

$$\tilde{f}_1(\tau) = O(v^{-1}\log v), \quad \text{and } \tilde{f}_1|_2 w(\tau) = O(v^{-1}\log v), \quad \text{as } v\to\infty. \tag{8.5}$$

Notice also that $\tilde{f}_1$ and $\tilde{f}$ have the same holomorphic projection. Now recall that the Poincare series for $m > 0$,

$$P^s_{m,\epsilon_p}(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} \epsilon_p(d)e^{2\pi i m\tau}v^{s-1}|_2\gamma,$$

is absolutely convergent for $\operatorname{Re} s > 1$ and has holomorphic continuation to $s = 1$, which gives a holomorphic modular form of weight 2, level p, Nebentypus character ϵ_p. So by the definition of holomorphic projection and (8.5), one has for $m > 0$

$$\lim_{s\to 1}\langle f, P^{\bar{s}}_{m,\epsilon_p}\rangle = \lim_{s\to 1}\langle \tilde{f}_1, P^{\bar{s}}_{m,\epsilon_p}\rangle,$$

that is,

$$a_m = \lim_{s\to 1}\left(\tilde{a}_m(s) - \tilde{C}'(m,0)I_1(s) - \tilde{C}(m,0)I_2(s)\right), \tag{8.6}$$

where

$$\tilde{a}_m(s) = \frac{(4\pi m)^s}{\Gamma(s)}\int_0^\infty a_m(v)e^{-4\pi m v}v^s\frac{dv}{v},$$
$$I_1(s) = \frac{(4\pi m)^s}{\Gamma(s)}\int_0^\infty \mathcal{W}_0(4\pi m v)e^{-2\pi m v}v^s\frac{dv}{v},$$
$$I_2(s) = \frac{(4\pi m)^s}{\Gamma(s)}\int_0^\infty \mathcal{W}_0'(4\pi m v)e^{-2\pi m v}v^s\frac{dv}{v}.$$

By ([EMOT] p. 216 (16)), one sees that

$$I_1(s) = 1, \quad \text{and} \quad I_2(s) = \frac{1}{s-1}. \tag{8.7}$$

Next, one has by Theorem 7.2

$$\tilde{a}_m(s) = -4b_m - 4\sum_{t=\frac{n+m\sqrt{D}}{2p}\in d_{K/F}^{-1,+}}\rho(td_{K/F})\alpha_m\left(\frac{t'p}{m\sqrt{D}}, s\right)$$
$$-4\sum_{t=\frac{n+m\sqrt{D}}{2p}\in(d'_{K/F})^{-1,+}}\rho(t'd_{K/F})\alpha_m\left(\frac{t'p}{m\sqrt{D}}, s\right).$$

Here for $c > 0$

$$\alpha_m(c,s) = \frac{(4\pi m)^s}{\Gamma(s)}\int_0^\infty \beta_1(4\pi m c v)e^{-4\pi m v}v^s\frac{dv}{v}$$
$$= \frac{(4\pi m)^s}{\Gamma(s)}\int_1^\infty\frac{dr}{r}\int_0^\infty e^{-4\pi m v(1+cr)}v^s dv$$
$$= \int_1^\infty\frac{dr}{r(1+cr)^s}.$$

The integral is studied in page 218 of [GZ1] and satisfies

$$\alpha_m(c,s) = \frac{2\Gamma(2s)}{\Gamma(s)\Gamma(s+1)}Q_{s-1}(1+2c) + \mathrm{err}(c,s)$$

for some error function $\mathrm{err}(c,s)$ satisfying

$$\mathrm{err}(c,1) = 0, \tag{8.8}$$

$$\mathrm{err}(c,s) = O\left(\tfrac{1}{c^{1+s}}\right) \quad \text{when } c\to\infty. \tag{8.9}$$

One checks that $1+\frac{2t'p}{m\sqrt{D}}=\frac{n}{m\sqrt{D}}$. So

$$\begin{aligned}\tilde{a}_m(s) &= -4b_m - \frac{8\Gamma(2s)}{\Gamma(s)\Gamma(s+1)} \sum_{t=\frac{n+m\sqrt{D}}{2p}\in d_{K/F}^{-1,+}} \rho(td_{K/F})Q_{s-1}(\frac{n}{m\sqrt{D}}) \\ &\quad - \frac{8\Gamma(2s)}{\Gamma(s)\Gamma(s+1)} \sum_{t=\frac{n+m\sqrt{D}}{2p}\in (d'_{K/F})^{-1,+}} \rho(t'd_{K/F})Q_{s-1}(\frac{n}{m\sqrt{D}}) + \mathrm{Err}(s) \\ &= -4b_m - \frac{8\Gamma(2s)}{\Gamma(s)\Gamma(s+1)} \sum_{t=\frac{n\pm m\sqrt{D}}{2p}\in d_{K/F}^{-1,+}} \rho(td_{K/F})Q_{s-1}(\frac{n}{m\sqrt{D}}) + \mathrm{Err}(s).\end{aligned}$$

Combining this with (8.6) and (8.7), one has

$$\begin{aligned}a_m &= -4b_m - \tilde{C}'(m,0) \\ &\quad - \lim_{s\to 1}\left[8 \sum_{t=\frac{n\pm m\sqrt{D}}{2p}\in d_{K/F}^{-1,+}} \rho(td_{K/F})Q_{s-1}(\frac{n}{m\sqrt{D}}) + \frac{\tilde{C}(m,0)}{s-1}\frac{\Gamma(s)\Gamma(s+1)}{\Gamma(2s)}\right].\end{aligned}$$

