Regularized Determinants & KLFs

1. Regularized Determinants

Bsp. (Regularized Product)

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots := ?$$

Consider Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\mathrm{Re}(s) > 1)$$

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log(n)}{n^s}$$

Formally: $$e^{-\zeta'(0)} \text{ „=" } \prod_{n=1}^{\infty} n$$

Laurent exp. at $s=0$:

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2}\log(2\pi) \cdot s + O(s^2)$$

$\rightsquigarrow$ Define $$1 \cdot 2 \cdot 3 \cdot \ldots := e^{-\zeta'(0)} = \sqrt{2\pi}$$

More generally: $0 =: \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$
eigenvalues of operator L

$$\prod_{j=1}^{\infty} \lambda_j := ?$$

[$L = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\det(L) = \prod \lambda_j$]

Define spectral zeta function

$$\zeta_L(s) := \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s} \qquad \mathrm{Re}(s) \gg 0$$

If

- convergence for $\mathrm{Re}(s) \gg 0$
- merom. continuation to $\mathbb{C}$
- $\zeta_L(s)$ is holomorphic at $s=0$

then

$$\underset{\sim\sim\sim}{\det{}^{*}(L)} = e^{-\zeta'_L(0)} \text{ „=" } \prod_{j=1}^{n} \lambda_j$$

regularized determinant of L.

__1. Ex.__ $X = (0,1)$ 

$$L = -\frac{\partial^2}{\partial x^2}$$ with Dirichlet condition (D)

$\det'(L) = ?$ Solve $L\psi_j = \lambda_j \psi_j$

$\rightsquigarrow$ $\psi_j = A \sin(\sqrt{\lambda_j}\, x) + B \cdot \cos(\sqrt{\lambda_j}\, x)$

- $\psi(0) = \psi(1)$ $\Rightarrow$ $B = 0$
- (D) $= \psi(1) = A \sin(\sqrt{\lambda_j}\, x) \overset{!}{=} 0$

$\rightsquigarrow$ Formula for eigenvalues

$$\lambda_j = \pi^2 j^2 \qquad (j \in \mathbb{N})$$

$$\rightsquigarrow \quad \zeta_L(s) = \sum_{j=1}^{\infty} \frac{1}{(\pi^2 j^2)^s} = \pi^{-2s} \underline{\zeta(2s)}$$

2. Ex. $E_\tau \cong \mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}}$

$$\frac{|dz|^2}{\mathrm{Im}(\tau)} \rightsquigarrow \Delta_\tau = -\mathrm{Im}(\tau)\cdot\frac{\partial^2}{\partial z\,\partial\bar z}$$

$\det'(\Delta_\tau) = ?$

For $(c,d)\in\mathbb{Z}^2$: eigen fct.

$$\psi_{c,d} = \exp\left(\frac{\pi}{\mathrm{Im}(\tau)}\left(c(\bar\tau z-\tau\bar z)+d(z-\bar z)\right)\right)$$

$$\lambda_{c,d} = \frac{\pi^2}{\mathrm{Im}(\tau)}\cdot|c\tau+d|^2$$

$$\rightsquigarrow \zeta_{\Delta_\tau}(s) = \sum_{\lambda_{c,d}\neq 0}\frac{1}{\lambda_{c,d}^{\,s}} = \pi^{-2s}\sum_{\substack{c,d\in\mathbb{Z}^2\\(c,d)\neq(0,0)}}\frac{\mathrm{Im}(\tau)^s}{|c\tau+d|^{2s}}$$

$$= \pi^{-2s}\,\zeta(2s)\cdot 2\cdot E_\infty(\tau,s)$$

(1863)

