A theta lift related to the Shintani lift

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Modular forms

- Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the complex upper-half plane.
- $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$.
- A weakly holomorphic modular form of weight $k \in \mathbb{Z}$ is a function $F : \mathbb{H} \to \mathbb{C}$ such that
  1. F is holomorphic on $\mathbb{H}$.
  2. $F(\frac{az+b}{cz+d}) = (cz+d)^k F(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.
  3. F has a Fourier expansion

$$F(z) = \sum_{n \gg -\infty} a(n) q^n, \qquad (q = e^{2\pi iz}).$$

- The space of such forms is denoted by $M_k^!$.
- F is a holomorphic modular form, if $F(z) = \sum_{n \geq 0} a(n) q^n$, and the corresponding space is denoted by M_k.
- F is a cusp form, if $F(z) = \sum_{n \geq 1} a(n) q^n$, and the corresponding space is denoted by S_k.

Harmonic weak Maass forms

- A harmonic weak Maass form of weight k is a function $F : \mathbb{H} \to \mathbb{C}$ with
  1. F is annihilated by the weight k Laplace operator
  $$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$
  2. $F(\frac{az+b}{cz+d} z) = (cz+d)^k F(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$.
  3. F has a Fourier expansion of the form
  $$F(z) = F^+(z) + F^-(z) = \sum_{n \gg -\infty} a^+(n) q^n + \sum_{n<0} a^-(n) \Gamma(1-k, 4\pi|n|y) q^n$$
  with the incomplete Gamma function $\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} dt$.
- The space of such forms is denoted by H_k^+.
- We have the inclusions
$$S_k \ \subset \ M_k \ \subset \ M_k^! \ \subset \ H_k^+.$$

Differential operators

- Write $z = x + iy \in \mathbb{H}$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.
- The Maass lowering and raising operators

$$L_k = -2iy^2\frac{\partial}{\partial \bar{z}}, \qquad R_k = 2i\frac{\partial}{\partial z} + ky^{-1}$$

  lower or raise the weight from k to $k-2$ or $k+2$.
- The ξ-operator

$$\xi_k F = 2iy^k\overline{\frac{\partial}{\partial \bar{z}}F} \qquad (= y^{k-2}\overline{L_k F} = R_{-k}v^k\overline{F})$$

  of Bruinier and Funke defines a *surjective* map

$$\xi_k : H_k^+ \to S_{2-k}$$

  with kernel $M_k^!$.
- $\xi_k F$ is explicitly given by

$$\xi_k F = \xi_k F^- = -2\sum_{n>0}\overline{a^-(-n)}(4\pi n)^{1-k}q^n,$$

  i.e. $\xi_k F$ has essentially the same Fourier coefficients as F^-.

Binary Integral Quadratic forms

- A (binary integral) quadratic form is a polynomial

$$Q(x, y) = ax^2 + bxy + cy^2$$

  with $a, b, c \in \mathbb{Z}$.
- The discriminant of Q is defined as $D = b^2 - 4ac$.
- Let $\mathcal{Q}_D$ be the set of all quadratic forms of fixed discriminant D.
- $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ by $(M, Q) \mapsto M^t QM$.
- For $D < 0$ and $Q \in \mathcal{Q}_D$, the quadratic equation

$$Q(z, 1) = az^2 + bz + c = 0$$

  has a unique solution $z_Q \in \mathbb{H}$, the CM point associated to Q.
- For $D > 0$ and $Q \in \mathcal{Q}_D$, the set of solutions of

$$a|z|^2 + bx + c = 0$$

  defines a geodesic C_Q in $\mathbb{H}$, i.e. a vertical line or a semicircle around some point on the real axis.