A simple calculation, using the fact $\frac{\Gamma(s)\Gamma(s+1)}{\Gamma(2s)}=1-(s-1)+O((s-1)^2)$, reveals

$$a_m = -4b_m - 4c_m - 2C(m,0)\Lambda(0,\chi)\alpha(K/F).$$

This finishes the proof of the Theorem. □

Remark 8.2. It is easy to check by Theorems 7.2 and 7.4 that $\tilde{f}|_2W_p=\sqrt{p}\tilde{f}|_2U_p$ for the usual U_p operator and the modular form $\tilde{f}$ in Section 7. This implies that $f|_2W_p=\sqrt{p}f|_2U_p$ for the holomorphic form f in Theorem 8.1. Now [BB], Lemma 3 implies that $f\in S_2^+(p,\epsilon_p)$ is in the *plus* space. In particular, we have

$$b_m + c_m = 0 \tag{8.10}$$

whenever $\epsilon_p(m)=-1$. It is amusing to compare this identity with Theorem 1.4, which deals with the situation $\epsilon_p(m)\neq -1$.

9. Proof of Theorem 1.4 and possible generalizations

Now we are ready to prove Theorem 1.4. For the convenience of the reader, we restate the theorem here as follows:

Theorem 9.1. *Let (K,Φ) be a non-biquadratic quartic CM number field with CM type Φ and maximal totally real subfield $F=\mathbb{Q}(\sqrt{p})$ such that $p\equiv 1 \mod 4$ is prime and*

$$d_{K/F}\cap\mathbb{Z}=q\mathbb{Z},\quad N_{F/\mathbb{Q}}d_{K/F}=q,\quad q\equiv 1 \pmod 4 \text{ is prime.} \tag{9.1}$$

Let $\tilde{K}$ be the reflex field of (K,Φ) with maximal totally real subfield $\tilde{F}=\mathbb{Q}(\sqrt{q})$. Let Ψ be a normalized integral Hilbert modular form for $\Gamma=\mathrm{SL}_2(\mathcal{O}_F)$ of weight $c(0)$ such that

$$\mathrm{div}(\Psi)=\sum_{m>0}\tilde{c}(-m)T_m,$$

with integral coefficients $\tilde{c}(-m) \in \mathbb{Z}$. *Then*

$$\log \|\Psi(\mathcal{CM}(K))\|_{\mathrm{Pet}} = \frac{W_{\tilde{K}}}{4} \sum_{m>0} \tilde{c}(-m) b_m - \frac{W_{\tilde{K}}}{4} c(0) \alpha(\tilde{K}/\tilde{F})$$

with

$$\alpha(\tilde{K}/\tilde{F}) = \Lambda(0,\chi) \left(\Gamma'(1) + \frac{\Lambda'(0,\chi)}{\Lambda(0,\chi)} - \log 4\pi \right).$$

Here $b_m = b_m(\tilde{K}/\tilde{F})$ *and* $W_{\tilde{K}}$ *are given in Theorem 1.1 (or Theorem 7.2),* $\|\Psi\|_{\mathrm{Pet}}$ *denotes the Petersson metric of* Ψ *normalized as in* (2.27)*,* χ *is the quadratic Hecke character of* $\tilde{F}$ *associated to* $\tilde{K}/\tilde{F}$*, and* $\Lambda(s,\chi)$ *is the completed L-function of* χ*.*

Proof. First, one has by the definition of G_m in Section 2, Theorem 5.1, and Corollary 5.4:

$$\begin{aligned}
2G_m(\mathcal{CM}(K)) &= \Phi_m(\mathcal{CM}(K)) - \#\mathcal{CM}(K)\mathcal{L}_m \\
&= \lim_{s\to 1} \left(\Phi_m(\mathcal{CM}(K), s) + \frac{\#\mathcal{CM}(K) C(m,0)}{2(s-1)} - \#\mathcal{CM}(K)\mathcal{L}_m \right) \\
&= \frac{W_{\tilde{K}}}{2} \lim_{s\to 1} \Bigg[2 \sum_{\mu = \frac{n - m\sqrt{q}}{2p} \in d_{\tilde{K}/\tilde{F}}^{-1,+}} Q_{s-1}\Big(\frac{n}{m\sqrt{q}}\Big) \rho(\mu d_{\tilde{K}/\tilde{F}}) \\
&\qquad + 2 \sum_{\mu = \frac{n + m\sqrt{q}}{2p} \in d_{\tilde{K}/\tilde{F}}^{-1,+}} Q_{s-1}\Big(\frac{n}{m\sqrt{q}}\Big) \rho(\mu d_{\tilde{K}/\tilde{F}}) \\
&\qquad + \frac{2\#\mathcal{CM}(K)}{W_{\tilde{K}}} \frac{C(m,0)}{2(s-1)} - \frac{2\#\mathcal{CM}(K)}{W_{\tilde{K}}} \mathcal{L}_m \Bigg] \\
&= \frac{W_{\tilde{K}}}{2} c_m + \frac{W_{\tilde{K}}}{2} \lim_{s\to 1} \left(\frac{C(m,0)}{2(s-1)} - \mathcal{L}_m \right) \left(\frac{2\#\mathcal{CM}(K)}{W_{\tilde{K}}} - \Lambda(0,\chi) \right).
\end{aligned}$$

