Kudla-Rapoport divisors on unitary Shimura varieties

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Special cycles and modular forms

Special cycles on Shimura varieties associated to orthogonal and unitary groups are related to Fourier coefficients of modular forms.

- Hirzebruch–Zagier: Hilbert modular surfaces
- Gross–Kohnen–Zagier: Modular curves
- Kudla–Millson: On $\mathrm{O}(p, q)$, $\mathrm{U}(p, q)$
- Borcherds: Explicit relations
- Funke–Millson: Higher weight
- Kudla–Rapoport–Yang: Arithmetic situation
- ...

The modular curve $X(1)$

$X(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^*$ parametrizes elliptic curves over $\mathbb{C}$:

$$\begin{aligned} X(1) &\xrightarrow{\sim} \{\text{elliptic curves over } \mathbb{C}\}/\sim \\ z &\longmapsto E_z := \mathbb{C}/(\mathbb{Z}z + \mathbb{Z}). \end{aligned}$$

- $X(1) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C}),\ \ z \mapsto j(E_z)$.
- Geometry is easy.
- Arithmetic is much richer.

CM points

Recall: $[z] \in X(1)$ is called a CM point if $\mathrm{End}(E_z) \supsetneq \mathbb{Z}$.

- If $-m < 0$ is a discriminant, let

$$Z(m) = \sum_{[z] \text{ with } \mathrm{End}(E_z) \supset \mathcal{O}_{-m}} [z].$$

- $\deg(Z(m)) = H(m)$.

Theorem (Zagier)

The generating series

$$\phi(\tau) = -\frac{1}{12} + \sum_{m>0} \deg(Z(m)) q^m$$

is a (mock) modular form of weight $3/2$ (where $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$).

Slightly more general

- Consider the modular curve $X_0(N) = \Gamma_0(N)\backslash\mathbb{H}^*$.
- $Z(m)$: Heegner divisor of discriminant m on $X_0(N)$.

Theorem (Gross–Kohnen–Zagier)

The generating series

$$A(\tau) = \sum_{m \geq 0} Z(m) q^m$$

is a modular form of weight $3/2$ *with values in* $\mathrm{CH}^1(X_0(N))$.

Theorem (variant of Gross–Zagier)

For a newform $g = \sum_n b(n) q^n$ *in* $S_{3/2}(\Gamma_0(4N))$ *let* $\phi(g) = (g, A(\tau)) \in \mathrm{CH}^1(X_0(N))_{\mathbb{C}}$. *If* $-m_0 < 0$, *we have*

$$\langle \phi(g), Z(m_0) \rangle \doteq b(m_0) \cdot L'(\mathrm{Sh}(g), 1).$$

Ball quotients

Goal: Generalize this to $\mathrm{U}(n-1,1)$.

- $k/\mathbb{Q}$ imaginary quadratic of discriminant $d < 0$.
- For simplicity: d odd and $h_k = 1$.
- $(V, \langle\cdot,\cdot\rangle)$: hermitian space over k of signature $(n-1,1)$.
- $\mathcal{D} = \{z \subset V_{\mathbb{R}} : \dim(z) = 1,\ Q\,|_z < 0\}$
  $\cong \{w \in \mathbb{C}^{n-1} : \|w\|^2 < 1\}$, the complex $(n-1)$-ball.

Let $L \subset V$ be a $\mathcal{O}_k$-self-dual hermitian lattice. The group $\Gamma = \mathrm{U}(L)$ acts on $\mathcal{D}$. We consider

$$X_\Gamma = \Gamma \backslash \mathcal{D}.$$

Example: If $n = 2$ then $\mathrm{SU}(1,1) \cong \mathrm{SL}_2(\mathbb{R})$ and $\mathcal{D} \cong \mathbb{H}$.

Heegner divisors

For $x \in V$ with $\langle x, x \rangle > 0$, let $\mathcal{D}(x) = \{z \in \mathcal{D} : \; z \perp x\}$.

The special divisor on X_Γ of discriminant $m > 0$ is

$$Z(m) = \sum_{\substack{x \in L/\Gamma \\ \langle x,x \rangle = m}} \mathcal{D}(x).$$

Theorem (Kudla–Millson)

The generating series

$$A(\tau) = \sum_{m \geq 0} Z(m) q^m$$

is a modular form of weight n with values in $H^2(X_\Gamma, \mathbb{C})$.

Kudla's program

- There should be similar modularity results for arithmetic special cycles on integral models of Shimura varieties for unitary and orthogonal groups.
- Arithmetic intersection numbers of special cycles should be given by coefficients of derivatives of Eisenstein series and derivatives of L-functions.

Integral models

$\mathcal{M}_{p,q}$: moduli space (stack) for triples (A, λ, κ), where

- A: abelian scheme of dimension $p + q$,
- $\lambda : A \to A^\vee$ principal polarization,
- $\kappa : \mathcal{O}_k \to \mathrm{End}(A)$ an $\mathcal{O}_k$-action of signature (p, q).

Example

$\mathcal{M}_{1,0}$: moduli space for elliptic curves with CM by $\mathcal{O}_k$.

Define $\mathcal{M} = \mathcal{M}_{1,0} \times_{\mathcal{O}_k} \mathcal{M}_{n-1,1}$.