<u>KLF</u> At $s=0$

$$E_\infty(\tau,s) = 1 + \log\left(|\Delta(\tau)|^{1/6}\cdot \mathrm{Im}(\tau)\right)s + O(s^2)$$

$$\Rightarrow \quad \det{}'(\Delta_\tau) = e^{-\zeta'_{\Delta_\tau}(0)}$$

$$\overset{\text{KLF}}{=} 4\,\mathrm{Im}(\tau)\,|\Delta(\tau)|^{1/6}$$

$$= 4\,\mathrm{Im}(\tau)\,|\eta(\tau)|^4.$$

2. Reg. Det for hyperbolic Riemann surfaces

$X = \Gamma \backslash \mathbb{H}$ $\qquad z = x + iy \in \mathbb{H}$

- $$ds^2_{hyp} = \frac{dx^2 + dy^2}{y^2}$$ hyperbolic metric

  is <u>singular</u> near p_j and e_j

- $$\Delta_{hyp,X} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

- Question: $\det{}'(\Delta_{hyp,X})$?

- <u>Problem</u> If $n > 0$, then

  $\Delta_{hyp,X}$ has <u>continuous</u> spectrum

  (• KLF possible?)

(joint with G. Freixas)

<u>Motivation</u>:

<u>Thm</u> (Deligne, Quillen, Gillet–Soulé)

- ω_X
- $\lambda(\mathcal{O}_X)$

There is isometry

$$\overline{\lambda(\mathcal{O}_X)}^{\otimes 12} \xrightarrow{\sim} \langle \omega_X, \omega_X \rangle$$

Quillen metric on $\lambda(\mathcal{O}_X)$

$$\lambda(\mathcal{O}_X) = \det H^0(X, \mathcal{O}_X) \otimes \det H^1(X, \mathcal{O}_X)^{-1}$$

is following renormalization

$$\|\cdot\|_Q = \underline{\det{}'(\Delta_{hyp,X})}^{-1/2} \cdot \|\cdot\|_{L^2}$$

holds in case $n=0$, $m=0$

Idea: to define $\det'(\Delta_{hyp,X})$

$ds^2_{hyp,\varepsilon}$ nice ε-truncated metric on X

$\rightsquigarrow$ $\Delta_{hyp,\varepsilon}$ = Laplacian

Def.

$$\det'(\Delta_{hyp,X}) = \lim_{\varepsilon \to 0} \frac{\det'(\Delta_{hyp,\varepsilon})}{f(\varepsilon)}$$

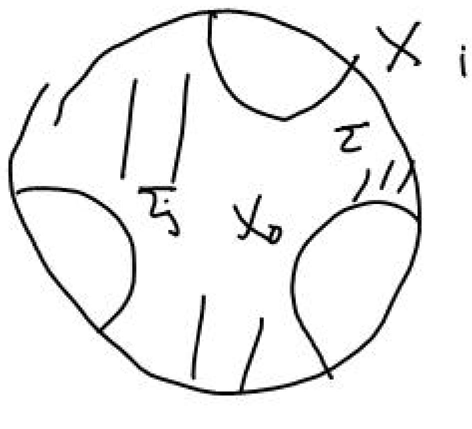
$$f(\varepsilon) := \varepsilon^{\frac{1}{6} \sum_{\omega_j < 0} \left(1 - \frac{1}{\omega_j}\right)^2} \cdot \log\left(\frac{1}{\varepsilon}\right)^{\frac{\#\text{cusps}}{3}}$$

Theorem $\det'(\Delta_{hyp,\varepsilon})$ has expression in terms of other regularized determinant, as, e.g.

- Δ_{hyp} on X_i with (D) at boundary ε
- $\Delta_{hyp,\varepsilon}$ on X_i with (D)
-
-

Key-ingredient: Mayer-Vietoris formula

Classically: Burghelea-Friedlander-Kappeler



Δ on X
Δ_i on X_i with (D)

$$\frac{\det'(\Delta)}{\prod_i \det'(\Delta_i)} = \det'(\underbrace{R_{hyp}}_{\text{information from boundary}})$$