Here $c_m = c_m(\tilde{K}/\tilde{F})$ is the number in Theorem 8.1 but related to $\tilde{K}/\tilde{F}$, and $\chi = \chi_{\tilde{K}/\tilde{F}}$. So one has to have

$$\Lambda(0,\chi) = \frac{2\#\mathcal{CM}(K)}{W_{\tilde{K}}}, \tag{9.2}$$

and

$$4G_m(\mathcal{CM}(K)) = W_{\tilde{K}} c_m. \tag{9.3}$$

On the other hand, Proposition 2.5 and Theorem 8.1 (applying to $\tilde{K}/\tilde{F}$) assert

$$0 = \sum_{m>0} \tilde{c}(-m) a_m,$$

that is,

$$-\sum_{m>0} \tilde{c}(-m) c_m = \sum_{m>0} \tilde{c}(-m) b_m + \frac{1}{2} \alpha(\tilde{K}/\tilde{F}) \sum_{m>0} \tilde{c}(-m) C(m,0).$$

Here $a_m = a_m(\tilde{K}/\tilde{F})$ and $b_m = b_m(\tilde{K}/\tilde{F})$ are the numbers in Theorem 8.1 but related to $\tilde{K}/\tilde{F}$. Therefore,

$$\begin{aligned}\log \|\Psi(\mathcal{CM}(K))\|_{\mathrm{Pet}} &= -\sum_{m>0} \tilde{c}(-m) G_m(\mathcal{CM}(K)) \\ &= -\frac{W_{\tilde{K}}}{4} \sum_{m>0} \tilde{c}(-m) c_m \\ &= \frac{W_{\tilde{K}}}{4} \sum_{m>0} \tilde{c}(-m) b_m + \frac{W_{\tilde{K}}}{8} \alpha(\tilde{K}/\tilde{F}) \sum_{m>0} \tilde{c}(-m) C(m,0).\end{aligned}$$

By Proposition 2.6, one has

$$\sum_{m>0} \tilde{c}(-m) C(m,0) = -2c(0).$$

This completes the proof of the Theorem 9.1 (i.e., Theorem 1.4). $\square$

Theorem 1.1 is the special case of Theorem 9.1 where $c(0) = 0$.

Remark 9.2. A natural question is whether our main result just proved can be extended to a general non-biquadratic CM number field. If we still require p to be prime and relax q to be odd and square-free, i.e., $K = F(\sqrt{\Delta})$ with Δ an odd totally negative primitive element which is a square modulo 4 (Lemma 7.1), then every step needed to prove the main theorem is true and already in this paper except Lemma 5.3, which is not true anymore. We expect that there is still a constant κ depending only on K and $\tilde{K}$ such that

$$C(\mu) = \kappa \rho_{\tilde{K}/\tilde{F}}(\mu d_{\tilde{K}/\tilde{F}}) \tag{9.4}$$

for every totally positive $\mu \in d_{\tilde{K}/\tilde{F}}^{-1}$, where $C(\mu)$ is given by (5.7). When q is prime, we proved in Section 5 that $\kappa = W_{\tilde{K}}$. If (9.4) is true in general, then Theorem 1.4 holds in this slightly more general case with the minimal change of a factor. If Δ is not primitive, or not a square modulo 4, the Eisenstein series and thus the modular form we constructed might have higher level than p. One might get down to level p by taking the trace. This would yield a similar result if (9.4) holds. The level at 2 seems to be complicated in general.

Finally, if we also relax p to be a general fundamental discriminant, the Borcherds lifting theory is only stated in terms of vector valued modular forms [Bo1]. However, the results of [BB] will probably generalize to this case, yielding a smooth description of the lifting in terms of scalar valued modular forms. The automorphic Green functions are already constructed in a rather general setting in [Br1, Br2]. In that way, Section 2 could be generalized. The holomorphic projection in Section 8 is more complicated, too, but could probably be done by looking at all Atkin-Lehner involutions. In that way the whole argument of the paper would carry over except for Lemma 5.3. Here the same problems arise as discussed above. It would be very interesting to see any of these generalizations worked out.

Remark 9.3 (Application to Siegel modular forms). Let $\mathcal{H}_2 = \{Z \in \mathrm{Mat}_2(\mathbb{C});\ Z = {}^tZ, \mathrm{Im}(Z) > 0\}$ be the Siegel upper half plane of genus two. The Siegel modular group

$\Gamma_2 = \mathrm{Sp}_2(\mathbb{Z}) \subset \mathrm{GL}_4(\mathbb{Z})$ acts on $\mathcal{H}_2$, and the quotient $X_2 = \Gamma_2\backslash\mathcal{H}_2$ is a normal quasi-projective algebraic variety over $\mathbb{C}$. It can be viewed as the complex points of the coarse moduli space of principally polarized abelian surfaces. Forgetting the real multiplication induces a morphism φ_D from the Hilbert modular surface X associated to the real quadratic field $F = \mathbb{Q}(\sqrt{D})$ (where $D \equiv 1 \pmod 4$ is a positive fundamental discriminant) to X_2. The image $G_D \subset X_2$ of X is known as the *Humbert surface* of discriminant D. It turns out that φ_D induces a birational morphism $X^{sym} \to G_D$, where X^{sym} is the symmetric Hilbert modular surface associated to F. Moreover, one can define G_D (for any positive integer $D \equiv 1 \pmod 4$) by means of equations on $\mathcal{H}_2$ analogously to (2.18). For details we refer to [Ge] Chapter IX, [Ru] Section 4, [Fr] Chapter 3.