Traces of CM values and geodesic cycle integrals

- For $D < 0$ and a Γ-invariant function F we define the D-th trace of F by

$$t_F(D) = \sum_{Q \in \mathcal{Q}_D/\Gamma} \frac{F(z_Q)}{|\overline{\Gamma}_Q|}.$$

- For $D > 0$ and a cusp form $F \in S_{2k+2}$ we define the D-th trace of F by

$$t_F(D) = \sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{\Gamma_Q \backslash C_Q} F(z) Q(z,1)^k dz.$$

- Zagier: For $F(z) = J(z) = j(z) - 744$ the function

$$q^{-1} - \sum_{D<0} t_F(D) q^{-D}$$

  is a weakly holomorphic modular form of weight $3/2$.

- The Shintani lift of a cusp form $F \in S_{2k+2}$,

$$\Lambda_{Sh}(F, z) = \sum_{D>0} t_F(D) q^D,$$

  is a cusp form of weight $k + 3/2$.

Theta lifts

- **Idea:** Construct generating series of such traces as a 'theta lift'

$$\Lambda(F,\tau) = \int_{\Gamma\backslash\mathbb{H}} F(z)\overline{\Theta(\tau,z)}y^k d\mu(z) \qquad (d\mu(z) = y^{-2}dx\, dy)$$

  where $\Theta(\tau,z)$ has weight k in z and $\overline{\Theta(\tau,z)}$ has weight ℓ in τ.
- Write $\tau = u + iv$ and $z = x + iy$.
- We consider the Millson theta function

$$\Theta_M(\tau,z) = v\sum_{D\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_D} Q_z e^{-4\pi v|Q(z,1)|^2/y^2} e^{-2\pi i D\tau},$$

  where

$$Q(z,1) = az^2 + bz + c, \qquad Q_z = a|z|^2 + bx + c.$$

- It has weight 0 in z and weight $1/2$ in τ (by Poisson summation).
- **Theorem.** The Millson theta lift

$$\Lambda_M(F,\tau) = \int_{\Gamma\backslash\mathbb{H}} F(z)\Theta_M(\tau,z)d\mu(z)$$

  converges for $F \in H_0^+$ and is in $H_{1/2}^+$.

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- *Idea of proof:*
  1. Show that the theta function $\Theta_M(\tau,z)$ decays square exponentially as $y \to \infty$ (using Borcherds) to see that the integral converges.
  2. Use the differential equation

  $$\Delta_{0,z}\Theta_M(\tau,z) \doteq \Delta_{1/2,\tau}\Theta_M(\tau,z)$$

     and Stokes' Theorem to show that $\Lambda_M(F,\tau)$ is harmonic.
  3. Compute the Fourier expansion (that's the difficult part).

Higher weight Millson lift

- We generalize the Millson lift to higher weights by
$$\Lambda_{M,k}(F,\tau) = L_{1/2,\tau}^{k/2} \int_{\Gamma\backslash\mathbb{H}} (R_{-2k,z}^{k} F)(z)\Theta_M(\tau,z)d\mu(z)$$
for $F \in H_{-2k}^{+}$ (Bruinier and Ono, Alfes).
- One can also consider a higher weight Millson theta function and the corresponding lift
$$\widetilde{\Lambda}_{M,k}(F,\tau) = \int_{\Gamma\backslash\mathbb{H}}^{\text{reg}} F(z)\Theta_{M,k}(\tau,z)y^{-2k}d\mu(z).$$
- The higher weight Millson theta function is no longer square exponentially decreasing at $i\infty$, so the integral has to be regularized.
- **Proposition.** The two lifts agree on H_{-2k}^{+} up to multiplication of some elementary constant.
- **Theorem.** a) For $k \geq 0$ the Millson theta lift defines a map
$$\Lambda_{M,k} : H_{-2k}^{+} \to H_{1/2-k}^{+}.$$
- b) For $k > 0$ we have
$$\Lambda_{M,k} : M_{-2k}^{!} \to M_{1/2-k}^{!}.$$
- c) For $k = 0$, the image of $M_0^{!}$ only lies in $H_{1/2}^{+}$.

