# Borcherds lifts of harmonic Maass forms

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March 2017

## Eisenstein series

- For $k \geq 4$ even, the weight $k$ Eisenstein series

$$E_k(z) = \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

  (with $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$) is a holomorphic modular form of weight $k$.
- For $k = 2$, the non-holomorphic weight 2 Eisenstein series

$$E_2(z) = \lim_{s \to 0} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{y^s}{(cz+d)^2 |cz+d|^{2s}} = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n - \frac{3}{\pi y}$$

  transforms like a modular form of weight 2, but is harmonic rather than holomorphic.
- Note that $E_2(z)$ has a holomorphic and a non-holomorphic part.

# Harmonic Maass forms

- A harmonic Maass form of weight $k$ is a function $f : \mathbb{H} \to \mathbb{C}$ with
  1. $f$ is harmonic.
  2. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma = \mathrm{PSL}_2(\mathbb{Z})$.
  3. $f(z)$ grows at most linearly exponentially as $y \to \infty$.
- A harmonic Maass form $f$ splits into a holomorphic and a non-holomorphic part $f = f^+ + f^-$.
- More precisely, it has a Fourier expansion $f = f^+ + f^-$ with

$$f^+(z) = \sum_{n \gg -\infty} a^+(n) q^n$$

$$f^-(z) = a^-(0) y^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} a^-(n) \Gamma(1-k, 4\pi|n|y) q^n$$

- The Eisenstein series $E_2$ is a harmonic Maass form of weight 2.

# Quadratic forms and geodesics

- A (binary integral) quadratic form is a polynomial

$$Q(x, y) = ax^2 + bxy + cy^2 \qquad (a, b, c \in \mathbb{Z}).$$

- Let $\mathcal{Q}_D$ be the set of all quadratic forms of discriminant $D = b^2 - 4ac$.
- For $D > 0$ and $Q \in \mathcal{Q}_D$, the set of solutions of

$$a|z|^2 + bx + c = 0$$

defines a geodesic $C_Q$ in $\mathbb{H}$.

# Generating series of traces of cycle integrals

- Let
$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots$$
be the modular $j$-function, and $J = j - 744$.
- For $D > 0$ not being a square we define its trace by
$$\mathrm{tr}_D(J) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \int_{\Gamma_Q \backslash C_Q} J(z) \frac{dz}{Q(z,1)}.$$
- For $D > 0$ being a square, the integral does not converge and has to be regularized in a suitable way.
- **Theorem.** (Duke-Imamoglu-Tóth/Bruinier-Funke-Imamoglu) The generating series
$$h^+(z) = \sum_{D>0} \mathrm{tr}_D(J) q^D$$
defines the holomorphic part of a harmonic Maass form $h = h^+ + h^-$ of weight $1/2$.
- The $\xi_{1/2}$-image of $h$ is not a cusp form, but weakly holomorphic.

## Borcherds' regularized theta lift

- Let $f$ be a harmonic Maass form of weight $1/2$ which is holomorphic on $\mathbb{H}$:

$$f(z) = f^+(z) = \sum_{n \gg -\infty} a(n) q^n.$$

- Borcherds' regularized theta lift of $f$ is defined as an integral of the shape

$$\Phi(z, f) = \int_{\Gamma \backslash \mathbb{H}}^{\mathrm{reg}} f(\tau) \overline{\Theta(\tau, z)} v^{1/2} \frac{du\, dv}{v^2} \qquad (\tau = u + iv \in \mathbb{H}).$$

- Here $\Theta(\tau, z)$ is a Siegel theta function which has weight $1/2$ in $\tau$ and weight $0$ in $z$, so $\Phi(z, f)$ also has weight $0$ in $z$.
- The integral does not converge and has to be regularized in a suitable way. Defining the regularization and proving that the regularized integral exists is the main difficulty.
- It turns out that the theta lift has logarithmic singularities at Heegner points corresponding to the principal part $\sum_{n<0} a(n) q^n$ of $f$.
- **Example.** Let $\theta = \sum_{n \in \mathbb{Z}} q^{n^2}$ be the Jacobi theta function. Then

$$\Phi(z, 12\,\theta) = -4 \log |\Delta(z) y^6| + \text{constant},$$

where $\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ is the unique normalized cusp form of weight 12.

# Borcherds products

- Let $f(z) = \sum_{n \gg -\infty} a(n) q^n$.
- **Theorem.** (Borcherds) The product

$$\Psi(z, f) = q^\rho \prod_{n=1}^{\infty} (1 - q^n)^{a(n^2)}$$

converges for $y \gg 0$ large enough, and extends to a meromorphic modular form of weight $a(0)$ on $\mathbb{H}$ with roots/poles at Heegner points in $\mathbb{H}$.

- For the proof, one checks on the Fourier expansions that

$$\Phi(z, f) = -4 \log |\Psi(z, f) y^{a(0)/2}| + \text{constant}.$$

- **Example:** Let $\theta = \sum_{n \in \mathbb{Z}} q^{n^2}$ be the Jacobi theta function. Then

$$\Psi(z, 12\,\theta) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \Delta(z)$$

$$\Phi(z, 12\,\theta) = -4 \log |\Delta(z) y^6| + \text{constant}$$

# Generalization to harmonic Maass forms

- So far: theta lifts and Borcherds products associated to holomorphic forms.
- **Theorem.** (Bruinier-Ono) Let

$$f(z) = \sum_{n \gg -\infty} a^+(n) q^n + \sum_{n<0} a^-(n) \Gamma(1/2, 4\pi|n|y) q^n$$

be a harmonic Maass form whose non-holomorphic part $f^-$ is rapidly decreasing as $y \to \infty$.

