Shimura varieties mod p

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Siegel moduli problem

- Define $\mathsf{A}_{g,N}(\mathbb{C}) := \{(A, \lambda, \phi_N)\}/\cong$ with
  - (A, λ) principally polarized abelian $\mathbb{C}$-variety
    $(g = 1 \implies$ every A has a unique pp)
  - $\phi_N : ((\mathbb{Z}/N\mathbb{Z})^{2g}, \text{std. sympl. form}) \to (A(\mathbb{C})[N], \text{Weil pairing})$ is a symplectic similtude. (This is called a level N structure.)
    $(g = 1 \implies$ every isom. $(\mathbb{Z}/N\mathbb{Z})^2 \cong A(\mathbb{C})[N]$ is a sympl. similt.)
- With $K(N)$ defined by
  $1 \to K(N) \to \mathrm{GSp}_{2g}(\hat{\mathbb{Z}}) \to \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \to 1$, we have

$$\mathsf{A}_{g,N}(\mathbb{C}) \cong \coprod_{\varphi(N) \text{ copies}} \Gamma(N)\backslash\mathbb{H}_g \quad (\Gamma(N) \subseteq \mathrm{Sp}_{2g}(\mathbb{Z}))$$
$$= \mathrm{Sh}_{K(N)}(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})(\mathbb{C})$$

Shimura varieties

This is one way of thinking about Shimura varieties[1] and one reason why to care about them at all:

> Shimura varieties arise as moduli spaces of abelian varieties (with extra structure (e.g. polarization, level structure)).

Fact: Shimura varieties actually are algebraic varieties (resp. schemes in the limit); "canonically" defined over a number field, the so-called **reflex field** ($= \mathbb{Q}$ in the Siegel case with the $\varphi(N)$ connected components each defined over $\mathbb{Q}[\zeta_N]$).

[1] Shimura varieties of Hodge type anyway

The integral model

- Let $p \nmid N \geq 3$. We define a model of $\mathrm{Sh}_{K(N)}(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$ over $\mathbb{Z}[1/N]$. Let

$$\begin{aligned} \mathrm{Sh}^{\mathrm{int}}_{K(N)} \colon (\mathbb{Z}[1/N]\text{-schemes})^{\mathrm{op}} &\to (\text{sets}), \\ S &\mapsto \{\text{isom. classes of } (A, \lambda, \phi_N)\}, \end{aligned}$$

  where (A, λ) is a principally polarized abelian S-scheme and $\phi_N \colon (\underline{\mathbb{Z}/N\mathbb{Z}}_S)^{2g} \to A[N]$ a symplectic level N structure.
- This is (representable by) a smooth quasi-projective $\mathbb{Z}[1/N]$-scheme.

A remark on $K(N)$

- We have $K(N) = K_p K^p$, where $K^p = \prod_{\ell \neq p} K_\ell \subseteq \mathrm{GSp}_{2g}(\mathbb{A}_f^p)$ and

$$K_\ell = \begin{cases} \mathrm{GSp}_{2g}(\mathbb{Z}_\ell), & \ell \nmid N \text{ (e.g., } \ell = p), \\ \{g \in \mathrm{GSp}_{2g}(\mathbb{Z}_\ell) \mid g \equiv I_2 \mod \ell^{r_\ell}\}, & r_\ell = v_\ell(N). \end{cases}$$

- The following considerations will almost exclusively depend on K_p, the part at p, and not on K^p, the part away from p.

Parahoric subgroups

- Consider the lattice chain
$$\cdots \subset \mathbb{Z}_p^{2g} \subset p^{-1}\mathbb{Z}_p \oplus \mathbb{Z}_p^{2g-1} \subset p^{-1}\mathbb{Z}_p^2 \oplus \mathbb{Z}_p^{2g-2} \subset \cdots$$
and define I to be the stabilizer of this lattice chain in $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$.
Then I is an **Iwahori** subgroup. It corresponds to $\Gamma_0(p)$.
Roughly, working with it instead of $\mathrm{GSp}_{2g}(\mathbb{Z}_p)$ means replacing abelian schemes by isogeny chains of abelian schemes.
- In a more general setting, one deals with the Bruhat-Tits building of a reductive group over a local (p-adic) field.
A **parahoric** subgroup then is the connected stabilizer of a facet. Special cases:
    - facet is a hyperspecial vertex $\rightsquigarrow$ hyperspecial subgroup (example: $K_p = \mathrm{GSp}_{2g}(\mathbb{Z}_p) \subseteq \mathrm{GSp}_{2g}(\mathbb{Q}_p)$);
    - facet is an alcove $\rightsquigarrow$ Iwahori subgroup.