As for Hilbert modular surfaces, there is a Borcherds lift from weakly holomorphic modular forms for $\Gamma_0(4)$ of weight $-1/2$ to meromorphic Siegel modular forms for Γ_2 whose divisor is supported on Humbert surfaces. This follows from [Bo1], using the exceptional isomorphism relating $\mathrm{O}(2,3)$ and Sp_2. See also [GN] for an intrinsic description in terms of Sp_2. It turns out that for every G_D, there is a (up to multiplication by ± 1 unique) normalized integral Siegel Borcherds product $\Psi_{D,Siegel}$ whose divisor is equal to G_D. (Notice that $\Psi_{D,Siegel}$ may have a character of order 2.) Some of these functions were known before. For instance $\Psi_{1,Siegel}^2$ is up to a power of 2 the Siegel cusp form χ_{10} of weight 10 constructed by Igusa as a product of theta functions, and $\Psi_{4,Siegel}$ is up to a power of 2 the Igusa cusp form χ_{35} of weight 35.

Moreover, it is known that the pullback to the Hilbert modular surface X of a Humbert surface $G_{D'}$ under the morphism φ_D is a linear combination of Hirzebruch-Zagier divisors. More precisely, we have:

$$\varphi_D^*(G_{D'}) = \sum_{\substack{x\in\mathbb{Z}_{\geq 0} \\ x<\sqrt{DD'} \\ x^2\equiv DD'\ (4)}} T_{(DD'-x^2)/4}, \tag{9.5}$$

see [Fr] Theorem 3.3.5. Therefore, by Theorem 2.8, the pullback to X of the Borcherds product $\Psi_{D',Siegel}$, is a Hilbert Borcherds product with divisor (9.5). Consequently, the CM values corresponding to quartic CM fields satisfying the conditions of Theorem 9.1 of all Siegel Borcherds products can be computed by means of Theorem 9.1. It will be interesting to compare such values with the results of [GL].

10. Examples

Here we give some examples of Borcherds products, and express them as Doi-Naganuma lifts. In that way, the formula in Theorem 1.1 can be checked numerically.

Recall that p is a prime congruent to 1 modulo 4. By a result due to Hecke [He] the dimension of $S_2^+(p,\epsilon_p)$ is equal to $[\frac{p-5}{24}]$. In particular there exist three such primes for which $S_2^+(p,\epsilon_p)$ is trivial, namely $p = 5, 13, 17$. In these cases $W_0^+(p,\epsilon_p)$ is a free module of rank $\frac{p+1}{2}$ over the ring $\mathbb{C}[j(p\tau)]$. Therefore it is not hard to compute explicit bases. For any $m \in \mathbb{N}$ with $\epsilon_p(m) \neq -1$ there is a unique $f_m = \sum_{n\geq -m} c_m(n)q^n \in W_0^+(p,\epsilon_p)$ whose

Fourier expansion starts with

$$f_m = \begin{cases} q^{-m} + c_m(0) + O(q), & \text{if } p \nmid m, \\ \frac{1}{2}q^{-m} + c_m(0) + O(q), & \text{if } p \mid m. \end{cases}$$

The f_m ($m \in \mathbb{N}$) form a base of the space $W_0^+(p, \epsilon_p)$. The Borcherds lift Ψ_m of f_m is a Hilbert modular form for Γ of weight $c_m(0) = -C(m,0)/2$ with divisor T_m. Here $C(m,0)$ denotes the m-th coefficient of the Eisenstein series $E_2^+(\tau, 0)$ as before.

To check the formula for the CM-values of Borcherds products, one wants to evaluate at CM-points. Unfortunately, by Theorem 2.4 the infinite product expansion of any Borcherds lift only converges near the cusp ∞ (respectively near other cusps if one considers the corresponding Fourier expansions). The CM-points usually do not lie in this domain of convergence. Therefore one has to find a different expression for the Borcherds product one wants to evaluate. In some cases this can be done as follows.

Recall that the Doi-Naganuma lift is a $\mathbb{C}$-linear map from $M_k(p, \epsilon_p)$ to the space $M_k(\Gamma)$ of holomorphic Hilbert modular forms of weight k for the group Γ [DN], [Na], [Za]. It takes cusp forms to the subspace $S_k(\Gamma)$ of cusp forms in $M_k(\Gamma)$. It is injective on $M_k^+(p, \epsilon_p)$ and vanishes identically on $M_k^-(p, \epsilon_p)$.

If $g = \sum_{n \geq 0} b(n) q^n$ is an element of $M_k^+(p, \epsilon_p)$, then its Doi-Naganuma lift is given by

$$\mathrm{DN}(g)(z_1, z_2) = -\frac{B_k}{k} b(0) + \sum_{\substack{\nu \in \partial_F^{-1} \\ \nu \gg 0}} \sum_{d \mid \nu} d^{k-1} \tilde{b}\left(\tfrac{p\nu\nu'}{d^2}\right) q_1^{\nu} q_2^{\nu'}. \tag{10.1}$$

Here B_k denotes the k-th Bernoulli number and the latter sum runs through all positive integers d for which $\nu/d \in \partial_F^{-1}$. If $g \in S_k^+(p, \epsilon_p)$, then $\mathrm{DN}(g)$ is in $S_k(\Gamma)$ by [Za] §5. For general g it follows by [Bo1] Theorem 14.3 (combined with Theorem 5 of [BB]) that $\mathrm{DN}(g) \in M_k(\Gamma)$. Notice that $\mathrm{DN}(g)$ is a symmetric Hilbert modular form, that is, $\mathrm{DN}(g)(z_1, z_2) = \mathrm{DN}(g)(z_2, z_1)$.

Sometimes it happens that a holomorphic Borcherds product can also be written as a Doi-Naganuma lift. Such identities between the multiplicative Borcherds lift and the additive Doi-Naganuma lift are of independent interest and we will state some examples below. It would be very interesting to understand conceptually when that occurs. Does it happen infinitely often?