1. Borcherds' regularized theta lift

$$\Phi(z, f) = \int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} f(\tau)\overline{\Theta(\tau, z)} v^{1/2} \frac{du\, dv}{v^2} \qquad (\tau = u + iv \in \mathbb{H})$$

still exists.

2. The product

$$\Psi(z, f) = q^\rho \prod_{n=1}^{\infty} (1 - q^n)^{a^+(n^2)}$$

converges for $y \gg 0$ large enough, and extends to a meromorphic modular form of weight $a(0)$ on $\mathbb{H}$ with roots/poles at Heegner points in $\mathbb{H}$.

- This result does not apply to the DIT generating series of traces of cycle integrals of $J$.

- **Theorem.** (S.) Let

$$f(z) = \sum_{n \gg -\infty} a^+(n)q^n + a^-(0)\sqrt{v} + \sum_{\substack{n \ll \infty \\ n \neq 0}} a^-(n)\Gamma(1/2, 4\pi|n|y)q^n$$

  be a harmonic Maass form.

  1. Borcherds' regularized theta lift

  $$\Phi(z,f) = \int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} f(\tau)\overline{\Theta(\tau,z)}v^{1/2}\frac{du\,dv}{v^2} \qquad (\tau = u + iv \in \mathbb{H})$$

     still exists.
  2. As before, it has logarithmic singularities at Heegner points corresponding to the holomorphic principal part $\sum_{n<0} a^+(n)q^n$
  3. Additionally, it is not differentiable along certain geodesics corresponding to the non-holomorphic principal part $\sum_{n>0} a^-(n)\Gamma(1/2, 4\pi|n|y)q^n$.

- We also consider twisted Borcherds lifts $\Phi_\Delta(z,f)$ where $\Delta$ is a fundamental discriminant (as Bruinier and Ono).
- We can apply this extension of the Borcherds lift to the DIT harmonic Maass form.

# The Borcherds lift of the Duke-Imamoglu-Tóth form

- Let $h$ be the DIT harmonic Maass form of weight $1/2$ with holomorphic part

$$h^+ = \sum_{D>0} \mathrm{tr}_D(J) q^D, \qquad \mathrm{tr}_D(J) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \int_{\Gamma_Q \backslash C_Q} J(z) \frac{dz}{Q(z,1)}.$$

- **Proposition.** Let $\Delta > 1$ be a fundamental discriminant. The Fourier expansion of the twisted Borcherds lift of $h$ is given by

$$\begin{aligned} \Phi_\Delta(z,h) = & -4 \sum_{m=1}^{\infty} \mathrm{tr}_{\Delta m^2}(J) \sum_{b(\Delta)} \left(\frac{\Delta}{b}\right) \log\left|1 - e(mz + b/\Delta)\right| \\ & - 8\pi \,\mathrm{tr}_\Delta(1) y \\ & - 2 \sum_{\substack{[a,b,c] \in \mathcal{Q}_\Delta \\ a>0}} \mathbf{1}_Q(z) \left( \arctan\left( \frac{\sqrt{\Delta} y}{a|z|^2 + bx + c} \right) + \frac{\pi}{2} \right) \end{aligned}$$

  where $\mathbf{1}_Q$ is the characteristic function of the bounded component of $\mathbb{H} \setminus C_Q$.

- The first two lines are harmonic on $\mathbb{H}$, whereas the third line is continuous, but not differentiable along geodesics $C_Q$ for $Q \in \mathcal{Q}_\Delta$.

## The derivative of the Borcherds lift of the Duke-Imamoglu-Tóth form

- **Proposition.** The derivative of $\Phi_\Delta(z,h)$ is given by

$$\Phi'_\Delta(z,h) = -\operatorname{tr}_\Delta(1) - \sum_{m=1}^{\infty}\sum_{d\mid m}\left(\frac{\Delta}{m/d}\right) d\operatorname{tr}_{\Delta d^2}(J)q^m + \frac{1}{2\pi}\sum_{\substack{[a,b,c]\in\mathcal{Q}_\Delta\\ a>0}}\frac{\mathbf{1}_Q(z)}{az^2+bz+c}$$

- The first line is holomorphic $\mathbb{H}$, whereas the second line has jumps along geodesics $C_Q$ for $Q \in \mathcal{Q}_\Delta$.
- **Corollary.**(Duke-Imamoglu-Tóth) The generating series

$$F_\Delta(z) = -\operatorname{tr}_\Delta(1) - \sum_{m=1}^{\infty}\sum_{d\mid m}\left(\frac{\Delta}{m/d}\right) d\operatorname{tr}_{\Delta d^2}(J)q^m$$

  is holomorphic on $\mathbb{H}$ and transforms as

$$F_\Delta(z) - z^{-2}F_\Delta(-1/z) = \frac{1}{\pi}\sum_{\substack{[a,b,c]\in\mathcal{Q}_\Delta\\ c<0<a}}\frac{1}{az^2+bz+c}.$$

- This generalizes to higher level.