Siegel Modular Forms

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Overview

1. Siegel modular forms: Basic definitions and properties
2. Hecke operators and the spinor zeta function
3. Böcherer's conjecture
4. Fourier-Jacobi expansions

Holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$

- Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half plane.
- $M \in \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ via

$$z \mapsto Mz := \frac{az+b}{cz+d}.$$

- A modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ with
  - $f(Mz) = (cz+d)^k f(z)$ for all $M \in \mathrm{SL}_2(\mathbb{Z}), z \in \mathbb{H}$.
  - f is holomorphic at ∞, i.e. f has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}, \qquad a_n \in \mathbb{C}\,.$$

- If $a_0 = 0$ then f is called a cusp form.
- The spaces of modular forms and cusp forms of weight k are denoted by M_k and S_k.
- Now we replace $\mathbb{H}$ and $\mathrm{SL}_2(\mathbb{Z})$ by 'higher dimensional' analogs.

The Siegel upper half plane

- Let $g \in \mathbb{N}$ and let

$$\mathbb{H}_g = \{Z = X + iY \in \mathbb{C}^{g\times g} : Z = Z^t, Y \text{ positive definite}\}$$

be the Siegel upper half plane of genus n.

- An element $Z \in \mathbb{H}_g$ has the form

$$\begin{pmatrix} z_{11} & z_{12} & \dots & z_{1g} \\ * & z_{22} & \dots & z_{2g} \\ \vdots & & \ddots & \vdots \\ * & \dots & * & z_{gg} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1g} \\ * & x_{22} & \dots & x_{2g} \\ \vdots & & \ddots & \vdots \\ * & \dots & * & x_{gg} \end{pmatrix} + i \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1g} \\ * & y_{22} & \dots & y_{2g} \\ \vdots & & \ddots & \vdots \\ * & \dots & * & y_{gg} \end{pmatrix}$$

with Y positive definite.

- For $g = 1$ we have $\mathbb{H}_1 = \mathbb{H}$.
- For $g = 2$ we have

$$\mathbb{H}_2 = \left\{ \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathbb{C}^{2\times 2} : \mathrm{Im}(\tau), \mathrm{Im}(\tau') > 0, \mathrm{Im}(\tau)\mathrm{Im}(\tau') - \mathrm{Im}(z)^2 > 0 \right\}$$

The symplectic group

- The symplectic group of genus g is
$$\mathrm{Sp}_{2g}(\mathbb{R}) = \left\{ M \in \mathbb{R}^{2g\times 2g} : M^t J M = J \right\}, \qquad J = \begin{pmatrix} 0_g & 1_g \\ -1_g & 0_g \end{pmatrix},$$
i.e. $\mathrm{Sp}_{2g}(\mathbb{R})$ is the isometry group of the symplectic form J on $\mathbb{R}^{2g}$.
- We write
$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$
with $g \times g$ blocks $A, B, C, D \in \mathbb{R}^{g\times g}$.
- $M \in \mathrm{Sp}_{2g}(\mathbb{R})$ is equivalent to the conditions:
$$A^t D - C^t B = 1_g \text{ and } A^t C,\ B^t D \text{ are symmetric.}$$
- In particular, $\mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$.
- If $M \in \mathrm{Sp}_{2g}(\mathbb{R})$, then
$$M^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$
- Symplectic matrices have determinant 1.
- $\mathrm{Sp}_{2g}(\mathbb{R})$ acts on $\mathbb{H}_g$ by 'fractional linear transformations'
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}.$$