The Fourier expansion of the Doi-Naganuma lift $\mathrm{DN}(g)$ converges rapidly on the whole domain $\mathbb{H}^2$ and can be used for numerical computations.

10.1. **The case $\mathbf{p = 5}$.** We denote the fundamental unit by $\varepsilon_0 = \frac{1}{2}(1 + \sqrt{5})$. Here the first few f_m were computed in [BB]. For completeness and for the convenience of the reader we repeat the result:

$$\begin{aligned}
f_1 &= q^{-1} + 5 + 11\,q - 54\,q^4 + 55\,q^5 + 44\,q^6 - 395\,q^9 + 340\,q^{10} + 296\,q^{11} - 1836\,q^{14} + \dots, \\
f_4 &= q^{-4} + 15 - 216\,q + 4959\,q^4 + 22040\,q^5 - 90984\,q^6 + 409944\,q^9 + 1388520\,q^{10} + \dots, \\
f_5 &= \tfrac{1}{2}\,q^{-5} + 15 + 275\,q + 27550\,q^4 + 43893\,q^5 + 255300\,q^6 + 4173825\,q^9 + \dots, \\
f_6 &= q^{-6} + 10 + 264\,q - 136476\,q^4 + 306360\,q^5 + 616220\,q^6 - 35408776\,q^9 + \dots,
\end{aligned}$$

$$f_9 = q^{-9} + 35 - 3555\,q + 922374\,q^4 + 7512885\,q^5 - 53113164\,q^6 + 953960075\,q^9 + \dots,$$
$$f_{10} = \tfrac{1}{2}\,q^{-10} + 10 + 3400\,q + 3471300\,q^4 + 9614200\,q^5 + 91620925\,q^6 + 5391558200\,q^9 + \dots.$$

The Eisenstein series $E_2^+(\tau,0) \in M_2^+(5,\epsilon_5)$ has the Fourier expansion

$$E_2^+(\tau,0) = 1 - 10q - 30q^4 - 30q^5 - 20q^6 - 70q^9 - 20q^{10} - 120q^{11} - 60q^{14} - 40q^{15} - 110q^{16} - \dots.$$

From this and a little estimate one concludes that there exist precisely 3 holomorphic Borcherds products in weight 10, namely

$$\Psi_1^2 = q_1^{2\rho_1} q_2^{2\rho_1'} \prod_{\substack{\nu\in\partial_F^{-1}\\ \operatorname{tr}(\nu\rho_1')>0}} \left(1 - q_1^{\nu} q_2^{\nu'}\right)^{2\tilde{c}_1(5\nu\nu')},$$

$$\Psi_6 = \prod_{\substack{\nu\in\partial_F^{-1}\\ \nu\gg 0}} \left(1 - q_1^{\nu} q_2^{\nu'}\right)^{\tilde{c}_6(5\nu\nu')},$$

$$\Psi_{10} = \prod_{\substack{\nu\in\partial_F^{-1}\\ \nu\gg 0}} \left(1 - q_1^{\nu} q_2^{\nu'}\right)^{\tilde{c}_{10}(5\nu\nu')}.$$

Here the Weyl vector for Ψ_1 is $\rho_1 = \varepsilon_0/\sqrt{5}$ by [BB]. From the fact that T_6 and T_{10} do not meet the boundary it follows that the Weyl vectors of Ψ_6 and Ψ_{10} are 0.

The dimension of $M_{10}^+(5,\epsilon_5)$ is 3, and it turns out that all the three Borcherds products lie in the image of the Doi-Naganuma lift. This can be seen as follows. Let h_1, h_2 be a basis of $S_{10}^+(5,\epsilon_5)$. Then $\mathrm{DN}(h_j) \in S_{10}(\Gamma)$. The restriction of $\mathrm{DN}(h_j)$ to T_1 can be viewed as an elliptic cusp form of weight 20 for the group $\mathrm{SL}_2(\mathbb{Z})$. This implies that there is a linear combination $\alpha_1 h_1 + \alpha_2 h_2$ whose Doi-Naganuma lift H vanishes on T_1. Consequently H/Ψ_1 is a holomorphic Hilbert modular form of weight 5 for the group Γ. On the other hand, it is easily seen that Ψ_1 is a multiple of the modular form Θ constructed by Gundlach, and hence is antisymmetric, i.e. $\Psi_1(z_1,z_2) = -\Psi_1(z_2,z_1)$ (see [Gu], Theorem 2). Since H is symmetric as a DN-lift, the function H/Ψ_1 is antisymmetric, too. But this implies that it also vanishes on T_1. Thus H/Ψ_1^2 is a holomorphic Hilbert modular form of weight 0 and therefore constant. Consequently, Ψ_1^2 is a multiple of the DN-lift H. All occurring constants can be determined by computing the first terms of the Fourier expansions.

Since Γ has just one cusp, the difference of Ψ_6 and a suitable multiple of the DN-lift of $E_{10}^+(\tau,0)$ will be in $S_{10}(\Gamma)$. Arguing as before, we can find a linear combination of $E_{10}^+(\tau,0)$, h_1, h_2 whose Doi-Naganuma lift H' has the property that $\Psi_6 - H'$ is in $S_{10}(\Gamma)$ and vanishes on T_1. Consequently, $\Psi_6 - H'$ is a multiple of Ψ_1^2 and hence a Doi-Naganuma lift. Thus Ψ_6 is a Doi-Naganuma lift itself. For Ψ_{10} one can argue in the same way.

