Mahler's measure and L-values

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Let $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ be a Laurent polynomial. In 1961 Mahler introduced the (logarithmic) Mahler measure

$$m(P) = \int_0^1 \ldots \int_0^1 \log |P(e(t_1), \ldots, e(t_n))| dt_1 \ldots dt_n$$
$$= \frac{1}{2\pi i} \int_{T^n} \log |P(z_1, \ldots, z_n)| \frac{dz_1}{z_1} \ldots \frac{dz_n}{z_n}$$

where $e(t) = e^{2\pi i t}$ and T^n is the real n-torus $S^1 \times \ldots S^1$.

- Studied by Mahler to give (lower and upper) bounds for the height and length of a polynomial.
- Soon number theoretic connections were discovered.

Polynomials in 1 variable

If $n = 1$ and $P(X) = a_0 \prod(X - \alpha_i) \in \mathbb{C}[X]$, we can use Jensen's formula to find

$$m(P) = \log|a_0| + \sum_{|\alpha_i|>1} \log|\alpha_i|.$$

- Was studied by Lehmer in order to find large primes numbers.
- If $P \in \mathbb{Z}[X]$ is monic and satisfies $m(P) = 0$, then P is a product of powers of X and cyclotomic polynomials.
- For $P \in \mathbb{Z}[X]$ $m(P)$ is the logarithm of an algebraic number.

Polynomials in 2 variables

For $n > 1$ it's still true that $m(P) = 0$ implies a certain decomposition of P in terms of cyclotomic polynomials.
The values $m(P)$ aren't logarithms of algebraic numbers anymore but they are interesting numbers.

Theorem (Smyth 1981)

$$m(1 + X + Y) = L'(\chi_3, -1)$$

where χ_n is the real odd Dirichlet character of conductor n.

- $m(1 + X + X^2 + Y) = L'(\chi_4, -1)$.
- Formulas for many other $L'(\chi_f, -1)$ exist in terms of Mahler measures.
- **Conjecture (Chinburg)**: Such formulas should exist for all f.

Deninger's work

Numerical observation by Boyd:

$$m(X + \frac{1}{X} + Y + \frac{1}{Y} + 1) = L'(E, 0)$$

where E is the elliptic curve defined by the projective closure of the zero locus of $X + \frac{1}{X} + Y + \frac{1}{Y} + 1$.
This translates to:

$$\int_0^{2\pi} \int_0^{2\pi} \log(1 + 2\cos(s) + 2\cos(t)) dsdt = 15L(E, 2) = 15 \sum \frac{a_n}{n^2}.$$

Deninger's work

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In 1997 Deninger connected the Mahler measure of P to a Deligne period of the mixed motive associated to the variety defined by P.

- Several conditions have to be posed on P: Deninger's formula works best if P does not vanish on T^n.
- Then relations between Mahler measures and L-values follow naturally from the Deligne-Beilinson conjectures.

Boyd's calculations

Boyd performed computer calculations and produced thousands of examples where P defines an elliptic curve E_P and

$$m(P) = c \cdot L'(E_P, 0),$$

with an explicit $c \in \mathbb{Q}$.

Examples: The family of polynomials $P_k(X, Y) = Y^2 + kXY + Y - X^3$ satisfies Deninger's conditions and

$$m(P_{-1}) = 2L'(E_{-1}, 0), \quad (1)$$

$$m(P_{-2}) = L'(E_{-2}, 0), \quad (2)$$

$$m(P_{-3}) = L'(E_{-3}, 0). \quad (3)$$

By Deninger's work we have

$$m(P_k) = \frac{1}{2\pi i} \int_{\gamma_k} \eta_k$$

The Deninger cycle is

$$\gamma_k = \{(X, Y) \in E_k(\mathbb{C}) : \ |X| = 1, |Y| \geq 1\}.$$

The cocycle η_k is given by the cup product of $\log|X|$ and $\log|Y|$ in Deligne cohomology.

Let $\phi_k : X_1(N_k) \to E_k$ be a modular parametrisation for E_k.

The Deninger cycle is the push-forward of a modular symbol γ'_k on $X_1(N_k)$.

