Restriction of Coherent Eisenstein Series

1. Classical Eisenstein Series

$$G_k(z) := \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^k}$$

$k\geq 4$ even
$z\in\mathfrak{H}$

$$= 2\zeta(k)\cdot\underbrace{\left(\frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus(0,0)\\ \gcd(c,d)=1}} \frac{1}{(cz+d)^k}\right)}$$

$$E_k(z) := \ldots \in M_k(1)$$

$$E_k(z) = \sum_{\gamma\in\Gamma_\infty\setminus\Gamma_0(1)} 1\big|_k\gamma$$

$$\Gamma_\infty = \left\{\pm\begin{pmatrix}1 & u\\ & 1\end{pmatrix} : u\in\mathbb{Z}\right\}\subset\Gamma_0(N)$$

<u>Rmk</u>: For N prime, $k=2$, one can define $E_{2,N}(z)\in M_2(N)$.

Adelically, we can define $E_k(z)$ as follows.

Let $G = GL_2$, P = parabolic of G

F local field

$$P(F) = \left\{ \begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} \in GL_2(F) \right\}.$$

$\mathbb{A}$: adele of $\mathbb{Q}$

$$\Delta : P(F) \longrightarrow \mathbb{C}^\times$$
$$\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} \longmapsto \left| \frac{a_1}{a_2} \right|.$$

Let $I(s) := \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} (\Delta^{s+\frac{1}{2}})$ for $s \in \mathbb{C}$

Want to choose $\Phi_s \in I(s)$.

All $\Phi_s \in I(s)$ satisfy

$$\Phi_s\left(\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} \cdot g\right) = \Delta\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix}^{s+\frac{1}{2}} \Phi_s(g).$$

Write $\Phi_s = \otimes_v \Phi_{s,v}$

If $\underset{\substack{\| \\ p}}{v} < \infty$, we know that

$$GL_2(\mathbb{Q}_p) = P(\mathbb{Q}_p) \cdot \underbrace{GL_2(\mathbb{Z}_p)}_{\text{compact}}$$

Let $\Phi_{s,v} : GL_2(\mathbb{Q}_p) \longrightarrow \mathbb{C}$ be the unique function s.t.

$$\Phi_{s,v}\Big|_{GL_2(\mathbb{Z}_p)} = \text{identity}$$

$$\Phi_{s,v}\left(\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} \cdot g_v \right) = \left| \frac{a_1}{a_2} \right|_v^{s+\frac{1}{2}} \Phi_{s,v}(g_v)$$

$$\forall\, g_v \in Gl_2(\mathbb{Q}_p)$$

For $v = \infty$, one has $Gl_2(\mathbb{R}) = P(\mathbb{R}) \cdot SO_2(\mathbb{R})$

Define $\Phi_{s,\infty} : Gl_2(\mathbb{R}) \longrightarrow \mathbb{C}$ by

$$\Phi_{s,\infty}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) = e^{i \cdot k \cdot \theta}$$

$$\Phi_{s,\infty}\left(\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} \cdot g_\infty \right) = \left| \frac{a_1}{a_2} \right|^{s+\frac{1}{2}} \Phi_{s,\infty}(g_\infty)$$

$$\forall\, g_\infty \in Gl_2(\mathbb{R}).$$

Now for this $\Phi_s = \otimes \Phi_{s,v}$, define the Eisenstein series

$$E(g, \Phi_s) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \Phi_s(\gamma g).$$

$\forall\ g \in G(\mathbb{A})$.

Let $g_z = \otimes_v g_{z,v} \in G(\mathbb{A})$ where $z = x+iy \in \mathfrak{H}$ and

$$g_{z,v} = \begin{cases} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & v < \infty \\ \frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} & v = \infty \end{cases}$$

Exercise:

$$E(g_z, \Phi_s)\Big|_{s=\frac{k-1}{2}} = y^{k} \cdot E_k(z).$$

2. Coherent Eisenstein series

$G = SL_2, \quad B = P \cap SL_2$

$K = \mathbb{Q}(\sqrt{D})$ for $\quad \chi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ nontrivial, quad

suppose $D<0$.

$\chi : B(\mathbb{A}) \longrightarrow \mathbb{C}^\times$

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \longmapsto \chi(a)$$

Then $\quad I(\chi, s) := \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \left(\chi \cdot \Delta^s \right)$

has a nice description at $s=0$.

$$(V, Q) = (K, u \cdot \mathrm{Nm}) \quad u \in \mathbb{Q}^\times$$

p prime, $\quad V_p$ local quadratic space

$\quad \varepsilon_p(V_p)$ Hasse invariant.

$p = \infty$,	signature of V_∞	$(2,0)$	$(0,2)$
	$\varepsilon_\infty(V_\infty)$	1	-1

p split in K $\qquad \varepsilon_p(V_p) = 1$

p non-split in K $\qquad V_p = K_p, \quad Q = \mathrm{Nm}$ or $u_p \cdot \mathrm{Nm}$ $\quad u_p \notin \mathrm{Nm}\, K_p$

$\qquad \varepsilon_p(V_p) \qquad +1 \qquad -1$

Let $\mathcal{C} = (\mathcal{C}_p)$ be a collection of local quad spaces where $\epsilon_p(V_p) = 1$ for all but finitely many p.