A base for $M_{10}(5,\epsilon_5)$ is given by the eta quotients $\eta(\tau)^{25-6a}\eta(5\tau)^{6a-5}$, where $a = 0,\dots,5$. It turns out that there is a base of $M_{10}^+(5,\epsilon_5)$, consisting of modular forms whose Fourier expansion begins with

$$\begin{aligned}
g_1 &= q^4 - q^5 - q^6 - 18\,q^9 + 19\,q^{10} + 20\,q^{11} + 133\,q^{14} + O(q^{15}),\\
g_6 &= -132 - 264\,q + 306360\,q^4 - 271512\,q^5 - 236400\,q^6 + 1613256\,q^9 + O(q^{10}),\\
g_{10} &= -132 - 3400\,q + 4047800\,q^4 - 3834200\,q^5 - 5106800\,q^6 - 55543800\,q^9 + O(q^{10}).
\end{aligned}$$

Comparing the first terms of the Fourier expansions, one finds that

$$\mathrm{DN}(g_1) = \Psi_1^2, \qquad \mathrm{DN}(g_6) = \Psi_6, \qquad \mathrm{DN}(g_{10}) = \Psi_{10}. \tag{10.2}$$

Now we compute the values of the rational functions Ψ_6/Ψ_1^2 and Ψ_{10}/Ψ_1^2 at a selection of CM-points, and verify the first table in the introduction numerically.

We first consider the cyclic CM extension $K = \mathbb{Q}(\zeta_5)$ of F, where $\zeta_5 = e^{2\pi i/5}$. If σ denotes the complex embedding of K taking ζ_5 to ζ_5^2 then $\Phi = \{1, \sigma\}$ is a CM type of K. We have $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F\zeta_5$ and the corresponding CM cycle $\mathcal{CM}(K, \Phi)$ is represented by the point $\tau_1 = (\zeta_5, \zeta_5^2) \in \mathbb{H}^2$. Using (10.1) and (10.2) one can compute the Fourier expansion of Ψ_6, Ψ_1^2, and Ψ_{10} and evaluate at τ_1 numerically. It confirms the case $p = q = 5$ in the table. As another example we consider the case $p = 5$ and $q = 41$. Here one has the non-Galois CM extension $K = F(\sqrt{\Delta})$ with $\Delta = -\frac{13+\sqrt{5}}{2}$. So $\Delta' = -\frac{13-\sqrt{5}}{2}$. Recall that $\sigma(\sqrt{\Delta}) = \sqrt{\Delta'}$, and $\sigma^{-1}(\sqrt{\Delta}) = -\sqrt{\Delta'}$. It turns out that $h_K = 1$ and $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F z$ with $z = \frac{1}{2}(1 + \sqrt{\Delta} + \Delta)$. The CM cycles $\mathcal{CM}(K, \Phi, \mathcal{O}_K)$ and $\mathcal{CM}(K, \Phi', \mathcal{O}_K)$ are represented by $\tau_1 = (z, \sigma(z))$ and $\tau_3 = (\varepsilon_0 z, \varepsilon_0' \frac{1}{2}(1 - \sqrt{\Delta'} + \Delta'))$, respectively. The numerical calculation confirms the case $p = 5$, $q = 41$. The case $q = 61$ was confirmed the same way.

Remark 10.1. The formula of [HZ] for the intersection of two Hirzebruch-Zagier divisors on $\Gamma\backslash\mathbb{H}^2$ implies that Ψ_1, Ψ_6, Ψ_{10} have precisely one common zero on $\Gamma\backslash\mathbb{H}^2$, represented by the point $(e^{2\pi i/3}, e^{2\pi i/3})$. Arguing more carefully one actually finds that the restrictions of Ψ_6, Ψ_{10}, Ψ_{15}, Ψ_5^2 to T_1 generate the ring of elliptic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight divisible by 20. (In particular, Ψ_1, Ψ_5, Ψ_{10} have no common zero on $\Gamma\backslash\mathbb{H}^2$.) An inductive argument shows that the ring of symmetric Hilbert modular forms for Γ of weight divisible by 10 is generated by Ψ_1^2, Ψ_6, Ψ_{10}, Ψ_{15}, Ψ_5^2. Hence every rational function on the symmetric Hilbert modular surface X^{sym} associated to Γ can be written as a rational function in these five Borcherds products.

10.2. **The case $\mathbf{p = 13}$.** We denote the fundamental unit by $\varepsilon_0 = \frac{1}{2}(3 + \sqrt{13})$. Here the f_m can be computed as follows. The space $S_6(13, \chi_0)$ has dimension five. It contains an element h whose Fourier expansion begins with $h = q^5 - q^7 - 2q^8 + 2q^9 - 4q^{10} + \dots$. It turns out that $f_1(\tau) = h(\tau)\eta(\tau)^{-1}\eta(13\tau)^{-11}$. The other f_m can be obtained in a similar way as in [BB]. We list the first few f_m:

$$\begin{aligned}
f_1 &= q^{-1} + 1 + q + 3\,q^3 - 2\,q^4 - q^9 - 4\,q^{10} + 4\,q^{12} + 3\,q^{13} + 6\,q^{16} - 8\,q^{17} - 4\,q^{22} + \dots,\\
f_3 &= q^{-3} + 4 + 9\,q - 2\,q^3 + 12\,q^4 + 12\,q^9 - 60\,q^{10} - 68\,q^{12} + 51\,q^{13} + 108\,q^{14} + \dots,\\
f_4 &= q^{-4} + 3 - 8\,q + 16\,q^3 + 29\,q^4 - 88\,q^9 + 24\,q^{10} - 85\,q^{12} + 152\,q^{13} - 352\,q^{14} + \dots,
\end{aligned}$$