Siegel modular forms

- A Siegel modular form of genus g and weight k is a holomorphic function $F : \mathbb{H}_g \to \mathbb{C}$ such that
  1. $F(MZ) = \det(CZ + D)^k F(Z)$ for all $M \in \mathrm{Sp}_{2g}(\mathbb{Z}), Z \in \mathbb{H}_g$,
  2. F has a Fourier expansion of the form

$$F(Z) = \sum_{\substack{N=N^t \geq 0 \\ \text{half-integral}}} a_N e^{2\pi i \, \mathrm{tr}(NZ)}, \qquad a_N \in \mathbb{C},$$

  where the sum is over all half-integral symmetric positive semi-definite $g \times g$-matrices N.
- If

$$N = \begin{pmatrix} N_{11} & N_{12}/2 & \dots & N_{1g}/2 \\ * & N_{22} & \dots & N_{2g}/2 \\ \vdots & & \ddots & \vdots \\ * & \dots & * & N_{gg} \end{pmatrix} \text{ and } Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1g} \\ * & z_{22} & \dots & z_{2g} \\ \vdots & & \ddots & \vdots \\ * & \dots & * & z_{gg} \end{pmatrix}$$

  with $N_{ij} \in \mathbb{Z}$ then

$$\mathrm{tr}(NZ) = \sum_{i,j=1}^{g} N_{ij} z_{ij}.$$

- F is a cusp form if $a_N = 0$ whenever N is not positive definite.
- For $g = 1$ this is the usual definition of an elliptic modular form.
- For $g > 1$ the so-called Koecher principle ensures that F automatically has a Fourier expansion as in point 2 of the definition.

What is similar as in the $g = 1$ case?

- The spaces M_k^g and S_k^g are finite dimensional.
- One can construct (different types of) Eisenstein series, Poincaré series, Theta series, ...
- There is an invariant measure $d\mu_g(Z) = \det(Y)^{-(g+1)} dXdY$ on $\mathbb{H}_g$, so we can define the Petersson scalar product on S_k^g by

$$\langle F, G \rangle = \int_{\mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g} F(Z) \overline{G(Z)} \det(Y)^k d\mu_g(Z)$$

- There are natural self-adjoint Hecke operators T_m, so there is a basis of S_k^g of simultaneous Hecke eigenforms.
- One can attach different kinds of L-series to $F \in M_k^g$ which often have analytic continuation and functional equations.

The Siegel operator: from genus g to $g-1$

- Let $F \in M_k^g$ be a Siegel modular form of genus g and weight k.
- The Siegel operator is defined by

$$(F|\Phi)(Z') = \lim_{y\to\infty} F\left(\begin{pmatrix} Z' & 0 \\ 0 & iy \end{pmatrix}\right)$$

  where $Z' \in \mathbb{H}_{g-1}$.
- It defines a linear map $\Phi : M_k^g \to M_k^{g-1}$.
- This makes it possible to do inductive proofs on the genus g.
- For $g = 1$, this is just the map

$$\sum_{n=0}^{\infty} a_n e^{2\pi i n z} \mapsto a_0$$

- $F \in M_k^g$ is a cusp form if and only if $F|\Phi = 0$.
- For even $k > 2g$ the Siegel operator is surjective.
- A preimage of a cusp form $F \in S_k^{g-1}$ can be constructed via the Klingen-Eisenstein series $E_F \in M_k^g$ built from F.

Hecke operators

- For $m \in \mathbb{N}$ we consider the sets

$$\Gamma = \{M \in \mathbb{Z}^{2g\times 2g} : M^t JM = J\} = \mathrm{Sp}_{2g}(\mathbb{Z}),$$
$$\Delta_m = \{M \in \mathbb{Z}^{2g\times 2g} : M^t JM = mJ\}$$

- The double coset $\Gamma\Delta_m\Gamma$ consists of finitely many left cosets mod Γ, so we have a finite disjoint union

$$\Gamma\Delta_m\Gamma = \bigcup_{j=1}^{t} \Gamma M_j$$

with $M_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \Delta_m$.

- The Hecke operator T_m on M_k^g is defined by

$$F|T_m = \sum_{j=1}^{t} F|M_j = \sum_{j=1}^{t} \det(C_j Z + D_j)^{-k} f(M_j Z)$$

(up to some missing normalising factors).

- This defines a linear map $T_m : M_k^g \to M_k^g$.