The pull-back of the rational functions X, Y along ϕ_k are modular units $u = \phi^* X, v = \phi^* Y$ and

$$\phi^* \eta_k = \eta(u, v) = \log |u| d(\arg(v)) - \log |v| d(\arg(u))$$

Finally

$$m(P_k) = \int_{\phi_{k,*}\gamma'_k} \eta_k = \int_{\gamma'_k} \eta(u, v).$$

L-functions

Let $f = \sum a_n q^n \in \mathcal{M}_k(\Gamma_1(N))$. The Dirichlet series

$$L(f, s) = \sum \frac{a_n}{n^s}$$

converges for $\Re(s) > k$ and define the completed L-function

$$\Lambda(f, s) = \Gamma(s)(2\pi)^s N^{s/2} L(f, s) = N^{s/2} \int_0^\infty (f(iy) - a_0) y^s \frac{dy}{y} = N^{s/2} \mathcal{M}(f, s).$$

It has meromorphic continuation to $\mathbb{C}$ with possible poles at $s = 0, k$ and satisfies the functional equation

$$\Lambda(f, s) = i^k \Lambda(f|W_N, k - s),$$

where $f|W_N(\tau) = N^{k/2}(\tau)^{-k} f(-1/N\tau)$.

The Rogers-Zudilin method

Let $E_k = a_k + \sum_{m,n\geq 1} m^{k-1} q^{mn} = a_k + \tilde{E}_k$. To calculate the L-function of a product of Eisenstein series Rogers-Zudilin introduced the following trick.

$$\Lambda(E_k E_l, j) = \Lambda(E_k(E_l|W_1), j) \overset{\text{up to easy additive terms}}{=} \mathcal{M}(\tilde{E}_k(\tilde{E}_l|W_1), j) \tag{4}$$

$$= i^l \int_0^\infty \Big(\sum_{m_1, n_1 \geq 1} m_1{}^{k-1} e^{-2\pi m_1 n_1 y} \Big) \Big(\sum_{m_2, n_2 \geq 1} m_2{}^{l-1} e^{-2\pi \frac{m_2 n_2}{y}} \Big) y^{j-l} \frac{dy}{y} \tag{5}$$

$$= i^l \sum_{m_1, n_1, m_2, n_2 \geq 1} \int_0^\infty m_1{}^{k-1} m_2{}^{l-1} e^{-2\pi (m_1 n_1 y + \frac{m_2 n_2}{y})} y^{j-l} \frac{dy}{y} \tag{6}$$

$$\overset{y' = m_1 y / m_2}{=} i^l \sum_{m_1, n_1, m_2, n_2 \geq 1} \int_0^\infty m_1{}^{k+l-j-1} m_2{}^{j-1} e^{-2\pi (m_2 n_1 y' + \frac{m_1 n_2}{y'})} y'^{j-l} \frac{dy'}{y'} \tag{7}$$

The Rogers-Zudilin method

$$= i^{k+l} \int_0^\infty \tilde{E}_j(iy)\tilde{E}_{k+l-j}(i/y) y^{j-l} \frac{dy}{y} \tag{8}$$

Finally

$$\Lambda(E_k E_l, j) \overset{\text{up to easy additive terms}}{=} \Lambda(E_j E_{k+l-j}, k), \tag{9}$$

for $j \in \{4, \ldots k + l - 4\}$ even.
More interesting: For $j = k + l$ one of the factors is

$$\sum_{m,n \geq 1} m^{-1} e^{-2\pi mny} = \log\left(\eta(iy) e^{-2\pi y/24}\right).$$

Rogers-Zudilin method

Proposition (Diamantis-N-Strömberg)

Let χ_1, χ_2 and ψ_1, ψ_2 be pairs of non-trivial primitive Dirichlet characters modulo M_1, M_2 and N_1, N_2, respectively. Then for an integer $j \in \{1, \dots, k+l-1\}$ such that $(\chi_1 \cdot \psi_1)(-1) = (-1)^{k-j}$ we have

$$\Lambda(E_l^{\chi_1,\chi_2} \cdot E_k^{\bar{\psi}_2,\bar{\psi}_1,M_1M_2}, j) = C \cdot \Lambda(E_j^{\chi_1,\psi_2} \cdot E_{k+l-j}^{\bar{\chi}_2,\bar{\psi}_1,M_1N_2}, l)$$

where C is an explicit algebraic number.