Mod p reduction

- The mod p reduction of $\mathrm{Sh}_{K(N)}(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$ is

$$\mathrm{Sh}_{K(N)}^{\mathrm{red}} := \mathrm{Sh}_{K(N)}^{\mathrm{int}} \otimes_{\mathbb{Z}[1/N]} \mathbb{F}_p.$$

- By definition (+ Yoneda lemma), it comes with a universal abelian scheme

$$\mathcal{A} \to \mathrm{Sh}_{K(N)}^{\mathrm{red}},$$

hence with a universal p-divisible group

$$\mathcal{G} := \mathcal{A}[p^{\infty}] \to \mathrm{Sh}_{K(N)}^{\mathrm{red}} =: S.$$

Newton stratification

- Using the p-divisible group $\mathcal{G} \to S$, we can partition $S(\bar{\mathbb{F}}_p)$ according to the isogeny class of

$$\mathcal{G}_{\bar{s}}, \quad \bar{s} \in S(\bar{\mathbb{F}}_p),$$

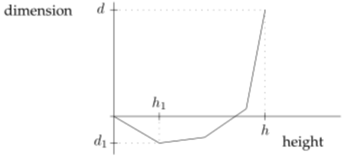
  which is a p-divisible group over $\bar{\mathbb{F}}_p$.
- This partition is called the **Newton (polygon) stratification**.

Rational Dieudonné modules

- Recall: A p-divisible group (always over $\bar{\mathbb{F}}_p$) is (up to isomorphism) determined by its Dieudonné module M, a free $\mathcal{O}_{\breve{\mathbb{Q}}_p}$-module (where $\breve{\mathbb{Q}}_p := \widehat{\mathbb{Q}_p^{\mathrm{ur}}} = W(\bar{\mathbb{F}}_p)[\frac{1}{p}]$) of rank $\mathrm{rk}(M) = \mathrm{ht}(G) = 2g$ together with a σ-linear map $F\colon M \to M$ and a σ^{-1}-linear map $V\colon M \to M$ such that $FV = p$ (so that V is uniquely determined by F).
- $X \to Y$ of p-divisible groups is an isogeny iff the induced map between Dieudonné modules becomes an isomorphism after inverting p.
- Thus: A p-divisible group is *up to isogeny* determined by its **rational Dieudonné module** $M[\frac{1}{p}] = M \otimes_{\mathcal{O}_{\breve{\mathbb{Q}}_p}} \breve{\mathbb{Q}}_p$.

Newton polygons

- Rational Dieudonné modules are F-isocrystals and these form a semisimple category with simple objects indexed by $\mathbb{Q}$.
- The decomposition of an F-isocrystal into simple objects is the slope decomposition, best visualized by the associated **Newton polygon**: Let $\lambda_1 = \frac{d_1}{h_1} \leq \cdots \leq \lambda_r = \frac{d_r}{h_r}$ be the slopes, then the Newton polygon is the convex polygon starting in $(0,0) \in \mathbb{Q}^2$, then going right h_1 and up d_1, and so on.
- Partial order: $\mathrm{NP}_1 \leq \mathrm{NP}_2$ iff they have the same endpoints and NP_1 lies below NP_2.



The image of the Newton map

- We get the Newton map $\nu\colon S(\bar{\mathbb{F}}_p) \to \{\text{Newton polygons}\}$.
- Its image consists of the Newton polygons with:
  - slopes between 0 and 1,
  - endpoint $(2g, g)$,
  - the symmetry condition: the i'th slope and the $(i-1)$'th to last slope add up to 1 for all i.
- In particular the Newton polygons with slope sequences $(0^{(g)}, 1^{(g)})$ (minimal, ordinary) resp. $((\frac{1}{2})^{(2g)})$ (maximal, supersingular) are in the image.

Grothendieck's semicontinuity theorem

Theorem

For every fixed Newton polygon ν_0 the set

$$\{\bar{s} \in S(\bar{\mathbb{F}}_p) \mid \nu(\bar{s}) \geq \nu_0\} \subseteq S(\bar{\mathbb{F}}_p)$$

is Zariski-closed.