Relations of Hecke operators

- The Hecke operators commute, i.e. $T_m T_n = T_n T_m$ for all $m, n \in \mathbb{N}$.
- T_m is self-adjoint w.r.t. the Petersson scalar product, i.e.

$$\langle T_m F, G\rangle = \langle F, T_m G\rangle$$

- In particular, M_k^g has a basis of simultaneous Hecke eigenforms.
- It holds $T_m T_n = T_{mn}$ for $(m, n) = 1$.
- For $g = 1$ we have the relation

$$T_p T_{p^\delta} = T_{p^{\delta+1}} + p^{k-1} T_{p^{\delta-1}}$$

  so it suffices to know the action of T_p for every prime p on M_k^1 to know the action of every T_m.
- For $g > 1$ there is no such simple relation, i.e. knowing the action of T_p for every prime p is not enough to know every T_m.

Multiplicative properties of Fourier coefficients

- Suppose that $F(Z) = \sum_{N \geq 0} a_N e^{2\pi i \operatorname{tr}(NZ)} \in M_k^g$ is a Hecke eigenform with $F|T_m = \lambda_m F$.
- The relations $T_m T_n = T_{mn}$ for $(m, n) = 1$ imply

$$\lambda_{mn} = \lambda_m \lambda_n \quad \text{for } (m, n) = 1.$$

- For $g = 1$ we have $a_m = \lambda_m a_1$, so the coefficients of F are multiplicative if we normalise $a_1 = 1$.
- But for $g > 1$, the coefficients a_N are labelled by half-integral matrices, so what should 'multiplicative' mean?
- The relation between the eigenvalues and the Fourier coefficients can be described using L-series.
- For $g = 1$ we have

$$a_1 \sum_{m=1}^{\infty} \frac{\lambda_m}{m^s} = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

- For $g = 1$ the multiplicative properties of λ_m are equivalent to the Euler product expansion

$$\sum_{m=1}^{\infty} \frac{\lambda_m}{m^s} = \prod_p \sum_{\delta=0}^{\infty} \frac{\lambda_{p^\delta}}{p^{\delta s}} = \prod_p \frac{1}{1 - \lambda_p p^{-s} + p^{k-1-2s}}.$$

Siegel modular forms of genus 2 and quadratic forms

- From now on, let the genus be $g = 2$ and let k be even.
- Let $F(Z) = \sum_{N \geq 0} a_N e^{2\pi i \operatorname{tr}(NZ)} \in M_k^g$.
- The indices $N = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ in the Fourier expansion represent binary integral positive definite quadratic forms $N(x, y) = nx^2 + rxy + my^2$.
- The group $\mathrm{SL}_2(\mathbb{Z})$ acts on these forms by $U.N = U^t NU$.
- $\mathrm{SL}_2(\mathbb{Z})$ acts with finitely many orbits on the set $\mathcal{Q}_D^+$ of forms N with fixed discriminant $D = r^2 - 4nm$.
- The transformation behaviour of $F \in M_k^2$ shows that $a_{U^t NU} = a_N$ for $U \in \mathrm{SL}_2(\mathbb{Z})$.

Some L-series in genus 2

- Let $F(Z) = \sum_{N \geq 0} a_N e^{2\pi i \,\mathrm{tr}(NZ)} \in M_k^2$ be a Hecke eigenform with $F|T_m = \lambda_m F$.
- The spinor (or Andrianov) zeta function of F is defined by

$$Z_F(s) = \zeta(2s - 2k + 4) \sum_{m=1}^{\infty} \frac{\lambda_m}{m^s}.$$

- The Koecher-Maass series of F is defined by

$$D_F(s) = \sum_{N > 0/\sim} \frac{a_N}{\det(N)^s}$$

  where N runs through representatives of the $\mathrm{SL}_2(\mathbb{Z})$-classes of binary integral positive definite quadratic forms.
- For N a binary integral positive definite quadratic form, we define the ray class series of F by

$$R_{N,F}(s) = \sum_{m=1}^{\infty} \frac{a_{mN}}{m^s}.$$

Results of Andrianov on the spinor zeta function in genus 2

Let $F(Z) = \sum_{N \geq 0} a_N e^{2\pi i \operatorname{tr}(NZ)} \in M_k^2$ be a Hecke eigenform with $F|T_m = \lambda_m F$.