Def $\mathcal{C}$ is $\left\{\begin{matrix} \text{coherent} \\ \text{incoherent} \end{matrix}\right\}$ if $\prod_{p \le \infty} \epsilon_p(V_p) = \begin{cases} +1 \\ -1 \end{cases}$.

$S(\mathcal{C}_p)$ Schwartz func.

$\circlearrowleft$ Weil rep from ψ_p

$G(\mathbb{Q}_p)$

$$S(\mathcal{C}_{\mathbb{A}}) = \bigotimes_{p \le \infty} S(\mathcal{C}_p)$$

$$\lambda_{\mathcal{C}} : S(\mathcal{C}_{\mathbb{A}}) \longrightarrow I(0, \chi)$$

$$\varphi = \bigotimes_p \varphi_p \longmapsto g \mapsto \Big(\bigotimes_p \omega_p(g_p)\varphi_p\Big)(0)$$

Let $\Pi_{\mathcal{C}}$ = Image of $\lambda_{\mathcal{C}}$. This is an irred rep of $G(\mathbb{A})$.

$$I(0,\chi) = \Big(\bigoplus_{\mathcal{C} \text{ coherent}} \Pi_{\mathcal{C}}\Big) \oplus \Big(\bigoplus_{\mathcal{C} \text{ incoherent}} \Pi_{\mathcal{C}}\Big)$$

For $\Phi \in I(0,\chi)$, let $\Phi_s \in I(s,\chi)$ be the func defined by $\Phi_s(g) = \Phi(g) \cdot |a(g)|^{2s}$

One can generalize the construction before by replacing $\mathbb{Q}$ w/ tot. real field F, and K/F CM extension.

Suppose K/F is unramified @ all finite places. $d := [F:\mathbb{Q}]$

∂_F: different of F

$\mathcal{N}$: sq-free integral ideal in F s.t. all prime factors are ~~~~ unramified in K

<u>Thm</u> (Yang '05) There is a function $E^*(z, s, \Phi^{\mathcal{N}})$ on $\mathfrak{H}^d \times \mathbb{C}$ s.t.

(1) As a function of s, $E^*(z, s, \Phi^{\mathcal{N}})$ is meromorphic w/ finitely many poles & holomorphic along $\mathrm{Re}(s) = 0$.

(2) As a function of $z = (z_1, \dots, z_d) \in \mathfrak{H}^d$ $E^*(z, s, \Phi^{\mathcal{N}})$ is a Hilbert mod form of wt 1.

(3) Define $\varepsilon(\mathfrak{N}) := (-1)^{\omega(N m \mathfrak{N})} \cdot \prod_{v | \partial_F} \chi_v(\partial_F)$

then $\mathcal{E}^*(\mathfrak{z}, 0, \Phi^{\mathfrak{N}}) \neq 0$

$$\Updownarrow$$

$$\varepsilon(\mathfrak{N}) = 1$$

in which case

$\mathcal{E}^*(\mathfrak{z}, 0, \Phi^{\mathfrak{N}})$ is a hol. Hilbert modular form of wt 1.

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3. Diagonal Restriction of $\mathcal{E}^*(\mathfrak{z}, 0, \Phi^{\mathfrak{N}})$

Special case

Fix N odd prime

$d_1, d_2 < 0$ fund. disc

$D = d_1 d_2 > 0$ disc of real quad field F

$K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) / F$ CM

Assume
- $(d_1, d_2) = 1$
- $\left(\frac{d_1}{N}\right) = \left(\frac{d_2}{N}\right) = -1$ $\quad (*)$

~~Th~~ Then

$$f_{d_1, d_2, N}(z) := E^*((z,z), 0, \Phi^N)$$

$$= L(0, \chi_{K/F}) + 2\sum_{m=1}^{\infty} a_m(d_1, d_2, N) q^m$$

is a hol ell mod form of wt 2
level N, trivial neben
where

$$a_m(d_1, d_2, N) = \sum_{\substack{t = \frac{a + m\sqrt{D}}{2} \in \mathcal{N} \\ |a| < m\sqrt{D}}} \rho_{K/F}(t \cdot \mathcal{N}^{-1})$$

$$\rho_{K/F}(\beta) = \#\{\mathcal{A} \subset \mathcal{O}_K : Nm_{K/F}\mathcal{A} = (\beta)\}$$

Yang's Question : For a fixed N,
what is the span of

$$\mathcal{F} := \left\{ f_{d_1, d_2, N} : \begin{array}{l} d_1, d_2 < 0 \text{ fund disc} \\ \text{satisfying } (*) \end{array} \right\}.$$

Thm: Let $M_2^{+,0}(N) \subset M_2(N)$ be the $\mathbb{C}$-span of $E_{2,N}(\tau)$ and eigenforms $G \in S_2(N)$ s.t. $L(G,1) \neq 0$.

Then $M_2^{+,0}(N)$ is the span of $\mathbb{F}$.