For the case $j = k + l$ it's best to use Eisenstein series with rational Fourier coefficients. Let $N \geq 1$ and $a, b \in \mathbb{Z}/N\mathbb{Z}$, then the series

$$G_k^{a,b} = c_{a,b} + \sum_{\substack{m,n\geq 1 \\ n\equiv b \bmod N \\ m\equiv a \bmod N}} m^{k-1}q^{nm} + (-1)^k \sum_{\substack{m,n\geq 1 \\ n\equiv -b \bmod N \\ m\equiv -a \bmod N}} m^{k-1}q^{nm}$$

is in $\mathcal{M}_k(\Gamma_1(N^2))$.

Theorem (Brunault 2015)

$$\pi\Lambda^*(G_1^{a,b}G_1^{c,d} + G_1^{a,-b}G_1^{c,-d}, 0) = \int_0^{i\infty} \eta\left(g_{a,c}, g_{-d,b}\right)$$

where $g_{x,y}$ is a modular unit for $\Gamma_1(N^2)$

$$g_{x,y} = q^{B(x/N)/2} \prod_{n\geq 0}(1 - q^n q^{x/N}\zeta_N^y) \prod_{n\geq 1}(1 - q^n q^{-x/N}\zeta_N^{-y}).$$

Brunault's solution to some of Boyd's conjectures

Method for E_{-1} defined by $Y^2 - XY + Y - X^3 = 0$ of conductor 14:

- Find the pullback of X, Y under the parametrisation $X_1(14) \to E_{-1}$. It's a modular unit and so a quotient of Siegel units.
- Find modular symbol that maps to the Deninger cycle.
- Apply Brunault's theorem to get $m(P_k)$ as an L-value of a linear combination of products of Eisenstein series.
- Hope that the linear combination of products of Eisenstein series equals a multiple of f_{-1}, the newform associated to E_{-1}.

$$m(P_{-1}) = 2L'(E_{-1}, 0).$$

Mahler measures of three variable polynomials

Let $P = XYZ(X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} - 2)$ and $V(P) = \{(X, Y, Z) \in \mathbb{C}^3 : \; P(X, Y, Z) = 0\}$ be the complex affine variety defined by P.

Projective closure + Desingularisation defines a K3-surface Y with

$$L(Y, s) = \zeta(s-1)^{20} L(f_8, s)$$

for the unique newform $f_8 \in \mathcal{S}_3(\Gamma_1(8))$.

Theorem (Bertin)

$$m(P) = 4\Lambda(f_8, 3).$$

Getting the Rogers-Zudilin method to work in higher weights

Let's imagine that $V(P)$ is non-singular.

- The Deninger 2-cycle is

$$\gamma_k = \{(X, Y, Z) \in : \ |X| = |Y| = 1, |Z| \geq 1\}.$$

- Deninger's 2-cocycle is the cup product of the rational functions $\log|X|, \log|Y|, \log|Z|$ in Deligne cohomology.

Let $E_1(8)(\mathbb{C})$ be the complex points of the universal elliptic curve associated to $\Gamma_1(8)$.

$$E_1(8)(\mathbb{C}) \cong \Gamma_1(8) \times \mathbb{Z} \times \mathbb{Z} \backslash \mathcal{H} \times \mathbb{C},$$

where the action is given by $(\gamma, m, n)(\tau, z) = (\gamma\tau, j(\gamma, \tau)^{-1}(z + m\tau + n)$.