Relation to the Shintani lift

- **Theorem.** For $k > 0$ and $F \in H^+_{-2k}$ we have

$$\xi_{1/2-k,\tau}\Lambda_{M,k}(F,\tau) \doteq \Lambda_{Sh,k}(\xi_{-2k,z}F,\tau),$$

  so we have a commutative diagram

$$\begin{array}{ccc} H^+_{-2k} & \xrightarrow{\ \xi_{-2k}\ } & S_{2k+2} \\ \big\downarrow{\scriptstyle \Lambda_M} & & \big\downarrow{\scriptstyle \Lambda_{Sh}} \\ H^+_{1/2-k} & \xrightarrow{\ \xi_{1/2-k}\ } & S_{3/2+k} \end{array}$$

- For $k = 0$ the theorem holds up to addition of an explicit linear combination of unary theta series.

Fourier coefficients of the Millson lift

- The Millson theta lift of $F \in H^{+}_{-2k}$ has a Fourier expansion of the form

$$\Lambda_{M,k}(F,\tau) = \sum_{D \gg -\infty} a^{+}(D)q^{D} + \sum_{D<0} a^{-}(D)\Gamma(1-k, 4\pi|D|y)q^{D},$$

  where D runs only over discriminants.
- The holomorphic Fourier coefficients $a^{+}(D)$ for $D > 0$ are given by the traces

$$\sum_{Q \in \mathcal{Q}_D/\Gamma} \frac{R^{k}_{-2k}F(z_Q)}{|\overline{\Gamma}_Q|}$$

- The non-holomorphic Fourier coefficients $a^{-}(D)$ for $D < 0$ are given by the traces

$$\sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{\Gamma_Q \backslash C_Q} (\xi_{-2k}F)(z)Q(z,1)^{k}dz$$

- Thus $\Lambda_{M,k}(F,\tau)$ realises both the CM values and the cycle integrals of F as coefficients of a harmonic weak Maass form.

Generalizations

- Consider higher level N, i.e. replace $\mathrm{SL}_2(\mathbb{Z})$ by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}.$$

  We do not need N to be square free.
- Allow more general harmonic weak Maass forms F as inputs for the Millson lift (F^- may also grow exponentially at $i\infty$).
- Consider twisted theta lifts $\Lambda_{M,k,d}(F,\tau)$ for fundamental discriminants d, and obtain generating series of *twisted* traces.

A funny application

Theorem. (Alfes, Bringmann-Guerzhoy-Kane)
Let $G \in S_{2k+2}(N)$ be a newform and $F \in H^{+}_{-2k}(N)$ such that $\xi_{-2k} F = G$. Further, let $d < 0$ be a fundamental discriminant. Then

$$L(G, \chi_d, k) = 0 \quad \Leftrightarrow \quad \Lambda_{M,k,d}(F, \tau) \in M^{!}_{1/2-k}.$$

Here $\chi_d = \left(\frac{d}{\cdot}\right)$ is the Kronecker symbol.

Identities of cycle integrals

- Recall the identity of theta lifts

$$L^{k/2}_{1/2,\tau} \int_{\Gamma\backslash\mathbb{H}} (R^k_{-2k,z}F)(z)\Theta_M(\tau,z)d\mu(z) \doteq \int^{\text{reg}}_{\Gamma\backslash\mathbb{H}} F(z)\Theta_{M,k}(\tau,z)y^{-2k}d\mu(z).$$

- Thus there are two ways of computing the Fourier coefficients, which leads to the following identity of traces of cycle integrals:
- If $k \geq 0$ is even and $D > 0$ not a square, then we have

$$\sum_{Q\in\mathcal{Q}_D/\Gamma} \int_{\Gamma_Q\backslash C_Q} R^{k+1}_{-2k}F(z)dz \doteq \sum_{Q\in\mathcal{Q}_D/\Gamma} \overline{\int_{\Gamma_Q\backslash C_Q} \xi_{-2k}F(z)Q(z,1)^k dz}.$$

- Motivated by these identities, we showed that the same relations also hold without the sum in front, i.e. they hold for each individual summand (this generalizes work of Bringmann, Guerzhoy and Kane).