Theorem

The spinor zeta function $Z_F(s) = \zeta(2s-2k+4)\sum_{m=1}^{\infty} \frac{\lambda_m}{m^s}$ has the Euler product expansion $Z_F(s) = \prod_p Q_{p,F}(p^{-s})^{-1}$ where

$$Q_{p,F}(t) = 1 - \lambda_p t + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})t^2 - \lambda_p p^{2k-3} t^3 + p^{4k-6} t^4.$$

Theorem

Let $D < 0$ be a fundamental discriminant and let $N_1, \ldots, N_h$ be representatives of the $\mathrm{SL}_2(\mathbb{Z})$-classes of binary integral positive definite quadratic forms of discriminant D. Then

$$\left(\sum_{j=1}^{h} a_{N_j}\right) Z_F(s) = L_D(s+k-2) \sum_{j=1}^{h} R_{N_j,F}(s)$$

where $L_D(s) = \zeta(s)L\big(\big(\frac{D}{\cdot}\big), s\big)$ denotes the Dedekind L-function of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ and $R_{N,F}(s) = \sum_{m=1}^{\infty} \frac{a_{mN}}{m^s}$ is a ray class series. $R_{N,F}(s)$ and thereby also $Z_F(s)$ have analytic continuation and satisfy a functional equation under $s \mapsto 2k-2-s$.

Böcherer's conjecture

- Let $F \in M_k^2$ be a Hecke eigenform.
- Let $D < 0$ be a fundamental discriminant and $\chi_D = \left(\frac{D}{\cdot}\right)$ be the corresponding quadratic character (Kronecker symbol).
- The twisted spinor zeta function of F is defined by

$$Z_F(\chi_D, s) = \prod_p Q_{p,F}(\chi(p)p^{-s})^{-1} = L(\chi_D^2, 2s - 2k + 4) \sum_{m=1}^{\infty} \frac{\chi_D(m)\lambda_m}{m^s}.$$

- **Böcherer's conjecture:** There is some constant c_F depending only on F such that

$$Z_F(\chi_D, k-1) = c_F |D|^{-k+1} \left(\sum_{j=1}^{h} a_{N_j} \right)^2 .$$

- The conjecture is trivially true if k is odd, since then both sides are 0.
- Böcherer proved his conjecture in the case that F is a Klingen-Eisenstein series or a Saito-Kurokawa lift of an elliptic modular form $f \in S_{2k-2}^1$.
- For Saito-Kurokawa lifts we have

$$Z_F(\chi_D, s) = L(\chi_D, s - k + 1) L(\chi_D, s - k + 2) L_f(\chi_D, s)$$

  and the result then follows easily from the formula of Waldspurger for $L_f(\chi_D, k-1)$.
- For k even, F a cusp form and F not a lift, not much is known.

On the Fourier-Jacobi expansion

- Let $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathbb{H}_2$ and $F \in M_k^2$.
- We rewrite the Fourier expansion:

$$F(Z) = \sum_{\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right) \geq 0} a(n,r,m) e^{2\pi i(n\tau + rz + m\tau')}$$

$$= \sum_{m \geq 0} \left(\sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq 4nm}} a(n,r,m) e^{2\pi(n\tau + rz)} \right) e^{2\pi i m\tau'} = \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i m \tau'}.$$

- This is called the Fourier-Jacobi expansion of F, and $\phi_m : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is the m-th Fourier-Jacobi coefficient of F.
- The transformation rules of F imply some transformation rules for ϕ_m, namely ϕ_m is a Jacobi form of weight k and index m.
- It is therefore useful to study Jacobi forms: for example, estimates of their Fourier coefficients gives estimates for the growth of the coefficients of F.
- In genus 1, if $F \in M_k^1$ is a nonzero eigenform, then $a_m = \lambda_m a_1$ implies $a_1 \neq 0$.
- **Conjecture:** If $F \in M_k^2$ is a nonzero eigenform, then $\phi_1 \neq 0$.
- This is only known for Saito-Kurokawa lifts (trivial) and for small weights by direct computations of M_k^2.