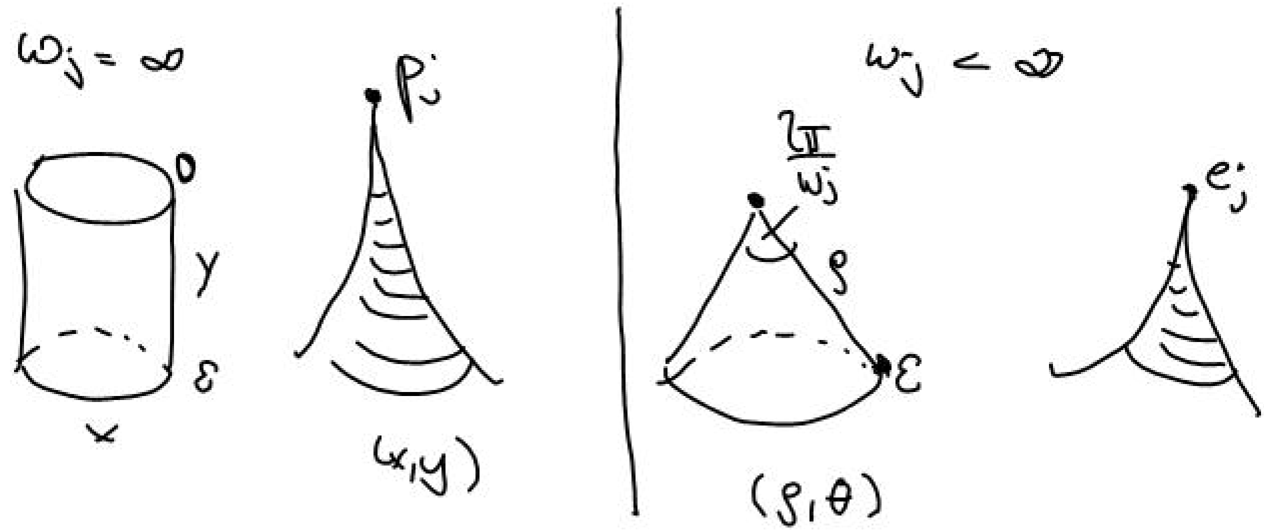
Thm. Quillen metric defined via

$$\|\cdot\| = \left(\det{}'\Delta_{hyp,X}\right)^{-1/2} \cdot \|\cdot\|_{L^2}$$

gives desired analogue of ARR - isometry..

For applications we want to compute the regularized determinants explicity...

$$\det{}'(\Delta_{hyp}, X_j) = ?$$



$$\det{}'(\Delta_{hyp,X_j}) = e^{-\zeta'_{X_j}(0)}$$

where

$$\zeta_{X_j}(s) = \sum_{k>0} \sum_j \frac{1}{\left(\frac{1}{4} + r_{j,k}^2\right)^s}$$

where $r_{j,k}$ is a zero of $F_k^{X_j}(r)$

$$F_k^{X_j}(r) = \begin{cases} K_{ir}(2\pi|k|\varepsilon), & X_j \text{ cusp} \\ P^{-|k|\omega_j}_{-\frac{1}{2}+ir}(\cosh(\varepsilon)) & X_j = \text{cone} \end{cases}$$

Result

$$\zeta_{x_j}'(0) = \begin{cases} -\zeta(0)\log(\varepsilon) + \dfrac{\zeta(-1)\,4\pi}{\varepsilon} + g(\varepsilon) & \text{„}x_j = \text{cusp"} \\[2ex] -\dfrac{1}{6}\left(\omega_j + \dfrac{1}{\omega_j}\right)\log(\varepsilon) - 2\zeta(0)\left(\log(\omega_j) - 1\right) \\ \quad 2\omega\,\zeta'(-1) + \displaystyle\sum_{k=1}^{\infty}\sum_{m=2}^{\infty} \frac{m}{(m+1)(m+2)}\,\zeta(1+m, k\omega_j) + g(\varepsilon) \end{cases}$$

$g(\varepsilon), h(\varepsilon) \longrightarrow 0$ as $\varepsilon \to 0$.