Corollary (Weak stratification property)

$\overline{\nu^{-1}(\nu_0)} \subseteq \bigcup_{\nu_1 \leq \nu_0} \nu^{-1}(\nu_1)$
In particular, the ordinary Newton stratum is dense and the supersingular Newton stratum is closed.

The Newton strata are locally closed

Corollary

The Newton strata are locally closed.
(Hence they are the sets of $\bar{\mathbb{F}}_p$-valued points of reduced subschemes $S_\nu \subseteq S$.)

Proof: Only finitely many Newton polygons in the image of the Newton map lie below a given Newton polygon.
Hence, by Grothendieck's semicontinuity theorem, the Newton strata are *finite* intersections of closed sets and open sets.

Oort's central leaves

- We can also partition by isomorphism classes instead of isogeny classes, i.e., consider the central leaves

$$\{\bar{s} \mid \mathcal{G}_{\bar{s}} \cong X\}, \quad X/\bar{\mathbb{F}}_p \text{ } p\text{-divisible group.}$$

- The central leaves are closed within their respective Newton strata. In particular, they are locally closed, hence the sets of $\bar{\mathbb{F}}_p$-valued points of reduced subschemes. Proof uses results on slope filtrations of p-divisible groups with constant Newton polygon.
- Remark: Actually one should take polarizations into account here, but for simplicity we won't.

Nicer index sets

- Choosing a basis, the information contained in the rational Dieudonné module $M[\frac{1}{p}] = M \otimes_{\mathcal{O}_{\breve{\mathbb{Q}}_p}} \breve{\mathbb{Q}}_p$ really just is an element

$$[b] \in \mathrm{GL}_{2g}(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_{2g}(\breve{\mathbb{Q}}_p)_\sigma =: B(\mathrm{GL}_{2g})$$

  $([b] = [b'] \text{ if } b' = gb\sigma(g)^{-1})$.
- And the information contained in M is an element

$$[b] \in \mathrm{GL}_{2g}(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_{2g}(\mathcal{O}_{\breve{\mathbb{Q}}_p})_\sigma =: C(\mathrm{GL}_{2g}).$$

Nicer index sets (cont.)

So the Newton stratification is indexed by $B(\mathrm{GL}_{2g}) = \mathrm{GL}_{2g}(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_{2g}(\breve{\mathbb{Q}}_p)_\sigma$ and Oort's foliation by $C(\mathrm{GL}_{2g}) = \mathrm{GL}_{2g}(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_{2g}(\mathcal{O}_{\breve{\mathbb{Q}}_p})_\sigma$. The Newton strata resp. leaves are the fibers of maps δ resp. Υ.

$$\begin{tikzcd} S(\bar{\mathbb{F}}_p) \arrow[r, "\Upsilon"] \arrow[rr, bend right, "\delta"'] & C(\mathrm{GL}_{2g}) \arrow[r] & B(\mathrm{GL}_{2g}) \end{tikzcd}$$

This point of view leads to group-theoretical and combinatorial considerations and generalizes nicely.

Generalization

- Two directions of generalization:
  - Consider a wider class of Shimura varieties.
  - Consider a wider class of level structures (at the prime in consideration).
- First we need an integral canonical model at the prime in consideration. For parahoric level structures and Shimura data of Hodge type where the underlying reductive group is (at p) split over a tamely ramified extension, Kisin-Pappas have constructed these.
- They come with a "universal" isogeny chain of abelian schemes with additional structure.

Generalization (cont.)

- Now one can apply the Dieudonné module functor etc. and arrive at

$$\begin{tikzcd} S(\bar{\mathbb{F}}_p) \arrow[r, "\Upsilon"] \arrow[rr, bend right, "\delta"'] & G(\breve{\mathbb{Q}}_p)/G(\breve{\mathbb{Q}}_p)_\sigma \arrow[r] & G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\mathcal{O}_{\breve{\mathbb{Q}}_p})_\sigma \end{tikzcd}$$

  where $\mathcal{G}$ is the parahoric group scheme over $\mathbb{Z}_p$ associated with the parahoric level $K_p \subseteq G(\mathbb{Q}_p)$.
- Are the leaves locally closed? smooth? Are the Newton strata locally closed? What are the dimensions? What are the images of δ, Υ? How do things behave under change of the parahoric level? ...