- It is regular, flat over $\mathcal{O}_k$, of relative dimension $n - 1$.
- We have
$$\mathcal{M}(\mathbb{C}) = \coprod_i \Gamma_i \backslash \mathcal{D}.$$

Kudla–Rapoport divisors

For $(A_0, A) \in \mathcal{M}(S)$ let $L(A_0, A) = \mathrm{Hom}_{\mathcal{O}_k}(A_0, A)$.

- There is a positive definite hermitian form on this projective $\mathcal{O}_k$-module:

$$\begin{aligned} \langle x, y \rangle &= \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \\ &\in \mathrm{End}_{\mathcal{O}_k}(A_0) \cong \mathcal{O}_k \end{aligned}$$

$$\begin{array}{ccc} A_0 & \xrightarrow{\;x\;} & A \\ {\scriptstyle \lambda_0}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\ A_0^\vee & \xleftarrow[\;y^\vee\;]{} & A^\vee. \end{array}$$

$\mathcal{Z}(m)$: moduli stack for triples (A_0, A, x), where

- $(A_0, A) \in \mathcal{M}(S)$.
- $x \in L(A_0, A)$ and $\langle x, x \rangle = m$.

$\mathcal{Z}(m)(\mathbb{C})$ is given by special divisors on the $\Gamma_i \backslash \mathcal{D}$.

An arithmetic theta lift of harmonic Maass forms

For a harmonic Maass form $f \in H^+_{2-n}$ with $f^+ = \sum_m c^+(m) q^m$ let

$$\mathcal{Z}(f) = \sum_{m>0} c^+(-m)\mathcal{Z}(m),$$

$$\Phi(z, f) = \int_{\Gamma(1)\backslash\mathbb{H}}^{reg} f(\tau)\theta_L(\tau, z)\, d\mu(\tau).$$

Theorem (B.–Howard–Yang)

$\Phi(z, f)$ is a (sub-)harmonic logarithmic Green function for $\mathcal{Z}(f)$.

- We obtain an arithmetic theta lift $H^+_{2-n} \to \widehat{\mathrm{Z}}^1(\mathcal{M})$,

$$f \mapsto \widehat{\mathcal{Z}}(f) := (\mathcal{Z}(f), \Phi(z, f)).$$

- Lift of $f_m = q^{-m} + O(1) \in H_{2-n}$ is $\widehat{\mathcal{Z}}(m) = (\mathcal{Z}(m), \Phi(z, f_m))$.

Arithmetic generating series

In view of Kudla's conjectures, look at the generating series

$$\widehat{A}(\tau) = \sum_{m \geq 0} \widehat{\mathcal{Z}}(m) q^m.$$

Theorem (B.–Howard–Kudla–Rapoport–Yang)

The generating series $\widehat{A}(\tau)$ is a modular form in $M_n(\Gamma_0(d), \chi_k^n)$ with values in $\widehat{\mathrm{CH}}^1(\mathcal{M})$.

Corollary

We can define an arithmetic theta lift on $S_n(\Gamma_0(d), \chi_k^n)$ by

$$g \mapsto \hat{\phi}(g) = (g, \hat{A}(\tau)) \in \widehat{\mathrm{CH}}^1(\mathcal{M})_{\mathbb{C}}.$$

Height pairings

Consider the height pairing $\widehat{\mathrm{CH}}^1(\mathcal{M}) \times \mathrm{Z}^{n-1}(\mathcal{M}) \to \mathbb{R}$.

Corollary

If $\mathcal{Y} \in \mathrm{Z}^{n-1}(\mathcal{M})$ is any cycle, then

$$\langle \widehat{A}(\tau), \mathcal{Y}\rangle = \sum_{m\geq 0} \langle \widehat{\mathcal{Z}}(m), \mathcal{Y}\rangle q^m \in M_n(\Gamma_0(d), \chi_k^n).$$

- For a positive definite self dual $\mathcal{O}_k$-module Λ of rank $n-1$, there is a *small CM cycles* $\mathcal{Y}_\Lambda$ on $\mathcal{M}$.

Theorem (B.–Howard–Yang+ ε)

Assume that n is even. Let $g \in S_n(\Gamma_0(d), \chi_k^n)$ and let Λ be a positive definite self dual $\mathcal{O}_k$-module of rank $n-1$. Then

$$\langle \hat{\phi}(g), \mathcal{Y}_\Lambda\rangle \doteq -\deg(Y_\Lambda) \cdot L'(g, \theta_\Lambda, s_{center}).$$

The $n = 2$ case

If $n = 2$, we have:

- $\mathrm{SU}(1,1) \cong \mathrm{SL}_2(\mathbb{R})$.
- $\mathcal{M}(\mathbb{C}) \cong \coprod \Gamma_i \backslash \mathbb{H}$, a union of modular curves of level $|d|$.
- $\Lambda \subset k$ is a fractional ideal.

Corollary

Let $g \in S_2(\Gamma_0(d))$ and let $\Lambda \subset k$ be a fractional ideal. Then

$$\langle \hat{\phi}(g), \mathcal{Y}_\Lambda \rangle \doteq -\deg(Y_\Lambda) \cdot L'(g, \theta_\Lambda, 1).$$