$$
\begin{aligned}
f_9 &= q^{-9} + 13 - 9\,q + 36\,q^3 - 198\,q^4 + 2419\,q^9 + 2304\,q^{10} - 8160\,q^{12} + 5967\,q^{13} + \dots,\\
f_{10} &= q^{-10} + 4 - 40\,q - 200\,q^3 + 60\,q^4 + 2560\,q^9 - 2410\,q^{10} + 13260\,q^{12} + 10880\,q^{13} + \dots,\\
f_{12} &= q^{-12} + 12 + 48\,q - 272\,q^3 - 255\,q^4 - 10880\,q^9 + 15912\,q^{10} + 5270\,q^{12} + \dots,\\
f_{13} &= \tfrac{1}{2}q^{-13} + 7 + 39\,q + 221\,q^3 + 494\,q^4 + 8619\,q^9 + 14144\,q^{10} + 35360\,q^{12} + \dots,\\
f_{14} &= q^{-14} + 6 + 504\,q^3 - 1232\,q^4 - 9240\,q^9 - 32571\,q^{10} + 64428\,q^{12} + 89432\,q^{13} + \dots,\\
f_{26} &= \tfrac{1}{2}q^{-26} + 6 + 208\,q + 3432\,q^3 + 10296\,q^4 + 790920\,q^9 + 1627418\,q^{10} + \dots.
\end{aligned}
$$

The Eisenstein series $E_2^+(\tau,0) \in M_2^+(13,\epsilon_{13})$ has the Fourier expansion

$$E_2^+(\tau,0) = 1 - 2\,q - 8\,q^3 - 6\,q^4 - 26\,q^9 - 8\,q^{10} - 24\,q^{12} - 14\,q^{13} - 12\,q^{14} - 22\,q^{16} + \dots.$$

One finds that there are precisely 3 holomorphic Borcherds products of weight 6 (with trivial character), namely

$$
\begin{aligned}
\Psi_1^6 &= q_1^{6\rho_1} q_2^{6\rho_2'} \prod_{\substack{\nu\in\partial_F^{-1}\\ \operatorname{tr}(\nu\rho_1')>0}} \left(1 - q_1^\nu q_2^{\nu'}\right)^{6\tilde c_1(13\nu\nu')},\\
\Psi_{14} &= \prod_{\substack{\nu\in\partial_F^{-1}\\ \nu\gg0}} \left(1 - q_1^\nu q_2^{\nu'}\right)^{\tilde c_{14}(13\nu\nu')},\\
\Psi_{26} &= \prod_{\substack{\nu\in\partial_F^{-1}\\ \nu\gg0}} \left(1 - q_1^\nu q_2^{\nu'}\right)^{\tilde c_{26}(13\nu\nu')}.
\end{aligned}
$$

Here the Weyl vector for Ψ_1 is $\rho_1 = \frac{\varepsilon_0}{3\sqrt{13}}$ by [BB]. From the fact that T_{14} and T_{26} do not meet the boundary it follows that the Weyl vectors of Ψ_{14} and Ψ_{26} are 0. (The Borcherds product Ψ_4^2 of weight 6 has a non-trivial character.)

It turns out that the dimension of $M_6^+(13,\epsilon_{13})$ is 4, and all the three Borcherds products lie in the image of the Doi-Naganuma lift. This can be seen in a similar way as in the first example. A base for $M_6(13,\epsilon_{13})$ is given by the eta quotients $\eta(\tau)^{13-2a}\eta(13\tau)^{2a-1}$, where $a = 0,\dots,7$. One finds that there are elements in $M_6^+(13,\epsilon_{13})$, whose Fourier expansion begins as follows:

$$
\begin{aligned}
g_1 &= q^4 - 6\,q^9 - 5\,q^{10} + 15\,q^{12} - 10\,q^{13} + 9\,q^{14} - 5\,q^{16} + 18\,q^{17} + \dots,\\
g_{14} &= -252 - 504\,q^3 + 1232\,q^4 + 9240\,q^9 + 10472\,q^{10} + 78456\,q^{12} + 37576\,q^{13} + \dots,\\
g_{26} &= -252 - 208\,q - 3432\,q^3 + 17888\,q^4 - 77064\,q^9 - 81224\,q^{10} + 412776\,q^{12} + \dots.
\end{aligned}
$$

Comparing the first terms of the Fourier expansions, we see that

$$\mathrm{DN}(g_1) = \Psi_1^6, \qquad \mathrm{DN}(g_{14}) = \Psi_{14}, \qquad \mathrm{DN}(g_{26}) = \Psi_{26}. \tag{10.3}$$

By means of (10.1) and (10.3), the second table in the introduction was verified numerically.

References

[AbSt] *M. Abramowitz and I. Stegun*, Pocketbook of Mathematical Functions, Verlag Harri Deutsch (1984).

[Bo1] *R. E. Borcherds*, Automorphic forms with singularities on Grassmannians, Invent. Math. **132** (1998), 491–562.

[Bo2] *R. E. Borcherds*, The Gross-Kohnen-Zagier theorem in higher dimensions, Duke Math. J. **97** (1999), 219–233.

[Br1] *J. H. Bruinier*, Borcherds products and Chern classes of Hirzebruch-Zagier divisors, Invent. math. **138** (1999), 51–83.