Theorem

There is a birational map $E_1(8) \to V(P)$ given by $(\tau, z) \mapsto (X, Y, Z)$ defined by

$$X = \frac{(\wp_\tau(z)) - \wp_\tau(1/2))(\wp_\tau(1/8) - \wp_\tau(1/4))^2}{(\wp_z(1/8) - \wp_\tau(1/2))(\wp_\tau(z + 1/4) - \wp_\tau(1/4))(\wp_\tau(z) - \wp_\tau(1/4))}$$

$$Y = \frac{(\wp_\tau(-z) - \wp_\tau(1/2))(\wp_\tau(1/8) - \wp_\tau(1/4))^2}{(\wp_\tau(1/8) - \wp_\tau(1/2))(\wp_\tau(-z + 1/4) - \wp_\tau(1/4))(\wp_\tau(-z) - \wp_\tau(1/4))}$$

$$Z = \frac{\wp_\tau(1/8) - \wp_\tau(1/2)}{\wp_\tau(1/4) - \wp_\tau(1/2)}.$$

What's the Deninger cycle in $E_1(8)$? Weight 3 modular symbols correspond to 2-cycles on $E_1(8)$ defined as follows. If α, β are two cusps of $\Gamma_1(8)$ let $\widetilde{\alpha\beta}$ be a path connecting them. The modular symbol $X \otimes \{\alpha, \beta\}$ corresponds to the 2-cycle

$$\{(\tau, z) : \ \tau = \widetilde{\alpha\beta}(t_1), \ z = t_2\tau, \text{ for } t_1, t_2 \in [0, 1]\}.$$

After some numerical and theoretical considerations we arrived at the modular symbol $X\{-\frac{1}{2}, \frac{1}{2}\}$.

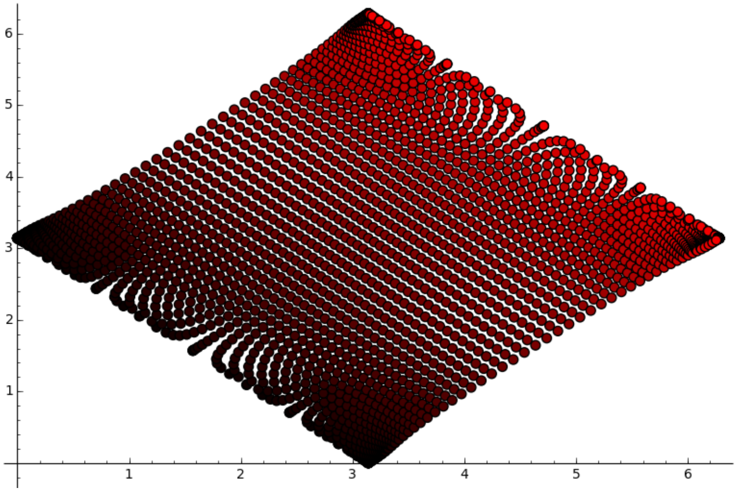

Figure: Angles of X,Y in the image of Shokurov cycle in V(P)

What's the pull-back of Deninger's cocycle?

Beilinson constructed explicit cocycles on Kuga-Sato varieties called Eisenstein symbols. The 1-cocycles correspond to Siegel modular units.

We proved that the pull-back of $\log|X| \cup \log|Y| \cup \log|Z|$ is the Eisenstein symbol $\frac{64}{3}\mathrm{Eis}^1(0,2)$. So

$$m(P) \overset{\text{should be}}{=} \frac{64}{3} \int_{X\{-\frac{1}{2},\frac{1}{2}\}} \mathrm{Eis}^1(0,2).$$

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Brunault generalised the Roger's Zudilin method to deal with integrals like this.
We finally obtain

$$m(P) = -4\pi^2\Lambda^*\left(\begin{array}{l} G^{(1)}_{3,4}G^{(2)}_{2,-6} - G^{(1)}_{3,-4}G^{(2)}_{2,6} + G^{(1)}_{3,4}G^{(2)}_{2,6} + G^{(1)}_{3,-4}G^{(2)}_{2,-6} \\ +G^{(1)}_{1,4}G^{(2)}_{2,2} - G^{(1)}_{1,-4}G^{(2)}_{2,-2} - G^{(1)}_{1,4}G^{(2)}_{2,-2} + G^{(1)}_{1,-4}G^{(2)}_{2,2} \end{array}, 0\right)$$

The linear combination of Eisenstein series in the L-function is ...

$$4f_8(8z)$$