[Br2] *J. H. Bruinier*, Borcherds products on $O(2, l)$ and Chern classes of Heegner divisors, Lect. Notes Math. **1780**, Springer-Verlag (2002).

[Br3] *J. H. Bruinier*, Infinite products in number theory and geometry, Jahresber. Dtsch. Math. Ver. **106** (2004), Heft 4, 151–184.

[BB] *J. H. Bruinier and M. Bundschuh*, On Borcherds products associated with lattices of prime discriminant, Ramanujan J. **7** (2003), 49–61.

[BBK] *J. H. Bruinier, J. Burgos, and U. Kühn*, Borcherds products and arithmetic intersection theory on Hilbert modular surfaces, preprint (2003). `http://arxiv.org/abs/math.NT/0310201` to appear in Duke Math. Journal. `http://arxiv.org/abs/math.NT/0212286`

[BK] *J. H. Bruinier and U. Kühn*, Integrals of automorphic Green's functions associated to Heegner divisors, Int. Math. Res. Not. **2003:31** (2003), 1687–1729.

[Co] *H. Cohn*, A Classical Invitation to Algebraic Numbers and Class fields, Springer-Verlag, New York (1978).

[CH] *P. E. Conner and J. Hurrelbrink*, Class Number Parity, Series in Pure Math. vol. **8**, World Scientific (1988).

[DN] *K. Doi and H. Naganuma*, On the functional equation of certain Dirichlet series, Invent. Math. **9** (1969), 1–14.

[EMOT] *A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi*, Tables of Integral Transforms, vol. I, McGraw-Hill (1954).

[Fr] *H.-G. Franke*, Kurven in Hilbertschen Modulflächen und Humbertsche Flächen im Siegelraum, Bonner Math. Schriften **104** (1978).

[Ge] *G. van der Geer*, Hilbert Modular Surfaces, Springer-Verlag (1988).

[Go] *E. Z. Goren*, Lectures on Hilbert Modular Varieties and Modular Forms, CRM Monograph Series **14**, American Mathematical Society (2002).

[GL] *E.Z. Goren and K. Lauter*, Class invariants for quartic CM fields, preprint (2004), pp14.

[GN] *V. A. Gritsenko and V. V. Nikulin*, Automorphic forms and Lorentzian Kac-Moody algebras. II, Internat. J. Math. **9** (1998), no. 2, 201–275.

[GZ1] *B. Gross and D. Zagier*, On singular moduli, J. Reine Angew. Math. **355** (1985), 191–220.

[GZ2] *B. Gross and D. Zagier*, Heegner points and derivatives of L-series, Invent. Math. **84** (1986), 225–320.

[Gu] *K.-B. Gundlach*, Die Bestimmung der Funktionen zur Hilbertschen Modulgruppe des Zahlkörpers $\mathbb{Q}(\sqrt{5})$, Math. Annalen **152** (1963), 226–256.

[He] *E. Hecke*, Analytische Arithmetik der positiv definiten quadratischen Formen, Kgl. Danske Vid. Selskab. Math. fys. Med. XIII **12** (1940). Werke, 789–918.

[Hej] *D. A. Hejhal*, The Selberg Trace Formula for $\mathrm{PSL}(2, \mathbb{R})$, Lect. Notes Math. **1001**, Springer-Verlag (1983).

[HZ] *F. Hirzebruch and D. Zagier*, Intersection Numbers of Curves on Hilbert Modular Surfaces and Modular Forms of Nebentypus, Invent. Math. **36** (1976), 57–113.

[Ku1] *S. Kudla*, Central derivatives of Eisenstein series and height pairings, Ann. of Math. **146** (1997), 545–646.

[Ku2] *S. Kudla*, Integrals of Borcherds forms, Compositio Math. **137** (2003), 293–349.

[KRY1] *S. Kudla, M. Rapoport, and T.H. Yang*, On the derivative of an Eisenstein series of weight one, Int. Math. Res. Not. **7** (1999), 347–385.

[KRY2] *S. Kudla, M. Rapoport and T. Yang*, Derivatives of Eisenstein series and Faltings heights, Compos. Math. **140** (2004), 887–951.

[La] *K. Lauter*, Primes in the denominators of Igusa Class Polynomials, preprint (2003), 3 pages, http://www.arxiv.org/math.NT/0301240/

[Na] *H. Naganuma*, On the coincidence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hilbert modular forms over a real quadratic field, J. Math. Soc. Japan **25** (1973), 547–555.

[Sc] *J. Schofer*, thesis, University of Maryland (2005).

[Ru] *B. Runge*, Endomorphism rings of abelian surfaces and projective models of their moduli spaces, Tohoku Math. J. **51** (1999), 283–303.

[Sh] *G. Shimura*, Abelian varieties with complex multiplication and modular functions, Princeton Univ. Press (1998).

[St] *J. Sturm*, Projections of C^∞ automorphic forms, Bull. Amer. Math. Soc. **2** (1980), 435–439.

[Ya1] *T.H. Yang*, CM number fields and modular forms, preprint (2003), pp34.

[Ya2] *T.H. Yang*, Local densities of 2-adic quadratic forms, J. Number Theory **108** (2004), 287–345.

[Za] *D. Zagier*, Modular Forms Associated to Real Quadratic Fields, Invent. Math. **30** (1975), 1–46.

Mathematisches Institut, Universität zu Köln, Weyertal 86–90, D-50931 Köln, Germany
E-mail address: `bruinier@math.uni-koeln.de`

Department of Mathematics, University of Wisconsin Madison, Van Vleck Hall, Madison, WI 53706, USA
E-mail address: `thyang@math.wisc.edu`